Mathematical Study of Socio-Ecological-Economic Interaction: Sustainability Perspective

Thesis submitted for the award
of the degree of

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by

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Declaration

I, Santanu Bhattacharya, solemnly declare that this thesis represents my own work which has been done after registration for the degree of PhD at Jadavpur University and has not been previously included in any thesis or dissertation for the purpose of earning a degree, diploma, or any other credential.

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This is to certify that the thesis entitled "**Mathematical Study of Socio-Ecological-Economic Interaction: Sustainability Perspective**" submitted by **Shri Santanu Bhattacharya**, who got his name registered **(Index No.: 51/19/Maths./26)** on 29/08/19 for the award of PhD (Science) degree of Jadavpur University, is based upon his work under the supervision of myself and co-supervision of Prof. Biswajit Sarkar of Yonsei University, Seoul, South Korea and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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This thesis is dedicated to my parents

Smt. Tanushree Bhattacharya

&

Shri Anil Bhattacharya

for their endless love, support and encouragement.

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Contents

1 Introduction

1

Introduction

1.1 Socio-ecological-economic interactions

The intricate linkages and interdependencies between social, ecological, and economic systems are called socio-ecological-economic interactions [1]. These interactions recognize that humans, societies, and the economy are inextricably linked to the natural environment in which they live [2]. Understanding and controlling these connections is critical for long-term growth and the well-being of people and the environment.

Understanding socio-ecological-economic interconnections is essential for developing comprehensive and sustainable policies that balance human demands and economic growth with environmental and ecological system maintenance. It necessitates multidisciplinary methods and collaborative efforts from various stakeholders, including governments, corporations, non-governmental organisations, and communities.

1.2 Fishery: A sustainable livelihood

The fishery is a socio-ecological interaction where human interacts with nonhuman species. Global fishing significantly contributes towards the sustainable development goals (SDG), which the United Nations are accomplishing [3, 4]. In particular, the inclusion of global fisheries services can help to meet the Sustainable Development Goals of eradicating extreme poorness (SDG 1), no appetite (SDG 2), pure water and sanitation (SDG 6), sustainable consumption and production (SDG 12), and life on land (SDG 15). Millions of people living along coastlines rely on the world's fisheries for a living and supporting national economies [5]. They also serve as a safety net for some of the world's poorest people, supplying them with food and cash revenue, particularly during times of crisis. Sustainable fisheries contribute to a country's prosperity by providing jobs in the fishing, processing, and related service industries and subsistence-based activities at grassroots levels [6].

Another fascinating part of aquaculture that enhances a nation's socio-ecological-economic culture is recreational fishing. Recreational fishing or "anglin" is the term used to describe fishing when the main goal is not to provide sustenance or make money via the sale or trade of fishing products but to engage in the activity for leisure, relaxation, and enjoyment [7]. Recreational fishing has the potential to play an important role in ecotourism, as it offers a sustainable and nature-focused activity that draws tourists to explore and appreciate the natural environment [8]. Ecotourism contributes to recreational fishing in the following ways: catch and release fishing tours, watching with fishing, conservation-oriented fishing tours, fishing expeditions in protected areas, educational fishing tours, fishing in national parks and reserves, fishing for invasive species management, photography and observational fishing [9]. Such a recreational fishing industry can contribute significantly to revenue generation, benefiting local economies, businesses, and conservation efforts. Although revenue generation related to recreational fishing is well understood in developed countries, it is significantly less understood in developing nations. Due to its dualist economy, which combines elements of both the developed and developing worlds, South Africa is a useful microcosm for applying economic evaluation frameworks Potts et al. [10]. The first stage in using recreational fishing as a tool for economic growth is comprehending participation rates and duality's role in related economic activity. A total of 1320 face-to-face and online surveys were utilised to estimate the annual monetary expenditure by recreational anglers, and Social Accounting Matrix analysis was performed to simulate the economic activity connected with this spend. An estimated 1,327,633 people participated in recreational fishing, which supported 94,070 full-time jobs and generated US$2.2 billion in economic activity annually [10]. However, less than 10% of economic activity benefited lower-income households, indicating a divide between the top two sectors in this dualist economy.

1.3 Overexploitation in fisheries

Overfishing is the main factor contributing to the global loss of numerous fish and other aquatic resources [11]. Pollution, habitat loss, and environmental change can also impact these resources, but drastic fishing frequently has an equal or even higher impact [12]. Overexploitation, explained as fishing at a pace larger than the species can replace itself via growth and reproduction [13], is a fundamental issue that fishery managers have battled to regulate for many years. If the mismatch continues long enough, population decline is unavoidable. Several instances of decline illustrate this phenomenon, including the collapse of northern cod stock in eastern Canada, the devastation of certain rockfish and groundfish populations on the US west coast, the North Sea's collapse of cod, whiting, and haddock stocks, and the prolonged decline and recent partial recovery of cod, haddock, and flounder stocks in New England [14].

Many aquatic species go extinct due to over-harvesting, and many more are currently in threat [15, 16, 17]. According to Polidoro et al. [18], in the last five decades, two marine mammals, namely the Japanese Sea Lion (Zalophus japonicus) and the Caribbean Monk Seal (Monachus tropicalis), have become extinct primarily due to drastic harvesting activities. Many scientists today claim that overfishing is the immediate cause of the loss of the white abalone, a completely oceanic species [19]. Using COSEWIC's (Committee on the Status of Endangered Wildlife in Canada) classifications for threatened species, Venter et al. [20] quantified threats facing 488 species in Canada. Overexploitation is to blame for the demise of 32% of those species. As a result of Mijkherjee et al. [21]'s identification, 39 regional fish species would disappear from West Bengal's natural environment that includes Butterfish or pabda fish *Ompok pabo*, Kuria labeo *Labeo gonius*, & tire track eel Mastocembelus armatus. Fisheries collapse can also have traumatic socio-economic effects. In Canada, the cod fishery collapse caused drastic changes in the social dynamics of remote areas [22, 23] resulting in the mass layoff of over ten thousand fishery-related individuals [24]. The energy flow of the food chain may change as fish species gradually disappear. The majority of fish taken are ambidextrous predators. They eat zooplankton, but bigger fish eat them. As a result of overfishing, which has lowered ecosystem services, top-down and bottom-up cascading impacts have been seen [25, 26].

1.3.1 Is overfishing inescapable?

The problem of overfishing can be successfully tackled through collaborative efforts aimed at its proper management [27]. When Fish and other aquatic animals are captured from oceans or lakes at a rate that surpasses their ability to reproduce and grow, this leads to overfishing. This can lead to a decrease in the number of fish, causing problems for the balance of underwater habitats and affecting the people who depend on fishing for their livelihood. The issue of overfishing is prevalent in numerous locations, yet feasible solutions can be discovered. Here are a couple of key elements to reflect upon:

(i) *Sustainable harvesting strategies [28]:* A paramount aspect in ensuring fish's long-term health and reproductive viability is the strict enforcement of rules designed for their protection. This means making rules about how much fish can be caught, what kind of equipment and methods can be used for fishing, how big the fish should be before they can be caught, and creating protected areas in the ocean where fish can have babies and increase their population.

(ii) *Superior governance of fisheries [29]:* Proper fishery management is necessary to prevent the depletion of fish populations. This means gathering and studying information about fish in the water, keeping track of fishing, and changing the rules and limits on fishing based on what we know. Governments, scientists, and fishing communities must work together to create good plans for managing fishing. Policy and regulations should be used to bring fishing capacity to sustainable levels, including the wise use of subsidies and the eradication of illicit, unreported, and unregulated (IUU) fishing. The regulatory authority should implement the proper strategies to minimise the harvesting effort and its effects on ecosystem services. Taxing landed fish [30], fishing vessels [31], setting a fishing quota [32], and fishing days [33] are some common and effective ways to lessen harvesting pressure. In this thesis, the sole controlling factor we consider is the tax on landed fish.

(iii) *Collaboration on a global scale [34]:* Lots of fish move from one place to another and go across country borders, so countries need to work together to take care of and control the fishing activities. International agreements and organisations can work together to ensure that countries that share fish care for them well.

(iv) *Cognition and responsible decision-making [35]:* Teaching people about the value of fishing in a way that doesn't harm the environment, and encouraging them to choose seafood that is responsibly caught, can make a big difference. Supporting fishing practices that are certified as sustainable and avoiding catching too many of certain species can create economic incentives for fisheries to adopt sustainable practices.

(v) *Innovation and technological support [36]:* Improved technology can aid in the improvement of fishing techniques and reduce the unintentional capture of unintended marine species. Introducing innovative concepts such as specialised fishing equipment, real-time data monitoring, and improved tracking systems can play a vital role in minimising the environmental damage caused by fishing.

Although overfishing is still a big problem, some fishing areas have recovered by using sustainable methods of managing fish populations. By putting these actions into practice and working together, we can reduce and possibly even undo the harm caused by overfishing. This will help ensure that marine resources can thrive in the long run.

1.4 Disease dynamics in fishery

The health of aquatic animals received very little attention in epidemiological studies until recently, which concentrated on human and terrestrial animal systems. In the waters, for instance, viruses are the most prevalent type of life. Based on approximations, every millilitre of marine water contains an abundance of virus-like particles [37] and 1023 viral infections occur in the marine environment per second [38]. Researching diseases in marine species is crucial to safeguard a priceless nutrition and recreational resource because of the relationship between peoples and the marine environment [39].

Table 1.1: Symptoms and causes of some fish infections.

Disease	Symptoms	Cause
Ulcers [40]	Pinky white open wounds with a white margin that are occasionally secondary infected by fungus and other bacteria.	Polluted water or an abnormally high pH level. In unfavourable circumstances, minor scratches may become infected.
Cloudy eyes [41]	Whole surface or lens of the eye becomes clouded and opaque. The mucus may accumulate on the skin's surface.	Water contamination, vitamin deficiency. Digenetic flukes, such as Diplostomum, might create problems on rare instances.
White spots [41]	Small white patches on the skin, fins, and gills, roughly the size of a salt grain.	It's stress-related. bad or inappropriate water conditions, shifting temperature, and overall bad husbandry are usually to blame.
Fungus [42]	Freshwater fish are affected by fuzzy growths on their skin and fins.	These are usually secondary infections that attack wounds left by ulcers and parasites, with white spots being the most common.
Fin rot [43]	Fins that are deteriorated with a pale pinky-white margin and blood in the fin tissue.	Most fish contains bacteria, frequently activated by stress from unfavorable water conditions. If the water is dirty, nipped fins may develop secondary infections, and fungus, that may also infect certain wounds.
Swim bladder [44]	Fish find it challenging to swim to the surface or to the deeper parts of the water.	Occasionally brought on by bad water quality. Selected breeding has a genetic problem.
Lymphocystis [45]	On the skin and fins, the virus causes crusty, grey-white lumps to appear. They sometimes take on the colour of the basal skin and may affect freshwater or marine fish. Clusters of swollen cells make up these masses.	Stress, unscientific regulating, or subpar water can all cause the condition, which is crucial. While not displaying any symptoms, some fish may still contain the virus.
Hole in the head [46]	Sores on the head. Small holes found along the line that goes around the jaw and eyes. Fish can become lighter and lose their natural color. Can become tired or lacking in energy.	Cichlids, like oscars and discus, are more likely to get this fish disease. The parasite Hexamita sp. often causes it. Hexamitosis is another name for this condition.
Argulus (Fish lice) [47]	Tiny dark oval lice around the pectoral fins. Fish might rub against things to try to get rid of the parasite on their body. Red, irritated spots where the lice have attached.	Argulus spp. cause direct harm to their host's integument through their connection and nourishing instruments. This harm can result from either dopey activities or chemical emit.
Gill mites [48]	Blushing around the gills. Angle may pant for discuss at the surface. Open gill covers. Angle may scratch against objects.	Gill bugs are ectoparasites meaning they live on the outside surface of their have. The bugs are undetectable to the bare eye and nourish on the blood and tissue of the fish's gills.

The utilization of exotic species has been pivotal in the advancement of aquaculture [49], such as in the aquatic culture of Atlantic salmon in Chile and the land-based systems for tilapia and sea bass in the UK. New cultural systems and practises have emerged due to the diversity and quantity of species being generated, and these practises have the potential to impact the establishment and spread of pathogens as well as the occurrence of disease [50]. For instance, polyculture, which involves maintaining multiple species on the same property, creates the possibility of disease transfer between species [51]. Fish raised in farms may come into contact with viruses from wild aquatic species that they wouldn't ordinarily encounter. Aquaculture could introduce unknown pathogens to wild fish populations. Tuna farming in southern Australia serves as an illustration for both situations. When juvenile tuna are harvested in the open access fisheries and brought ashore, they come into touch with parasites to which they are typically not exposed or only minimally exposed, which causes the disease to appear [52, 53, 54]. Additionally, imported frozen fish is used as feed for tuna, which is highly likely how the pilchard herpes virus, which caused outbreaks in fish populations, was introduced [55]. The movement of live fish for aquaculture, food, and ornamental purposes, along with the influence of climate change, has played a significant role in broadening the geographic distribution of various aquatic animal species and their infections [56]. Additionally, climate change has made it easier for diseases to emerge by promoting host-switching [57].

The expansion of the aquaculture industry coincides with an increase in disease outbreaks, which has a detrimental impact on the yield, financial success, and long-term viability of the aquaculture sector globally. To fulfil rising demand, farming practises have been transformed from vast to super-intensive, resulting in abrupt disease epidemics [58]. Other significant factors of growing aquatic fish disease include global warming, climate change, and industrial contaminants [59, 60]. Bacterial infections in aquaculture have been shown to impact economic and social growth in several nations [61]. A sound infection control plan can drastically cut down on financial loss. Table 1.1 lists some of the most destructive diseases in fisheries and their symptoms and causes. An illustration of those diseases is shown in Fig. 1.1.

Figure 1.1: The most common fish diseases. Picture courtesy: simplyaquarium.com.

1.5 Bioeconomics

Bioeconomics is a field that combines economics, ecology, and natural resource regulation to study how the economy and the environment interact with each other [62]. The basic idea of bioeconomics is that the economy is part of the earth's natural environment and depends on it for resources and facilitation [63]. Essential ideas and principles in bioeconomics include carefully using resources, understanding the value of nature's benefits, managing natural resources, dealing with problems caused by businesses, and working together across different fields [64].

Bioeconomics is vital for managing and conserving fisheries. Fishery bioeconomics looks at how to fish in a profitable way and preserves fish populations for the future [65]. It considers the relationship between the environment and the fishing industry. A fishery can be defined as a stock or stocks of fish and businesses that can exploit them [66]. A fleet of similar boats from a single port can exploit a single fish stock in a relatively simple system. It can also be more sophisticated with ships from different ports harvesting fish from several populations that are environmentally connected [67]. The following are some critical features of bioeconomics in fishery management:

(i) *Acquire the optimal path [68]:* Bioeconomics supports figuring out the best way to catch fish by considering how fast they grow and reproduce and how much it costs to fish in different ways. This means deciding between making money now and taking care of the environment for the future to get the most value from fish resources.

(ii) *Sustainable fishing strategies [69]:* This analysis helps in creating superior harvesting strategies for fishing, like catch limits and specific times for fishing. This is done by studying how different ways of managing fisheries affect the environment and economy. It seeks to stop catching too many fish and ensure there are enough for the future while also thinking about the money and jobs for fishing communities.

(iii) *Commercial rationale [70]:* Bioeconomics studies how money and financial rewards can affect the way people fish. This study examines how fishing subsidies, taxes, and market systems affect how much money fishing can make and how well fish resources are protected. By making sure that making money and being sustainable go hand in hand, bioeconomics can promote responsible ways of fishing.

(iv) *Model formulation & simulation [71]:* Scientists and government officials often use computer models to study and predict how fish populations and fishing will change over time. These models use information about living things, money, and ways of managing things to see how different choices affect things and help people make decisions.

(v) Socio-economic interactions [72]: In bioeconomics, it is recognized that fishing has effects on the environment and other people who are not fishing. Examples include catching unwanted fish, harming habitats, and affecting people not directly involved in fishing.

It aims to have these extra costs and benefits when making decisions, ensuring that fishing policies consider wider environmental and social impacts.

(vi) *Resource evaluation [73]:* Bioeconomics too includes the valuation of fishery assets and the biological system administrations they give. Financial valuation procedures, such as cost-benefit investigation and market-based approaches, offer assistance in evaluating the financial worth of fisheries and their commitments to neighbourhood economies, nourishment security, and social values. This data is vital in decision-making forms and fishery administration.

By and large, bioeconomics gives an intriguing approach that combines biological and financial viewpoints to address the challenges of fishery administration.

1.5.1 Maximum sustainable yield (MSY)

Fisheries managers use the term maximum sustainable yield (MSY) to refer to the maximum level of fishing that can be absorbed over the long run without jeopardizing the ability of the fish population to populate itself [74]. It is the biggest capture that can be taken from a stock of fish or other marine species over an uncertain period without draining the stock [75]. The population can maintain itself indefinitely by somatic growth, spawning, and recruitment, making it the biggest catch that still enables this. The purpose of managing a fishery to realize MSY is to strike an adjustment between optimizing the fishery's efficiency and guaranteeing the fish populace's long-term supportability [76]. It points to supplying the most extreme benefits from the fishery while anticipating overfishing and keeping up the environmental keenness of the marine environment. Accomplishing MSY includes deciding the ideal level of fishing effort or catch that permits the fish populace to develop and replicate at its greatest rate [77].

Discussions of the connections between effort, harvest, and stock size are made possible by the biological study. However, it is vital to comprehend the degree of effort produced under specific conditions to understand how a commercial fishery operates [78]. This is what a bioeconomic model is meant to do. Most people who engage in commercial fishing do so for financial gain. It is feasible to create a model that can aid in forecasting anticipated levels of effort and output by including information on prices, costs, and how the profit level will fluctuate with output. The one presented in this thesis is relatively straightforward and is based on the sustainable yield curve. Even though it is straightforward, it can be used to introduce the fundamentals of fisheries bioeconomics.

1.5.2 Maximum economic yield (MEY)

Maximising a fishery's sustainable catch, or maximum sustainable yield (MSY), has been a frequent objective in global fishery management. While this goal maximizes a fishery's gross value of production, it does not guarantee that the fishery is maximizing financial gains. MEY, a point associated with a jointly occurring level of sustainable catch, fishing effort, and stock biomass that maximises the economic yield of a harvested stock in a fishery, is used to determine the optimum rate of exploitation if the management goal is to maximise the economic benefit from the fishery. In 1911, Warming [79] (followed by Gordon [80], Scott [81], Andersen [82]) introduced this idea. By creating steady-state formulae for dynamic problem formulations, several study [83, 84] made a substantial contribution to the continued development of the notion.

MEY indicates that it goes beyond an anticipated catch level, unlike its name. In order to maximize earnings, the fishery must effectively utilize boat capital and other resources in combinations that minimize harvest costs at the MEY catch level [85]. To minimize the fishing cost, boat must utilize the ideal equipment, engine power, gasoline, hull size, and manpower. In other words, the fishery cannot be over-capitalized [86]. Therefore, in this case, managing a fishery at MEY has the potential to achieve "fishery-level productivity”, but achieving such productivity also demand “vessel-level productivity” (vessels generate in a way that maximizes profit) and “administration efficiency” [87].

1.5.3 Societal revenue

The term "Societal revenue" has garnered considerable attention among researchers, particularly in the context of the fishery [88, 89, 90, 91]. It serves as a crucial indicator, representing the total accumulation of revenue generated from all fishing-related activities. This includes not only the income earned by fishers but also the revenue obtained by regulatory agencies involved in managing and overseeing fishing practices, as well as the income derived from recreational fishing. This comprehensive measure takes into account the financial gains originating from various aspects of the fishing industry, working together to contribute significantly to the overall economic impact on society. As a result, it provides valuable insights into the broader implications of fishing-related economic activities and their effects on the well-being of communities and society as a whole.

1.5.4 Profit at equilibrium

A business is in an equilibrium state when it earns its maximum profit, which is typically denoted by Π. The difference between total cost (TC) and total revenue (TR) is the profit. As a result, it can be expressed as $\Pi = \mathrm{TR} - \mathrm{TC}$.

Both stock size and effort can be used to characterise the fishery’s equilibrium profitability. In this context, when profits at equilibrium are represented as a function of the stock, the total revenue (TR) is determined by the product of price and the equilibrium harvest, while the total cost (TC) is calculated as the product of the unit cost of harvest and the equilibrium harvested stock. Notably, the unit cost of harvest exhibits a decline with increasing catchability and stock size. On the other hand, when equilibrium profits are expressed as a function of effort, TR is the result of the product between price and the equilibrium harvested stock, and TC is determined by the product of imposed effort and the unit cost per unit of effort, which is considered constant.

1.5.5 Demand-supply phenomenon

Due to the strong demand for fishery goods, the employment of sophisticated fishing gear and vessel technology, and expanding trade across the oceans, the world's fisheries have risen significantly during the past fifty years [92]. From 89.6 million tonnes in 2016 [93] to the largest amount ever, 96.4 million tonnes in 2018 [94], worldwide fish production (inland plus marine) has significantly risen. Modern fishery management faces complex problems as a result of this high demand. For instance, they aim to maintain the health of the aquatic ecosystem while planning for long-term sustainable fish production. On the other hand, arrangements are made to collect enough fish biomass to satisfy market demand. The objective is to increase fish consumption to 21.4 kg by 2030, up from 9.9 kg per person in the 1960s, according to Food and Agriculture Organization [95]. The current over-harvesting issues in fisheries and aquaculture are predicted to get worse as a result of the demand for fish to feed the world's rising population's needs for protein and amino acids. Monitoring demand for fisheries products and adjusting production targets poses a new challenge for fishery management strategy. It is difficult for the management authorities to keep the cost of fish and fisheries products reasonable. To maintain the stock availability, price regulation is even more important in developing and impoverished nations. According to many research [96, 97], there is an inverse relationship between fish price and accessible fish stocks. Surprisingly, the price of high-valued fish in the UK market changes greatly depending on how much fish is landed [98]. Due to variations in the fish's availability, there is a large disparity in the market price of the in-demand Hilsa fish (it Tenualosa ilisha) across South Asian nations. For the betterment of society, dynamic price regulation based on the current demand and supply for fish may therefore offer insights.

According to classical economic theory [99], an essential commercial factor in any bioeconomic study is maintaining the balance between the landed biomass (or supply quota) and market demand. Fig. 1.2 shows how the interactions of demand (red curve), and supply (black curve) determine the price. First, we define the equilibrium price, referring to the price, where the quantity demanded is equal to the quantity supplied. The market equilibrium position is represented by the point (P_E, Q_E) in Fig. 1.2, where P_E and Q_E stand for the equilibrium price and quantity, respectively. At this moment the quantity demanded is equals the quantity supplied. The question then arises, why is this the equilibrium price? Because a price that is higher or lower than equilibrium price is subjected to the forces that will push it towards the equilibrium price. Similar to a ball in a bowl, the ball always returns to its stable position. The stability in price is found solely at the equilibrium price, where buyers compete against other buyers, and sellers compete against other sellers. In a simplified manner, this can be compared

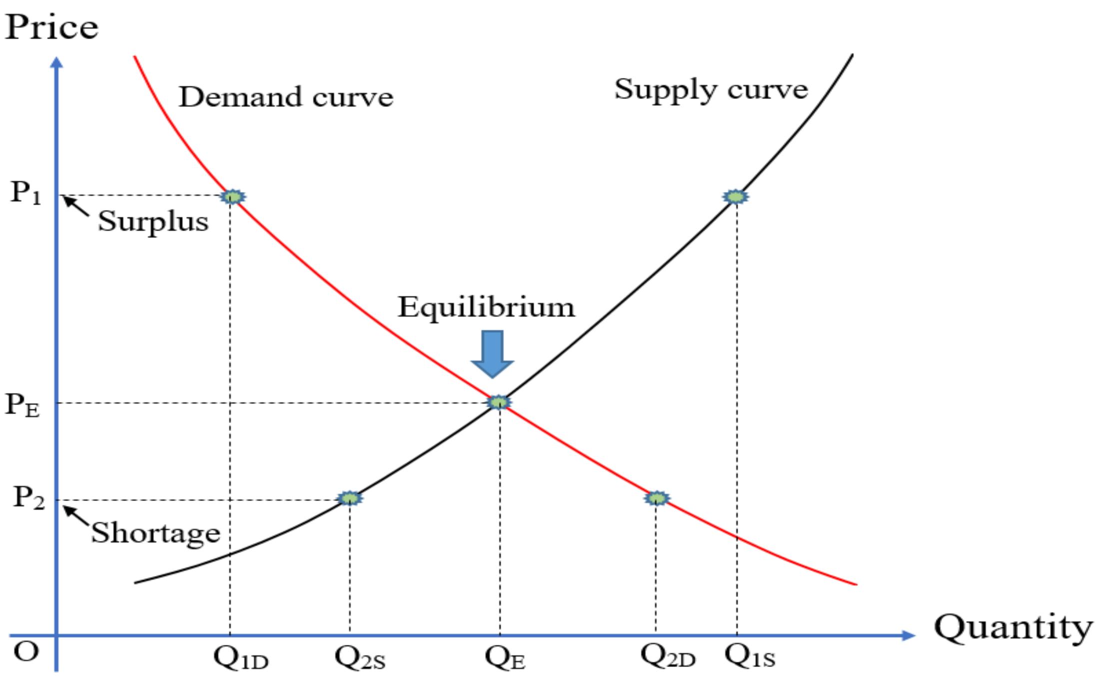

Figure 1.2: This figure shows how the intricacy between demand and supply determine the market equilibrium.

to an auction, where the highest bidder acquires the item, and the seller who sets the lowest price makes the sale. For instance, if the current price (P_1) is above the equilibrium price (P_E), there will be a surplus, with quantity supplied (Q_{1S}) exceeding quantity demanded (Q_{1D}). This surplus prompts sellers to lower their prices, to outperform competitors and increase sales. The price will continue to decline until reaching equilibrium. Conversely, if the price (P_2) is below the equilibrium (P_E), a shortage occurs, as quantity demanded (Q_{2D}) surpasses quantity supplied (Q_{2S}). This scarcity leads buyers to bid up the price, and sellers, finding ready buyers, are incentivized to raise prices. The price will continue to rise until equilibrium is restored. Thus, any price other than the equilibrium price will be influenced by buyer and seller incentives, eventually driving the price towards equilibrium. Thus, only the equilibrium price is stable.

1.6 Ecotourism

Securing a steady stream of funding to meet maintenance expenditures is always a crucial issue for the effectiveness of biodiversity preservation and ecosystem stewardship [100]. Although there may be numerous issues, ecotourism is a promising way to do this [101]. Ecotourism is defined as leisure travel that introduces visitors to complex and fascinating ecosystems and the cultures and customs that go along with them [102]. Initially developed in the late 1980s, ecotourism gained prominence in 2002 as part of the UN’s "International Year of Ecotourism” [103]. Ecotourist visits the location to take in the local biodiversity, historical context, and cultural activities [104]. Ecotourism can also boost local economies by generating revenue and employment, inspiring residents to care for the environment towards sustainable development [105]. Sustainable tourism encompasses various eco-friendly activities like boating, whale and dolphin watching, snorkelling, scuba diving, fishing, and more, all designed to minimize negative impacts on the environment [106]. Therefore, theoretical investigation of the foundations of effective ecotourism is an essential research issue since it is one of the key ways the native people can achieve sustainable prosperity and preserve biodiversity and the natural ecology.

Tourism as a whole is one of the modern world's fastest-growing businesses, and it is becoming increasingly crucial to economies all over the world [107, 108]. According to Ceballos-Lascurain et al. [109] & Lee and Iwasa [110], ecotourism is one form of travel that emphasizes a location's natural history, biodiversity, ecosystems, and geography as well as its cultural assets. Due to the growing popularity of ecotourism, the obligations of local communities to operate tourist attractions in a sustainable and environmentally responsible manner has come under increased scrutiny [104, 111]. Fishery-based ecotourism has increasingly gained attention in recent years. For example, marine mammal watching in Australia [112]. Boat-based dolphin watching at Chilika Lake, Odisha, India [113]. Recreational fishing has become a significant source of income for the coastal people in countries like Greece [114] Taiwan [115].

1.7 Mathematical modelling

Mathematical modeling is a process by which a real-world problem can be described in the language of mathematics. The concept of modeling is used in all fields such as engineering, economics, physics, chemistry, environmental science, etc. In this thesis, we apply mathematical modelling to understand the (i) socio-ecological-economic interaction of aquatic species (in particular fish), (ii) transmission of aquatic disease dynamics, (iii) the relationship between fish and fishers, (iv) demand-supply dynamics of open market, (v) some suitable policies that can help to maintain the overexploitation, (vi) a sustainable method to find maximum revenue, and (vii) the effect of ecotourism in aquatic socio-economic and environmental sustainability.

In general, there are two categories of mathematical models used in socio-ecological-economic system: (1) deterministic models that take into account non-random rate flows in a population or any system variable; and (2) stochastic models that take into account probabilities in the movements between the compartments of the model, such as the probability of a susceptible fish becoming infected, the probability of catching mature fish, or the probability of a harvested stock becoming vulnerable [116]. Each of this category mainly focused on the biological interaction between fish species, and the sustainable harvesting strategies. This thesis is based on a deterministic socio-ecological-economic system.

1.8 Basic mathematical tools

In this section, we present some basic definitions, theorems and mathematical tools that have been used throughout this thesis.

Definition 1.1. ***(Dynamical system [117])*** *A dynamical system is characterized as a set of states (the phase space) whose trajectory is determined by an evolution rule, where the trajectory is a function of a single parameter, typically represented as time.*

Definition 1.2. ***(Deterministic system [117])*** *A dynamical system is classified as deterministic when every state within the phase space corresponds to a unique subsequent state, meaning that the evolution rule of the system is a well-defined function, mapping each state to a unique succeeding state.*

In deterministic systems, for each time t, the evolution rule is a mapping from the phase space to the phase space given by

$$\psi(p,t) \equiv \psi_t(p) : U \longrightarrow U,$$

where $t \in \mathbb{R}$ is the continuous time variable, U is the phase space, $p(t) = \psi_t(p_0)$ denotes the position of the system at time t that started at p_0. Moreover, we assume that $t \geq 0$ and at $t = 0$, $\psi_t(x_0) = p_0$.

Definition 1.3. ***(Orbits or trajectories [117])*** *An orbit or trajectory refers to the sequence of states that follows from or leads to a specific initial state. The positive or forward orbit, in particular, is defined as the following set of subsequent states*

$$\Gamma_p^+ \equiv \{\psi_t(p) \ : \ t \geq 0\}.$$

Likewise, the negative or backward orbit consists of sequences of states that, as per the evolution rule, lead back to the initial state. If the function ψ_t is injective, the negative orbit can be represented by the following set

$$\Gamma_p^- \equiv \{\psi_t(p) \ : \ t \leq 0\}.$$

Alternatively, there may be situations where multiple prior points could result in the same p.

Summing up, the full orbit of a point p is given by $\Gamma_p = \Gamma_p^+ \cup \Gamma_p^-$.

Definition 1.4. ***(Invariant set [118])*** *A set Θ is said to be invariant under an evolution rule ψ_t if*

$$\psi_t(\Theta) = \Theta, \textit{ for all } t.$$

Consequently, for every p belonging to the invariant set Θ, $\psi_t(p)$ also lies within Θ for any t. Hence, the entire orbit of any point p within the set Θ will remain within Θ. Furthermore, a set Θ is termed forward invariant if $\phi_t(\Theta) \subset \Theta$ for all $t > 0$.

Definition 1.5. ***(Autonomous system of differential equations [119])*** *A set of differential equations of the form*

$$\dot{p} = f(p)$$

where $p \in \mathbb{R}^n$ and vector field $f : E(\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is said to be autonomous if f does not depend on t explicitly. Unless stated otherwise, we will assume

$$f \in C^1(E) = \{\textit{Set of all continuously differentiable functions on } E\}.$$

Definition 1.6. ***(Initial value problem [118])*** *An autonomous system of differential equations is referred to as an initial value problem (IVP) when it meets the initial condition*

$$p(t_0) = p_0.$$

Therefore, an IVP for an autonomous system of differential equations is expressed by

$$\begin{cases} \dot{p} = f(p), \\ p(t_0) = p_0. \end{cases} \tag{1.1}$$

Definition 1.7. ***(Lipschitz function [118])*** *Suppose G is an open subset of $\mathbb{R}^n$. A function $f : G \to \mathbb{R}^n$ is considered Lipschitz if, for all $p,\ q \in G$, there exists a positive real constant M satisfying the following condition*

$$|f(p) - f(q)| \le M|p - q|.$$

Theorem 1.8. ***(Picard-Lindelöf existence and uniqueness [118])*** *Suppose that for $p_0 \in \mathbb{R}^n$ there is real number $b > 0$ such that there is a closed ball $B_b(p_0)$ and $f : B_b(p_0) \to \mathbb{R}^n$ is Lipschitz with constant M. Then the IVP (1.1) has a unique solution $p(t)$ for $t \in [t_0 - a, t_0 + a]$ whenever the condition*

$$a = \frac{b}{K}, \textit{ where } K = \max_{p \in B_b(p_0)} |f(p)|$$

satisfied.

