Bachelor of Engineering in Computer Science and Engineering Examination, 2018

(1st Year, 2nd Semester)

Mathematics - III

Time : Three hours Full Marks : 100

(50 marks for each part)

Use a separate Answer-Script for each part

PART - I

Answer ***any five*** questions 5×10=50

1. a) Define group. Let G be a group and ξ be the identify of G. Suppose that for $x, y, z \in G$, $xyz = \xi$ holding G. Show that $yzx = \xi$. Does it follow that $yxz = \xi$? Justify your answer. 2+1+2

   b) Define cyclic group. Show that even cyclic group is commutative. Is the converse true ? Justify your answer. 1+2+2

2. a) Define normal subgroup of a group. Let H and k be two normal subgroups of a group G such that $H \cap K = \{\xi\}$. Show that ab = ba for all $a \in H$ and $b \in K$. 2+3

   b) Define quotient group of a group. Show that the quotient group of a commutative group is also commutative. Is the converse true ? Justify your answer. 2+2+1

[Turn over

3. a) State and prove Lagrange's Theorem for group. 2+3

   b) Define centre of a group. Let G be a noncommutative group and Z(G) be the centre of G. Show that $|Z(G)| \leq \frac{1}{4}|G|$. 1+4

4. a) State and prove first isomorphism theorem of group. 2+3

   b) Define $GL(x, \mathbb{R})$ and $SL(x, \mathbb{R})$. Show that $GL(x, \mathbb{R})/_{SL(n\mathbb{R})} \simeq \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R}\{0\}$, the set of non zero real numbers. 2+3

5. a) Define subring of a ring. With proper justification give an example of a ring R without identify but having a subring S with identity. 1+4

   b) Define integral domain and field. Show that every finite integral domain is a field. 2+3

6. a) Define ideal of a ring. Give an example of a left ideal in a ring R which is not a right ideal of R. Also give an example of a right ideal of R which is not a left ideal R. 1+2+2

   b) Define prime ideal and maximal ideal of a commutative ring R. Show that every maximal ideal of R is a prime ideal of R. 2+3

14. a) Let U and W be two subspaces of a finite dimensional vector space V over a field F. Then prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

   b) When are two subspaces of a vector space said to be complement of each other ? Prove that every subspace of a finite dimensional vector space possesses a complement.

   Find a complement subspace of U in $\mathbb{R}^3$ where U = L{(1, 2, 1), (2, 1, 3)} 5+5

PART - II

Answer ***any five*** questions 10×5=50

All questions carry equal marks.

8. a) Express the following determinant as a square of a determinant and find its value :

$$\begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ca \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix}$$

b) Show that

$$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(bc+ca+ab)^3.$$ 5+5

9. a) Find the inverse of the following matrix :

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

and hence solve the set of equations

$$x-y+2z=1$$
$$x+y+z=2$$
$$2x-y+z=5$$

b) If $A=\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, show that $A^3=A^{-1}$ 6+4

10. a) Define rank of a matrix. Determine the rank of the following matrix :

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & 4 & -6 \\ 0 & 0 & 5 & -2 \\ 3 & 6 & 8 & -1 \end{bmatrix}$$

b) Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

and verify that for a symmetric matrix, with unequal eigen values the eigen vectors are mutually orthogonal. 4+6

11. a) State and prove Cayley-Hamilton Theorem.

b) Find the characteristic equation of the matrix.

$A=\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence compute A^{-1}. Also find the matrix represented by

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I.$$ 5+5

12. a) Define vector space and give one example. Let S be the set of all solutions of the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0, \ a_{ij} \in \mathbb{R}.$$

Show that S is a subspace of $\mathbb{R}^3$. Also, find its dimension.

b) In a vector space V over a field F, show that

i) $O\,\alpha = \theta$ for all $\alpha \in V$

ii) $C\,\theta = \theta$ for all $C \in F$

iii) $(-1)\alpha = -\alpha$;

iv) $C\alpha = \theta \Rightarrow C = O$ or $\alpha = \theta$ 6+4

13. a) If U, W be two subspaces of a vector space V over a field F, then prove that their linear sum U + W form a subspace of V and is the smallest subspace of V containing the subspaces U and W.

b) Prove that the set S={(1,1,0), (1,0,1),(0,1,1)} is a basis set for the vector space $\mathbb{R}^3$. Show that the vector (1,1,1) may replace any one of the vectors of the set S to form a new basis of $\mathbb{R}^3$, but the same is not true for the vector (3, 1, 2). 5+5

7. a) Let R be a commutative ring with identity. Show that $R[x]/_{\langle x\rangle} \simeq R$. Hence show that <x> is a prime ideal of $\mathbb{Z}[x]$ but not a maximal ideal of $\mathbb{Z}[x]$. 3+2

b) Define field extension. Let F/K be a field extension. Define the degree or dimension of F/K. Find $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]$ and $[\mathbb{C}:\mathbb{R}]$. 1+2+2