Definition 1.9. ***(Equilibrium solutions [117])*** *An equilibrium solution (also known as a steady state solution, fixed point, or critical point) of the system (1.1) is a constant solution $\bar{p}$ that fulfills the condition*

$$f(\bar{p}) = 0.$$

Definition 1.10. ***(Linearization [117])*** *For the system (1.1), we assume $f \in C^1(E)$ and $\bar{p}$ is an equilibrium point. Then the linearization of $\dot{p} = f(p)$ at the equilibrium $\bar{p} \in E$ is the system of differential equations*

$$\dot{q} = Df(\bar{p})q,$$

where

$$q(t) = p(t) - \bar{p} \text{ and } Df(\bar{p}) = \begin{pmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} & \cdots & \frac{\partial f_1}{\partial p_n} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial p_2} & \cdots & \frac{\partial f_2}{\partial p_n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial f_n}{\partial p_1} & \frac{\partial f_n}{\partial p_2} & \cdots & \frac{\partial f_n}{\partial p_n} \end{pmatrix}_{p=\bar{p}}.$$

The matrix $Df(\bar{p})$ is called the Jacobian matrix or variational matrix of f at $\bar{p}$.

Definition 1.11. ***(Generalized eigenspaces [118])*** *The equilibrium solutions of system (1.1) are classified by their generalized eigenspaces according to the sign of the real part of the eigenvalues of the variational matrix $Df(\bar{p})$. Let, σ_n, $n \in \mathbb{N}$ be the eigenvalues associated with the equilibrium $\bar{p}$ of the system (1.1). Then*

- E^u = *Unstable eigenspace spanned by the eigenvectors of the eigenvalues σ_n with*
$$Re(\sigma_n) > 0.$$
- E^c = *Center eigenspace spanned by the eigenvectors of the eigenvalues σ_n with*
$$Re(\sigma_n) = 0.$$
- E^s = *Stable eigenspace spanned by the eigenvectors of the eigenvalues σ_n with*
$$Re(\sigma_n) < 0.$$

Therefore, the complete eigenspace E with respect to the equilibrium $\bar{p}$ is given by the following direct sum:

$$E = E^u \oplus E^c \oplus E^s.$$

Definition 1.12. ***(Hyperbolic equilibrium [117])*** *An equilibrium $\bar{p}$ of system (1.1) is hyperbolic if none of the eigenvalues of $Df(\bar{p})$ is zero or purely imaginary. In this case E^c is empty. Hyperbolic equilibrium can be categorized into following three classes.*

1. ***Sink:*** *An equilibrium $\bar{p}$ of system (1.1) is a sink if all of the eigenvalues of $Df(\bar{p})$ have negative real parts. In this case, $E = E^s$ and the equilibrium is called stable. Sink can be classified as stable node or stable focus.*
    - ***Stable node:*** *If the eigenvalues are negative real then the sink is called a stable node.*
    - ***Stable focus:*** *If the eigenvalues are complex conjugates with negative real part then it is called a stable focus.*

2. ***Source:*** *An equilibrium $\bar{p}$ of system (1.1) is a source if all of the eigenvalues of $Df(\bar{p})$ have positive real parts. In this case, $E = E^u$ and the equilibrium is called unstable. Source can be classified as unstable node or unstable focus.*

    - ***Unstable node:*** *If the eigenvalues are real and positive then a source is called an unstable node.*
    - ***Unstable focus:*** *If the eigenvalues are complex conjugates with positive real part then it is called an unstable focus.*

3. ***Saddle****: An equilibrium $\bar{p}$ of system (1.1) is saddle if it is hyperbolic but not a sink or a source. Here, $E = E^s \oplus E^u$. A saddle point is also an unstable equilibrium.*

Definition 1.13. ***(Non-hyperbolic equilibrium [120])*** *An equilibrium $\bar{p}$ of system (1.1) is non hyperbolic or degenerate if at least one of the eigenvalues of $Df(\bar{p})$ have zero real part. In this case E^c is non empty.*

- ***Center:*** *It is a non hyperbolic equilibrium where eigenvalues are complex conjugates with zero real part.*

Definition 1.14. ***(Local stability [120])*** *An equilibrium solution $\bar{p}$ of (1.1) is termed locally stable if for each positive ε there exists a positive δ such that every solution $p(t)$ of* (1.1) *with initial condition $p(t_0) = p_0$ and $||p_0 - \bar{p}|| < \delta \;\Rightarrow\; ||p(t) - \bar{p}|| < \varepsilon$ for all $t \geq t_0$, where $||.||$ is the Euclidean norm. An equilibrium solution that fails to be locally stable is referred to as unstable.*

Definition 1.15. ***(Local asymptotic stability [120])*** *An equilibrium solution $\bar{p}$ of (1.1) is said to be locally asymptotically stable if it is locally stable and if there exists a $\sigma > 0$ such that $||p_0 - \bar{p}|| < \sigma \;\Rightarrow\; \lim_{t\to\infty} ||p(t) - \bar{p}|| = 0$.*

Definition 1.16. ***(Instability [120])*** *An equilibrium solution $\bar{p}$ of (1.1) is called unstable if it is not stable.*

Theorem 1.17. ***(Hartman-Grobman theorem [119])*** *If $\bar{p}$ is a hyperbolic equilibrium point of the system (1.1), then there is a homeomorphism h (i.e., h is a continuous, injective mapping with a continuous inverse) defined on some neighborhood $\Omega_{\bar{p}}$ in $\mathbb{R}^n$, locally taking orbits of the nonlinear system $\dot{p} = f(p),\; p \in \mathbb{R}^n$ to those of the linear system $\dot{q} = Df(\bar{p})q,\; q \in \mathbb{R}^n$, where $q = p - \bar{p}$. The mapping h maintains the direction of orbits and can be selected to maintain the time parameterization as well. Additionally, if the mapping h is a homeomorphism, then the stability (or instability) of the linear system implies local asymptotic stability (or lack thereof) of the nonlinear system.*

Theorem 1.18. ***(Routh-Hurwitz criteria [120])*** *Given the polynomial,*

$$P(\sigma) = \sigma^n + a_1\sigma^{n-1} + \ldots + a_{n-1}\sigma + a_n, \tag{1.2}$$

where the coefficients a_i are real constants, $i = 1,2,...,n$. n Hurwitz matrices are defined by using the coefficients of $P(\sigma)$ as

$$H_k = \begin{pmatrix} a_1 & 1 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & 1 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ . & . & . & . & \dots & . \\ 0 & 0 & 0 & 0 & \dots & a_k \end{pmatrix}, \; k = 1,2,...,n,$$

where $a_k = 0$ if $k > n$. All the roots of the polynomial $P(\sigma)$ will have negative real part if and only if the determinants of all k Hurwitz matrices are positive, i.e., $det(H_k) > 0,\ k = 1,2,...,n$. Following are the Routh-Hurwitz criteria for $n = 2$ and 3.

- $n = 2 \; : \; a_1 > 0,\ a_2 > 0.$
- $n = 3 \; : \; a_1 > 0,\ a_3 > 0,\ a_1a_2 - a_3 > 0.$
- $n = 4 \; : \; a_1 > 0,\ a_3 > 0,\ a_4 > 0,\ a_1a_2a_3 > a_3^2 + a_1^2a_4.$

Theorem 1.19. ***(Local stability using Routh-Hurwitz criteria [120])*** *Suppose $\bar{p}$ is an equilibrium of the system (1.1), and the characteristic equation of the variational matrix $Df(\bar{p})$, as given by (1.2), satisfies the Routh-Hurwitz criteria, meaning $det(H_k) > 0$ for $k = 1,2,...,n$. In such a case, the equilibrium $\bar{p}$ is considered locally asymptotically stable.*

Definition 1.20. ***(Global asymptotic stability [120])*** *An equilibrium solution $\bar{p}$ of (1.1) is said to be globally asymptotically stable if it is locally asymptotically stable and if $||p_0 - \bar{p}|| < \infty$ implies $\lim_{t\to\infty} ||p(t) - \bar{p}|| = 0$.*

Definition 1.21. ***(Positive definite function [120])*** *Consider an open subset G of $\mathbb{R}^n$ that contains the equilibrium $\bar{p}$ of system (1.1). A real-valued $C^1(G)$ function W, given by $W : G \to \mathbb{R}$, is labeled positive definite [120] within the set G if the following two conditions are satisfied:*

$$(i)\; W(\bar{p}) = 0,$$
$$(ii)\; W(p) > 0 \text{ for all } p \in G \text{ with } p \neq \bar{p}.$$

If the function W satisfies the condition that $-W$ is positive definite, then W is termed negative definite.

Theorem 1.22. ***(Lyapunov stability theorem [120])*** *Consider an equilibrium $\bar{p}$ of the system (1.1), and let W be a positive definite C^1 function defined as $W : G \to \mathbb{R}$, where G is an open subset of $\mathbb{R}^n$ containing the equilibrium $\bar{p}$.*

1. *If* $\frac{dW}{dt} \leq 0$ *for all* $p \in G \setminus \{\bar{p}\}$ *then* $\bar{p}$ *is said to be locally stable. W, in this case, is called a "weak Lyapunov function".*
2. *If* $\frac{dW}{dt} < 0$ *for all* $p \in G \setminus \{\bar{p}\}$ *then* $\bar{p}$ *is said to be locally asymptotically stable. In this case, W is called a "strict Lyapunov function".*
3. *If* $\frac{dW}{dt} > 0$ *for all* $p \in G \setminus \{\bar{p}\}$ *then* $\bar{p}$ *is unstable.*

Theorem 1.23. ***(LaSalle's invariance principle [121])*** *Let* $\bar{p}$ *be an equilibrium of system (1.1) and L be a weak Lyapunov function given by* $L : E \to \mathbb{R}$*, where E is an open, forward invariant subset of* $\mathbb{R}^n$ *containing the equilibrium* $\bar{p}$*. Also consider that L is non-decreasing in* $Z = \left\{p \in E \ : \ \frac{dL}{dt} = 0\right\}$*. Suppose* $\bar{p}$ *represents the largest forward invariant subset of Z. In this case,* $\bar{p}$ *attracts every point in E and eventually achieves global stability within E.*

Definition 1.24. ***(Periodic solution)*** *A solution* $\psi(p,t)$ *of (1.1) is called a periodic solution if there exists* $T > 0$ *such that*

$$\psi(p_0, t+T) = \psi(p_0, t) \text{ for all } t \text{ and}$$
$$\psi(p_0, t+s) \neq \psi(p_0, t) \text{ for all } 0 < s < T.$$

It is obvious that if $\psi(p_0,t)$ *has a period T then such solutions has period* $2T$, $3T$, *... If T is the smallest, we call this solution* $\psi(p,t)$ *as* $T-$ *periodic.*

Theorem 1.25. ***(Bendixson's criteria [120])*** *Consider the system (1.1) in* $\mathbb{R}^2$*. Suppose D is a simply connected open subset of* $\mathbb{R}^2$*. If divergence of f,* $\nabla.f = \Sigma_{n=1}^{2} \frac{\partial f_n}{\partial p_n}$ *is non-zero and maintains a constant sign in D, then there are no periodic orbits of the autonomous system (1.1) within the region D.*

Theorem 1.26. ***(Dulac's criteria [120])*** *Consider the system (1.1) in* $\mathbb{R}^2$*. Suppose D is a simply connected open subset of* $\mathbb{R}^2$ *and* $B(p,q)$ *is a real valued* C^1 *function in D. If divergence of Bf,* $\nabla.(Bf) = \Sigma_{n=1}^{2} \frac{\partial(Bf_n)}{\partial p_n}$ *is non-zero and remains constant in sign within the region D, then there are no periodic orbits of the autonomous system (1.1) within D.*

Theorem 1.27. ***(Bifurcation theorem [120])*** *Consider an autonomous system of ordinary differential equations*

$$\dot{p} = f(p, \xi), \ p \in \mathbb{R}^n, \ \xi \in \mathbb{R}, \text{ and } f \text{ is continuously differentiable.} \tag{1.3}$$

- ***Hopf bifurcation theorem [120]****: Suppose, the system (1.3) has an equilibrium* $\bar{p}(\xi)$*. Moreover, the Jacobian matrix* $Df(\bar{p}(\xi), \xi)$ *has one pair of complex eigenvalues*

$$\sigma_{1,2}(\xi) = A(\xi) \pm iB(\xi)$$

such that for some $\xi = \xi^*$ *it becomes purely imaginary, i.e.,*

$$A(\xi^*) = 0 \text{ and } B(\xi^*) \neq 0.$$

Then the eigenvalues will cross the imaginary axis with nonzero speed if (transversality condition)

$$\left.\frac{dA(\xi)}{d\xi}\right|_{\xi=\xi^*} \neq 0.$$

The system of differential equations (1.3) will undergo a Hopf bifurcation around $\bar{p}(\xi)$ *for* $\xi = \xi^*$ *and will possess a periodic solution with approximate period* $T = \frac{2\pi}{B(\xi^*)}$ *as* ξ *crosses* ξ^*.

- ***Saddle-node bifurcation theorem*** *[119]: Consider that* $f(p_0, \xi_0) = 0$. *Then the Jacobian of* f *at* p_0 *can be represented by the* $n \times n$ *matrix* $B \equiv Df(p_0)$. *Suppose that* B *has a simple eigenvalue* $\lambda = 0$ *with eigenvector* v, *and the transpose of* B, (B^T) *has an eigenvector* w *corresponding to the simple eigenvalue* $\lambda = 0$. *In addition, assuming that* B *has* k *eigenvalues with negative real parts and* $(n-k-1)$ *eigenvalues with positive real parts and that the aforementioned criteria are met.*

  $$w^T f_\xi(p_0, \xi_0) \neq 0,\ w^T D^2 f_\xi(p_0, \xi_0)(v, v) \neq 0. \tag{1.4}$$

  The equilibrium points of (1.3) *in* $R^n \times R$ *then form a smooth curve that passes through* (p_0, ξ_0) *and is tangent to the hyperplane* $R^n \times \xi_0$. *There are either no equilibrium points of* (1.3) *near* p_0 *when* $\xi < \xi_0$ *(or when* $\xi > \xi_0$*) or two equilibrium points of* (1.3) *near* p_0 *when* $\xi > \xi_0$ *(or when* $\xi < \xi_0$*), depending on the signs of the expressions in* (1.4). *The system* (1.3) *experiences a saddle-node bifurcation at the equilibrium point* x_0 *as the parameter* ξ *passes through the bifurcation value* $\xi = \xi_0$. *The two equilibrium points of* (1.3) *near* p_0 *are hyperbolic and have stable manifolds of dimensions* k *and* $k+1$, *respectively. The open, dense subset in the Banach space of all* C^∞, *one-parameter vector fields with an equilibrium point at* p_0 *and a simple zero eigenvalue make up the set of* $C^\infty-$*vector fields satisfying the aforementioned criterion.*

- ***Transcritical bifurcation theorem*** *[119]: Whenever the relations given in Eq.* (1.4) *are transformed into*

  $$\begin{aligned} w^T f_\xi(p_0, \xi_0) &= 0, \\ w^T Df_\xi(p_0, \xi_0)v &\neq 0, \text{ and} \\ w^T D^2 f_\xi(p_0, \xi_0)(v, v) &\neq 0, \end{aligned} \tag{1.5}$$

  then the system (1.3) *undergoes a transcritical bifurcation at the critical point* p_0 *as the* *bifurcating parameter ξ passes through the critical value $\xi = \xi_0$.*

1.9 The optimal control theory

Optimal control (OC) involves the process of determining control and state trajectories for a dynamic system over a specific time frame, with the goal of optimizing an objective functional, cost functional, or performance index [122]. For instance, the dynamic system might represent a spacecraft with control inputs linked to rocket thrusters, aiming to minimize fuel usage while reaching the moon; alternatively, it could signify a nation's economy, seeking to minimize unemployment through fiscal and monetary policies. Another example could be optimizing the percentage of vaccinated individuals over time in an epidemic model to minimize infections and vaccination costs.

The ability to manipulate controls in a system allows one to pursue specific objectives, with the underlying system encompassing various forms, such as ordinary differential equations, partial differential equations, discrete equations, stochastic differential equations, integro-difference equations, and combinations of discrete and continuous systems. In this study, we focus on applying the OC theory to ordinary differential equations with a fixed time.

1.10 Basic mathematical tools for optimal control problem

In a general optimal control problem, there is an objective functional or cost functional, denoted as $J(x(t),u(t))$, associated with a set of state variables $(x(t) \in X)$ and a set of control variables $(u(t) \in U)$, within a given time interval $t_0 \leq t \leq t_f$. The primary aim is to determine a piecewise continuous control $u(t)$ and the corresponding piecewise differentiable state variable $x(t)$ that maximizes the given objective function. Prior to defining the fundamental optimal control problem, we will establish some necessary definitions.

Definition 1.28. ***(Piecewise continuous [118])*** *Consider an interval $I \subseteq \mathbb{R}$, whether finite or infinite. A function $k : I \to \mathbb{R}$ is considered piecewise continuous if it is continuous at each $t \in I$, except for a finite number of points, if any, and at every $t \in I$, k is equal to either its left or right limit.*

Definition 1.29. ***(Piecewise differentiable [118])*** *Consider an interval $I \subseteq \mathbb{R}$, whether finite or infinite. Let $k : I \to \mathbb{R}$ be a finite-valued function that is continuous on I and differentiable at all but a finite number of points within I. Moreover, if k', the derivative of k is continuous wherever it is defined, then k qualifies as a piecewise differentiable function.*

Definition 1.30. ***(Continuously differentiable [118])*** *Let $I \subseteq \mathbb{R}$ be an interval, whether finite or infinite. A finite-valued function $k : I \to \mathbb{R}$ is said to be continuously differentiable if its derivative k' exists and is continuous on the entire interval I.*

Definition 1.31. ***(Concave [118])*** *let $J = [l_1, l_2] \subseteq \mathbb{R}$ be an closed and bounded interval. Then a real-valued function $M : J \to \mathbb{R}$ is called concave on J if*

$$\beta M(t_1) + (1-\beta)M(t_2) \leq M(\beta t_1 + (1-\beta)t_2)$$

for all $0 \leq \beta \leq 1$ and for any $l_1 \leq t_1, t_2 \leq l_2$.

A function M is considered convex on the interval $[l_1, l_2]$ if it fulfills the reverse inequality or, equivalently, if $-M$ is concave.

Definition 1.32. ***(Optimal control problem [123])*** *Let us consider a piecewise continuous control function $h(t)$ and the corresponding continuous and piecewise differentiable state function $x(t)$, defined on a bounded time interval $[t_0, t_f]$. Let that function solves the state equation*

$$\frac{dx}{dt} = g(t, x(t), u(t)), \tag{1.6}$$

with the initial condition $x(t_0) = x_0$, the ultimate objective is to determine the value of the control variable $h(t)$ that maximizes the objective functional:

$$\Pi(u^*) = \max_u \; J(x(t), u(t)) = \max_u \int_{t_0}^{t_f} f(t, x(t), u(t))dt \tag{1.7}$$

subject to

$$\begin{aligned} \dot{x}(t) &= g(t, x(t), u(t)), \\ x(t_0) &= x_0, \end{aligned}$$

and $x(t_f)$ could be free, which means that the value of $x(t_f)$ is unrestricted.

The functions f and g are always assumed to be continuously differentiable in all three cases. We also assume that the control set U consists of Lebesgue measurable functions. Consequently, as the control(s) will always be piecewise continuous, the corresponding states will also be piecewise continuous.

Our primary focus is to find the maximum of a function. However, we can effortlessly switch between maximization and minimization by merely negating the cost functional:

$$min\{J\} = -max\{-J\}.$$

1.10.1 Pontryagin's maximum principle

Pontryagin and his colleagues created the first-order criteria required to determine the optimal control. This result is regarded as one of the most significant mathematical results of the twentieth century. Pontryagin proposed using adjoint functions to add the differential equation to the objective functional. Adjoint functions serve a similar role in multivariate calculus as Lagrange multipliers, which attach restrictions to the function of many variables to be maximised or minimised.

Here's a step-by-step description of how to solve optimum control problems using Pontryagin's Maximum Principle [124, 125]:

(i) *Defining the problem:* Start by proposing the optimal control problem. This includes describing the state variables, control variables, objective function (the cost to be minimised or maximised), and any system and control constraints.

(ii) *Hamiltonian construction:* Build the Hamiltonian function, denoted by H. This function defined as the sum of the objective function (cost function) and the inner product of the adjoint variables (also called costate variables or Lagrange multipliers) and the state equations of the system. Thus the Hamiltonian is calculated as follows

$$H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u),$$

where x, u and λ are the state vector, the control vector, and the adjoint (costate) vector respectively. The functions $L(x,u)$ and $f(x,u)$ are respectively indicates the objective function or the Lagrange function, and the state equations.

(iii) *Obtaining the adjoint equation:* Determine the adjoint differential equation, which figures out the development of the adjoint variables. The adjoint equation is derived from the Hamiltonian's maximisation condition with regard to the adjoint variables and is presented by

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x}.$$

(iv) *Creating transversality condition:* Employ a transversality boundary condition to limit the adjoint variables at the final time. This condition ensures that the costate variables are consistent with the terminal constraints.

(v) *Employing optimality condition:* Apply the optimality condition, which is derived by maximizing the Hamiltonian with respect to the control variables, to solve for the optimal control. The optimality condition is given by $\frac{\partial H}{\partial u} = 0$.

(vi) *Obtaining the adjoint variables:* Remove the control variable from the state equation using the optimal control equations obtained from the optimality condition. Then, solve the resulting differential equations for the state and adjoint variables, subject to their respective initial conditions and the transversality boundary condition.

(vii) *Find optimal control:* Obtain the optimal state and adjoint variables. Then, substitute the values into the optimal control expression obtained from the optimality condition to find the optimal control u^*.

Following these procedures yields the optimum state trajectory, adjoint trajectory, and optimal control for the specified optimal control problem. The Pontryagin's Maximum Principle is a strong tool for systematically determining the essential criteria for optimality. However, for increasingly complex systems and cost functions, numerical techniques may be necessary to solve the optimal control problem [126].

1.10.2 The optimal control theory towards sustainable fishery

For sustainable development, optimal control theory can be applied to a variety of fishery models. Marine protected zones and seasonal fisheries, for example, are natural outcomes of optimal control. Obtaining an optimal system for a food chain model might also be useful in optimal control problems. The most common use of OCT in fishery models is to calculate the maximum economic yield (MEY). This can be accomplished by varying several control parameters, such as a harvesting control parameter (e.g., fishing effort) or a regulatory parameter (e.g. fishing tax). It should be noted that control parameters might be either bounded or unbounded. Fishing effort and fishing tax are taken as the main controlling parameters in this thesis, and the parameters are also bounded to make the system more realistic. We will now present the solution methods linked with the optimal control problem in order to achieve sustainable fisheries.

1.10.3 Solution methodology

Optimal control with bounded control: We here consider a simple fishery harvesting problem. More particularly, let $X(t)$ with $t \in [0,L]$, be the biomass fish population at ant time t. Then the dynamic equation of the population can be given by

$$\frac{dX}{dt} = G(X(t)) - h(t), \quad t \in (0,L) \tag{1.8}$$

where G represents the growth function and $h(t)$ is the rate of fishing. Considering the logistic growth rate one must have

$$G(X) = rX\left(1 - \frac{X}{K}\right).$$

Here, r and K respectively denotes the intrinsic growth rate and environmental carrying capacity. Now consider the harvesting rate is proportion to the population level, then one can write $h(t) = E(t)X(t)$, where $E(t)$ is the imposed harvesting effort. We want to maximize the net revenue (Π) which is defined as the difference between total revenue and total cost. The total revenue can be obtained as TR= ph, where $p > 0$ is the market price of per unit harvested biomass, and total cost (TC)= cE, where $c > 0$ is the cost per unit of fishing effort. Note that total cost is proportional to the fishing effort.

Then the associate optimal control problem can be written in the form

$$\max \Phi(E) = \int_0^L e^{-\delta t} \left(pEX^E - cE \right) dt \tag{1.9}$$

subject to

$$E \in M = \left\{ w \in L^2(0,L); 0 \leq w(t) \leq \bar{E} \; a.e. \; \& \; t \in (0,L) \right\}, \quad \bar{E} > 0,$$

where X^E is the solution of the problem:

$$\begin{aligned} &\dot{X} = G(X(t)) - E(t)X(t), \quad t \in (0,L) \\ &X(0) = X_0 > 0. \end{aligned} \tag{1.10}$$

The control restriction means that the effort is bounded and δ is the annual discount rate.

Now the Hamiltonian of the system can be given by

$$H(t,X(t),E(t),\lambda(t)) = e^{-\delta t} \left(pEX^E - cE \right) + \lambda(t) \left(G(X(t)) - E(t)X(t) \right), \tag{1.11}$$

where $\lambda(t)$ is the adjoint variable. To solve the optimal control problems with bounds on the control, the following necessary condition must be arisen.

Proposition 1.33. *(necessary conditions) Suppose E^* and X^* represent the optimal levels for problem* (1.9)*. In such a case, there exists a piecewise differentiable adjoint variable $\lambda(t)$ so that*

$$H(t,X^*(t),E(t),\lambda(t)) \leq H(t,X^*(t),E^*(t),\lambda(t))$$

for all controls E at each time t, where

$$\begin{aligned} &\lambda'(t) = -\frac{\partial H(t,X^*(t),E^*(t),\lambda(t))}{\partial X} \quad \textit{(adjoint condition)}, \\ &\lambda(L) = 0 \qquad\qquad \textit{(transversality condition)}. \end{aligned}$$

By an adaptation of the Pontryagin Maximum Principle, the optimal control must satisfy (optimality condition):

$$E^* = \begin{cases} a & \textit{if } \dfrac{\partial H}{\partial E} < 0, \\ a \leq \tilde{E} \leq b & \textit{if } \dfrac{\partial H}{\partial E} = 0, \\ b & \textit{if } \dfrac{\partial H}{\partial E} > 0, \end{cases} \tag{1.12}$$

i.e., the maximization is over all admissible controls, and $\tilde{E}$ is obtained by the expression $\frac{\partial H}{\partial E} = 0$. In particular, the optimal control E^* maximizes H pointwise with respect to $a \leq E(t) \leq b$.

Proof. The proof of this result can be found in [123]. □

It is important to acknowledge that in this context, we are demonstrating the profit maximization technique. However, if we were dealing with a cost minimization problem, the value E^* would be chosen to minimize H pointwise. Consequently, this reversal would lead to a change in the signs $<$ and $>$ in the first and third lines of the optimality condition (1.12).

Remark 1.34. *To numerically address a problem incorporating control bounds, we can express the optimal control $\tilde{E}$ obtained without truncation in a concise manner, constrained within the limits of a and b:*

$$E^*(t) = min(a, max(b, \tilde{E})).$$

Not all optimal control problems can be solved analytically. Most harvesting problem are complicated to solve analytically, so it is necessary to employ numerical methods. There are many numerical methods to solve optimal control problems, some of which are the Shooting method and Multiple shooting method (can be found in [127]). In addition, a very well-known method, the forward-backwards sweep method, can be found in the book by Lenhart and Workman [126].

1.11 Literature review and motivations

World fisheries have increased tremendously in the last fifty years due to the high demand for fishery products, the use of sophisticated fishing gear & vessel technology, and growing trade [92]. Global fish production (inland plus marine) has increased from 89.6 million tonnes in 2016 [93] to the highest ever, 96.4 million tonnes in 2018 [94]. Overexploitation has led many fisheries under stress, or its extinction [128, 129]. Different policies have been implemented regionally, nationally, and globally to protect world fisheries and promote sustainable development. To this effect, FAO (Food and Agriculture Organization) introduced the Code of Conduct for Responsible Fisheries (CCRF) in 1995 [130]. CCRF was further intensified in 2015 by implementing Sustainable Development Goal (SDG) 14 to conserve, protect, and sustainably use the oceans, seas, and marine resources [3, 4].

A regime shift is an essential phenomenon in many ecological and physical systems. A regime shift is a change in the average value of a particular data series within a year to a decadal scale [131]. A more recent definition of regime shift is defined as a large, abrupt, and persistent change in the system behaviour that causes significant impacts on human well-being [132, 133]. In the case of fisheries, regime shift means a change in the harvested species due to a change in the non-harvested species or other factors [134]. For example, different driving forces, like eutrophication, pollution, climate change, etc., may alter the distribution of biotic and abiotic factors, which may cause a difference in the spatial distribution of planktonic invertebrates. Therefore, a regime shift due to the change in plankton distribution may occur in the economic species [134]. Intensive harvesting is the primary reason for a regime shift in the fishery [135, 136]. Climate change is closely associated with it [137, 138]. Due to a regime shift, a fishery may abruptly shift from a harvesting state to a non-harvesting state. The above studies acknowledge regime shifts in marine environments but often neglect the interconnectedness between fish ecology and market demand-supply of fish and associated products. On the other hand, many studies [139, 140, 141, 142] recognize this mutual impact yet fail to detect regime shifts due to simplistic model systems. It is essential to consider both ecological dynamics and market forces in an integrated manner to comprehensively understand the complex interaction of fish, fisheries and demand-supply. This will enable a more informed approach to the sustainable management of marine resources. An important question is – does intensive harvesting the only reason for a regime shift in fisheries? Can demand also be a regulatory factor for the regime shift if the open market theory of price variation due to a change in demand and supply is considered in the fishery? This thesis, therefore, aims to integrate economic theory into current fishery management policies to find the occurrence of unwanted regime shifts in fisheries.

Fish is a renewable resource; most fisheries contain susceptible and infected fishes [143]. Global warming, climate change and industrial pollutants have been attributed to increasing aquatic fish disease [59, 60]. Water pollution is considered one of the significant causes of fish infection in the coastal areas [144, 145, 146]. Some other reasons behind the increasing infection rate are water temperature variation, changes in coastal dynamics, and lack of proper governance [59]. New and transboundary diseases have recently augmented epidemiological studies of aquatic fish in the presence of infection [147]. Infection may cause a low level of fish productivity [148]. Disease caused by a virus, bacteria, protists, and metazoans in fish is ubiquitous and known for a very long period [55, 149]. It is, therefore, essential to analyze mathematically the socio-ecological-economic aspects of the fishery model in the presence of infection. Earlier bioeconomic studies ignored the effect of disease on the harvested stock [139, 150, 151, 152, 153]. On the other hand, some studies [154] considered harvesting in an eco-epidemiological model and showed that infection could be eliminated through proper harvesting strategy, but they ignored the economic effect of disease in the harvested stock. Similar studies were done in a ratio-dependent predator-prey-parasite model [155] and Leslie-Gower ecoepidemic model [156] with prey harvesting. Harvested stock and revenue generation may be severely affected due to infection [157, 158]. However, the reason and distribution of fish infection must be better understood, particularly for marine fish [159].

Contagious diseases cause a significant economic loss in fishery either by reducing the biological productivity of the diseased fish and/or by lowering the commercial value of the infected fish [149, 160]. White spot syndrome virus (WSSV) is a predominant infectious disease in shrimp. Since 1992, this disease has devastated shrimp production and related industries in many countries, like Thailand [161], Ecuador [162], India [163], Iran [164] and USA [165]. This virus spreads rapidly from one infected shrimp to another susceptible shrimp and can kill them within seven to ten days [166, 167]. Thus, the infected shrimps have no chance to reproduce but to die. The economic loss due to WSSV in the last two-three decades has been reported to be huge. During 2010-2016, Thailand reported a financial loss of 11.58 B$ and working days loss of 0.1 million [168]. India reported 250 M$ loss due to WSSV in 2006-2008, along with 2.15 million man-days employment loss [163]. Shrimp production in Bangladesh dropped to 18,630 tonnes in 1998 from 25,742 tonnes in 1997, causing a significant economic loss [169]. This viral infection drastically reduced the income from shrimp export in Iran. The frozen shrimp export of Iran declined to 2,290 tonnes in 2007 compared to 7,680 tonnes in 2004, causing a reduction in income from 32.8 M$ in 2004 to 8.7 M$ in 2007 [164]. Peterman and Posadas [170] reported a total $16.9M$$ loss in 2016 due to the catfish disease in the east Mississippi catfish industry. Therefore, a global challenge is protecting fish and fishery from diseases and reducing economic loss by maintaining sustainable production. This thesis will analyze how infection may affect the productivity of a fishery. Another question is – will there be any regime shift in an infected fishery when economic theory is incorporated. This thesis will demonstrate such a possibility of regime shift.

There are numerous mathematical models that consider harvesting of renewable resources like fish. The single-species bioeconomic fishery model with harvesting has garnered significant interest among researchers [171, 172]. Kar and Matsuda [173] examined the economic and biological consequences of implementing marine protected areas (MPAs) and how they affect a single-species fishery model, with a specific focus on the dynamics of harvesting. Cid et al. [174] presented a discrete single-species fishery modelling framework emphasizing how temporal variation in harvesting efforts significantly influences population dynamics. Mansal et al. [175] explored a fishery model encompassing both variable harvesting effort and variable price, and they extended their study to include an age-structured fishery model. On the other hand, Moussaoui and Auger [142] examined a bioeconomic single-species fishery model with saturated harvesting function, variable fishing effort, and price. However, neither of these studies considered the presence of infection within the harvested stock. A two-dimensional predator-prey model under impulsive constant fishing pressure was considered, and its ecological and economic consequences were reported by Bischi et al. [176]. Similar ecological models with constant harvesting were explored in numerous studies [177, 178, 179] and the references therein. A fishery model with one predator and two prey was analyzed in Raymond et al. [180], where they discussed the stability properties of different equilibrium points and bio-economic harvesting with constant harvesting effort. The most popular harvesting strategy in bioeconomic models is the traditional catch per unit effort (CPUE) harvesting strategy [139, 181, 140]. However, the CPUE type harvesting has several abridgements, such as the harvesting rate becoming infinite as the harvested stock is infinite or the effort is infinite. This is quite unrealistic from the applicability and theoretical viewpoints [142, 179, 182]. To remove such unrealistic features of the harvesting rate with a finite effort, Moussaoui and Auger [142] proposed a bioeconomic fishery model considering a nonlinear saturated type harvesting effort. Krishna et al. [183] considered saturation on both effort and stock levels.

According to classical economic theory [99], an essential commercial factor in any bioeconomic study is maintaining the balance between the landed biomass and market demand. Also, the demand for a product plays one of the most significant roles in any bioeconomic model. Every fluctuation in market price depends on the demand. Auger et al. [139] described a fishery model considering the demand as a linear function of price. Mansal et al. [175] considered a modification in the previous demand function and considered demand as a linearly decreasing function of price. The variation in market price was assumed to be proportional to the difference between demand and supply. A similar linear function for the price-demand relationship was considered recently by Moussaoui and Auger [142]. While the majority of existing studies have predominantly focused on linear demand functions [175, 142], it is widely acknowledged that nonlinear demand functions offer a more realistic representation of market dynamics [184, 185]. Given this understanding, the present research endeavours to contribute significantly by conducting multiple comparative studies. These investigations investigate the impact and implications of various demand functions within socioeconomic interactions. By exploring and analyzing the effects of these diverse demand functions, this study seeks to unveil critical insights that can potentially revolutionize our understanding of market scenarios and decision-making processes.

Fishing has a direct effect on the harvested biomass. Fixing a harvesting quota for a particular fish species may protect the species from being overharvested [186, 187]. A fishing license or vessel buy-back policy is another means to reduce overharvesting [188]. Some popular and practical approaches to reduce harvesting pressure are to levy a tax on the landed fish [30], put a tax on the fishing vessels [31], fix a fishing quota [32] and fishing days [33]. Furthermore, a fishing fee or tax is usually considered one of the crucial measures for controlling overharvesting. These regulatory measures help protect fish and fisheries and achieve the SDG 14 targets at large [189]. Taxation is more critical in recreational fisheries run by various non-governmental agencies, where the management imposes taxes for access to the fishing zone, doing fishing or cultural activities as a part of the collective management [190, 191].

Policymakers may use the tax revenue earned through such fiscal policy for the socio-economic upliftment of the fishers and the marine ecosystem. Iceland is one of such countries that successfully implemented fishing fees for pelagic and demersal fishes [192]. On the other hand, many socio-ecological-economic studies have shown a conflict between conservation policy and socio-economic objectives [193]. For example, a higher fishing tax may relieve the fish stock from over-harvesting but may jeopardize the livelihood of local fishing people. It is particularly true in underdeveloped countries where fishermen have limited alternatives for their livelihood. Therefore, imposing a fishing tax scientifically and sensibly is essential. While the studies mentioned above have made notable strides in examining the implications of taxation policies on various aquatic aspects and the improvement of fishers' livelihoods, there remains a significant research void concerning the impact of these policies on aquatic epidemiology. This thesis, with unwavering commitment, aims to address this crucial gap in knowledge comprehensively and sustainably. By delving into the unexplored territory of aquatic epidemiology in the context of taxation policies, this research endeavours to shed light on vital connections and intricate dynamics that potentially revolutionize our understanding of aquatic ecosystems and their interplay with human activities. Through rigorous investigation and analysis, this thesis seeks to contribute to a more holistic and informed approach to policy-making, fostering sustainable practices and the well-being of aquatic environments and communities reliant upon them.

Ecotourism is one of the leading and potential branches of the tourism industry because of its significant role in sustainable development [194, 195, 196]. Ecotourism was advocated in 1968 when Hetzer [197] integrated culture, education, and tourism in a string, later becoming a conservation and sustainable development pillar. According to The International Ecotourism Society (TIES), ecotourism is a liable excursion into the natural environment that must conserve the area's ecology, maintain the territorial inhabitant's prosperity, and the consciousness to preserve the ecosystem [198]. Recreational fishing and non-extractive recreational activities may be an integral part of ecotourism. Coral reef ecotourism and fishing are one of the most crucial nature-based tourism having potential ecological and economic value [199, 200]. Marine wildlife, including marine mammals, maybe another potential player in the modern fishery that could play a significant role in achieving the goal of fishery-based ecotourism [201, 202, 203].

One of the ecotourism principles is providing direct financial benefits for conservation. For this, tourist entry fees at the ecotourism spot may be one step forward to maintaining the preservation of the tourist spot and the economic development of the people associated with such program [8]. The preceding studies have discussed sustainable ecotourism principles; however, they may not fully address the primary objectives of the blue economy, which encompass safeguarding marine species for sustainable use, enhancing local livelihoods and economies, and preserving the health of marine ecosystems and their resources. To achieve these goals, a scientific integration of fishery and ecotourism could prove valuable in reducing pressure on fish and fisheries while fostering long-term economic and social development for coastal communities. A combination of market-based fishing strategies in a multi-species fishery and a fair taxation policy holds promise for sustainable growth. Additionally, fishery-based ecotourism emerges as a potential component that can significantly improve the financial well-being of local populations by safeguarding the aquatic ecosystem. Implementing an entry tax on tourists may further support the development of ecotourism in the region. One of the significant purposes of this thesis is to propose and analyze a harvesting model that harmonizes the ecological dynamics between predator and prey fish with ecotourism and the principles of an open market economy, thus bridging the research gaps.

Through the comprehensive literature review, several research questions emerge, presenting unexplored avenues in bioeconomic modelling. These thought-provoking inquiries include:

(i) What precise environmental carrying capacity is imperative for sustaining harvesting, and at what level of enrichment can we optimize fish harvest and generate maximum revenue?

(ii) To what extent does the open market philosophy contribute to the potential occurrence of disastrous regime shifts within the fishing industry?

(iii) How does the interplay of supply and demand intricacies influence the dynamics of fisheries, particularly in attaining the elusive Maximum Economic Yield (MEY)?

(iv) Is fish demand a potential factor for regime shift?

(v) What specific harvesting rate and demand function combinations show a regime shift?

(vi) What is the extent of taxation's beneficial impact on a fishery, particularly in the presence of infection?

(vii) Are there discernible economic trade-offs that necessitate exploration and understanding?

(viii) Is there any trade-off that maximizes the societal benefit?

(ix) How do the complexities of demand patterns, tax policies, and disease dynamics collectively influence fishery dynamics and revenue generation?

(x) When an economic trade-off exists, what specific harvesting rate allows for attaining MEY with the least imposition of fishing effort?

(xi) In what multifaceted ways does the scientific integration of commercially harvested and recreational fishery effectively contribute to reducing pressure on fisheries and fostering coastal communities' economic and social development?

These pivotal research questions beckon further investigation and hold significant implications for advancing our comprehension of sustainable fishery management and the overarching goals of the blue economy.

1.12 Aim of the thesis

Fishing is a social-ecological interaction where human and non-human species (including the harvested and non-harvested species) interact. Such social-ecological systems are dynamic and complex [204]. The complexity may multiply if the social-ecological phenomenon of interest is connected with the economic ingredients. This thesis aims to comprehensively explore the intricate interactions between social, ecological, and economic factors in fishing practices. The primary objectives of this thesis are as follows:

- Investigate the coupling of social-ecological interactions in fish harvesting with dynamic fish market prices, focusing on understanding how market demand influences fish and fisheries dynamics.

- If economic theory is added to describe dynamic fish prices based on the demand-supply for fish, do the extended socio-ecological-economic model exhibit a regime shift?
- Investigate different functions for expressing harvesting rate and demands showing regime shift.
- Perform qualitative studies on bioeconomic fishery management in the presence of infections, analyzing the effects of various bioeconomic parameters on infection control and disease eradication.
- Propose and examine regulatory policies like taxation to mitigate overexploitation while considering their ecological and economic effects.
- Integrate fishing-based tourism with optimal fishing tax policies to promote sustainable resource management and revenue generation in ecotourism sites.
- Optimize societal revenue by determining an optimal tax level using Pontryagin's maximum principle, aiming to achieve a win-win situation for all stakeholders in sustainable fishery management.

Throughout the thesis, we will conduct analytical and numerical studies to explore various social, ecological, and economic phenomena, including species extinction, persistence, stability, economic trade-offs, and bionomic equilibrium. Additionally, we will investigate the occurrence of MSY and MEY to understand their significance in fishery dynamics and revenue generation.

Furthermore, the thesis aims to offer valuable insights into several critical socio-ecological-economic questions. These include determining the environmental carrying capacity needed for sustainable harvesting, analyzing the influence of the open-market philosophy on regime shifts in the fishing industry, and understanding the intricate interplay of supply and demand in fisheries dynamics to achieve the (MEY).

The study will also explore the interplay of taxation, infection, and demand in fishery dynamics and revenue generation. In cases where economic trade-offs exist, we will determine the optimal harvesting rate to achieve MEY with minimal fishing effort. Additionally, we will analyze how integrating commercially harvested and recreational fisheries can reduce pressure on fisheries and enhance economic and social development in coastal communities.

By addressing these research questions, this thesis aims to provide valuable insights into sustainable fishery management, ecological conservation, and socioeconomic development in complex social-ecological-economic systems.

1.13 Thesis overview

In this thesis, we study social-ecological-economic interaction and management of some commercial species in the presence or absence of an infectious disease. The basis of this thesis is to know how the dynamics of a renewable species are affected by human intervention and the demand-supply theory of the open market.

In **Chapter 2**, we analyzed a basic two-dimensional bioeconomic model representing the time evolutions of fish population in the presence of harvesting and market price that depends on supply and demand. We explored the dynamic behaviours of the system with four different harvesting functions and eight demand functions. Different analytical results were presented for a pair of saturated demand and saturated harvesting functions, but simulation results were given for all the thirty-two pairs. It is shown for all thirty combinations that the system may shift from a harvesting state to a non-harvesting state through a saddle-node bifurcation for some values of harvesting effort and demand. However, a disastrous regime shift with unbounded prices may be observed in some combinations. For example, there will be no regime shift for quadratic demand, whatever the harvesting rate. However, a regime shift may be observed in all four harvesting rates if the demand function is saturated. A trade-off between the harvesting effort and net revenue always occurs for the quadratic and exponential demand functions but never for saturated and mixed demand functions.

Intense harvesting and emerging infectious diseases are potential threats to the global fishery. A proper management policy with a scientific understanding of species interaction is a footstep in a long-term sustainable fishery. Considering those factors in **Chapter 3**, we performed a qualitative study of the bioeconomic management of a fishery in the presence of infection and dynamic harvesting. The model narrates the rate equations of the healthy fish, infected fish, fishing effort and market price, where the fishing effort is considered to be dependent on the fish price, and the demand-supply theory of the open market regulates the fish price. Routh-Hurwitz criterion is utilized for the local stability analysis, whereas the high-dimensional Bendixson criterion is used for the global stability analysis. The one-and-two-parameters bifurcation analysis explains various switching in equilibrium states, which include infection-free, infected and harvesting-free states. The existence conditions of the bionomic equilibrium, where both the ecological and economic equilibrium exists, have been established. The harvested fish biomass is higher at the infection-free equilibrium state than at the infected equilibrium state under an increasing infection rate. However, the outcome is the opposite under increasing environmental carrying capacity. The total revenue is highest in the infection-free state when demand is high. An unintuitive result is that the infection persists higher if demand decreases.

Taxation policy for fishing received global consent to protect fisheries from drastic harvesting. Still, it should be applied sustainably for a greater ecological and economic benefit because over-taxation may impair fishers' earnings and reduce the overall societal revenue. The fish disease may alter the system dynamics and reduce the revenue generation from the fishery. In

Chapter 4, we propose a nonlinear bioeconomic harvesting model of a single-species fishery with infection, variable market price, and nonlinear demand to explore taxation's ecological and economic effects. We provide the stability results of the system's different ecological and economic equilibrium points. The analytical conditions for the existence of transcritical bifurcation are also established. The computational results show that the system exhibits three dynamical regimes depending on the fishing tax. It is shown that taxation might control intensive harvesting but augment disease spreading and price hiking. Higher regulatory tax may even cause a regime shift, where the system enters into a non-harvesting regime from the harvesting one, causing an ecological and economic imbalance. Using Pontryagin's maximum principle, we decipher that some optimal fishing tax exists for the maximum societal benefit in a disease-induced fishery.

The analysis is then improved in several ways in **Chapter 5**, including by considering more realistic demand-supply functions, improved harvesting techniques, and implementing a fishery regulatory policy to prevent drastic harvesting. A four-dimensional bioeconomic fishery model is considered and analyzed to explore the system's dynamic behaviour. The objective is to decipher the consequences of a single-species fishery model in the presence of infection, nonlinear saturated harvesting rate and market demand, and fishing tax. How increasing demand may cause a regime shift in the fish and fishery is the most crucial objective of this work. For this, we studied different equilibrium points of the system and analyzed their local and global stabilities. An extensive bifurcation analysis is also done to demonstrate the effect of single- and multi-parameter variations. Using Pontryagin's maximum principle, we further discussed optimal revenue generation. The one-parameter bifurcation analysis revealed that the demand parameter plays a vital role in the system dynamics. Demand can make the system stable from its unstable state. It also plays a role in removing the infection from the system. Increasing demand corresponds to increased harvesting effort, which helps eliminate the disease. On the contrary, the negative side of high demand could be severe. There may be a drastic change in the system's qualitative behaviour. A regime shift from a harvested state to a non-harvested state may occur, causing an imbalance between demand and supply and the socioeconomic condition of the people associated with the fishery. The two-parameter bifurcation results were presented to demonstrate the more extensive dynamical behaviour of the system. It shows that the non-harvesting regime, where the price is unbounded, is not observed in any bifurcation results where demand is not one of the bifurcation parameters. Such demand-induced fishery collapse is rare in the literature. This study further shows no trade-off between fishing tax and revenue generation, opposing the conventional observation. The tax revenue levied by the regulating authority and the societal revenue increase with increasing tax.

The blue economy advocates using marine ecosystems sustainably for economic development, improvement of the livelihood of local people, and overall national development while preserving aquatic health. A scientific integration of fishery and ecotourism may help to achieve the target. Intermingling a market-based fishing strategy in a multi-species fishery with a fair taxation policy may provide long-term sustainable growth. Fishery-based ecotourism with an entry fee for the tourist may further contribute to the financial improvement of the local people by preserving the aquatic ecosystem. In **Chapter 6**, we have proposed and analyzed a harvesting model that integrates the ecological interaction of predator and prey fish with ecotourism and the open market economy theory. In the ecological interaction, the prey fish is harvested commercially, whose market price is determined by the demand-supply relationship. The harvesting effort of the prey fish is proportional to the profit margin. The regulatory authority imposes a fishing tax on landed fish as a controlling measure to restrict overfishing. The predatory fish (dolphin) is banned from commercial harvesting but used for recreation purposes for visitors as a part of ecotourism. The mentioned social-ecological-economic interaction may match various commercial and fishery-based ecotourism sites, including the Chilika lagoon of Odisha state, India. We analyze the proposed model from the dynamic and economic points of view and provide the local and global stability conditions of the ecological and economic equilibrium points. The broader dynamics of the system are unveiled through one-and two-parameter bifurcation analysis. Using Pontryagin's maximum principle, we mathematically show that an optimal tax exists, maximizing overall revenue generation and societal benefit. The thesis ends with the conclusion and future direction in the last chapter.

2

Dynamic consequence of an ecological-economic model with different harvesting strategies and demand functions

2.1 Introduction

Over the past few decades, the world's seafood supply structure has undergone tremendous alteration [205]. Two significant trends are a standstill in wild fish harvesting and growth in associate production [206]. Fisheries rely heavily on harvesting rates for survival. Several authors described the bioeconomic model with constant harvesting effort [176, 177, 178, 179, 181, 207]. However, it is the most insignificant harvesting strategy and has been heavily criticized in recent times [183]. The most popular harvesting function associated with fishery is the traditional catch-per-unit-effort (CPUE) fishing strategy [125, 139, 140, 175, 208]. Several flaws are present in this functional form, such as, for each (fish), a random search is presumed, it assumes that each (fish) has an equal chance of being caught [209]. Secondly, if the stock size is constant, the harvesting rate may grow unboundedly with the effort level. Similarly, for a constant effort, the harvesting rate grows unboundedly with the stock size [90]. Nonetheless, saturated harvesting is more realistic than CPUE harvesting because it slows the harvesting rate and allows a maximum limit even when stock size is huge [183, 210, 211]. According to Moussaoui and Auger [142], the non-linear saturation type harvesting rate for a fixed fishing effort exhibits a plateau regarding rising fish biomass, which is reasonable given that boats have a finite capacity for stocking and engaging in subsequent fishing operations. We have considered all possible types of harvesting techniques in our study to provide a comparative analysis and demonstrate various ecological and economic relevance.

The fluctuation of aquatic production has significantly impacted the dynamics of the global aquaculture markets [212]. To understand the changes in fisheries supply that influence the demand for these resources from urban rich to rural poor, fish and aquaculture production trends have been observed and analyzed for decades [213]. As a result of their continuous depletion of wild stock, fishery demand-supply interaction patterns are changing alarmingly in poor and developed countries. As availability has risen, species such as prawns, salmon, sea bass, sea bream, and catfish have experienced significant price drops [214]. Thus, it is crucial to analyze a bioeconomic fishery model with dynamic price strategies that follow the supply-demand phenomenon of the open market [139]. Mansal et al. [175] advanced the concept further in two scenarios: an age-structured fisheries model and a fishery with resource storage. With the help of the fluctuating market price, Brochier et al. [184] illustrated an over-exploited fishery with the dynamic market price. Each of these models considers price-dependent demand. However, the demand might depend on both price and stock level [151]. In this chapter, many forms of price-dependent and stock-price-dependent demands are taken into account, and their effects on the dynamics of the fisheries have been observed.

Disastrous shifts in regimes happen in various social, ecological, and economic systems [132, 215, 216]. A regime shift occurs when a system switches from one set of self-reinforcing procedures and frameworks to another [132, 217, 218]. Researchers continue to face challenges in forecasting and counteracting this problem [219, 220]. A regime shift in fisheries refers to a drastic alteration in the dynamics of the fishery brought on by harvesting [221], climate change [137, 138], and eutrophications [222]. Overexploitation is assumed to be the primary reason for a regime shift in fishery [135, 136]. This fact is true if economics is not considered explicitly. However, if the dynamic price, i.e., the instantaneous price change, depends on the instantaneous demand and supply, demand may be the driver for regime shift. If demand increases, then price increases, which causes harvesting effort to rise. The increased harvesting eventually causes the regime shift, if any. If it is not the case, then the question is – will there be any regime shift if there is no fish demand. Indeed, the answer is no. Because there will be no intensive harvesting due to the lack of demand, and possibly no regime shift will occur.

Recently, Bairagi et al. [223] shows that market demand for fish can be the primary cause of regime shift in a fishery. However, they considered one particular supply and demand function: saturated demand and saturated supply. But they did not find an economic trade-off that would allow them to achieve the maximum economic yield (MEY). Now the question is, are these two results valid for all types of demand and supply functions? The answer is no. Further, as in many other bioeconomic models, such as those that consider polynomial demand functions with linear or nonlinear harvesting efforts, economic trade-offs between revenue generation and harvesting effort have been identified without catastrophic regime shifts [91, 208, 224]. None of these studies observed regime shifts and economic trade-offs with the variation of a single bioeconomic parameter. In this chapter, we use a simple bioeconomic model with a dynamic price of the harvested stock to demonstrate how the regime shift and the economic trade-off in fisheries are related to the demand-supply phenomenon of open markets. A further question arises: If the trade-off exists, what harvesting rate can the MEY be achieved with the least effort?

In this chapter, we considered a combination of four harvesting rates and eight demand functions and analyzed 32 ecological-economic interactions to answer the questions. Different analytical results were presented for a pair of saturated demand and saturated harvesting functions, but simulation results were given for all the thirty-two pairs. It is shown for all thirty-two combinations that the system may shift from a harvesting state to a non-harvesting state through a transcritical bifurcation for some values of harvesting effort and demand. However, a disastrous regime shift with unbounded prices hike may be observed in some combinations. For example, there will be no regime shift for quadratic demand, whatever the harvesting rate. However, regime shift may be observed in all four harvesting rates if the demand function is of saturated type. A trade-off between the harvesting effort and net revenue always occurs for the quadratic and exponential demand functions but never occurs for saturated and mixed demand functions.

The subsequent chapter is arranged as follows. Section 2.2 describes the ecological-economic model formulation for a single-species fishery with a dynamic market price. Section 2.3 contains the analysis of the bioeconomic model, such as equilibria, their local stability properties, and possible bifurcations. The existence of maximum economic yield is discussed in Section 2.4, and the economic trade-off is illustrated in 2.5. The chapter ends with a discussion in Section 2.6.

2.2 The model

A general socio-ecological-economic single-species fishery model is the first thing we look for in this section. Assume the commercial harvesting of a fish species, which has a demand in the market. If X is the stock size of the fish at any time t and P be its price per unit biomass, then their instantaneous rates change may be expressed as follows:

$$\begin{aligned} \frac{dX}{dt} &= f(X) - h_i(X,E), \\ \frac{dP}{dt} &= \phi P \left(D_j(X,P) - h_i(X,E) \right). \end{aligned} \tag{2.1}$$

It says that fish stock variation is balanced by the difference between the fish growth rate, $f(X)$, and harvesting rate h_i. The instantaneous change in fish's per capita market price is proportional to the difference between the instantaneous demand $D_j(X,P)$ and supply $h_i(X,E)$

with ϕ as the proportionality constant. The demand may depend both on the stock size & price, and the harvesting rate is a function of stock size & harvesting effort, E. Note that the supplied quantity equalizes with the amount harvested.

The expression $h_i(X,E)$ denotes the rate of fish harvesting (or the supplied quantity) with a parametric fishing effort E. Four different types of harvesting functions are typically prioritized in the fishery literature:

(i) The constant-effort harvesting, which has the form $h_1(X,E) = E$, and is independent of the fish stock [153].

(ii) The catch-per-unit-effort (CPUE) harvesting (or constant-yield harvesting), which has the form $h_2(X,E) = qXE$, where q is the catchability coefficient [208].

(iii) Saturated harvesting, $h_3(X,E) = \frac{qXE}{m_1X+m_2E}$ [90], where the saturation is on both the fish and effort level. Here, m_1 is the degree of competition among the boats, fishing net, fishermen and other technology used in fishing [225], and m_2 is the product of capture rate and handling time [226]. In this situation, the fisherman can impose a limitless amount of effort, which results in a maximum harvested amount $\frac{qX}{m_2}$, obtained from the limiting value of $\frac{qXE}{m_1X+m_2E}$ when $E \to \infty$ [183]. Similarly, when the stock size is enormous, one can obtain the maximum fish catch as $\frac{qE}{m_1}$.

(iv) Imposing an unlimited amount of effort is sometimes considered infeasible. In light of this, a straightforward saturated harvesting approach is considered, where the harvesting rate is expressed as $h_4(X,E) = \frac{qXE}{X+D}$, called a stock-dependent saturation, where D is the half-saturation constant. In this case, the anglers are subjected to a bounded fishing effort [142].

Different demand functions have been used in the literature. Usually, demand is considered to vary with the price (P) and therefore expressed as $D(P)$. It is customary to assume that demand decreases with increasing costs. However, demand may also vary with the stock size (X) of some products. Thus, a more general functional form of demand is $D(X,P)$. In the following, we present various demand functions used in the literature.

(a) The constant demand is expressed as $D_1(X,P) = A$, where A is a positive constant [227]. This is the simplest demand function and is unrealistic for most products. In such a case, demand elasticity, the change in the demand in response to the stock or price level [228], is zero.

(b) The linear demand function may be represented by $D_2(X,P) = A - \alpha P$, where A is the maximum demand and α (> 0) is the demand decreasing rate [208]. The demand for common mackerel fish follows a linear function of its price [229].

(c) The quadratic demand function has the form $D_3(X,P) = A - \alpha P - \beta P^2$ [230], where β (> 0) is a constant and A, α have the similar meanings as in $D_2(X,P)$. The demand for numerous small fish and shrimp adheres to a quadratic demand pattern [231].

(d) The saturated type demand function has the form $D_4(X,P) = \frac{A}{1+BP}$, where A indicates the maximum demand and B is the demand-sensitive parameter [232]. Thiao et al. [233] found that "Thiof" catch and abundance significantly declined from 1974 to 2006. The ex-vessel price boomed during this period, and the demand remained stable from 1994 to 2006. This situation persists today. Despite declining catches, the high price motivates fishermen to continue catching "Thiof" as their revenue through the landed value remains favourable.

(e) The exponential demand is represented as $D_5(X,P) = Ae^{-bP}$, where A is the maximum demand and b is the demand sensitive parameter [234]. The demand for various vegetables, fruits, and sweets can be described using this type of function [234].

(f) The logarithmic demand has the form $D_6(X,P) = A - b_1 \ln P$ [235], where A, and b_1 are some positive constants. Cosmetic products are an illustrative example of such price-dependent demand [236].

(g) The logistic type demand function is expressed as $D_7(X,P) = \frac{\hat{A}}{1+e^{bP}}$ [235], where $\frac{\hat{A}}{2}$ is the maximum demand and b is a positive constant. Two products, yoghurt and catsup, follow such a demand function [237].

(h) The price and stock dependent demand function has the form $D_8(X,P) = \frac{A+A_1X}{1+BP}$, where A is the constant demand, A_1 is the demand increasing rate with stock, and B is the demand decreasing rate with price [238]. The demand in Taiwanese vegetable farmhouses is strongly influenced by both price and stock levels Mishra et al. [239].

It is worth noting that in the constant demand scenario, demand is independent of market price, whereas in the remaining situations, demand is a diminishing function of market price. Only the last function depends on both the stock size and price. Thus, one may construct at most thirty-two models from Eq. (2.1) with different combinations of $h_i(X,E)$ and $D_j(X,P)$. In the following, we present an analysis of a single model out of these 32 models, considering a saturated harvesting function and a saturated demand function. One can perform a similar analysis for other models.

2.3 Model analysis

Consider that fish species grows logistically in the absence of harvesting and is represented by $f(X) = rX\left(1 - \frac{X}{K}\right)$, where r is the intrinsic growth rate (birth minus death), and K is the system's carrying capacity. Then the system (2.2) with saturated harvesting and demand functions reads

$$\begin{aligned}\frac{dX}{dt} &= rX\left(1-\frac{X}{K}\right)-h_4 = F(X,P),\\ \frac{dP}{dt} &= \phi P\left(D_4-\frac{qXE}{X+D}\right) = G(X,P),\end{aligned} \tag{2.2}$$

where $h_4(X,E)=\frac{qXE}{X+D}$, and $D_4(X,P)=\frac{A}{1+BP}$. The initial value is considered as $(X(0)=X_0>0, P(0)=P_0>0)$.

2.3.1 Positivity and boundedness

To establish that the considered system (2.2) is ecologically and economically well-posed, we need to show that the solution exists uniquely, and the solutions are positive and bounded. For such findings, we prove the following lemma.

Lemma 2.1. *Each solution* $(X(t),P(t))$ *of the system* (2.2) *having initial point* $(X_0,P_0)\in\mathbb{R}^2_+$ *exists uniquely. Furthermore, the solution remains positive and uniformly bounded in* $\mathscr{R}_{\mathscr{S}}$*, where*

$$\mathscr{R}_{\mathscr{S}}=\{(X,P):0<X<\zeta+\hat{\varepsilon},\ 0<\mathscr{Y}(X,P)<\frac{r}{s_1}+\varepsilon, \textit{for any positive}\ \ \hat{\varepsilon},\varepsilon\}$$

with $\mathscr{Y}(X,P)=\ln X+P$, *and* $s_1=\min\left\{\frac{r}{K},\frac{\phi A}{B}\right\}$.

Proof. Consider the set of all such Banach space of continuous functions as $\mathscr{B}$, and define the function f as

$$f:[0,t]\to\mathbb{R}^{2,0}_+.$$

The initial condition of the system (2.2) can be written in the form

$$X_\theta=\mathscr{N}_1(\theta)>0, \text{ and } P_\theta=\mathscr{N}_2(\theta)>0,\ \theta\in[0,t], \tag{2.3}$$

where $(\mathscr{N}_1(0),\mathscr{N}_2(0))\in\mathscr{B}$. Then the corresponding norm can be stated as

$$||\mathscr{N}||=\sup_{0<\theta<t}(|\mathscr{N}_1(\theta)|,|\mathscr{N}_2(\theta)|),$$

where $\mathscr{N}=(\mathscr{N}_1,\mathscr{N}_2)$. According to the fundamental theorem of functional differential equations [240], there exists a unique solution of the system (2.2) with the initial point (2.3).

One can easily write from (2.2)

$$\begin{aligned}X(t) &= X_0 e^{\int_0^t\left[r\left(1-\frac{X}{K}\right)-\frac{qE}{X+D}\right]ds}>0 \text{ if } X_0>0,\\ P(t) &= P_0 e^{\int_0^t\phi\left[\left(\frac{A}{1+BP}-\frac{qXE}{X+D}\right)\right]ds}>0 \text{ if } P_0>0.\end{aligned}$$

Thus, whenever the solutions start with a positive initial point, the system (2.2) remains positive for all future time, implying that the system is positively invariant.

From Eq. (2.2)a, one have

$$\frac{dX}{dt} \leq rX\left(1-\frac{X}{K}\right).$$

Then the standard comparison theorem gives

$$\limsup_{t\to\infty} X(t) \leq \zeta,$$

where $\zeta = \max\{X_0, K\}$.

Define

$$\mathscr{Y} = \ln X + P.$$

Its time derivative along the solution of (2.2) gives

$$\begin{aligned}
\frac{d\mathscr{Y}}{dt} &= \frac{1}{X}\frac{dX}{dt} + \frac{dP}{dt} \\
&= r\left(1-\frac{X}{K}\right) - \frac{qH}{X+D} + \phi P\left(\frac{A}{1+BP} - \frac{qXH}{X+D}\right) \\
&\leq r - \frac{r}{K}X - \frac{\phi A}{B}P \\
&\leq r - \left(\frac{r}{K}\ln X + \frac{\phi A}{B}P\right) \ (\text{as } X > \ln X \ \forall\, X > 0).
\end{aligned}$$

Choosing $s_1 = \min\{\frac{r}{K}, \frac{\phi A}{B}\}$, one can write

$$\frac{d\mathscr{Y}}{dt} + s_1\mathscr{Y} \leq r.$$

Following differential inequality theorem [241], one then have

$$0 < \mathscr{Y}(X,P) < \frac{r}{s_1} + \frac{\mathscr{Y}(X_0,P_0)}{e^{s_1 t}}.$$

Clearly, $t \to \infty$ provides $0 < \mathscr{Y}(X,P) < \frac{r}{s_1}$. Hence, solutions of the system (2.2) with positive initial values are bounded in

$$\mathscr{R}_{\mathscr{S}} = \{(X,P) : 0 < X < \zeta + \hat{\varepsilon},\ 0 < \mathscr{Y}(X,P) < \frac{r}{s_1} + \varepsilon,\ \text{for any positive } \ \hat{\varepsilon}, \varepsilon\}.$$

This completes the lemma. □

2.3.2 Equilibria and their stability

The system (2.2) has three equilibrium points:

(i) The trivial equilibrium $\Theta_0 = (0,0)$ always exists.

(ii) The only-fish equilibrium $\Theta_1 = (X_1, 0)$, where X_1 is the positive roots of the equation

$$r\bar{X}^2 - r(K-D)\bar{X} + K(qE - rD) = 0. \tag{2.4}$$

Solving it, one gets the roots as

$$\begin{aligned} X_1^+ &= \frac{1}{2r}\left\{r(K-D) + \sqrt{\{r(K-D)\}^2 - 4rK(qE-D)}\right\} \text{ and} \\ X_1^- &= \frac{1}{2r}\left\{r(K-D) - \sqrt{\{r(K-D)\}^2 - 4rK(qE-D)}\right\}. \end{aligned}$$

It is reasonable to assume that system's carrying capacity is always higher than the half-saturation constant, i.e., $K > D$. Then Eq. (2.4) has a unique real positive root X_1^+ if $E < \frac{rD}{q}$, two real positive roots, X_1^+ and X_1^-, if $\frac{rD}{q} < E < \frac{r(K+D)^2}{4qK}$, and no real root under the restriction $E > \frac{r(K+D)^2}{4qK}$. Hence, there may exist two only-fish equilibria of the form $\Theta_1^+ = (X_1^+, 0)$ and $\Theta_1^- = (X_1^-, 0)$ in some range of fishing effort, and a single only-fish equilibrium of the form $\Theta_1 = (X_1, 0)$ in some other range of E, where $X_1 = X_1^+$.

(iii) The interior equilibrium point $\Theta_2 = (X_2, P_2)$, where X_2 is again obtained from the Eq. (2.4), have the equilibrium components

$$X_2 = \frac{1}{2r}\left\{r(K-D) \pm \sqrt{\{r(K-D)\}^2 - 4rK(qE-D)}\right\}, \text{ and} \tag{2.5}$$

$$P_2 = \frac{1}{B}\left\{\frac{A(X_2+D)}{qX_2E} - 1\right\}. \tag{2.6}$$

Proceeding similarly, one gets the existence conditions of a single, double and no interior equilibrium points as $E < \min\left\{\frac{A(X_2+D)}{qX_2}, \frac{rD}{q}\right\}$, $\frac{rD}{q} < E < \min\left\{\frac{A(X_2+D)}{qX_2}, \frac{r(K+D)^2}{4qK}\right\}$ and $\frac{r(K+D)^2}{4qK} < E < \frac{A(X_2+D)}{qX_2}$, respectively.

It should be emphasized that for this demand-supply relationship, the fish-free equilibrium of the form $\Theta_3 = (0, P_3)$ does not exist. The saturated type demand function restricts to happen it. Since fish stock is zero in Θ_3, the last equation of system (2.2) has to satisfy $\frac{A}{1+BP_3} = 0$ for the existence of such equilibrium. So, for non-zero positive demand parameters A and B, the market price P_3 has to be infinite. As a result, instead of existing a fish-free equilibrium, an unbounded price-hike occurs. Such an equilibrium, however, may exist when demand functions are linear or quadratic.

2.3.3 Local stability analysis

The stability of an equilibrium point is determined by the sign of the eigenvalues of the Jacobian matrix of the system (2.2) evaluated at the equilibrium point. The Jacobian matrix evaluated at an arbitrary equilibrium point $\tilde{\Theta} = (\tilde{X}, \tilde{P})$ reads

$$J(\tilde{X}, \tilde{P}) = \left(\begin{array}{cc} r\left(1 - \frac{2X}{K}\right) - \frac{qE}{X+D} + \frac{qXE}{(X+D)^2} & 0 \\ -\frac{\phi qDEP}{(X+D)^2} & \phi \left(\frac{A}{1+BP} - \frac{qXE}{X+D} - \frac{ABP}{(1+BP)^2} \right) \end{array} \right)_{(\tilde{X},\tilde{P})}. \tag{2.7}$$

Then the following stability results hold.

Theorem 2.2. *(i) The trivial equilibrium* $\Theta_0 = (0,0)$ *is always unstable.*

(ii) Whenever the only-fish equilibrium $\Theta_1 = (X_1, 0)$ *exists, it is locally asymptotically stable (LAS) if* $\frac{\phi A(X_1+D)}{qX_1} < E < \frac{r(X_1+D)^2}{qK}$*, and it is an unstable point otherwise.*

(iii) The interior equilibrium point $\Theta_2(X_2, P_2)$*, whenever it exists, is LAS if and only if* $E < \frac{r(X_2+D)^2}{qK}$*. It is a saddle point if the inequality is reversed.*

Proof. (i) One eigenvalue of the matrix (2.7) corresponding to the equilibrium point $\Theta_0(0,0)$ is ϕA, which is always positive. This implies that the trivial equilibrium point $\Theta_0(0,0)$ is always unstable.

(ii) The Jacobian matrix corresponding to the only-fish equilibrium point $\Theta_1(X_1, 0)$ reads

$$J(X_1, 0) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.8}$$

where $a_{11} = -\frac{rX_1}{K} + \frac{qX_1E}{(X_1+D)^2}, a_{12} = 0,\ a_{21} = 0,\ a_{22} = \phi \left(A - \frac{qX_1E}{X_1+D} \right)$. Thus, the eigenvalues are $-\frac{rX_1}{K} + \frac{qX_1E}{(X_1+D)^2}$ and $\phi \left(A - \frac{qX_1E}{X_1+D} \right)$. Hence, $\Theta_1(X_1, 0)$ is LAS whenever the condition $\frac{A(X_1+D)}{qX_1} < E < \frac{r(X_1+D)^2}{qK}$ is satisfied, and unstable otherwise.

(iii) At the interior equilibrium point $\Theta_2(X_2, P_2)$, the variational matrix (2.7) reads

$$J(X_2, P_2) = \begin{pmatrix} f_{10} & f_{01} \\ g_{10} & g_{01} \end{pmatrix}, \tag{2.9}$$

where $f_{10} = -\frac{rX_2}{K} + \frac{qX_2E}{(X_2+D)^2}, f_{01} = 0,\ g_{10} = -\frac{\phi qDEP_2}{(X_2+D)^2},\ g_{01} = -\frac{\phi ABP_2}{(1+BP_2)^2}$. One eigenvalue of this matrix is $-\frac{\phi ABP_2}{(1+BP_2)^2}$, which is always negative. Therefore, the interior equilibrium will be LAS if the other eigenvalue is also negative, giving $E < \frac{r(X_2+D)^2}{qK}$. It will be a saddle point in the opposite case.

Hence the theorem. □

2.3.4 Bifurcations analysis

In the system (2.2), a number of local bifurcations can be identified with the variation in fishing effort, E. We show that the system can experience transcritical and saddle-node bifurcations at some critical values of this parameter.

2.3.4.1 Transcritical bifurcations

Transcritical bifurcation is associated with the interchange of stability of equilibrium points. Two equilibria overlap in this bifurcation for some critical value of a parameter, and one equilibrium point drives from stable to unstable while the other moves from unstable to stable state. This occurrence is known as the exchange of stability. The following theorem states the existence of transcritical bifurcation in system (2.2) when the effort, E, is varied.

Theorem 2.3. *The system* (2.2) *experiences a transcritical bifurcation at the only-fish equilibrium* $\Theta_1(X_1,0)$ *if the fishing effort E attains the critical value* $E^{TC} = \frac{A(X_1+D)}{qX_1}$.

Proof. (i) Observe that the Jacobian matrix (2.8) provides a simple zero eigenvalue when the fishing effort reaches the threshold $E = \frac{A(X_1+D)}{qX_1} = E^{TC}$. In this effort level E^{TC}, the right and left eigenvectors corresponding to the zero eigenvalue of the variational matrix $J(X_1,0)$ can, respectively, be found as $\zeta = (0,1)^T$, and $\eta = (0,1)^T$. For the existence of a degenerate transcritical bifurcation at $E = E^{TC}$, the following three transversality conditions [119] must be hold.

$$\begin{aligned}
\eta^T R_E\Big(\Theta_1(X_1,0);E=E^{TC}\Big) &= 0,\\
\eta^T DR_E\Big(\Theta_1(X_1,0);E=E^{TC}\Big)\zeta &= 0, \qquad (2.10)\\
\eta^T D^2R\Big(\Theta_1(X_1,0);E=E^{TC}\Big)(\zeta,\zeta) &\neq 0,
\end{aligned}$$

where $R = \Big(F(P,Q),G(P,Q)\Big)^T$ and $R_E = \Big(\frac{dF(P,Q)}{dE},\frac{dG(P,Q)}{dE}\Big)^T$ are two vector-valued functions, and the other two terms $DR_E\Big(\Theta_1(X_1,0);E=E^{TC}\Big)$ and $D^2R\Big(\Theta_1(X_1,0);E=E^{TC}\Big)$, respectively, indicate the Jacobian matrix of the vector-valued function R_E and $DR\Big(\Theta_1(X_1,0);E=E^{TC}\Big)\zeta$ at $E=E^{TC}$. It is to be noted that, in case of non-degenerate transcritical bifurcation, the second condition of (2.10) must be non-zero. Now,

$$\eta^T R_E\Big(\Theta_1(X_1,0);E=E^{TC}\Big) \;=\; (0\;\;1)\begin{pmatrix} -\frac{-qX_1}{X_1+D} \\ 0 \end{pmatrix} = 0,$$

$$\begin{aligned}\eta^T DR_E\Big(\Theta_1(X_1,0);E=E^{TC}\Big)\zeta &= (0\ \ 1)\begin{pmatrix} -\frac{qD}{(X_1+D)^2} & 0\\ 0 & 0\end{pmatrix}_{E=E^{TC}}\begin{pmatrix}0\\1\end{pmatrix}=0, \quad (2.11)\\ \eta^T D^2R\Big(\Theta_1(X_1,0);E=E^{TC}\Big)(\zeta,\zeta) &= (0\ \ 1)\begin{pmatrix} 0 & 0\\ -\frac{\phi qDE}{(X_1+D)^2} & -2AB\end{pmatrix}_{E=E^{TC}}\begin{pmatrix}0\\1\end{pmatrix}\\ &= -2AB\neq 0.\end{aligned}$$

Thus, following Sotomayars theorem [119], a degenerate transcritical bifurcation point always exists at the critical value $E=\frac{A(X_1+D)}{qX_1}=E^{TC}$. Hence the theorem. □

2.3.4.2 Saddle-node bifurcation

The saddle-node bifurcation is the fundamental mechanism by which equilibria are created and destroyed. Due to the variation in the control parameter, two equilibria move toward each other, collide, and mutually annihilate. Typically, a saddle-node bifurcation occurs if the Jacobian matrix at an equilibrium point exhibits a single zero eigenvalue, representing a neutral direction in the system's dynamics, and the other eigenvalues determine the stability of the equilibria. One can then prove the following theorem using Sotomayor's theorem [119].

Theorem 2.4. *Two only-fish equilibriums $\Theta_1^+=(X_1^+,0)$ and $\Theta_1^-=(X_1^-,0)$ are collide with each other at $E=\frac{r(X^{SN}+D)^2}{qK}=E^{SN}$ and then disrepairs. Thus, the system* (2.2) *experience a saddle-node equilibrium $\Theta^{SN}=(X^{SN},0)$ under the condition $E^{SN}\neq-\frac{2r(X^{SN}+D)^2}{qKD}$ and $E^{SN}>\frac{A(X^{SN}+D)}{qX^{SN}}$.*

Proof. Sotomayor's theorem [119] states that one of the Jacobian's eigenvalues must be zero, and the other must have a negative real component at the saddle-node equilibrium point. The Jacobian matrix at the equilibrium point $\Theta^{SN}=(X^{SN},0)$ can be calculated as

$$J(X^{SN},0)=\begin{pmatrix} -\frac{rX}{K}+\frac{qXE}{(X+D)^2} & 0\\ 0 & \phi\left(A-\frac{qXE}{X+D}\right)\end{pmatrix}_{(X^{SN},0)}. \tag{2.12}$$

The eigenvalues are $\lambda_1=-\frac{rX^{SN}}{K}+\frac{qX^{SN}E}{(X^{SN}+D)^2}$ and $\lambda_2=\phi\left(A-\frac{qEX^{SN}}{X^{SN}+D}\right)$. If E^{SN} represents the effort level when one eigenvalue (say, λ_1) becomes zero, we have $E^{SN}=\frac{r(X^{SN}+D)^2}{qK}$. The condition of another eigenvalue (λ_2) being negative is $E^{SN}>\frac{A(X^{SN}+D)}{qX^{SN}}$. Let $\bar{\eta}=[v_1,v_2]^T$ and $\bar{\zeta}=[w_1,w_2]^T$ are, respectively, the right and left eigenvectors of $J(\Theta^{SN})$ corresponding to the zero eigenvalue, where

$$v_1=1,\ v_2=0,\ w_1=1,\ w_2=0.$$

Then the transversality conditions of the saddle-node bifurcation [119] are

$$\begin{aligned}
\bar{\eta}^T R_E(\Theta^{SN}; E=E^{SN}) &= (1\ \ 0)\begin{pmatrix} -\frac{qX^{SN}}{(X^{SN}+D)} \\ 0 \end{pmatrix} = -\frac{qX^{SN}}{(X^{SN}+D)} \neq 0, \text{ and}\\
\bar{\eta}^T D^2R(\Theta^{SN};E=E^{SN})(\bar{\zeta},\bar{\zeta}) &= (1\ \ 0)\begin{pmatrix} -\frac{2r}{K}-\frac{qE^{SN}D}{(X^{SN}+D)^2} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= -\frac{2r}{K}-\frac{qE^{SN}D}{(X^{SN}+D)^2}\neq 0,
\end{aligned}$$

when $E^{SN} \neq -\frac{2r(X^{SN}+D)^2}{qKD}$ holds. Hence the proof. $\square$

2.4 Simulation results

For simulation purposes, we use the parameter set described in Table 2.1, which remains fixed unless mentioned.

Table 2.1: State variables and parameters with their descriptions and default values.

Variable	Description	Unit	
$X(t)$	Fish biomass at time t	metric tonnes (MT)	
$P(t)$	Market price per unit of biomass at time t	M\$**/metric tonne	
Parameter	Description	Default Value	Reference
r	Intrinsic growth rate of healthy fish	0.9 /year	[208]
K	Environmental carrying capacity	7 metric tonnes	[208]
q	Catchability coefficient	0.5 metric tonnes/SFU/year	[208]
E	Harvesting effort	SFU*	Variable
A	Maximum demand	metric tonnes/year	Variable
D	Positive constant	2	[208]
B	Demand sensitivity parameter	5 metric tonnes/M\$	
ϕ	Stiffness parameter	0.1 SFU/M\$	[208]

* SFU stands for Standardized Fishing Unit [242, 243] and ** M\$ indicates million USD.

We explored the switching phenomena of the system under the variation of the harvesting effort $E \in [0, 6.5]$. Fig. 2.1 shows three distinct dynamic regimes. A transition from the stable interior state at the lower value of E to the stable only-fish equilibrium state occurs through a transcritical bifurcation at $E = 3$, following Theorem 2.3. Observe that fish biomass steadily decreases with increasing effort. The price also drops sharply as the fish supply increases with the harvesting effort. The price becomes zero at the critical harvesting effort, $E = 3$, where a transcritical bifurcation occurs. The only-fish equilibrium $\Theta_1(X_1^{\pm}, 0)$ appears as E exceeds the value 3.

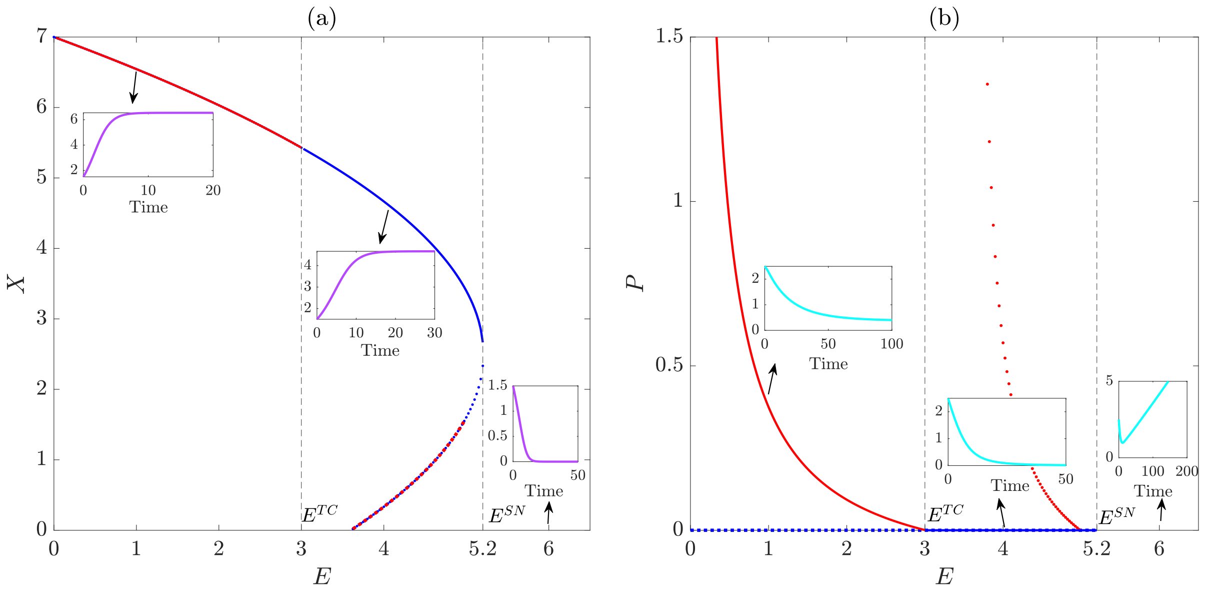

Figure 2.1: System dynamics under the variation of fishing effort, E. Here red and blue colours indicate the equilibrium components of $\Theta_2(X_2,P)$ and $\Theta_1(X_1^{\pm},0)$. The dotted line indicates the unstable branches and the solid line indicates the stable branch of the respective equilibrium points. The time series solutions for a fixed value of E taken from each region are given in the insets. Other parameters as in Table 2.1 with $A = 1.1$.

A pair of only-fish equilibrium points $\Theta_1(X_1^{\pm},0)$ exists and become stable in the range $3 < E < 5.2$, following Theorem 2.2 (ii). The equilibrium with higher fish density, Θ_1^{+}, is stable (blue solid curve), and the other one is unstable (blue dotted curve). The stable and unstable branches approach each other with increasing effort and colloids each other. These two equilibrium components disappear upon collision, following a saddle-node bifurcation (Theorem 2.4). No stable equilibrium state exists for the fishing effort $E > 5.2$; instead, the price becomes unbounded due to the unavailability of fish. Therefore, a catastrophic shift of the system's state occurs when the fishing effort crosses the upper threshold value, $E = 5.2$, where the system enters from a harvesting (no price) state to a non-harvesting (price unbounded) state, indicating a catastrophic regime shift. The inset figures visualize the corresponding time evolutions of the system for a particular value of E taken from each region. For instance, the system's time series for $E = 1 \in (0,3)$ shows stable behaviour of the interior equilibrium. For a particular value of the harvesting effort $E = 4 \in (3,5.2)$, the system stabilizes to the only-fish equilibria; and for $E = 6(> 5.2)$, an unbounded price hike with zero effort is observed.

2.4.1 Two-parameter bifurcation

In Fig. 2.1, we observed a catastrophic regime shift from an equilibrium fishing state to a non-equilibrium unbounded price-hike state when both the demand and harvesting (or supply) functions are of saturated types. Now the question is, under what demand-supply relationship such a regime shift may occur? To explore it and to observe broader dynamics, we presented two-parameter bifurcation results for the remaining 31 combinations of demand and harvesting functions and presented a few in Fig. 2.2 to Fig. 2.6. The bifurcation parameters chosen in each figure are A, representing the maximum demand, and E, the harvesting effort. We overlooked the constant demand situation since the non-trivial finite equilibrium price does not exist with this demand function. This is evident from the demand function D_4 and Eq. (2.6) when $B = 0$.

In Fig. 2.2, we presented the two-parameter bifurcation results where the saturation demand (D_4) combined with all four harvesting functions h_i, $i = 1,2,3,4$. It shows that each parametric plane is divided into three distinct regimes R_1, R_2 and R_3, representing the stability zone of the interior equilibrium, the only fish equilibrium, and the unbounded price-hike region, respectively. Interestingly, the unbounded price-hike area exists in each parametric plane when the fishers impose higher effort. Such a higher effort is responsible for overexploitation and regime shift. A transcritical bifurcation curve separates the magenta region from the green part. Each region is separated from the blue area by a saddle-node bifurcation line. Fig. 2.1 can be looked at as a particular case of Fig. 2.2(d). All three dynamic behaviours of Fig. 2.1 may be observed if one moves along the horizontal line $A = 1.1$.

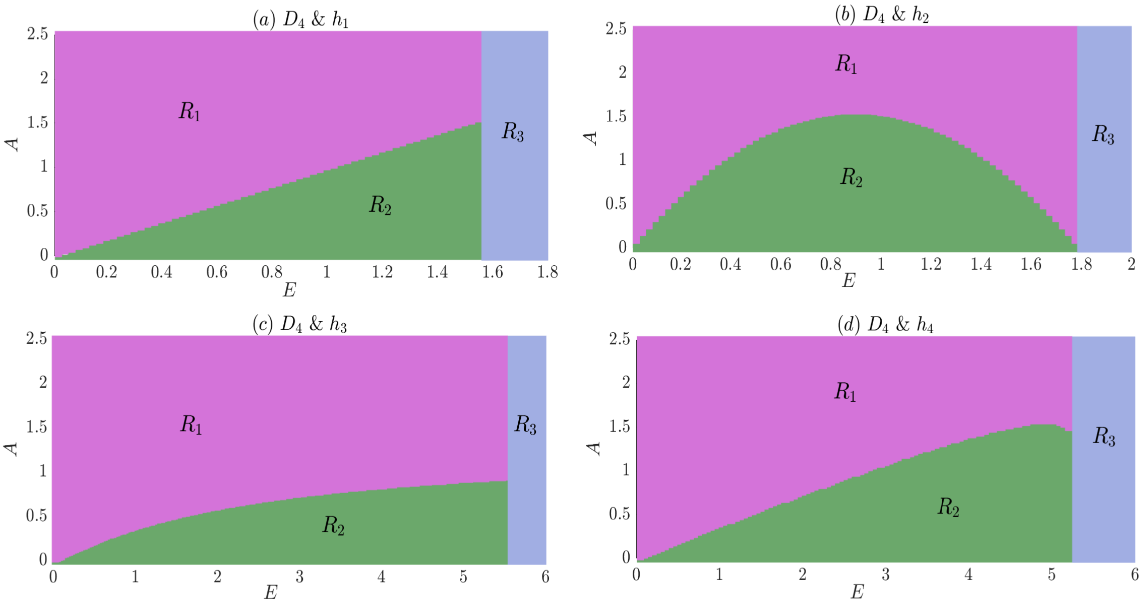

Figure 2.2: Two parameter bifurcation diagrams when the fishing effort (E) and maximum demand parameter (A) are varied simultaneously. We have considered saturated type demand (D_4) with all four harvesting rate functions $h_i (i = 1,2,3,4)$: (a) constant harvesting rate (h_1), (b) CPUE harvesting rate (h_2), (c) Saturated harvesting rate (h_3), and (d) the harvesting rate (h_4). Here R_1 and R_2 represent the regions where the interior equilibrium and the only-fish equilibrium are stable. In the region R_3, X goes to zero in finite time, and P becomes unbounded. Parameter are as in Fig. 2.1 with $m_1 = 1$ and $m_2 = 2$ for Fig. (c).

Different dynamic regions with the variation of E and A are plotted in Fig. 2.3 when the exponential demand function D_5 is combined with the four types of harvesting rate functions. One can notice that the regime shifts from a harvesting state to a non-harvesting state occurs much earlier than in the previous case when harvesting function is a constant (Fig. 2.3a) or

CPUE (Fig. 2.3b). Interestingly, no regime shift occurs due to overexploitation for the saturated harvesting functions h_3 and h_4 whatever be the harvesting effort. It shows that the demand and its functional form play a more critical role in the regime shift. One can get a similar bifurcation design with logistic type demand function $D_7(P)$ (see Table 2.2).

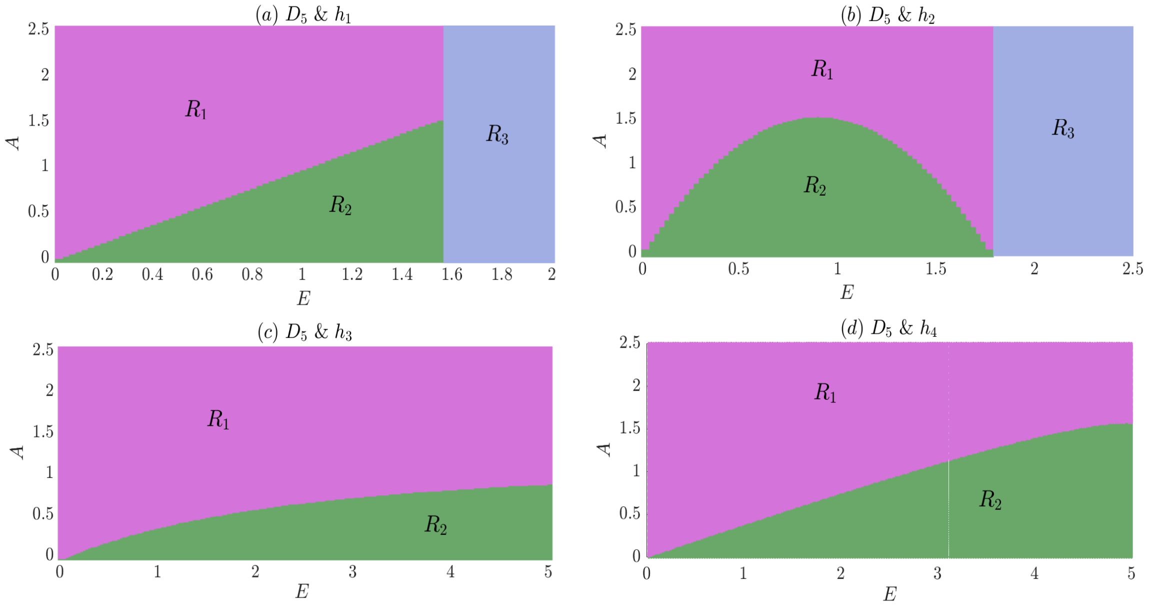

Figure 2.3: Two-parameter bifurcation diagrams of the system with demand function D_5 and harvesting function h_i $(i = 1,2,3,4)$. The parameters remain the same as in Fig 2.2 with $b = 5$. Catastrophic regime shift is observed in Figs. 2.3a,b.

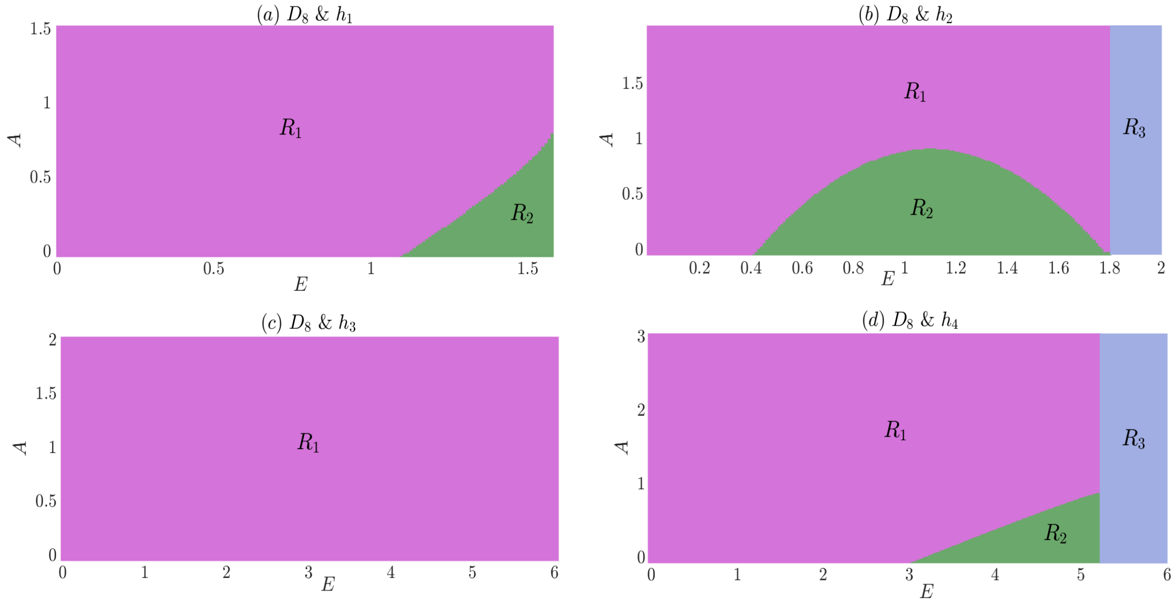

Figure 2.4: Two-parameter bifurcation diagrams with respect to E and A when the demand function D_8 is combined with the harvesting function h_i, $i = 1,\ldots,4$. Catastrophic regime shift is observed for the combinations (D_8,h_2) and (D_8,h_4). Parameters remain the same as in Fig 2.2 with $A_1 = 0.2$.

Fig. 2.4 represents the similar two-parameter bifurcation dynamics when the demand is both stock and price-dependent (D_8). This type of demand function has the largest stable interior equilibrium region. The combination of D_8 with h_3 shows the most robust dynamics. Here, the interior equilibrium is stable throughout the entire plane. No catastrophic shift in the system is observed in the case (D_8, h_1) also. Such unwanted behaviour, however, is observed for the combinations (D_8, h_2) and (D_8, h_4).

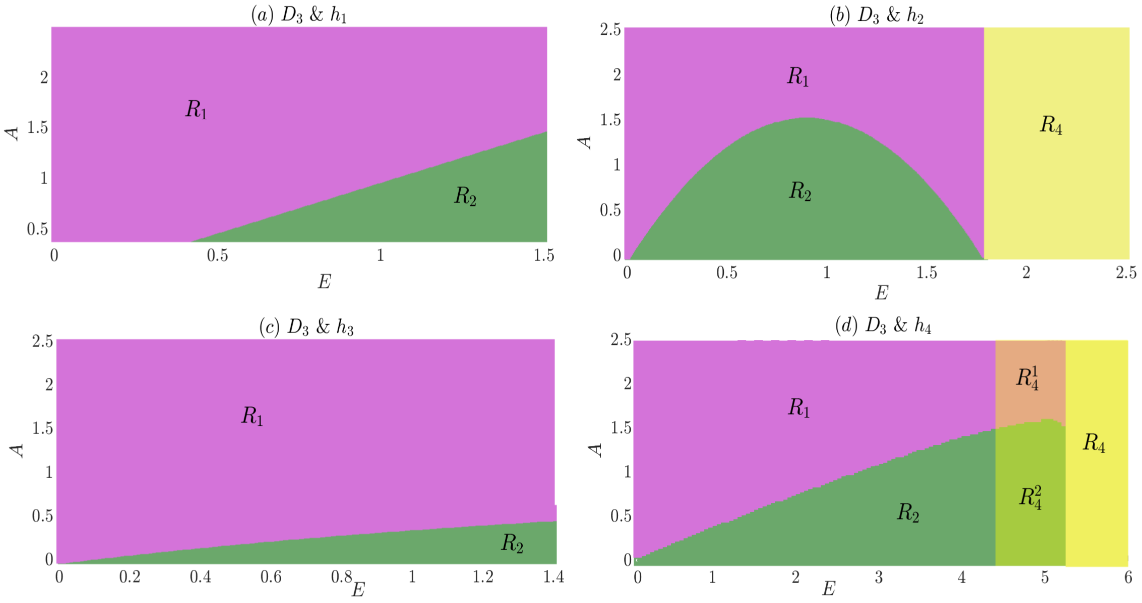

Figure 2.5: Similar two-parameter bifurcation diagrams with the quadratic demand function D_3. The choice of harvesting rates and parameter values remain the same as in Fig 2.2 with $(\alpha, \beta) = (0.61, 0.05)$ [208]. Here, R_1 is the stable region of the interior equilibrium Θ_2; R_2 is the stable region of the only-fish equilibrium Θ_1; R_4 is the stable region of the equilibrium point $\Theta_3(0, P_3)$; R_4^1 is the bistable region of the equilibrium points Θ_1 and Θ_3; R_4^2 is the bistable region of the equilibrium points Θ_2 and Θ_3.

Fig. 2.5 shows that no regime shift is observed in the $A-E$ parametric plane when the four harvesting functions are considered with the quadratic demand function, $D_3(X, P) = A - \alpha P - \beta P^2$. However, the most diverse dynamics are observed when D_3 is combined with the saturated harvesting function h_4 (Fig. 2.5). In this case, one may find five different dynamic behaviours of the system. It is mentionable that there exists an equilibrium point of the form $\Theta_3(0, P_3)$ when D_3 is combined with either $h_2(X, E) = qXE$ (Fig. 2.5b.) or $h_4(X, E) = \frac{qXE}{X+D}$ (Fig. 2.5d). The region R_4 delineates the parametric space where the equilibrium Θ_3 is stable. Two different bistable states may be observed in the regions R_4^1 and R_4^2. The equilibrium states Θ_3 and Θ_1 may be stable in the area R_4^1 depending on the initial values. On the other hand, the equilibrium points Θ_3, and Θ_2 may be stable in the bistable region R_4^2. The time evaluations of the system for five pairs of (E, A), taking one from each area of Fig. 2.5(d), are presented in Fig. 2.6 to show the monostability and bistability of the system. One can draw similar-two parameter bifurcation diagrams considering other combinations of demand and harvesting functions. We have tabulated the existence or nonexistence of the catastrophic regime shifts of all 32 combinations of $h_i (i = 1,..,4)$ and $D_j (j = 1,..,8)$ in Table 2.2.

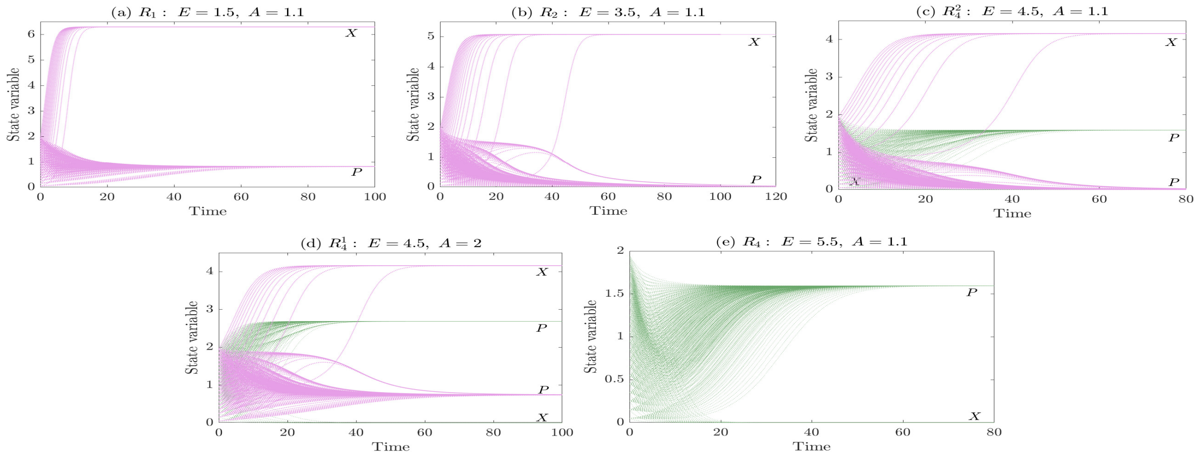

Figure 2.6: Representative time series behaviour of the system for D_3 and h_4 for some particular values of (E,A) taken one from each region of Fig. 2.5(d). Initial points are $X(0) \in (0,2)$ and $P(0) \in (0,2)$. (a) The coexistence equilibrium Θ_2 is stable when (E,A) is selected from the region R_1. (b) The only-fish equilibrium Θ_1 is stable when (E,A) is selected from the region R_2. (c) Two equilibrium points Θ_2 and Θ_3 may be stable when (E,A) is selected from the region R_4^2. (c) Two equilibrium points Θ_2 and Θ_3 may be stable when (E,A) is selected from the region R_4^2. (d) Two equilibrium points Θ_1 and Θ_3 may be stable when (E,A) is selected from the region R_4^1.

Table 2.2: This table shows the occurrence of catastrophic regime shift under all possible demand-supply combinations. Parameters are same as in Table 2.1 with $\alpha = 0.61$, $\beta = 0.05$, $A_1 = 0.2$, $\hat{A} = 1.1$, $B = 5$, $b = 5$, $b_1 = 0.7, m_1 = 1$ and $m_2 = 2$.

Demand function	Harvesting rate	Catastrophic regime-shift	Demand function	Harvesting rate	Catastrophic regime-shift
D_1	h_1	NA	D_5	h_1	Yes
	h_2	NA		h_2	Yes
	h_3	NA		h_3	No
	h_4	NA		h_4	No
D_2	h_1	No	D_6	h_1	No
	h_2	No		h_2	No
	h_3	No		h_3	No
	h_4	No		h_4	No
D_3	h_1	No	D_7	h_1	Yes
	h_2	No		h_2	Yes
	h_3	No		h_3	No
	h_4	No		h_4	No
D_4	h_1	Yes	D_8	h_1	No
	h_2	Yes		h_2	Yes
	h_3	Yes		h_3	No
	h_4	Yes		h_4	Yes

'NA' stands for not applicable.

2.5 Economic trade-off

Here, we look into whether there is an economic trade-off between fishing effort and net revenue. We aim to enhance the fishermen's profit or achieve the MEY. To do this, we first define the net revenue (Π), which is the difference between the total revenue (TR) earned by fishermen and their expenses (TC). Thus,

$$\begin{aligned}\text{Net revenue } (\Pi(X,P,E)) &= \text{TR} - \text{TC} \\ &= h_i(X,E)P - cE, \qquad (2.13)\end{aligned}$$

where c denotes the cost per unit of fishing effort. We find whether there exists an optimal value of the fishing effort for different h_i so that the net revenue is maximum. To maximize the net revenue, the optimal economic problem may be defined as

$$\mathscr{J} = \int_0^\infty \Pi(X,P,E)e^{-\delta t}dt, \qquad (2.14)$$

where δ indicates the annual discount rate and Π is defined in Eq. (2.13). The control variable E is subject to the constraints $0 < E < E^{\max}$, where $E^{\max}$ denotes the upper limit of the imposed effort. By virtue of Pontryagin's maximum principle [122], one can write the Hamiltonian as

$$\mathscr{H} = \Pi(X,P,E)e^{-\delta t} + \eta_1\left(rX\left(1-\frac{X}{K}\right) - h_i(X,E)\right) + \eta_2\phi P\left(D_j(P) - h_i(X,E)\right), \qquad (2.15)$$

subject to the system (2.1), where η_1, and η_2 are the adjoint variables. The optimal control variable E has to satisfy the following conditions to maximize $\mathscr{H}$ [51]:

$$\frac{\partial\mathscr{H}}{\partial E} = 0, \quad \frac{d\eta_1}{dt} = -\frac{\partial\mathscr{H}}{\partial X}, \quad \frac{d\eta_2}{dt} = -\frac{\partial\mathscr{H}}{\partial P}.$$

We considered a quadratic demand function with saturated harvesting to analyze the optimal effort level and the economic trade-off. One can select other combinations also.

With this choice of demand-supply, Eq. (2.15) reads

$$\mathscr{H} = \left[\frac{qXEP}{X+D} - cE\right]e^{-\delta t} + \eta_1\left[rX\left(1-\frac{X}{K}\right) - \frac{qXE}{X+D}\right] + \eta_2\phi P\left[A - \alpha P - \beta P^2 - \frac{qXE}{X+D}\right]. \qquad (2.16)$$

We are only interested in observing the optimal economic yield at the interior equilibrium point, say $\bar{\Theta}_2(\bar{X}_2,\bar{P}_2)$, as any other equilibria is insignificant from bioeconomic viewpoint. For optimal value, $\frac{\partial\mathscr{H}}{\partial E} = 0$ at $\bar{\Theta}_2(\bar{X}_2,\bar{P}_2)$, giving

$$\eta_1 + \eta_2\phi\bar{P}_2 = \left[\bar{P}_2 - \frac{c(\bar{X}_2+D)}{q\bar{X}_2}\right]e^{-\delta t}. \qquad (2.17)$$

Again, $\frac{d\eta_1}{dt} = -\left[\frac{\partial \mathscr{H}}{\partial X}\right]_{(\bar{X}_2,\bar{P}_2)}$ gives

$$\frac{d\eta_1}{dt} = G_1\eta_1 + G_2 e^{-\delta t}, \tag{2.18}$$

where $G_1 = \frac{2r\bar{X}_2}{K} - \frac{qE(\bar{X}_2-D)}{(\bar{X}_2+D)^2}$ and $G_2 = \frac{cDE}{\bar{X}_2(\bar{X}_2+D)} - \frac{2qDE\bar{P}_2}{(\bar{X}_2+D)^2}$. Solving (2.18), one gets

$$\eta_1 = -\frac{G_2}{G_1+\delta}e^{-\delta t}. \tag{2.19}$$

From (2.17) and (2.19), one have

$$\eta_2 = \frac{1}{\phi\bar{P}_2}\left[\bar{P}_2 - \frac{c(\bar{X}_2+D)}{q\bar{X}_2} + \frac{G_2}{G_1+\delta}\right]e^{-\delta t}. \tag{2.20}$$

Observe that each of these adjoint variables (η_1 and η_2) are bounded. Substituting the values of these adjoint variables in $\frac{d\eta_2}{dt} = -\left[\frac{\partial \mathscr{H}}{\partial P}\right]_{(\bar{X}_2,\bar{P}_2)}$, we have the maximum economic yield (MEY) equation as

$$\Gamma(E) = \left(\bar{P}_2 - \frac{c(\bar{X}_2+D)}{q\bar{X}_2} + \frac{D_2}{D_1+\delta}\right)\left(\alpha + 2\beta\bar{P}_2 - \frac{\delta}{\phi\bar{P}_2}\right) - \frac{q\bar{X}_2E}{\bar{X}_2+D} = 0, \tag{2.21}$$

for a suitable choice of the annual discount rate, δ. The positive value of E, say $E = E^c$, for which $\Gamma(E = E^c) = 0$ is the possible optimal effort, where Π is the locally maximum. We then numerically computed the optimum effort level $E = E^c$ and the corresponding MEY $\Pi(X,P,E^c)$. We choose the parameter values as in Fig. 2.5(d) with $A = 1.1$, $c = 0.01$, and $\delta = 0.001$. Fig. 2.7 shows that the Eq. (2.21) evaluated at the interior equilibrium has a unique optimal value $E = E^c = 1.35$. Then the MEY can be computed from (2.13) as $\Pi(E^c) = 0.447$ M$/year. The diagram of the associate net revenue is given in Fig 2.8(a), green curve. In this diagram, we have also plotted the net revenues with respect to the constant harvesting rate (h_1, magenta colour), CPUE harvesting rate (h_2, blue colour), and saturated harvesting rates (h_3, black colour and h_4, green colour). In each case, an economic trade-off exists between fishing effort and net revenue. Similar plots for three more demand functions are presented in Figs. 2.8(b)-(d).

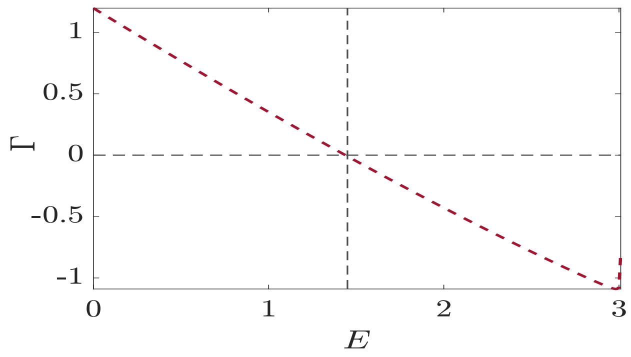

Figure 2.7: This figure shows that the optimal tax equation (2.21) has a unique positive root $E = 1.35$. Parameters are same as Fig. 2.5(d) with $A = 1.1$ and $c = 0.01$.

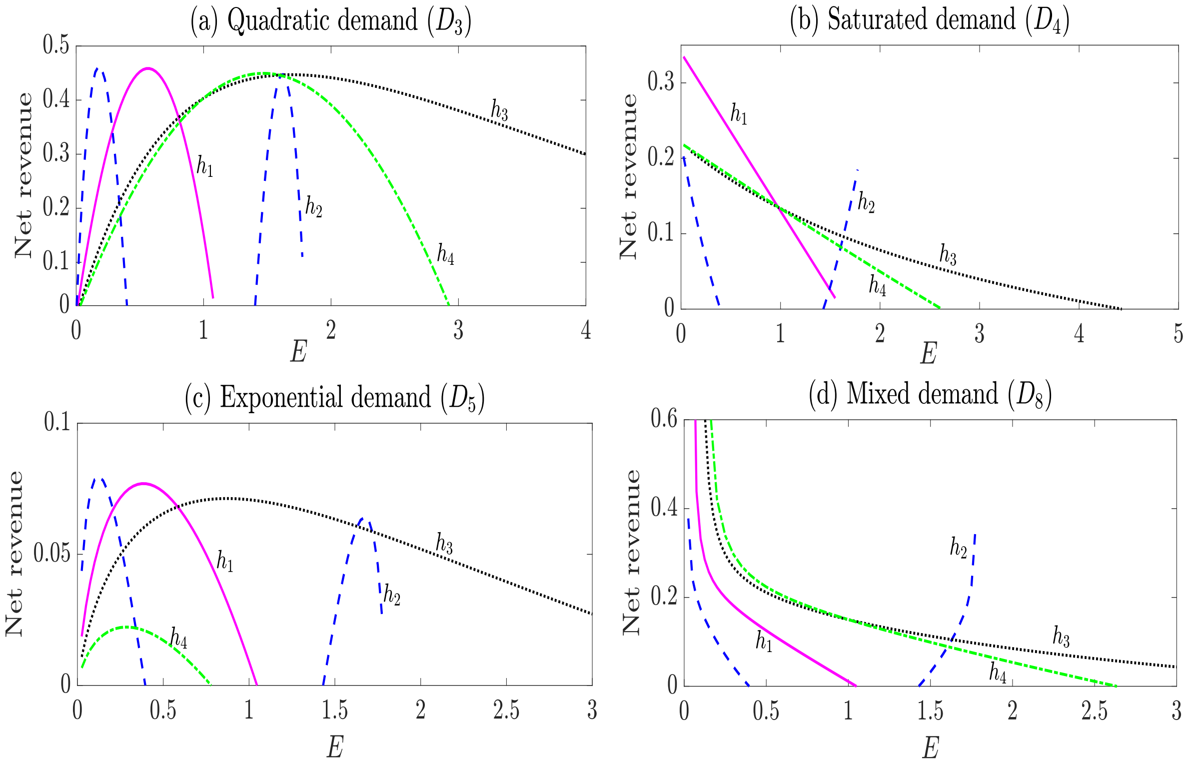

Figure 2.8: The net revenue with variable fishing effort for different demand functions, $h_i(i = 1,..,4)$. It shows that a trade-off exists for quadratic demand D_3 and exponential demand D_5 for all h_i. The parameters are as in Fig. 2.1 with $c = 0.01$.

In Table 2.3, we have tabulated the demand functions for which an economic trade-off exists for each type of harvesting function. The corresponding optimum effort, equilibrium fish biomass and price are also given. Note that the economic trade-off exists for the demand functions D_2, D_3, D_5, D_6, D_7, and there is no trade-off for the demand functions D_1, D_4, D_8. In the case of quadratic demand (D_3), the optimum fishing effort (E^c) varies with the harvesting rate functions, while the MEY ($\Pi(E^c)$) remains almost the same. However, there are clear variations in the MEY for the demand functions D_5 and D_6. For D_6, the MEY is maximum (4.166) corresponding to the harvesting function h_1 and it is minimum (1.237) for h_2. Another intriguing fact is that whenever the trade-off occurs, the CPUE harvesting rate (h_2) invariably requires the least effort to reach the MEY.

2.6 Discussion

The emerging world has driven growing trade, production, and consumption, demonstrating the need to establish forecasts for future patterns of production and consumption as well as their challenges. In a recent groundbreaking study, the International Food Policy Research Institute (IFPRI) and the World Fish Center examined global fish market forecasts over the next 20 years using sophisticated modelling tools [244]. As the model predicts, the global demand for fish is projected to grow faster than the supply, leading to a significant surge in fish prices.

Table 2.3: This table shows the optimal effort level (E^c) and the corresponding MEY $\Pi(E^c)$ for the demand-supply combinations depicted in Figure 2.8, where there is an economic trade-off. The corresponding equilibrium fish biomass and the market price are also given. Parameters are same as in Table 2.2.

Demand function	Harvesting rate	Optimum effort(E^c)	Equilibrium fish biomass	Price at equilibrium	MEY ($\Pi(E^c)$)
D_2	h_1	0.55	6.32	0.9	0.490
	h_2	0.17	6.34	0.92	0.494
	h_3	1.62	6.34	0.93	0.479
	h_4	1.18	6.46	1.07	0.448
D_3	h_1	0.57	6.29	0.81	0.458
	h_2	0.18	6.32	0.84	0.462
	h_3	1.7	6.32	0.84	0.447
	h_4	1.4	6.38	0.9	0.45
D_5	h_1	0.37	6.56	0.22	0.077
	h_2	0.12	6.5	0.2	0.08
	h_3	0.87	6.59	0.24	0.071
	h_4	0.27	6.88	0.47	0.022
D_6	h_1	0.3	6.65	3.14	4.166
	h_2	0.23	6.11	1.77	1.237
	h_3	0.7	6.66	3.18	4.157
	h_4	0.8	6.63	3.1	4.161
D_7	h_1	0.23	6.73	0.27	0.059
	h_2	0.07	6.73	0.26	0.061
	h_3	0.51	6.75	0.28	0.056
	h_4	0.57	6.75	0.28	0.055

The analyses of this chapter identified several issues that have yet to be given much thought in previous bioeconomic modelling. For instance, how may the open market philosophy contribute to the disastrous regime shift in the fishing industry? How does the complexity of supply and demand affect the dynamics of the fisheries to reach the MEY? Whenever the economic trade-off exists, at what harvesting rate can the MEY be achieved with the least amount of imposed fishing effort? We analyzed a basic two-dimensional bioeconomic model incorporating a rate equation for a fish species in the presence of harvesting and a rate equation for its price change in the market. We considered combinations of four harvesting functions and eight demand functions to find which combinations show a regime shift and economic trade-offs.

We observed that certain equilibria depend on market demand to exist. For instance, the fish-free equilibrium only occurs when the demand functions are polynomial or logarithmic. On the contrary, such a fish-free equilibrium does not exist for a saturated demand function, for which the price needs to be infinite. Still, the system may converge to this state in finite time when the harvesting effort exceeds some higher threshold value. Thus, the fishery shifted to a non-harvesting state due to the lack of fish, and a compensatory sharp price increase was observed. Such an unwanted regime shift could be dangerous for the environment and the underprivileged. Our bifurcation analysis showed that this regime shift occurs through a saddle-node bifurcation. However, a disastrous regime shift with an extreme price hike is not observed in the polynomial and logarithmic demand case. Again, when the demand is exponential, logistic or mixed, the regime shift depends on the demands' functional form and the supply term (or harvesting function).

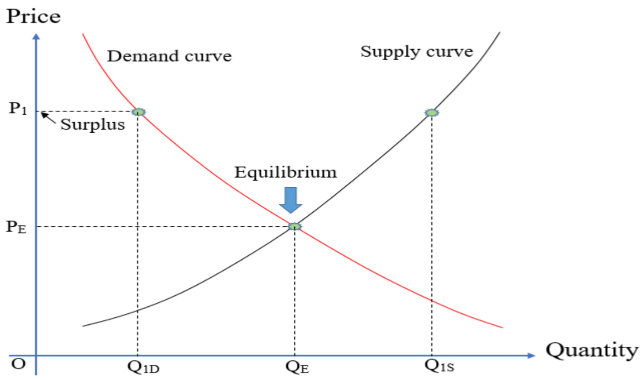

Figure 2.9: Price equilibrium (Q_E, P_E) exists where demand and supply become equal, i.e., where the demand curve meets the supply curve. A surplus occurs if the current price exceeds the equilibrium price, where the supplied quantity exceeds the demand.

An unusual situation arises for some intermediate range of the harvesting effort, where fish is available but its price is zero (the case where the only-fish equilibrium is stable). From the dynamical point of view, this situation appears through a transcritical bifurcation, where the interior equilibrium losses its stability to the only-fish equilibrium with zero price as the harvesting effort crosses the lower threshold value. From an economic viewpoint, an equilibrium price (P_1) exists in the market when the demand and supply become equal (see Fig. 2.9). However, if the quantity supplied Q_{1S} exceeds the amount demanded Q_{1D}, there will be a surplus. In this case, the interior equilibrium does not exist. The market price (P_1) always remains above the equilibrium price P_E.

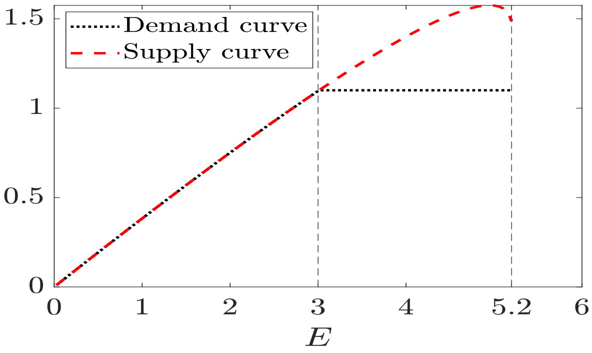

Figure 2.10: The demand curve (black dotted line) and supply curve (red dashed line). Equilibrium exists in the range $0 \leq E < 3$, where supply and demand are equal, and a surplus exists in the range $3 \leq E < 5.2$, where supply exceeds demand. For $E > 5.2$, there is unbounded price. Parameters remain the same as in Fig. 2.1.

Therefore, the system has to converge to the available only-fish equilibrium, where the price is zero, once the harvesting effort crosses the threshold value (see Fig. 2.1). Such a surplus situation occurred in the system for the intermediate harvesting range $3 < E < 5.2$. Fig. 2.10 demonstrates that the quantity supplied (red dashed curve) surpasses the amount demanded (black dotted curve) once the fishing effort exceeds the threshold of $E = 3$. The recent oil market price collapse in Saudi Arabia serves as a concrete example of such a scenario, where higher availability forced the oil market price to zero [245].

Understanding trade-offs is significant in economics because it helps in decision-making, figuring out costs and benefits. It also helps to see how well the resources are being used. Fishers and policymakers must understand that trade-offs are involved when they make choices. We used the optimum control theory to determine when economic trade-offs exist. Our study shows that a trade-off depends on demand functions and is independent of the type of harvesting functions. Such a trade-off exists for polynomial, exponential, logarithmic and logistic-type demand functions and does not exist for constant, saturated and mixed-type functions.

The chapter can indeed be extended even further. While we have explored a parametric variation of harvesting effort, an interesting avenue to explore would be to consider the time-dependent fishing effort that varies with the profit margin. Additionally, it is essential to acknowledge that fish disease can have significant consequences on the fish population and the economy, as highlighted by previous studies [55, 149]. Since most of the fisheries consist of both susceptible and infected fish populations [143], it becomes crucial to understand the potential impact of fish disease in this context. Considering these, we aim to proceed to Chapter 3, where we will delve deeper into these fishery-related phenomena and their implications.

3

Bioeconomics fishery model in presence of infection: Sustainability and demand-price perspectives[1]

3.1 Introduction

Fishery is an important sustainable livelihood. The recent tidings of the Food and Agriculture Organization (FAO) of the United Nations [14] show that about 39 million people are involved in fisheries and another 20.5 million people are engaged in aquaculture. Global fish production reached 96.4 million tonnes in 2018 compared to 93.1 million tonnes in 2017. The total revenue earned from fish exports in 2018 is about USD 164 billion [14]. Many factors put tremendous stress on the global fishery. Of course, overexploitation is the foremost factor of such stress. The fish disease [149] is another serious issue in the fish production and revenue earning [246]. Global warming, climate change and industrial pollutants have been attributed for increasing aquatic fish disease [59, 60]. It is, therefore, considered to be a global responsibility to maintain the health of the world fisheries for sustainable development.

Bioeconomics, which applies the biological knowledge for commercial and industrial purposes, has become a global trend in modern research [63]. Many remarkable works in bioeconomic modelling [247] have appeared in the literature, which consider both the ecological

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[1] *The bulk of this chapter has been published in Applied Mathematics and Computation, DOI:https://doi.org/10.1016/j.amc.2021.126225, (2021).*

and economic aspects [125]. Fish is necessarily a renewable resource and most of the fisheries contain susceptible as well as infected fishes [143]. It is, therefore, important to analyze mathematically the economical as well as biological aspects of the fishery model in presence of infection.

Most of the mathematical fishery models figure out the concept of an open-access fishery. They disregarded the concept of the free market, i.e., they ignored the balance between supply and demand. That is why it is not always possible to set a reasonable market price for the resource, thus influencing the total revenue. A fishery model with one predator and two-prey was analyzed in [180], where they discussed the stability properties of different equilibrium points and bio-economic harvesting with constant harvesting effort. A two-dimensional predator-prey model under impulsive constant fishing pressure was considered and its ecological and economic consequences were reported in [176]. Similar ecological models with constant harvesting were explored in numerous studies [177, 178, 179] and the references therein. Variable harvesting was considered in the ecological model to control the exploitation of resources and to determine optimal harvesting so that resource biomass and population maintain its optimal level [248]. Some mathematical models assumed a variable market price or demand to suit supply of the harvested stock [249]. But none of the above works considered infection in harvested stock. On the other hand, some authors described mathematical bioeconomic models in presence of infection, but they did not consider the variable market price [250] because of difficulty to construct and analyze a model with all such variables. This chapter aims to embed the theory of open market into a bioeconomic fishery model with infected fishes. Following the open market policy, demand has been considered as a decreasing function of the price. Another important point which has been described here is the mutual impact of biological equilibrium and the economic equilibrium. The available equilibrium fish biomass and the harvested fish biomass, corresponding to the infection-free and infected equilibrium states, are determined under the variable parametric condition and the corresponding total revenue is also determined.

The proposed chapter is structured as follows: In Section 3.2, a bioeconomic fishery management is proposed. Section 3.3 describes all possible equilibrium points and their local stability analysis. Section 3.4 gives the solution methodology for the global stability of the system. Time series analysis and bifurcation results for some important parameters are given in Section 3.5. Conditions for bionomic equilibrium solution are established and illustrated in Section 3.6. Section 3.7 discusses the major findings and further research scopes.

3.2 Model construction

A fish population is considered that grows logistically in the absence of infection and harvesting. The growth rate of fish stock then can be represented by

$$\frac{d\Delta}{dt} = r\Delta\left(1 - \frac{\Delta}{K}\right), \tag{3.1}$$

where Δ is the biomass of fish population at time t, r is its intrinsic growth rate and K is the environmental carrying capacity. Suppose that the fish population is harvested by catch-per-unit (CPU) effort hypothesis with E as the harvesting effort and q_1 as the catchability coefficient. Then the fish growth rate in presence of harvesting can be expressed as

$$\frac{d\Delta}{dt} = r\Delta\left(1 - \frac{\Delta}{K}\right) - q_1 E\Delta. \tag{3.2}$$

The dynamics of both the systems (3.1) and (3.2) are simple. However, one important characteristic of model (3.2) is that the fish population may go to extinction if harvesting effort is higher than some critical value, *i.e.*, $E > E_c$, where $E_c = \frac{r}{q_1}$. Otherwise, the fish population stabilizes to $\Delta^* = K(1 - \frac{q_1 E}{r})$, where r, q_1, E are positive constants with $r > q_1 E$. On the contrary, population in model (3.1) can never be extinct, but stabilizes at a fish density $\Delta^* = K$, which is higher than that of the second system.

Suppose that the fish species are infected by some parasites and, in that case, the fish population is divided into two subpopulations, viz. susceptible fish (S) and infected fish (I), such that the total fish stock at time t is $\Delta(t) = S(t) + I(t)$. In such a case, the interactive dynamics of susceptible and infected fishes are given by

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - q_1 SE, \\ \frac{dI}{dt} &= \lambda SI - \mu I - q_2 IE. \end{aligned} \tag{3.3}$$

The underlying assumptions of this model are: (i) the microparasitic infection occurs through horizontal transmission and follows the mass action law with λ as the force of infection, (ii) the infected fishes do not recover from the infection and also do not reproduce, but contribute to the carrying capacity [154]. Here the parameter μ represents the total death (natural plus virulence) rate of infected fish, and q_2 corresponds to the catchability coefficient of the infected fish. As the infection may affect the fish ability of swimming, it is reasonable to assume that $q_2 \geq q_1$.

In the models (3.2) and (3.3), harvesting effort E has been considered as a parameter. In reality, however, fishing effort is not a constant but a time-varying variable [175]. By convention, fishing effort is assumed to vary with the profit margin (the difference between the selling price and cost price). It then makes sense to assume effort to be an increasing function of profit. For example, fishermen will employ more boats in fishing if profit is there, otherwise, they will opt-out from the fishery. The market price of per unit biomass of fish $p(t)$ at any time t is dependent on the demand and supply. It is assumed that the market price at a particular time varies with the difference between demanded quantity and supplied quantity at that time with proportionality constant $\phi_2 > 0$. Demand is represented by a linear function, e.g., $D(p) = A - \alpha p(t)$, where A and α are positive constants which represent, respectively, the maximal demand and the rate at which the demand decreases with price [139]. Based on these assumptions, the sustainable bioeconomic management of a fishery in presence of infection can be represented as

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S+I}{K}\right) - \lambda SI - q_1 SE, \\ \frac{dI}{dt} &= \lambda SI - \mu I - q_2 IE, \\ \frac{dE}{dt} &= \phi_1\Big(p(q_1 S + q_2 I)E - cE\Big), \\ \frac{dp}{dt} &= \phi_2\Big((A - \alpha p) - (q_1 S + q_2 I)E\Big), \end{aligned} \tag{3.4}$$

where c represents the cost per unit of fishing effort and $\phi_1 > 0$ is a proportionality constant, called stiffness parameter. Variables and parameters used in this model are presented in Table 3.1.

Table 3.1: Variables and parameters with their descriptions

Variable	Description
$S(t)$	Susceptible fish biomass at time t
$I(t)$	Infected fish biomass at time t
$E(t)$	Harvesting effort at time t
$p(t)$	Price per unit of biomass at time t
Parameter	**Description**
r	Intrinsic growth rate of susceptible fish
K	Environmental carrying capacity
λ	Force of infection
q_1	Catchability coefficient of susceptible fish
μ	Total death (natural + virulance) rate of infected fish
q_2	Catchability coefficient of infected fish
c	Cost per unit of fishing effort
A	Maximal demand
α	The rate at which the demand decrease
ϕ_1	Stiffness parameter
ϕ_2	Proportionality constant

Model (3.4) is a generalization of many other models. For example, if there is no infection (i.e. $\lambda = 0$) and $\phi_1 = 1$ then it becomes a basic fishery model with variable price (see for reference [175]). If $\lambda = 0$ and $\phi_2 = 0$ then the submodel was studied in [140]. However, if

$\lambda = 0$ and $\phi_1 = 0 = \phi_2$ then the dynamic evolution of the fishery model was discussed in [125] and [251]. The model becomes a SI-type epidemic model if $E = 0$ and $\phi_1 = 0 = \phi_2$, which has been analyzed from different points of view [252]. Finally, if $E = 0$, $\lambda = 0$ and $\phi_1 = 0 = \phi_2$ then it represents a basic population model having density-dependent regulation [253]. The general model (3.4) will be analyzed from the epidemic, over exploitation and sustainable points of view as well as from the economic perspective.

3.3 Investigation for equilibrium points and their local stability analysis

The proposed model (3.4) has five equilibrium points: (i) $\xi_1 = (0,0,0,\frac{A}{\alpha})$, (ii) $\xi_2 = (K,0,0,\frac{A}{\alpha})$, (iii) $\xi_3 = \left(\frac{\mu}{\lambda}, \frac{r(\lambda K-\mu)}{\lambda(1+\lambda K)}, 0, \frac{A}{\alpha}\right)$, $\lambda K > \mu$, (iv) $\xi_4 = \left(\bar{S}, 0, \frac{r}{q_1}\left(1-\frac{\bar{S}}{K}\right), \frac{c}{q_1\bar{S}}\right)$, $K > \bar{S}$, where $\bar{S}$ is the positive root of the equation

$$r q_1 \bar{S}^3 - r q_1 K \bar{S}^2 + q_1 K A \bar{S} - K\alpha c = 0,$$

and (v) the interior equilibrium point $\xi^* = (S^*, I^*, E^*, p^*)$, where

$$\begin{aligned} S^* &= \frac{\mu + q_2 E^*}{\lambda}, \\ I^* &= \frac{r\lambda K - r\mu - (\lambda K q_1 + r q_2)E^*}{\lambda(r+\lambda K)}, \\ p^* &= \frac{c\lambda(r+\lambda K)}{r q_1 \mu + \lambda K q_1 \mu + \lambda K r q_2 - r q_2 \mu + r q_1 q_2 E^* - r q_2^2 E^*}, \end{aligned}$$

and E^* is the positive root of the equation

$$\bar{X}E^3 + \bar{Y}E^2 + \bar{Z}E + \bar{W} = 0, \tag{3.5}$$

where the coefficients are

$$\begin{aligned} \bar{X} &= r^2 q_2^2 (q_1 - q_2)^2, \\ \bar{Y} &= 2(r q_1 \mu + \lambda K q_1 \mu + q_2 r \lambda K - q_2 r \mu) r q_2 (q_1 - q_2), \\ \bar{Z} &= (r q_1 \mu + \lambda K q_1 \mu + q_2 r \lambda K - q_2 r \mu)^2 - A\lambda r q_2 (q_1 - q_2)(r + \lambda K), \\ \bar{W} &= c\alpha\lambda^2 (r+\lambda K)^2 - A\lambda(r+\lambda K)(r q_1 \mu + \lambda K q_1 \mu + \lambda K q_2 r - q_2 r \mu). \end{aligned}$$

The conditions of having only one positive root of the equation (3.5) are $\bar{X} > 0, \bar{W} < 0$ and $18\bar{X}\bar{Y}\bar{Z}\bar{W} - 4\bar{Y}^3\bar{W} + \bar{Y}^2\bar{Z}^2 - 4\bar{X}\bar{Z}^3 - 27\bar{X}^2\bar{W}^2 < 0$ [254]. Note that I^* and p^* are positive if

$L_1 > 0$ and $L_2 > 0$, where

$$\begin{aligned} L_1 &= r\lambda K - r\mu - (\lambda K q_1 + r q_2)E^*, \\ L_2 &= r q_1 \mu + \lambda K q_1 \mu + \lambda K r q_2 + r q_1 q_2 E^* - r q_2 \mu - r q_2^2 E^*. \end{aligned} \tag{3.6}$$

Thus, one can write the following lemma.

Lemma 3.1. *An interior equilibrium exists uniquely if (i)* $\bar{X} > 0$*, (ii)* $\bar{W} < 0$*, (iii)* $18\bar{X}\bar{Y}\bar{Z}\bar{W} - 4\bar{Y}^3\bar{W} + \bar{Y}^2\bar{Z}^2 - 4\bar{X}\bar{Z}^3 - 27\bar{X}^2\bar{W}^2 < 0$*, (iv)* $L_1 > 0$ *and (v)* $L_2 > 0$.

Table 3.2: Parametric conditions for the nature of roots of (3.5)

No. of positive root	$\bar{Y}$	$\bar{Z}$	$\bar{W}$	Υ	Ψ	Ω
0	−	+	+	+	+	+
				+	−	−
				−	−	−
				+	−	+
				−	+	−
				−	−	+
	+	−	+	+	−	−
				−	−	+
	−	−	+	+	−	−
				−	−	+
	+	+	+	Any sign	Any sign	Any sign
1	+	+	−	Any sign	Any sign	Any sign
	+	−	−	Any sign	Any sign	Any sign
	−	−	−	Any sign	Any sign	Any sign
	−	+	−	+	+	+
				+	−	−
				−	−	−
				−	−	+
				+	−	+
				−	+	−
2	−	+	+	+	+	−
				−	+	+
	+	−	+	+	+	+
				+	+	−
				−	−	−
				−	+	+
				+	−	+
				−	+	−
	−	−	+	+	+	+
				+	+	−
				−	−	−
				−	+	+
				+	−	+
				−	+	−
	−	+	−	−	+	+
				+	+	−
3	−	+	−	−	+	+
				+	+	−

This lemma guarantees the existence of a unique interior equilibrium point, however, the number of interior equilibrium points may increase as the cubic equation (3.5) may have more than one positive root. For the existence of more than one bioeconomic equilibrium points, one can follow Sturm's method. This method uses Sturm's functions [255], which are implicit functions of the coefficients of the given cubic equations [256, 257], given by

$$\begin{aligned} S_f &= \bar{X}E^3+\bar{Y}E^2+\bar{Z}E+\bar{W}, \\ S_{f_1} &= E^2+\frac{2\bar{Y}}{3\bar{X}}E+\frac{\bar{Z}}{3\bar{X}}, \\ S_{f_2} &= E+\frac{\Upsilon}{\Psi}, \\ S_{f_3} &= \Omega, \end{aligned}$$

where

$$\Upsilon=\bar{Y}\bar{Z}-9\bar{X}\bar{W},\ \ \Psi=2\bar{Y}^2-6\bar{X}\bar{Z},\ \ \Omega=2\bar{Y}\frac{\Upsilon}{\Psi}-3\bar{X}\left\{\frac{\Upsilon}{\Psi}\right\}^2-\bar{Z}.$$

Depending on the signs of the coefficients $\bar{Y}$, $\bar{Z}$, $\bar{W}$ and signs of the key quantities Υ, Ψ, Ω, the equation (3.5) will have zero, one, two or three positive roots and consequently no, one, two and three interior equilibrium points will be obtained if L_1 and L_2 be also positive. An essence of the number of positive roots of equation (3.5) is listed in Table 3.2. In the simulation section, we will illustrate further about the number of equilibrium points using these Sturm's functions.

Different stability properties of the equilibrium points are presented here using the linearization technique. In this method, an equilibrium point is said to be locally asymptotically stable if all the eigenvalues of the variational matrix evaluated at the equilibrium point have negative real parts [258]. At any arbitrary equilibrium point $\xi=(\hat{S},\hat{I},\hat{E},\hat{p})$, the Jacobian matrix of the system (3.4) is given by

$$J=\begin{pmatrix} r\left(1-\frac{2S+I}{K}\right)-\lambda I-q_1E & -\left(\frac{r}{K}+\lambda\right)S & -q_1S & 0 \\ \lambda I & \lambda S-\mu-q_2E & -q_2I & 0 \\ \phi_1pq_1E & \phi_1pq_2E & \phi_1[p(q_1S+q_2I)-c] & \phi_1(q_1S+q_2I)E \\ -\phi_2q_1E & -\phi_2q_2E & -\phi_2(q_1S+q_2I) & -\phi_2\alpha \end{pmatrix}_{(\hat{S},\hat{I},\hat{E},\hat{p})}. \tag{3.7}$$

We now prove the following theorems in relation to the stability of different equilibrium points.

Theorem 3.2. *The equilibrium point* $\xi_1(0,0,0,\frac{A}{\alpha})$ *is always unstable. The equilibrium* $\xi_2(K,0,0,\frac{A}{\alpha})$ *is locally asymptotically stable if* $\lambda K<\mu$ *and* $q_1AK<\alpha c$ *hold simultaneously, otherwise it is unstable. The equilibrium* $\xi_3=\left\{\left(\frac{\mu}{\lambda},\frac{r(\lambda K-\mu)}{\lambda(1+\lambda K)},0,\frac{A}{\alpha}\right),\lambda K>\mu\right\}$ *is locally asymptotically stable if* $\frac{A}{\alpha}\left(\frac{q_1\mu}{\lambda}+\frac{q_2r(K\lambda-\mu)}{\lambda(1+K\lambda)}\right)<c$*, otherwise it is unstable.*

Proof. The Jacobian matrix at $\xi_1=(0,0,0,\frac{A}{\alpha})$ reads

$$J(0,0,0,\frac{A}{\alpha})=\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & -\phi_1c & 0 \\ 0 & 0 & 0 & -\phi_2\alpha \end{pmatrix}. \tag{3.8}$$

Here eigenvalues are r (>0), $-\mu$ (<0), $-\phi_1 c$ (<0), $-\phi_2\alpha$ (<0). Thus, the equilibrium point $(0,0,0,\frac{A}{\alpha})$ is always unstable (saddle).

At ξ_2 = (K,0,0,$\frac{A}{\alpha}$), the variational matrix is evaluated as

$$J(K,0,0,\frac{A}{\alpha}) = \begin{pmatrix} -r & -r-\lambda K & -q_1K & 0 \\ 0 & \lambda K-\mu & 0 & 0 \\ 0 & 0 & \phi_1(q_1\frac{AK}{\alpha}-c) & 0 \\ 0 & 0 & -\phi_2 q_1 K & -\phi_2\alpha \end{pmatrix}. \tag{3.9}$$

In this case, four eigenvalues are given by $-r$, $\lambda K-\mu$, $\phi_1(q_1\frac{AK}{\alpha}-c)$, $-\phi_2\alpha$. It is easy to see that $(K,0,0,\frac{A}{\alpha})$ is locally asymptotically stable if $\lambda K<\mu$ and $q_1AK<\alpha c$, otherwise it is unstable.

At the equilibrium ξ_3 = $\left(\frac{\mu}{\lambda},\frac{r(\lambda K-\mu)}{\lambda(1+\lambda K)},0,\frac{A}{\alpha}\right)$, we have

$$J\Big(\frac{\mu}{\lambda},\frac{r(\lambda K-\mu)}{\lambda(1+\lambda K)},0,\frac{A}{\alpha}\Big) = \begin{pmatrix} -\frac{r\mu}{K\lambda} & -(\frac{r}{K}+\lambda)\frac{\mu}{\lambda} & -q_1\frac{\mu}{\lambda} & 0 \\ \frac{r(K\lambda-\mu)}{1+K\lambda} & 0 & -\frac{q_2r(K\lambda-\mu)}{1+K\lambda} & 0 \\ 0 & 0 & \phi_1\left[\frac{A}{\alpha}\left(\frac{q_1\mu}{\lambda}+\frac{q_2r(K\lambda-\mu)}{\lambda(1+K\lambda)}\right)-c\right] & 0 \\ 0 & 0 & -\phi_2\left[\frac{q_1\mu}{\lambda}+\frac{q_2r(K\lambda-\mu)}{\lambda(1+K\lambda)}\right] & -\phi_2\alpha \end{pmatrix}. \tag{3.10}$$

Here eigenvalues are

$$-\phi_2\alpha,\ \phi_1\left[\frac{A}{\alpha}\left(\frac{q_1\mu}{\lambda}+\frac{q_2r(K\lambda-\mu)}{\lambda(1+K\lambda)}\right)-c\right], \frac{1}{2}\left[-\frac{r\mu}{K\lambda}+\sqrt{\frac{r^2\mu^2}{K^2\lambda^2}-4\frac{\mu r(K\lambda-\mu)\big(\lambda+\frac{r}{K}\big)}{\lambda(K\lambda+1)}}\right]$$

and

$$\frac{1}{2}\left[-\frac{r\mu}{K\lambda}-\sqrt{\frac{r^2\mu^2}{K^2\lambda^2}-4\frac{\mu r(K\lambda-\mu)\big(\lambda+\frac{r}{K}\big)}{\lambda(K\lambda+1)}}\right].$$

Therefore, ξ_3 is locally asymptotically stable if $\frac{A}{\alpha}\left(\frac{q_1\mu}{\lambda}+\frac{q_2r(K\lambda-\mu)}{\lambda(1+K\lambda)}\right)<c$, otherwise it is unstable. Hence the theorem is proven. □

Theorem 3.3. *The equilibrium point* $\xi_4=\left(\bar{S},0,\frac{r}{q_1}\big(1-\frac{\bar{S}}{K}\big),\frac{c}{q_1\bar{S}}\right)$*, whenever it exists, is locally asymptotically stable if* $\mu>\lambda\bar{S}$ *and* $X_1Y_1Z_1>(Z_1^2+X_1^2W_1)$ *hold simultaneously, where* X_1,Y_1,Z_1 *and* W_1 *are defined in* (3.13).

Proof. At the equilibrium point $\xi_4=\left(\bar{S},0,\frac{r}{q_1}\big(1-\frac{\bar{S}}{K}\big),\frac{c}{q_1\bar{S}}\right)$, one can evaluate the variational matrix as

$$J\left(\bar{S},0,\frac{r}{q_1}(1-\frac{\bar{S}}{K}),\frac{c}{q_1\bar{S}}\right)=\begin{pmatrix} -\frac{r\bar{S}}{K} & -\frac{r\bar{S}}{K}-\lambda\bar{S} & -q_1\bar{S} & 0 \\ 0 & \lambda\bar{S}-\mu-\frac{q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right) & 0 & 0 \\ \frac{\phi_1rc}{q_1\bar{S}}\left(1-\frac{\bar{S}}{K}\right) & \frac{\phi_1q_2rc}{q_1^2\bar{S}}\left(1-\frac{\bar{S}}{K}\right) & 0 & \phi_1r\bar{S}\left(1-\frac{\bar{S}}{K}\right) \\ -\phi_2r\left(1-\frac{\bar{S}}{K}\right) & -\frac{\phi_2q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right) & -\phi_2q_1\bar{S} & -\phi_2\alpha \end{pmatrix} \tag{3.11}$$

and the corresponding characteristic equation reads

$$\beta^4+X_1\beta^3+Y_1\beta^2+Z_1\beta+W_1=0, \tag{3.12}$$

where

$$\begin{aligned} X_1 &= \frac{r\bar{S}}{K}+\phi_2\alpha-\lambda\bar{S}+\mu+\frac{q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right), \\ Y_1 &= -\frac{r\bar{S}}{K}\left\{\lambda\bar{S}-\mu-\frac{q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right)\right\}+\phi_2\alpha\left(\frac{r}{K}+\lambda\right)\bar{S}+\frac{r}{K}\phi_2\alpha\bar{S}+ \\ &\quad \phi_1\phi_2q_1r\bar{S}^2\left(1-\frac{\bar{S}}{K}\right)+\phi_1rc\left(1-\frac{\bar{S}}{K}\right), \\ Z_1 &= -\frac{\phi_2r\alpha\bar{S}}{K}\left\{\lambda\bar{S}-\mu-\frac{q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right)\right\}-\phi_1\phi_2q_1r\bar{S}^2\left(1-\frac{\bar{S}}{K}\right)\left\{\lambda\bar{S}-\mu-\right. \\ &\quad \left.\frac{q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right)\right\}-\phi_1rc\left(1-\frac{\bar{S}}{K}\right)\left\{\lambda\bar{S}-\mu-\frac{q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right)\right\}+\phi_1\phi_2r \\ &\quad \left(1-\frac{\bar{S}}{K}\right)\left(\bar{S}^2r(\phi_2q_2-q_1)+\alpha c\right), \\ W_1 &= -\left\{\lambda\bar{S}-\mu-\frac{q_2r}{q_1}\left(1-\frac{\bar{S}}{K}\right)\right\}\left(1-\frac{\bar{S}}{K}\right)\left\{\frac{r\phi_1\phi_2c\bar{S}^2}{K}+\phi_1\phi_2r^2q_1\bar{S}^2\left(1-\frac{\bar{S}}{K}\right)\right. \\ &\quad \left.+\phi_1\phi_2rc\alpha\right\}. \end{aligned} \tag{3.13}$$

All roots of (3.12) will be negative or have negative real parts [259] if $X_1>0, Z_1>0, W_1>0$ and $X_1Y_1Z_1>(Z_1^2+X_1^2W_1)$. The first three conditions hold if $\mu>\lambda\bar{S}$. Thus, the equilibrium ξ_4 becomes stable whenever $X_1Y_1Z_1>(Z_1^2+X_1^2W_1)$ holds. Hence the theorem. □

Theorem 3.4. *Suppose an internal equilibrium point $\xi^*=\left(S^*,I^*,E^*,p^*\right)$ exists uniquely. Then ξ^* is locally asymptotically stable if $\alpha p^{*2}>cE^*$ and $X_2Y_2Z_2>Z_2^2+X_2^2W_2$ hold together, where X_2,Y_2,Z_2, W_2 are defined in* (3.16).

Proof. Suppose the conditions of Lemma 3.1 hold and there exists a unique interior equilibrium

$\xi^* = (S^*, I^*, E^*, p^*)$ of the system (3.4). In this case, the Jacobian matrix is evaluated as

$$J(S^*, I^*, E^*, p^*) = \begin{pmatrix} -\frac{rS^*}{K} & -\left(\frac{r}{K}+\lambda\right)S^* & -q_1S^* & 0 \\ \lambda I^* & 0 & -q_2I^* & 0 \\ \phi_1 q_1 p^* E^* & \phi_1 q_2 p^* E^* & 0 & \frac{\phi_1 c E^*}{p^*} \\ -\phi_2 q_1 E^* & -\phi_2 q_2 E^* & -\phi_2 \frac{c}{p^*} & -\phi_2 \alpha \end{pmatrix}. \tag{3.14}$$

The corresponding characteristic equation can be expressed as

$$\beta^4 + X_2\beta^3 + Y_2\beta^2 + Z_2\beta + W_2 = 0, \tag{3.15}$$

where

$$\begin{aligned} X_2 &= \phi_2\alpha + \frac{rS^*}{K}, \\ Y_2 &= \frac{r\phi_2\alpha S^*}{K} + \phi_1 q_2^2 p^* E^* I^* + \left(\frac{r}{K}+\lambda\right)\lambda I^* S^* + \phi_1 q_1^2 S^* p^* E^* + \frac{c^*\phi_1\phi_2 E^*}{p^{*2}}, \\ Z_2 &= \phi_1\phi_2 E(I^* q_2^2 + S^* q_1^2)\left(\alpha p^* - \frac{cE^*}{p^*}\right) + \lambda\phi_2\alpha\left(\frac{r}{K}+\lambda\right)S^*I^* + \frac{\phi_1 q_2^2 r}{K}S^*I^*E^*p^* \\ &\quad + \lambda\phi_1 q_1 q_2 E^* I^{*2} p^* + \phi_1 q_1 q_2 \lambda p^* E^* S^* I^* + \frac{r\phi_1\phi_2 c^2 S^* E^*}{Kp^{*2}}, \\ W_2 &= \left(\frac{\phi_1\phi_2 q_2 \alpha r S^* I^* E^* p^*}{K} + \frac{\phi_1\phi_2 q_2 c r S^* I^* E^{*2}}{Kp^{*2}}\right)(q_2 - q_1) + \phi_1\phi_2 c^2\lambda\left(\frac{r}{K}+\lambda\right)\frac{S^*I^*E^*}{p^{*2}}. \end{aligned} \tag{3.16}$$

Here X_2 and W_2 are always positive. Note that Z_2 also becomes positive if $\alpha p^{*2} > cE^*$. So, by Routh-Hurwitz Criteria [259], the equilibrium point ξ^* is locally asymptotically stable if $\alpha p^{*2} > cE^*$ and $X_2Y_2Z_2 - (Z_2^2 + X_2^2 W_2) > 0$ hold simultaneously. This proves the theorem. □

3.4 Investigation for global stability

In this section the interest is to know about the basin of attraction of the interior equilibrium point ξ^*, i.e, whether ξ^* is globally asymptotically stable. The most common method for such studies is the construction of Lyapunov function [260]. But for the system (3.4), it is quite laboured to construct such global Lyapunov function. Hence high-dimensional Bendixson criterion is utilized [261].

The following assumption is assumed for the global stability of the internal equilibrium point of the system (3.4).

(**H**) There exist positive numbers ω, θ, ν, ρ and σ such that

$$\max\left\{c_{11} + \tfrac{c_{12}\omega}{\theta} + \tfrac{c_{14}\omega}{\nu}, \tfrac{c_{21}\theta}{\omega} + c_{22} + c_{23}\theta + \tfrac{c_{24}\theta}{\nu}, \tfrac{c_{31}}{w} + \tfrac{c_{32}}{\theta} + c_{33} + \tfrac{c_{35}}{\rho} + \tfrac{c_{36}}{\sigma}, \tfrac{c_{41}\nu}{\omega} + \tfrac{c_{42}\nu}{\theta} + \tfrac{c_{45}\nu}{\rho},\right.$$
$$\left.\tfrac{c_{51}\rho}{\omega} + c_{53}\rho + \tfrac{c_{54}\rho}{\nu} + c_{55} + \tfrac{c_{56}\rho}{\sigma}, \tfrac{c_{62}\sigma}{\omega} + c_{63}\sigma + \tfrac{c_{64}\sigma}{\nu} + \tfrac{c_{65}\sigma}{\rho} + c_{66}\right\} < 0.$$

Theorem 3.5. *The system* (3.4) *has no non-trivial periodic solutions under the hypothesis (**H**). Furthermore, the unique positive equilibrium* ξ^* *is globally stable in* $\mathbb{R}_+^{4,0}$.

Proof. Let $F : X \mapsto F(X) \in R^n$ be a C^1 function for $X \in D$, where $D \subset \mathbb{R}^n$ be an open set. Also consider the differential equation

$$\frac{dX}{dt} = F(X). \tag{3.17}$$

To find a high-dimensional Bendixson criterion, it is sufficient to show that the second compound equation

$$\frac{dZ}{dt} = \frac{\partial F^{[2]}}{\partial X}(X(t,X_0))Z(t) \tag{3.18}$$

with respect to a solution $X(t,X_0) \in D$ of system (3.17) is equi-uniformly asymptotically stable [261]. That means, for each $X_0 \in D$, the system (3.18) is uniformly asymptotically stable, and the exponential decay rate is uniform for X_0 in each compact subset of D, where $D \in \mathbb{R}^n$ is an open connected set. The second additive compound matrix of the Jacobian matrix $\frac{\partial F}{\partial X}$ is $\frac{\partial F^{[2]}}{\partial X}$, which is a $\binom{n}{2} \times \binom{n}{2}$ matrix, and thus the dimension of the linear system (3.17) is $\binom{n}{2}$ [262, 263]. So, if $A = (a_{ij})_{4\times4}$ be the general 4×4 matrix, then

$$A^{[2]} = \begin{pmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{pmatrix} \tag{3.19}$$

will be its second additive compound matrix. The equi-uniform asymptotic stability of (3.17) implies the exponential decay of the surface area of any compact two-dimensional surface in D. If D is simply connected, this excludes the existence of any invariant simple closed rectifiable curve within D, including periodic orbits. The following proposition will be used in the sequel.

Proposition 3.6. *Let* $D \subset \mathbb{R}^n$ *be a simply connected region. Assume that the family of linear system* (3.17) *is equi-uniformly asymptotically stable. Then*

(a) D *contains no simple closed invariant curves, including periodic orbits, homoclinic orbits, heteroclinic cycles;*

(b) each semi-orbit in D *converges to a single equilibrium.*

In particular, if D *is positively invariant and contains a unique equilibrium* X*, then* X *is globally asymptotically stable in* D.

For model (3.4), denote $X=(S,I,E,p)^T$ and

$$\begin{aligned} F[X] &= \Bigg(rS\Big(1-\frac{S+I}{K}\Big)-\lambda SI-q_1SE,\lambda SI-\mu I-q_2IE,\phi_1\Big(p(q_1S+q_2I)E-cE\Big),\\ &\quad \phi_2\Big(A-\alpha p-(q_1S+q_2I)E\Big)\Bigg)^T. \end{aligned}$$

We then have

$$\frac{\partial F}{\partial X}=\begin{pmatrix} r\big(1-\frac{2S+I}{K}\big)-\lambda I-q_1E & -\big(\frac{r}{K}+\lambda\big)S & -q_1S & 0\\ \lambda I & \lambda S-\mu-q_2E & -q_2I & 0\\ \phi_1pq_1E & \phi_1pq_2E & \phi_1[p(q_1S+q_2I)-c] & \phi_1(q_1S+q_2I)E\\ -\phi_2q_1E & -\phi_2q_2E & -\phi_2(q_1S+q_2I) & -\phi_2\alpha \end{pmatrix}$$

and assume

$$\frac{\partial F^{[2]}}{\partial X}=\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16}\\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26}\\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36}\\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46}\\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56}\\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{pmatrix}.$$

By (3.19), we obtain
$b_{11}=r\big(1-\frac{2S+I}{K}\big)-\lambda I-q_1E+\lambda S-\mu-q_2E$, $b_{12}=-q_2I$, $b_{13}=0$, $b_{14}=q_1S$, $b_{15}=0$, $b_{16}=0$, $b_{21}=\phi_1pq_2E$, $b_{22}=r\big(1-\frac{2S+I}{K}\big)-\lambda I-q_1E+\phi_1[p(q_1S+q_2I)-c]$, $b_{23}=\phi_1(q_1S+q_2I)E$, $b_{24}=-\big(\frac{r}{K}+\lambda\big)S$, $b_{25}=0$, $b_{26}=0$, $b_{31}=-\phi_2q_2E$, $b_{32}=-\phi_2(q_1S+q_2I)$, $b_{33}=r\big(1-\frac{2S+I}{K}\big)-\lambda I-q_1E-\phi_2\alpha$, $b_{34}=0$, $b_{35}=-\big(\frac{r}{K}+\lambda\big)S$, $b_{36}=-q_1S$, $b_{41}=-\phi_1pq_1E$, $b_{42}=\lambda I$, $b_{43}=0$, $b_{44}=\lambda S-\mu-q_2E+\phi_1[p(q_1S+q_2I)-c]$, $b_{45}=\phi_1(q_1S+q_2I)E$, $b_{46}=0$, $b_{51}=\phi_2q_1E$, $b_{52}=0$, $b_{53}=\lambda I$, $b_{54}=\phi_1[p(q_1S+q_2I)-c]$, $b_{55}=\lambda S-\mu-q_2E-\phi_2\alpha$, $b_{56}=-q_2I$, $b_{61}=0$, $b_{62}=\phi_2q_1E$, $b_{63}=\phi_1pq_1E$, $b_{64}=\phi_2q_2E$, $b_{65}=\phi_1pq_2E$, $b_{66}=\phi_1[p(q_1S+q_2I)-c]-\phi_2\alpha$.
The second compound system

$$(\dot{z_1},\dot{z_2},\dot{z_3},\dot{z_4},\dot{z_5},\dot{z_6})^t=\frac{\partial F^{[2]}}{\partial X}(z_1,z_2,z_3,z_4,z_5,z_6)^t$$

then becomes

$$\dot{z_1} = \left[r\big(1-\frac{2S+I}{K}\big)-\lambda I-q_1E+\lambda S-\mu-q_2E\right]z_1-q_2Iz_2+q_1Sz_4,$$

$$
\begin{aligned}
\dot{z_2} &= \phi_1 p q_2 E z_1 + \Big(r\big(1-\frac{2S+I}{K}\big) - \lambda I - q_1 E + \phi_1 (p(q_1 S + q_2 I) - c) \Big) z_2 + \\
&\quad \phi_1 (q_1 S + q_2 I) E z_3 - \big(\frac{r}{K} + \lambda\big) S z_4, \\
\dot{z_3} &= -\phi_2 q_2 E z_1 - \phi_2 (q_1 S + q_2 I) z_2 + \Big(r\big(1-\frac{2S+I}{K}\big) - \lambda I - q_1 E - \phi_2 \alpha \Big) z_3 \\
&\quad -\big(\frac{r}{K} + \lambda\big) S z_5 - q_1 S z_6, \qquad (3.20) \\
\dot{z_4} &= \lambda I z_2 - \phi_1 p q_1 E z_1 + \Big(\lambda S - \mu - q_2 E + \phi_1 \big(p(q_1 S + q_2 I) - c\big) \Big) z_4 + \phi_1 (q_1 S + q_2 I) E z_5, \\
\dot{z_5} &= \phi_2 q_1 E z_1 + \lambda I z_3 + \phi_1 [p(q_1 S + q_2 I) - c] z_4 + (\lambda S - \mu - q_2 E - \phi_2 \alpha) z_5 - q_2 I z_6, \\
\dot{z_6} &= \phi_2 q_1 E z_2 + \phi_1 p q_1 E z_3 + \phi_2 q_2 E z_4 + \phi_1 p q_2 E z_5 + \Big(\phi_1 \big(p(q_1 S + q_2 I) - c\big) - \phi_2 \alpha \Big) z_6.
\end{aligned}
$$

Now, set

$$W(Z) = max\{\omega|z_1|, \theta|z_2|, |z_3|, \nu|z_4|, \rho|z_5|, \sigma|z_6|\}.$$

Then from (3.20), one can get the following inequalities

$$
\begin{aligned}
&\frac{d^+}{dt}\omega|z_1| \le c_{11}\omega|z_1| + \frac{c_{12}\omega}{\theta}\theta|z_2| + \frac{c_{14}\omega}{\nu}\nu|z_4|, \\
&\frac{d^+}{dt}\theta|z_2| \le \frac{c_{21}\theta}{\omega}\omega|z_1| + c_{22}\theta|z_2| + c_{23}\theta|z_3| + \frac{c_{24}\theta}{\nu}\nu|z_4|, \\
&\frac{d^+}{dt}|z_3| \le \frac{c_{31}}{w}\omega|z_1| + \frac{c_{32}}{\theta}\theta|z_2| + c_{33}|z_3| + \frac{c_{35}}{\rho}\rho|z_5| + \frac{c_{36}}{\sigma}\sigma|z_6|, \qquad (3.21) \\
&\frac{d^+}{dt}\nu|z_4| \le \frac{c_{41}\nu}{\omega}\omega|z_1| + \frac{c_{42}\nu}{\theta}\theta|z_2| + \frac{c_{45}\nu}{\rho}\rho|z_5|, \\
&\frac{d^+}{dt}\rho|z_5| \le \frac{c_{51}\rho}{\omega}\omega|z_1| + c_{53}\rho|z_3| + \frac{c_{54}\rho}{\nu}\nu|z_4| + c_{55}\rho|z_5| + \frac{c_{56}\rho}{\sigma}\sigma|z_6|, \\
&\frac{d^+}{dt}\sigma|z_6| \le \frac{c_{62}\sigma}{\omega}\theta|z_2| + c_{63}\sigma|z_3| + \frac{c_{64}\sigma}{\nu}\nu|z_4| + \frac{c_{65}\sigma}{\rho}\rho|z_5| + c_{66}\sigma|z_6|,
\end{aligned}
$$

in which $\frac{d^+}{dt}$ denotes the right-hand derivative and

$$
\begin{aligned}
&c_{11} = -\frac{rS^*}{K}, c_{12} = -q_2 I^*, c_{13} = 0, c_{14} = q_1 S^*, c_{15} = 0, c_{16} = 0, \\
&c_{21} = \phi_1 p^* q_2 E^*, c_{22} = -\frac{rS^*}{K}, c_{23} = \frac{\phi_1 c E^*}{p^*}, c_{24} = -\big(\frac{r}{K} + \lambda\big) S^*, c_{25} = 0 = c_{26}, \\
&c_{31} = -\phi_2 q_2 E^*, c_{32} = -\frac{\phi_2 c}{p^*}, c_{33} = -\frac{rS^*}{K}, c_{34} = 0, c_{35} = -\big(\frac{r}{K} + \lambda\big) S^*, c_{36} = -q_1 S^*, \\
&c_{41} = -\phi_1 q_1 p^* E^*, c_{42} = \lambda I^*, c_{43} = 0, c_{44} = 0, c_{45} = \frac{\phi_1 c E^*}{p^*}, c_{46} = 0, \\
&c_{51} = \phi_2 q_1 E^*, c_{52} = 0, c_{53} = \lambda I^*, c_{54} = -\frac{\phi_2 c}{p^*}, c_{55} = -\phi_2 \alpha, c_{56} = -q_2 I^*, \\
&c_{61} = 0, c_{62} = \phi_2 q_1 E^*, c_{63} = \phi_1 q_1 p^* E^*, c_{64} = \phi_2 q_2 E^*, c_{65} = \phi_1 p^* q_2 E^*, c_{66} = -\phi_2 \alpha.
\end{aligned}
$$

Therefore,

$$\frac{d^+}{dt}W(Z(t)) \leq \psi W(Z(t))$$

with

$\psi = \max\{c_{11} + \frac{c_{12}\omega}{\theta} + \frac{c_{14}\omega}{\nu},\ \frac{c_{21}\theta}{\omega} + c_{22} + c_{23}\theta + \frac{c_{24}\theta}{\nu},\ \frac{c_{31}}{w} + \frac{c_{32}}{\theta} + c_{33} + \frac{c_{35}}{\rho} + \frac{c_{36}}{\sigma},\ \frac{c_{41}\nu}{\omega} + \frac{c_{42}\nu}{\theta} + \frac{c_{45}\nu}{\rho},\ \frac{c_{51}\rho}{\omega} + c_{53}\rho + \frac{c_{54}\rho}{\nu} + c_{55} + \frac{c_{56}\rho}{\sigma},\ \frac{c_{62}\sigma}{\omega} + c_{63}\sigma + \frac{c_{64}\sigma}{\nu} + \frac{c_{65}\sigma}{\rho} + c_{66}\}.$
Now, using the hypothesis (**H**) and since the system (3.4) is bounded, there exists a positive constant η such that $\psi \leq -\eta < 0$, implying

$$W(Z(t)) \leq W(Z(s))\exp(-\eta(t-s)),\ t \geq s > 0.$$

Hence the second compound system (3.19) is equi-uniform asymptotic stable, and therefore the interior equilibrium ξ^* of model (3.4) is globally stable, following Proposition 3.6. □

3.5 Simulation results

Based on the analytical results in Section 3.3, the dynamics of the system (3.4) is simulated. We first search for a parameter set that satisfies the basic model assumptions and gives at least one interior equilibrium point. The following set of hypothetical (but biologically reasonable) parameter values are considered:

$$r = 0.9, \lambda = 0.3, q_1 = 0.5, \mu = 0.05, q_2 = 0.7, c = 2, \phi_1 = 0.1, \phi_2 = 0.15, A = 2.1, \alpha = 0.61.$$

We then find the range of K by plotting L_1 and L_2 (see Eq. (3.6)) to ensure that I^* and P^* are positive. Subsequently, we plot the key quantities of Table 3.2 to find whether there exists any interior equilibrium point of the system (3.4). Existence and uniqueness of the equilibrium point is verified by checking the signs of the key quantities.

Figure 3.1 shows that the cubic equation (3.5) has a unique positive root as well as $I^* > 0$, $p^* > 0$ in the range $1.15 < K < 2.02$ and $K > 4.56$. Therefore, a unique interior equilibrium point of the system (3.4) exists for those ranges of K. For example, if we choose $K = 1.5$ then $L_1 = 0.1464 > 0$, $L_2 = 0.2543 > 0$ and the quantities mentioned in Table 3.2 are evaluated as

$$\bar{Y} = -0.0720,\ \bar{Z} = 0.1888,\ \bar{W} = -0.0429,\ \Upsilon = -0.0075,\ \Psi = -0.0076,\ \Omega = -0.3757.$$

From the 17th row of Table 3.2 and Lemma 3.1, one can notice that the sign restrictions of a unique equilibrium point are satisfied in this case, implying that the system (3.4) has exactly one interior equilibrium point for this parameter set. Similarly, for $K = 7$, $L_1 = 0.3693 > 0$, $L_2 = 1.2558 > 0$ and the key quantities

$$\bar{Y} = -0.03444,\ \bar{Z} = 2.1055,\ \bar{W} = -1.5945,\ \Upsilon = -0.4972,\ \Psi = -0.0366,\ \Omega = -1.5375,$$

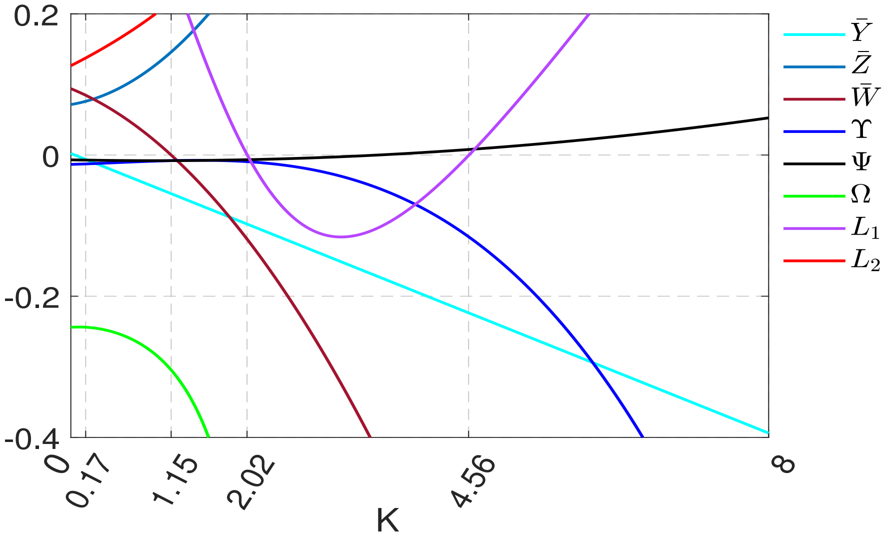

Figure 3.1: Determination of positive roots of the cubic equation (3.5) using the signs of the quantities $\bar{Y}$, $\bar{Z}$,$\bar{W}$ Υ, Ψ, and Ω (see Table 3.2) and positive value of I^*, p^* using L_1, L_2 with respect to K. This figure shows that there exists a unique interior equilibrium point of the model (3.4) for $1.15 < K < 2.02$ & $K > 4.56$ and no interior equilibrium point for $0 < K < 1.15$ & $2.02 < K < 4.56$.

follow the sign restrictions of the 20th row of Table 3.2, giving rise to a unique interior equilibrium point of the model (3.4). In fact, for all values K in the range $1.15 < K < 2.02$, the quantities $\bar{Y}$, $\bar{Z}$,$\bar{W}$ Υ, Ψ, Ω will satisfy the sign restrictions of the 17th row of Table 3.2, and the 20th row of Table 3.2 will be satisfied for all $K > 4.56$. Thus, there exists a unique positive root of the equation (3.5) and in both cases $L_1 > 0$, $L_2 > 0$, ascertaining a unique interior equilibrium of the system in these ranges. Similar observation shows that there is no interior equilibrium point for $0 < K < 1.15$ and $2.02 < K < 4.56$. Analogous study can be obtained for any other parameter. One may find a parameter set where more than one interior equilibrium exists. The parameter set considered here, however, provides a unique interior equilibrium only. After obtaining the existence range of the unique interior equilibrium point, we look for its stability range by verifying the stability criteria of the interior equilibrium point (see Theorem 3.4) for the same parameter set. From Fig. 3.2, one can be sure that the stability conditions $X_2 > 0$, $Z_2 > 0, W_2 > 0$ and $X_2Y_2Z_2 - (Z_2^2 + X_2^2W_2) > 0$ are satisfied in the region $1.15 < K < 2.02$ and $K > 4.56$, implying that the interior equilibrium point is stable in these ranges of K. Hence, the following values are taken with appropriate units

$$\begin{aligned}&r = 0.9,\ K = 7,\ \lambda = 0.3,\ q_1 = 0.5,\ \mu = 0.05,\ q_2 = 0.7,\ c = 2,\ \phi_1 = 0.1,\ \phi_2 = 0.15,\\ &A = 2.1,\ \alpha = 0.61\end{aligned} \tag{3.22}$$

as the baseline values and will remain fixed unless it is mentioned.

Fig. 3.3(a) with the parameter set (3.22) shows that all the state variables coexist in a stable

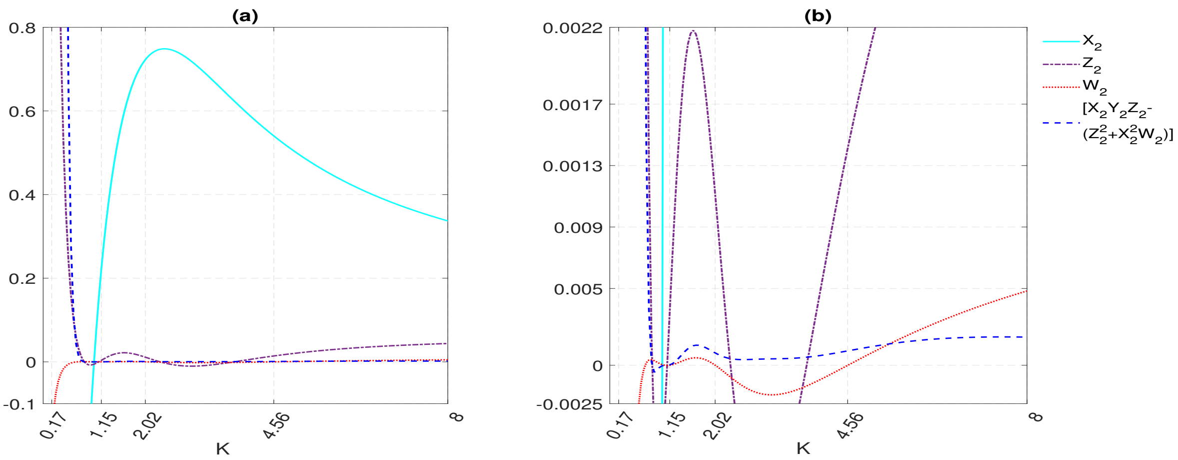

Figure 3.2: Stability of the interior equilibrium point for various ranges of K. (a) Following Theorem 3.4, ξ^* is stable in the ranges $1.15 < K < 2.02$ and $K > 4.56$ as X_2, Z_2, W_2 and $X_2Y_2Z_2-(Z_2^2+X_2^2W_2)$ are all positive there. (b) Positivity of these expressions are shown with a magnification. Parameters are as in (3.22).

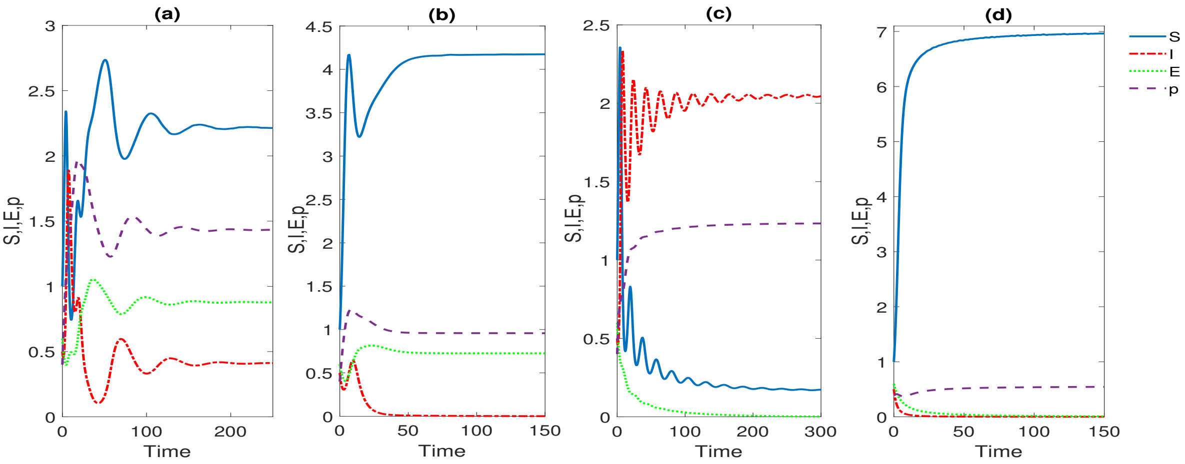

Figure 3.3: Time evolutions of the model (3.4) with initial conditions (1,0.5,0.6,0.4). (a) For $\lambda = 0.3$, the interior equilibrium ξ^* is locally asymptotically stable. (b) For $\lambda = 0.07$, ξ_4 is locally asymptotically stable. (c) The equilibrium ξ_3 is locally asymptotically stable for $\alpha = 1.7$. (d) For $\lambda = 0.007$ and $\alpha = 3.8$, the equilibrium ξ_2 is locally asymptotically stable. Parameter sets are as in (3.22).

state at their equilibrium values $S^* = 2.22, I^* = 0.41, E^* = 0.88, p^* = 1.43$. Total fish stock at the equilibrium level is around 2.63 units, out of which 0.41 units are infected and the remaining 2.22 units are healthy. Note that here price is high (1.43 per unit of fish) because supply is less due to low stock level. The harvesting effort is dependent on the profit margin. As the harvested stock is not high, the profit margin is also not high, and consequently, harvesting effort remains low even when the price is high enough. At lesser value of the force of infection, $\lambda = 0.07$, the system becomes infection-free (Fig. 3.3b). In this case, the system converges to the infection-free equilibrium $\xi_4 = (4.17, 0, 0.73, 0.96)$. Total fish production increases from 2.63 units in the presence of infection to 4.47 units in the absence of infection, and therefore fish price decreases due to its market availability. The fishery may be profitable in this case due to the higher availability of fishing stock even at the lower stock price in comparison to the previous case. However, if we increase the demand-decreasing parameter α from its base value 0.61 to 1.7, instead of reducing the force of infection λ, then the system stabilizes to the equilibrium $\xi_3 = (0.17, 2.04, 0, 1.23)$, where harvesting effort diminishes to zero but the other three state variables coexist in a stable state (Fig. 3.3c).

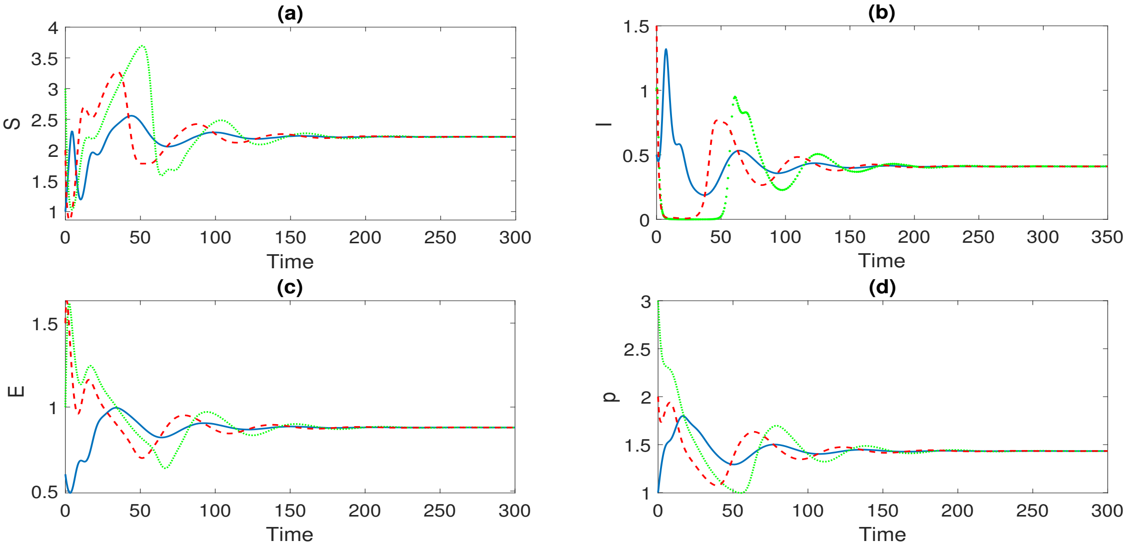

Figure 3.4: Global stability of the coexisting equilibrium ξ^*. Solutions starting from the initial points $(1,0.5,0.6,1),(3,1,1,3)$ and $(2,1.5,1.5,2)$ asymptotically converge to the steady-state values $S^* = 2.22, I^* = 0.41, E^* = 0.88, p^* = 1.43$. Parameters are as in (3.22) with $\omega = 0.0100,\ \theta = 0.10000,\ \rho = 9.0000,\ \sigma = 0.0001$.

As expected, demand decreases here at a faster rate and consequently price also decreases. As profit margin decreases due to lower fish price, fishermen gradually opt-out from the fishery and therefore E goes to zero. It is also to be noted that infected fish density increases in the absence of harvesting, implying that harvesting helps to reduce infection in the host population. The coexistence equilibrium might lose its stability and switch to the equilibrium point $\xi_2 = (6.97, 0, 0, 0.54)$, where the infected population and harvesting effort both are absent if the force of infection decreases and demand-decreasing rate parameter increases simultaneously (Fig. 3.3d). It is actually the combined case of Fig. 3.3(b) and Fig. 3.3(c). Due to a lower force of infection, the disease can not persist in its host population and fishing is also not profitable due to low demand.

To show that ξ^* is globally asymptotically stable, the same parameters are considered as in (3.22) and choose the positive numbers $\omega = 0.0100,\ \theta = 0.10000,\ \rho = 9.0000,\ \sigma = 0.0001$. In this case, the hypothesis (**H**) is satisfied as

$$\max\{-0.3342, -0.4564, -11383, -79.3863, -12.8048, -0.42\} = -0.3342 < 0$$

and hence, following Theorem 3.5, the unique positive equilibrium ξ^* is globally asymptotically stable in $\mathbb{R}^4_+$. Solutions starting from different initial points approach asymptotically to the interior equilibrium point $\xi^* = (2.22, 0.41, 0.88, 1.43)$, indicating global stability of ξ^* (see Fig. 3.4).

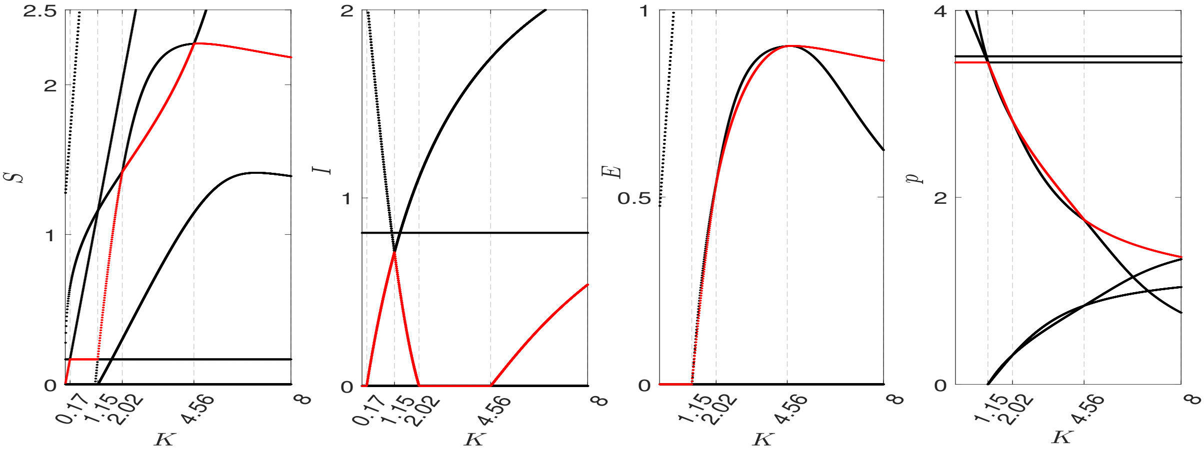

Figure 3.5: One parameter bifurcation diagrams with respect to the parameter K. The bifurcation points, where an exchange of stability occurs through transcritical bifurcation, are mentioned in the horizontal axis. For example, there exist four transcritical bifurcations in S population due to the variation of K at $0.17, 1.15, 2.02, 4.56$. Here red color represents the stable branch and black color represents the unstable branch. Other parameters remain fixed as in (3.22).

We have drawn different one-parameter bifurcation diagrams (see Fig. 3.5 and Fig. 3.6) to observe the response of the system with respect to a parameter when other parameters remain fixed. The environmental carrying capacity (K) is the most valuable parameter in this model and is assumed to be responsible for producing various dynamics including the paradox of enrichment [264] and species extinction [265]. We, therefore, explored the changes in system dynamics when K is varied in the range $0 < K < 8$. Fig. 3.5 demonstrates that the equilibrium ξ_2 is stable for lower values of K ($0 < K < 0.17$), where S and p components have non-zero equilibrium values, but I and E are zero. Note that red color represents the stable components of ξ_2 equilibrium, and black color represents the unstable component of other equilibrium points in the same range of K. In the range $0.17 < K < 1.15$, the equilibrium ξ_3 is stable, having non-zero components for S, I, p and zero for E. In the range $1.15 < K < 2.02$, the interior equilibrium ξ^* is stable, where all state variables have a non-zero value. For $2.02 < K < 4.56$, the equilibrium ξ_4 is stable, having non-zero components for S, E, p and zero for I. For $K > 4.56$, the interior equilibrium point ξ^* again becomes stable. Thus, there are four transcritical bifurcation points at $K = 0.17, 1.15, 2.02$ and 4.56, where two equilibrium points meet and interchange their stabilities. It is to be recalled that the environmental carrying capacity of the system measures how much individuals the system can support when a population grows there in an isolated way. Thus, an infection can not invade a host whose density is too low, i.e., the

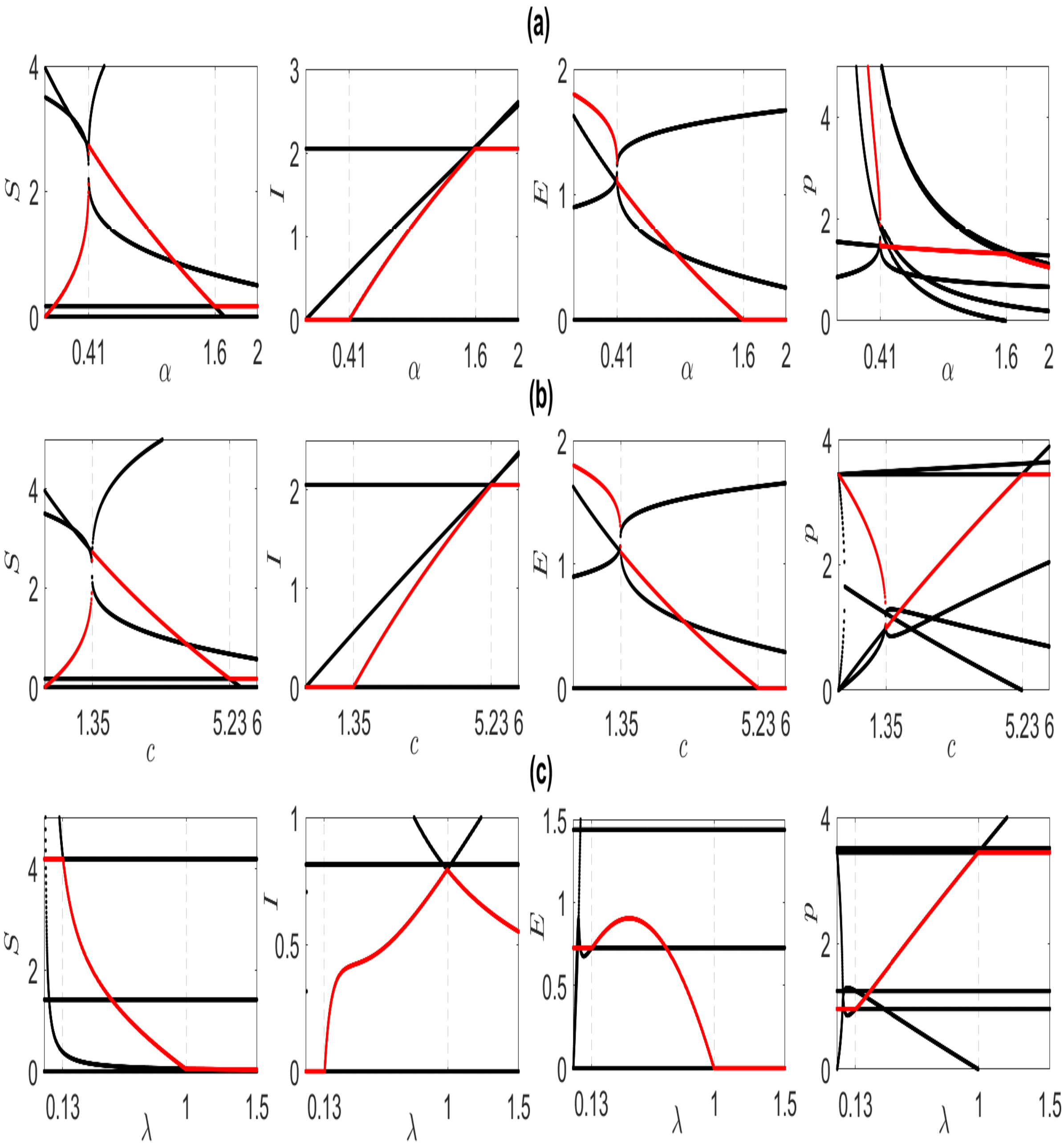

Figure 3.6: One parameter bifurcation diagrams with respect to the parameters α, c and λ: (a) For the variation of α, there exists two transcritical bifurcations at $\alpha = 0.14$ and $\alpha = 0.16$. (b) For the variation of c, there exists two transcritical bifurcations at $c = 1.35$ and $c = 5.23$. (c) For the variation of λ, there exists two transcritical bifurcations at $\lambda = 0.13$ and $\lambda = 1.0$. In each case, the transition occurs in the sequence $\xi_4 \longrightarrow \xi^* \longrightarrow \xi_3$. Here red color represents the stable branch and black color represents the unstable branch. Other parameters remain fixed as in (3.22).

environmental carrying capacity is low. Harvesting is also not profitable at low harvested stock, so E is also zero in the low range of K $(0 < K < 0.17)$. As K increases, I starts increasing, but still harvesting is not profitable so E remains zero in the range $0.17 < K < 1.15$. Thus, the first transcritical bifurcation occurs at $K = 0.17$. As K increases further, the system can support higher population and harvesting becomes possible in the range $1.15 < K < 2.02$, where all variables attain non-zero value and the equilibrium ξ^* becomes stable. Therefore, the second transcritical bifurcation occurs at $K = 1.15$. As harvesting increases, the infected fish density begins to decline and can not sustain when it is significantly high, causing the existence of third transcritical bifurcation at $K = 2.02$ with the appearance of infection-free equilibrium point ξ_4. The susceptible fish density continues to increase for increasing carrying capacity in the range $2.02 < K < 4.56$ and then saturates. Infection can again capture the host population, even at the same force of infection, as more fishes become available for infection and the interior equilibrium point ξ^* again appears following the fourth transcritical bifurcation at $K = 4.56$. Note that the price steadily decreases as long as harvesting is on (for $K > 1.15$). Similar bifurcations have been drawn (see Fig. 3.6) for α, c and λ representing, respectively, the rate at which demand decreases, cost per unit of fishing and the force of infection. Switching of equilibrium points through two transcritical bifurcations in the sequence $\xi_4 \longrightarrow \xi^* \longrightarrow \xi_3$ occurs with increasing value of these parameters.

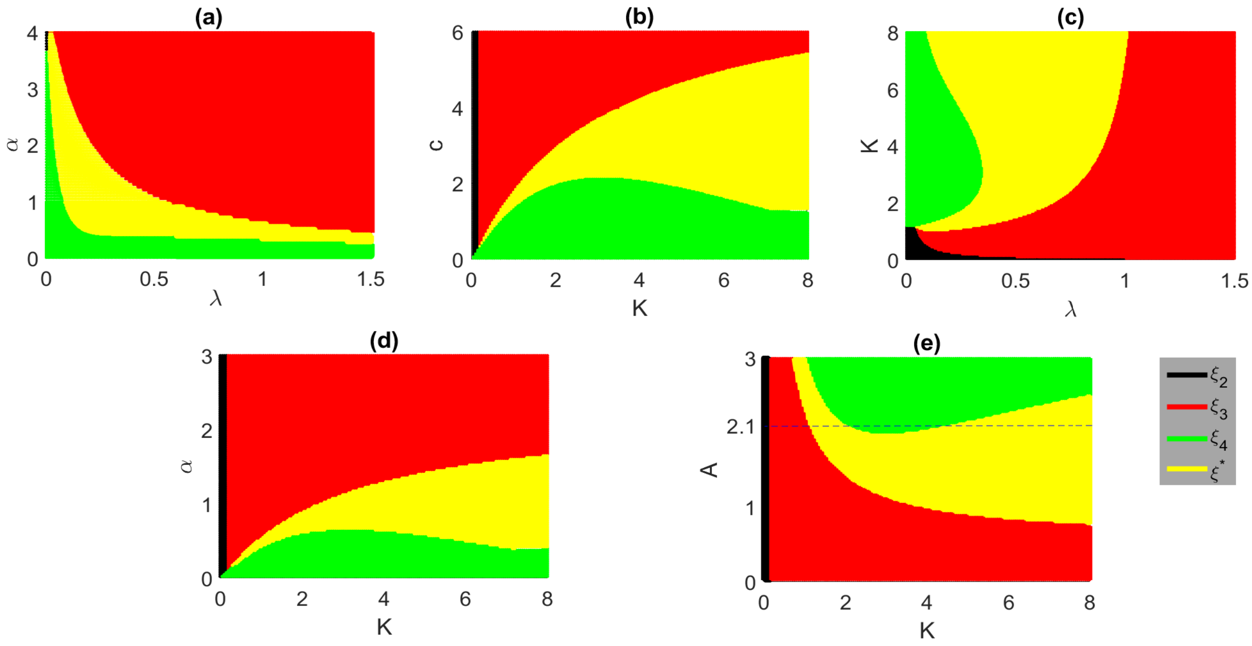

Figure 3.7: Two parameters bifurcation diagrams: (a) $\lambda - \alpha$ plane, (b) $K - c$ plane, (c) $\lambda - K$ plane, (d) $K - \alpha$ plane, (e) $K - A$ plane. Black, red, green, and yellow regions are, respectively, the stability regions of the equilibrium points ξ_2, ξ_3, ξ_4 and ξ^*. Parameters are as in (3.22).

We now look into the changes in the system dynamics when two parameters are varied simultaneously (Fig. 3.7). It gives a broader scenario in the dynamic changes of the system (3.4). Observe that each parametric plane is separated by four stability regions represented by four colours, viz. black, red, green and yellow, corresponding to the four equilibrium points ξ_2, ξ_3, ξ_4 and ξ^*, respectively. The stability domain of different equilibrium, however, differs significantly in each case. The equilibrium ξ_2 has the least stability domain in all parametric plane. One can easily observe the results of one-parameter bifurcation from this two-parameter bifurcation results. For example, consider a line $A = 2.1$ in Fig. 3.7e, which is parallel to the horizontal axis, the one-parameter results given in the Fig. 3.5 can be observed.

3.6 Bionomic Equilibrium

Bionomic equilibrium [266] is a combination of both biological equilibrium and economic equilibrium to inspect the maximum level of harvesting effort that matches its corresponding cost per unit of fishing effort. The economic rent or net revenue π at any time can be written as the difference between the total revenue earned by selling the harvested fishes (TR) and the total cost for the fishing effort (TC), i.e.,

$$\pi(S,I,E) = p_1(q_1S + q_2I)E - cE, \tag{3.23}$$

where the per-unit price of fish biomass, p_1, is assumed to be constant. The biological equilibrium is written as $\frac{dS}{dt} = \frac{dI}{dt} = 0$ and the economic equilibrium is obtained from $\pi = 0$. Thus it can be found as follows:

$$E = \frac{r}{q_1} - \frac{rS}{q_1K} - \left(\frac{r}{K} + \lambda\right)\frac{I}{q_1} \tag{3.24}$$

and

$$E = \frac{\lambda S}{q_2} - \frac{\mu}{q_2}. \tag{3.25}$$

From Eqs. (3.24) and (3.25), one gets the non-trivial biological equilibrium solution on the line segment

$$\left(\frac{\lambda}{q_2} + \frac{r}{q_1K}\right)S + \frac{1}{q_1}\left(\frac{r}{K} + \lambda\right)I - \left(\frac{r}{q_1} + \frac{\mu}{q_2}\right) = 0, \tag{3.26}$$

provided $E > 0$. Also, the economic equilibrium line can be written as

$$p_1(q_1S + q_2I) - c = 0, \tag{3.27}$$

provided $E > 0$. The biological equilibrium line (3.26) meets the S-axis and I-axis at $(\tilde{S}, 0)$ and $(0, \tilde{I})$, respectively, where

$$\tilde{S} = \frac{\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right)}{\left(\frac{\lambda}{q_2} + \frac{r}{q_1K}\right)} > 0 \text{ and } \tilde{I} = \frac{q_1\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right)}{\left(\frac{r}{K} + \lambda\right)} > 0.$$

The point of intersection of the line (3.26) and (3.27) in the first quadrant (if exists) is the bionomic equilibrium point provided the harvesting effort at that point of intersection (E_∞) is positive. Although, the bionomic equilibrium point can also exist when the lines (3.26) and (3.27) do not intersect within the first quadrant. The bionomic equilibrium point is the point where the biological equilibrium line (3.26) meets either $S = 0$ line or the line $I = 0$, i.e., when one of the worthy species S and I is completely fished out under the positivity maintaining conditions of E and π at that point. We have already seen that $\tilde{S}$ and $\tilde{I}$ are always positive. One can notice from Eq. (3.24) that $E = \frac{r}{q_1}\left(1 - \frac{\tilde{S}}{K}\right)$ at $(\tilde{S}, 0)$, which is always positive (since, $K > \tilde{S}$). Thus, the point $(\tilde{S}, 0)$ is bionomically feasible if $\pi > 0$, i.e., if $c < p_1 q_1 \tilde{S}$. Also, from Eq. (3.25), one can get $E = -\frac{\mu}{q_2}$ at the point $(0, \tilde{I})$, which is always negative and hence the point $(0, \tilde{I})$ is not bionomically feasible.
Solving (3.26) and (3.27), one obtains

$$S_\infty = \frac{\left\{\frac{p_1 q_1 q_2}{c}\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right) - \left(\frac{r}{K} + \lambda\right)\right\}\frac{c}{p_1 q_1}}{q_2\left(\frac{\lambda}{q_2} + \frac{r}{q_1 K}\right) - \left(\frac{r}{K} + \lambda\right)} \text{ and } I_\infty = \frac{\left\{\left(\frac{\lambda}{q_2} + \frac{r}{q_1 K}\right) - \frac{p_1 q_1}{c}\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right)\right\}cq_1}{\left(\frac{\lambda}{q_2} + \frac{r}{q_1 K}\right) - \frac{1}{q_2}\left(\frac{r}{K} + \lambda\right)}.$$

Therefore, $S_\infty > 0$ if any of the following two conditions hold

$$\left(\frac{r}{K} + \lambda\right) < Min\left\{\frac{p_1 q_1 q_2}{c}\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right), q_2\left(\frac{\lambda}{q_2} + \frac{r}{q_1 K}\right)\right\} \tag{3.28}$$

or

$$\left(\frac{r}{K} + \lambda\right) > Max\left\{\frac{p_1 q_1 q_2}{c}\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right), q_2\left(\frac{\lambda}{q_2} + \frac{r}{q_1 K}\right)\right\}. \tag{3.29}$$

Similarly, $I_\infty > 0$ provided

$$\left(\frac{\lambda}{q_2} + \frac{r}{q_1 K}\right) < Min\left\{\frac{p_1 q_1}{c}\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right), \frac{1}{q_2}\left(\frac{r}{K} + \lambda\right)\right\} \tag{3.30}$$

or

$$\left(\frac{\lambda}{q_2} + \frac{r}{q_1 K}\right) > Max\left\{\frac{p_1 q_1}{c}\left(\frac{r}{q_1} + \frac{\mu}{q_2}\right), \frac{1}{q_2}\left(\frac{r}{K} + \lambda\right)\right\}. \tag{3.31}$$

From (3.24) and (3.25), one can have $E_\infty > 0$ whenever

$$\frac{\mu}{\lambda} < S_\infty < K - \left(1 + \frac{\lambda K}{r}\right)I_\infty. \tag{3.32}$$

Therefore, if $c > p_1(q_1 S + q_2 I)$, the cost becomes more than the revenue and hence the fishery remains unexploited. On the other hand, if $c < p_1(q_1 S + q_2 I)$, the total revenue is more than the cost and hence it can be harvested. So, when $c \leq p_1(q_1 S + q_2 I)$, the bionomic equilibrium occurs either (i) at the point (S_∞, I_∞) in the first quadrant, where (3.26) and (3.27) intersect, or (ii) at the point $(\tilde{S}, 0)$ when (3.26) and (3.27) do not intersect. If neither (3.30) nor (3.31) holds, then I-species faces extinction in the bionomic equilibrium. If any one of the conditions (3.28) or (3.29), and any one of the conditions (3.30) or (3.31) hold together with the condition (3.32) then (S_∞, I_∞) exists. Thus, the collective harvesting of a fish species with infection may lead infected species to obliterate while the bionomic equilibrium of the open-access fishery continues with the support of the healthy fish species.

Example 3.7. *For illustration, consider the parameter set* (3.22) *with* $p_1 = 1.43$. *Then the Eqs.* (3.28), (3.31) *and* (3.32) *hold (i.e.,*$0.4286 < 0.4683 = Min\{0.4683, 0.4800\}$, $0.6857 > 0.6690 = Max\{0.6690, 0.6122\}$ *and* $0.1667 < 2.1622 < 5.488$*). Thus, there always exists a bionomic equilibrium point* $(S_\infty = 2.1622, I_\infty = 0.4536)$ *as shown in Fig. 3.8. From the Eq.* (3.24) *or* (3.25)*, one can obtain the corresponding harvesting effort* $E_\infty = 0.8552$. *One can similarly find the bionomic equilibrium* $(\tilde{S}, 0)$ *for a different set of parameter values.*

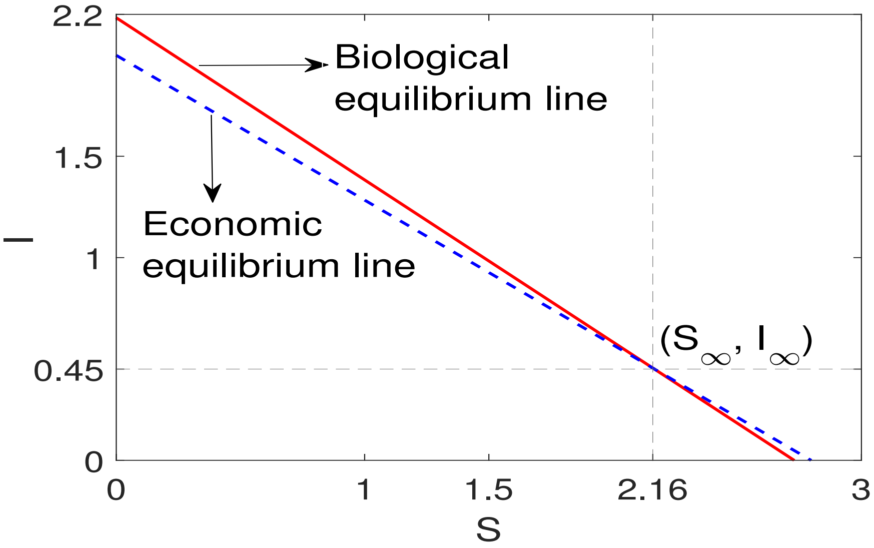

Figure 3.8: Existence of bionomic equilibrium point. The biological equilibrium line (solid) and the economic equilibrium line (dashed) intersect at the bionomic equilibrium point $(S_\infty = 2.1622, I_\infty = 0.4536)$.

3.7 Discussion

Millions of people live on fishing and many fisheries are under stress due to intense harvesting. Emerging diseases, caused due to global warming, water temperature rise, pollutants etc., is a potential threat to fisheries. A proper management policy equipped with the scientific understanding of species interaction is a footstep in a long-term sustainable fishery. In this chapter, we present such a theoretical study that club fish species interaction in presence of infection and the demand-supply theory based on open market policy. More specifically, we propose a four-dimensional bioeconomic model, which is a generalization of many other models, with variable harvesting effort and market-based stock price. It is assumed that a fish stock is infected by some microparasites, which otherwise grows logistically in absence of harvesting. Both the healthy and infected fishes are subject to harvesting following catch-per-unit effort hypothesis. Harvesting effort is considered as time-varying with respect to the profit margin, whereas fish price is regulated by the open-market theory of demand-supply relationship.

The proposed model has five feasible equilibrium points, which are time-independent values of the state variables. Out of these five equilibrium points, one is unachievable but the other four can be stable under different parametric conditions. Considering the ecological, epidemiological and economic importance, we have selected four important parameters and observed how their variational effect may cause stability switching. Our one-parameter bifurcation analysis revealed that harvesting is not possible at the lower value of environmental carrying capacity. Once its value crosses some threshold value such that the system can support a higher population then only harvesting will be feasible. It is interesting to note that infection is eliminated from the system once harvesting is on and the system remains infection-free for a long-range of K ($2.02 < K < 4.56$) (see Fig. 3.5). The ecological parameter K may cause five switching of equilibrium points, $\xi_2 \longrightarrow \xi_3 \longrightarrow \xi^* \longrightarrow \xi_4 \longrightarrow \xi^*$, as it moves from lower to higher value. For bioeconomic harvesting, we have to concentrate on two equilibrium points ξ_4 and ξ^*. For the other three parameters, α, c and λ, representing, respectively, the rate at which demand decreases, cost per unit of fishing, and force of infection, switching of equilibrium points occur in the sequence $\xi_4 \longrightarrow \xi^* \longrightarrow \xi_3$ (Fig. 3.6). It is to be mentioned that equilibrium analysis is important and has practical applications. State variables are maintained at a fixed level when such points are achieved. It is mentionable that each equilibrium point has its own significance. For example, if the equilibrium point $\xi_4 = \left(\bar{S}, 0, \frac{r}{q_1}\left(1 - \frac{\bar{S}}{K}\right), \frac{c}{q_1 \bar{S}}\right)$ is achieved then there will be no infection in the system. Similarly, all state variables, including the infected fish, are maintained at their respective positive values when the system arrives at the interior equilibrium point, ξ^*. Since infection has a detrimental effect on the system population, the system manager will try to modulate model parameters in such a way that the stability conditions of ξ_4 is achieved, and the system becomes disease-free. Another practical application of equilibrium points is that it may guide the fishery manager to maximize revenue at different equilibrium states.

Interestingly, the demand decreasing parameter α plays a very important role in the infection spreading. If α increases then demand decreases and therefore price decreases following open-market theory. This causes fishery unprofitable and the fishing effort gradually declines to zero. In the absence of harvesting, infected fishes grow at a faster rate by making susceptible fish infected. Thus, the number of healthy fish decreases sharply and the system stabilizes to a harvest-free state, where infected fish persists at a high density. The situation is exactly similar if per unit fishing cost is gradually increased, keeping other system parameters unaltered. Harvesting is also not sustainable when the infection rate is high. At the higher force of infection, susceptible fish density reduces to a very low level, where fishing is not profitable even though the stock price remains high. In a more general situation, where two parameters are varied simultaneously, the region of bioeconomic harvesting (where ξ_4 and ξ^* are stable) is large in the $\lambda - K$ and $K - c$ parametric planes (see Fig. 3.7). Interestingly, the system shows different dynamics like coexisting with infection, infection-free state and no-harvesting state due to simultaneous change in the two parameters.

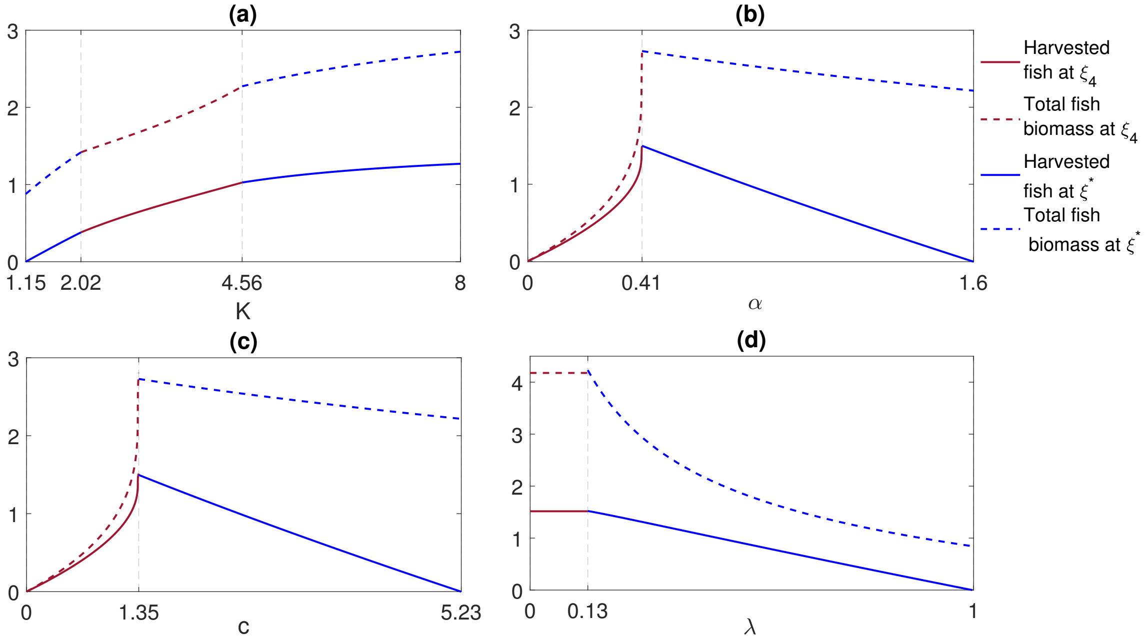

Figure 3.9: Available fish biomass (dotted line) and harvested fish biomass (solid line) corresponding to equilibrium state ξ_4 (red color) and ξ^* (blue color), respectively, for the variation of parameters K (Fig. a), α (Fig. b), c (Fig. c) and λ (Fig. d). Parameters are as in (3.22).

As mentioned previously that we are interested in two equilibrium points ξ_4 and ξ^*, where harvesting is possible. Now the question is which equilibrium has maximum equilibrium fish stock and maximum harvested stock. For this, we have plotted (see Fig. 3.9) the equilibrium fish stock and the corresponding harvested stock for the two equilibrium points ξ_4, where there is no infected fish, and ξ^*, where both the healthy and infected fishes are present. It shows that equilibrium harvest increases with the increased carrying capacity (Fig. 3.9a). The harvested stock is always higher at the infection-free equilibrium ξ_4 compare to that of the interior equilibrium at the lower value of K but always lower than the interior equilibrium at a higher value of K. On the other hand, the harvested biomass is maximum at the transcritical value, where ξ_4 and ξ^* interchanges their stability, while α and c vary (see Figs. 3.9b,c). In fact, the equilibrium harvested fish biomass is an increasing function of α and c at the disease-free equilibrium, but a decreasing function at the interior equilibrium. The equilibrium harvested fish stock remains unaltered at the disease-free equilibrium, however, it is a decreasing function of the force of infection, λ at ξ^* (Fig. 3.9d). The corresponding total revenue (TR) is presented in

Fig. 3.10. It shows that the revenue is highest in a disease-free system (at ξ_4 equilibrium point) when demand is high (see Fig. 3.10b). However, the total revenue is higher at ξ^* compare to ξ_4 at some intermediate force of infection (see Fig. 3.10d).

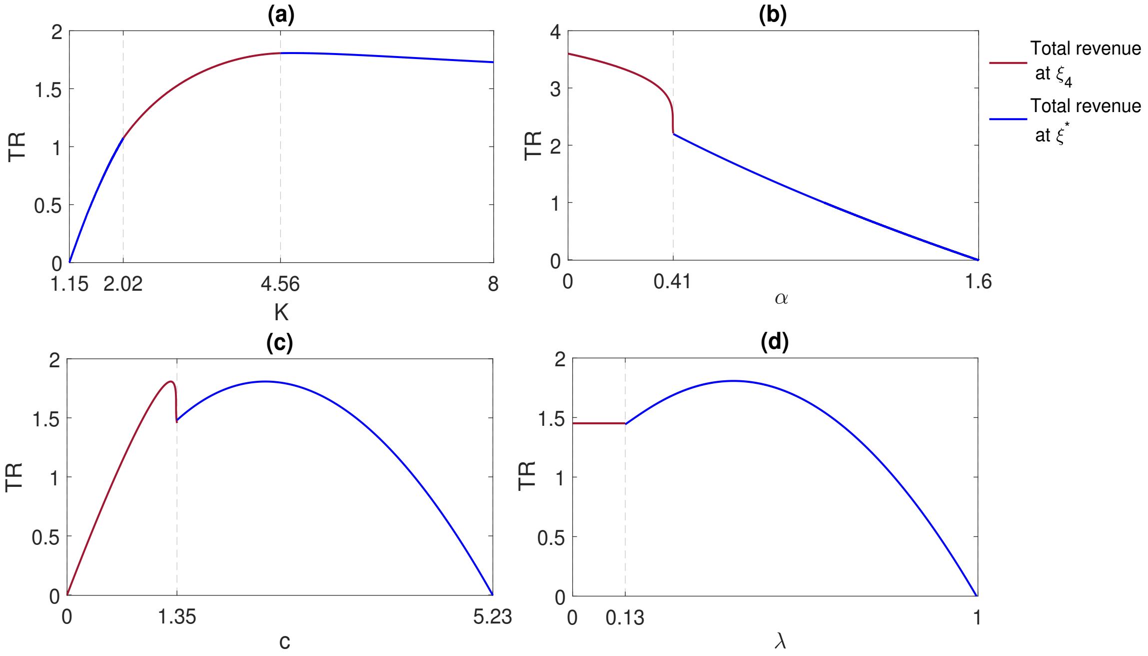

Figure 3.10: Total revenue (TR) at the equilibrium points ξ_4 and ξ^*. TR of harvested fish at the equilibrium state ξ_4 is represented by red color and the same at ξ^* is represented by blue color for the variation of different parameters. Parameters are as in (3.22).

The unique feature of this work is that it considers the open market theory in a bioeconomic fishery model with infection. To the best of our knowledge, no work has appeared by considering variable harvesting and market price in an infected fishery.

This conceptual work addresses a real-world ecological and economic issue, but there are several important aspects that require further investigation. For instance, incorporating non-linear demand, which is more realistic than linear demand. Additionally, it is crucial to address the issue of overharvesting and develop policies to mitigate this problem. One widely studied and effective approach for controlling overfishing is the implementation of taxation policies. Many researchers have utilized taxation models in the context of harvesting to promote sustainability and conservation [90, 267, 268, 269, 270, 271]. Therefore, incorporating a taxation policy in our study would be a significant step towards extending the research. With these considerations in mind, our objective is to proceed to Chapter 4, where we will delve deeper into these fishery-related phenomena, including non-linear demand, and the implementation of a taxation policy. This chapter will provide valuable insights into the implications of these factors and contribute to a more comprehensive understanding of sustainable fisheries management.

4

Dynamic behaviour of a single-species nonlinear fishery model with infection: The role of fishing tax and time-dependent market price [1]

4.1 Introduction

Infection in fish is ubiquitous and known for a very long period. Fish production and revenue generation may be severely affected due to disease [157, 158]. However, the reason and distribution of fish infection must be better understood, particularly for marine fish [159]. Water pollution is considered one of the significant causes of fish infection in the coastal areas [144, 145, 146]. Some other reasons behind the increasing infection rate are water temperature variation, changes in coastal dynamics, and lack of proper governance [59]. Recently, new and transboundary diseases have augmented epidemiological studies of aquatic fish in the presence of infection [147]. Infection may cause a low level of fish productivity [148]. The economic loss due to the production loss of fish for the disease may be huge despite complimentary price hikes due to short supply. Thailand reported a financial loss of US$ 7.38 billion during 2010-2017 for decreased shrimp production due to episodes of disease [168]. Peterman and Posadas

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[1] *The bulk of this chapter has been accepted in Journal of Nonlinear Science and Applications, (2023).*

[170] reported a total of 16.9M\$ loss in 2016 due to the catfish disease in the east Mississippi catfish industry. Therefore, a global challenge is protecting fish and fishery from diseases and reducing economic loss by maintaining sustainable production.

Several governing agencies apply many actions to protect overexploitation and preserve marine resources and habitats for sustainable use. For example, creating a marine protected area (MPA) is a well-accepted conservation policy for the fish, fisheries, and marine environment [272]. However, the success of MPA has been questioned. MPA is more likely to improve the biological goals (like increased fish abundance and improved fish habitat) but, in many cases, fails to revamp social benefit [273]. Fishing has a direct effect on the harvested biomass. Fixing a harvesting quota for a particular fish species may protect the species from being overharvested [186, 187]. A fishing license or vessel buy-back policy is another means to reduce overharvesting [188]. Furthermore, a fishing fee or tax is usually considered one of the crucial measures for controlling overharvesting. These regulatory measures help protect fish and fisheries and achieve the SDG 14 targets at large [189]. Policymakers may use the tax revenue earned through such fiscal policy for the socio-economic upliftment of the fishers and the marine ecosystem. Iceland is one of such countries that successfully implemented fishing fees for pelagic and demersal fishes [192].

Modern bioeconomic fishery received global attention as it can give insights into how to deal with the multi-difficulties of fisheries [63, 274] and prescribe suitable protective measures that could be ecologically and economically viable [275]. However, it is shown that a conflict exists between conservation policy and socio-economic objectives [193]. For example, a higher fishing tax may relieve the fish stock from over-harvesting but may jeopardize the livelihood of local fishing people. It is particularly true in underdeveloped countries where fishermen have limited alternatives for their livelihood. Therefore, imposing a fishing tax scientifically and sensibly is essential.

The price of many commodities, like fish, is determined by instantaneous demand and supply in an open market. Demand is an essential tool that enhances market price fluctuation. Price tends to increase if there is a shortfall in supply and vice-versa. The intricacy of demand, tax, and infection plays a role in the fishery system and revenue generation and needs better understood. Using an ecological model for the harvested species with the market-linked price might be more effective in deciding the control measure. Here, we propose and analyze a dynamic model of fish stock in the presence of infection, where harvesting effort depends on the profitability of the fishery. The model also considers a fishing tax on the landed fish, and the market price of fish depends on the difference between instantaneous demand and supply. Our analysis revealed that taxation might control intensive fish harvesting but augment disease spreading and price hiking. Higher regulatory tax may cause a regime shift, where the system enters a non-harvesting regime from the harvesting one. Using the optimal control theory, we show some trade-offs between revenue generation and regulatory tax. The overall societal revenue, defined here as the sum of fishers' income from selling fish plus the tax revenue earned by the regulatory body, is highest at the optimal tax level. However, the individual earnings in these heads are different at different tax levels.

The rest of the chapter is organized as follows. Section 4.2 describes the bioeconomic model formulation for a single-species fishery. Section 4.3 contains the positivity boundedness, the existence of equilibrium points of the model, and their local stability properties. The impacts of variation in the regulatory tax on the equilibrium values are also presented in this section. The existence of some optimal policies is discussed in Section 4.4. The chapter ends with a brief discussion in Section 4.5.

4.2 Model construction

Suppose $F(t)$ be the current stock level of a fish and $h(t)$ be the harvesting rate then the fish growth equation may be represented by

$$\frac{dF}{dt} = jF\left(1-\frac{F}{L}\right) - h(t), \tag{4.1}$$

where j is the intrinsic growth rate of the fish population and L is the environmental carrying capacity.

Many fish harvesting models [153, 180, 179] consider $h(t)$ as a constant and independent of the stock size. We, however, consider here that the harvesting rate follows the catch per unit of effort (CPUE) hypothesis, where harvesting at any time is proportional to the fish biomass of that time [139, 140, 175, 207]. Thus, $h(t) = q_1 H(t)F(t)$, where $H(t)$ is the harvesting effort at time t, measured in terms of the number of boats, fishing gears, individuals involved in the fishing; and q_1 is the catchability coefficient, measured in terms of the mesh size of the net, gear sophistication, etc. Then the rate equation (4.1) reads

$$\frac{dF}{dt} = jF\left(1-\frac{F}{L}\right) - q_1 HF. \tag{4.2}$$

Presume that the fish stock is infected by some parasites, giving rise to two fish sub-populations: a susceptible class, S, and an infected class, I. So, the net fish stock at any time t is $F(t) = S(t) + I(t)$, and at any time t, $S(t)+I(t) \leq L$, meaning that the entire fish population never exceeds the environmental carrying capacity. Then the interactive dynamics of the fish population can be represented as

$$\begin{aligned} \frac{dS}{dt} &= j(S+I)\left(1-\frac{S+I}{L}\right) - fSI - q_1 HS, \\ \frac{dI}{dt} &= fSI - \mu I - q_2 HI, \end{aligned} \tag{4.3}$$

where the rate parameters f, μ and q_2 represent, respectively, the disease transmission rate, death of the infected fish and catchability coefficient of the infected fish. It is assumed here that the mixing of susceptible and infected fishes is homogeneous, the disease spreads through horizontal transmission following the density-dependent rule, infected fishes do not recover, harvesting is non-selective, and all biological processes are instantaneous. Since infection may induce morbidity through hypoxia, reduce swimming ability, and the conspicuousness of the infected fish [276, 277], the catchability may be higher for infected fish compared to healthy fish under the same effort, i.e., $q_2 \geq q_1$.

The fishing agency assigns more manpower, boats, etc., to harvesting if there is a profit. However, the case will be the opposite if profitability reduces. Therefore, harvesting effort, which is usually assumed to be time-independent [155, 278, 279], should be time-dependent. Here we assume that the harvesting effort varies with time and is proportional to the profit margin (selling price − cost of fishing) [208]. If c is the cost of per unit harvesting effort and M is the market price per unit fish biomass at time t, then the system (4.3) with variable harvesting effort can be represented as

$$\begin{aligned}
\frac{dS}{dt} &= j(S+I)\left(1-\frac{S+I}{L}\right)-fSI-q_1HS,\\
\frac{dI}{dt} &= fSI-\mu I-q_2HI,\\
\frac{dH}{dt} &= \phi_1\left\{(q_1S+q_2I)M-c\right\}H,
\end{aligned} \tag{4.4}$$

where ϕ_1 is a proportionality constant.

Many authors have considered taxation policy in harvesting models [90, 267, 268, 269, 270, 271] to control overfishing. However, none of these has considered infection in the fish stock, which may cause a significant change in the system dynamics. If the fisherman pays a tax $\tau(>0)$ to the regulating agency for per unit biomass of the harvested fish, then the model (4.4) takes the form

$$\begin{aligned}
\frac{dS}{dt} &= j(S+I)\left(1-\frac{S+I}{L}\right)-fSI-q_1HS,\\
\frac{dI}{dt} &= fSI-\mu I-q_2HI,\\
\frac{dH}{dt} &= \phi_1\left\{(q_1S+q_2I)(M-\tau)-c\right\}H.
\end{aligned} \tag{4.5}$$

The fish price is adjusted daily in the open market, balancing demand and supply. In such a case, price should be regarded as a time variable [142, 176] instead of a constant as usually considered in many models [140, 208]. Then the per capita rate of price change should be proportional to the difference between the market demand (D) and the amount of supplied fish

(Q) at that time [139]. Considering a quadratic market demand

$$D(M) = A - A_1 M - A_2 M^2,$$

where A, A_1, A_2 are positive constants with $A_2 << A_1 << A$ [230], and noting that the supplied fish at any time t is

$$Q(t) = q_1 HS + q_2 HI,$$

the dynamic bioeconomic fishery model in the presence of infection, harvesting, and taxation can be expressed as

$$\begin{aligned}
\frac{dS}{dt} &= j(S+I)\left(1 - \frac{S+I}{L}\right) - fSI - q_1 SH = F_1(S,I,H,M),\\
\frac{dI}{dt} &= fSI - \mu I - q_2 IH = F_2(S,I,H,M),\\
\frac{dH}{dt} &= \phi_1\left(\big(q_1 S + q_2 I\big)(M-\tau) - c\right)H = F_3(S,I,H,M),\\
\frac{dM}{dt} &= \phi_2 M\big(D - Q\big) = F_4(S,I,H,M),
\end{aligned} \tag{4.6}$$

where ϕ_2 is a proportionality constant, and $F_i(S,I,H,M)$ $(i = 1,2,3,4)$ are the functional forms of the rate of change of the respective state variables. Table 4.1 represents the state variables and parameters considered to formulate the model (4.6) and their default parameter values to be used subsequently. Many authors have studied the harvesting model in the presence and absence of infection. For example, Hu and Cao [280] considered saturated harvesting in a predator-prey model and analyzed its stability and bifurcations. In [281], the authors considered a predator-prey model with constant harvesting and prey refuge to show the existence and uniqueness of the limit cycle. Juneja and Agnihotri [282] studied a predator-prey model with prey infection and predator harvesting. They mainly observed the infection recovery effect on the system dynamics and optimized the net profit taking tax as the controlling parameter. They, however, ignored the dynamic market price of the harvested species. The dynamics of a single-species fishery model, having variable harvesting effort and market price, were explored in [208]. The harvesting tax and its optimality were not considered here, and the per capita demand was considered constant. Ang and Safuan [283] analyzed a harvested predator-prey model with variable carrying capacity and in the presence of environmental toxicants. It is shown that bionomic equilibrium has a strong dependence on resource density. In addition, using the Pontryagin maximum principle, they prescribed the optimal harvesting policy. The effects of fear and refuge on the optimal harvesting in a predator-prey model with cross-diffusion were analyzed by Ma et al. [284]. The harvesting rate was considered a constant, and they did not consider the economic aspect of harvested species. Variable harvesting and the demand-dependent market price of the harvested stocks were considered in [139, 142**?** , 175].

These studies did not consider disease in the harvested fish and ignored the optimal tax policy and the corresponding societal revenue.

Table 4.1: State variables and parameters with their descriptions and default values.

State variable	Description	Unit	
$S(t)$	Healthy fish biomass at time t	metric tons	
$I(t)$	Infected fish biomass at time t	metric tons	
$H(t)$	Fishing effort at time t	SFU*	
$M(t)$	Market price per unit fish biomass at time t	M$**/metric ton	
Parameter	Description	Default Value	Reference
j	Intrinsic growth rate of healthy fish	0.9 /year	[208]
L	Environmental carrying capacity	5 metric tons	[285]
f	Disease transmission rate	0.04 /metric ton/year	[286]
q_1	Catchability coefficient of susceptible fish	0.8 /SFU/year	[287]
μ	Death rate of infected fish	0.05 /year	[208]
q_2	Catchability coefficient of infected fish	0.9 /SFU/year	[287]
c	Cost per unit of fishing effort	9 M$/SFU/year	[288]
A	Maximum demand	0.9 metric tons/year	Assumed
A_1	Demand sensitivity parameter	0.01 (metric tons)^2/M$/year	Assumed
A_2	Demand sensitivity parameter	0.005 (metric tons)^3/(M$)^2/year	Assumed
ϕ_1	Stiffness parameter	0.1 SFU/M$	[208]
ϕ_2	Proportionality constant	0.15 /metric ton	[208]
τ	Tax per unit biomass of harvested fish	M$/metric ton	Variable

* SFU stands for Standardized Fishing Unit [242, 243] and ** M$ indicates million USD.

4.3 Model analysis

4.3.1 Well-posedness of the system

The well-posedness of an ecological model can be justified by its positivity and boundedness results. Following similar technique as presented in Chapter 2, Section 2.3, one can prove the following lemma to show that the system (4.6) is positive and bounded.

Lemma 4.1. *With the initial condition* $\mathscr{I} = (S_0, I_0, H_0, M_0) \in \mathbb{R}_+^{4,0}$, *the positivity and boundedness of the system* (4.6) *is guaranteed in* $\mathscr{G}_{\mathscr{L}}$, *where* $\mathscr{G}_{\mathscr{L}} = \{(S,I,H,M) : 0 < (S+I) < \iota + \zeta_1,\ 0 < M < \hat{\iota} + \zeta_2,\ 0 < X(S,I,H,M) < \frac{s_4}{s_3} + \zeta$, *for any positive* $\zeta_1,\ \zeta_2,\ \zeta\}$. *Here* $\iota = \max\{S_0 + I_0, L\}$, $s_3 = \min\{\frac{j}{L}, q_3, \phi_1 q_3 \iota\}$ *and* $s_4 = j + \phi_2 A$.

4.3.2 Basic reproduction number

The basic reproduction number (BRN), defined by the number of secondary cases arising from a single infected individual introduced into a group of susceptible individuals [289], is an essential measure of disease dynamics. The success of a pathogen depends on the value of BRN, $\mathscr{R}_0$. If $\mathscr{R}_0 < 1$, then the epidemic cannot grow, and the system eventually becomes disease-free [290].

The system (4.6) contains only one infection state, I. Let $\mathscr{F}$ and $\mathscr{V}$, respectively, represent the rate of appearance of new infection and the rate of transitions [290]. Then

$$\mathscr{F} = \big(fSI\big)_{1\times 1} \text{ and } \mathscr{V} = (\mu I + q_2 IH)_{1\times 1}.$$

At the infection-free equilibrium point $E_1 = (S_1, 0, H_1, M_1)$, the transmission matrix $\hat{F}$ and the transition matrix $\hat{V}$ associated with system (4.6) are given by

$$\hat{F} = \left[\frac{\partial \mathscr{F}}{\partial I}\right]_{E_1=(S_1,0,H_1,M_1)} = fS_1, \text{ and } \hat{V} = \left[\frac{\partial \mathscr{V}}{\partial I}\right]_{E_1=(S_1,0,H_1,M_1)} = \mu + q_2 H_1.$$

Then

$$\mathscr{K} = \hat{F}\hat{V}^{-1} = \left(\frac{fS_1}{\mu + q_2 H_1}\right)_{1\times 1},$$

where $\hat{F}\hat{V}^{-1}$ is called the next generation matrix. The basic reproduction number ($\mathscr{R}_0$), which is the spectral radius of the next generation matrix $(\mathscr{K})$ [291], is given by

$$\mathscr{R}_0 = \frac{fS_1}{\mu + q_2 H_1},$$

where $H_1 = \frac{j}{q_1 L}(L - S_1)$ and S_1 is the equilibrium value of susceptible fish at the disease-free state.

4.3.3 Equilibrium points

The equilibrium points of the system (4.6) are the solutions of the simultaneous equations

$$\begin{aligned}
j(S+I)\left(1 - \frac{S+I}{L}\right) - fSI - q_1 SH &= 0,\\
fSI - \mu I - q_2 IH &= 0,\\
\phi_1\Bigg(\big(q_1 S + q_2 I\big)(M - \tau) - c\Bigg)H &= 0,\\
\phi_2 M\big(A - A_1 M - A_2 M^2 - q_1 SH - q_2 IH\big) &= 0.
\end{aligned}$$

The system (4.6) has seven equilibrium points:

(i) The trivial equilibrium $E_0 = (0,0,0,0)$, which always exists.

(ii) The disease-free equilibrium $E_1 = (S_1, 0, H_1, M_1)$, where the equilibrium components are

$$H_1 = \frac{j}{q_1 L}(L - S_1),\ M_1 = \tau + \frac{c}{q_1 S_1},$$

and S_1 is the positive root of the equation

$$S_1^4 + B_1 S_1^3 + B_2 S_1^2 + B_3 S_1 + B_4 = 0, \tag{4.7}$$

where
$B_1 = -L < 0, B_2 = \frac{L}{j}\left\{(A - A_1\tau - A_2\tau^2)\right\}, B_3 = -\frac{Lc}{jq_1}(A_1 + A_2\tau L) < 0, B_4 = -\frac{A_2c^2L}{jq^2} < 0.$
Since the number of sign change of the coefficients is exactly one under the restriction $A < A_1\tau + A_2\tau^2$, by Descartes' rule of sign, Eq. (4.7) has exactly one positive root. Note that H_1 is always positive as $S_1 < L$, and M_1 is also positive. Thus, the disease-free equilibrium point (E_1) uniquely exists if $A < A_1\tau + A_2\tau^2$.

(iii) The harvesting-free equilibrium has the form $E_2 = (S_2, I_2, 0, M_2)$, whose equilibrium components are given by $S_2 = \frac{\mu}{f}$, $M_2 = \frac{1}{2A_2}(-A_1 + \sqrt{A_1^2 + 4A_1A_2})$, and $I_2 = \frac{1}{2jf}\Big(-(2j\mu + j\mu - jfL) + \sqrt{(2j\mu + j\mu - jfL)^2 - 4j^2\mu(\mu - fL)}\Big)$. Since S_2 and M_2 are always positive, so E_2 exists if I_2 is positive and it holds whenever $fL > \mu$.

(iv) The harvesting-and disease-free equilibrium $E_3 = (S_3, 0, 0, M_3)$ always exists, where $S_3 = L > 0$ and $M_3 = \frac{1}{2A_2}(-A_1 + \sqrt{A_1^2 + 4A_1A_2}) = M_2 > 0$.

(v) The healthy and infected fish only equilibrium $E_4 = (S_4, I_4, 0, 0)$, whose state variables at the equilibrium level can be represented as $S_4 = \frac{\mu}{f}$ and $I_4 = \frac{1}{2jf}\Big(-(2j\mu + j\mu - jfL) + \sqrt{(2j\mu + j\mu - jfL)^2 - 4j^2\mu(\mu - fL)}\Big) = I_2$. This equilibrium exists if $fL > \mu$.

(vi) The only healthy fish equilibrium $E_5 = (S_5, 0, 0, 0)$ always exists with $S_5 = L$.

(vii) The coexisting equilibrium $E^* = (S^*, I^*, H^*, M^*)$, and the corresponding equilibrium components can be computed as

$$\begin{aligned} S^* &= \frac{1}{f}(\mu + q_2H^*), \\ M^* &= \tau + \frac{c}{\frac{q_1}{f}(\mu + q_2H^*) + q_2I^*}. \end{aligned}$$

Observe that both S^* and M^* are positive. The other two equilibrium components I^* and H^* are the positive roots of the equations

$$j\left(\frac{1}{f}(\mu + q_2H^*) + I^*\right)\left(1 - \frac{\frac{1}{f}(\mu + q_2H^*) + I^*}{L}\right) - (\mu + q_2H^*)\left(I^* - \frac{q_1H^*}{f}\right) = 0,$$

$$A - A_1\left[\tau + \frac{c}{\frac{q_1}{f}\left(\mu + q_2H^*\right) + q_2I^*}\right] - A_2\left[\tau + \frac{c}{\frac{q_1}{f}\left(\mu + q_2H^*\right) + q_2I^*}\right]^2 - \frac{q_1}{f}\left(\mu + q_2H^*\right)H^* + q_2I^*H^* = 0. \tag{4.8}$$

Our computational results for the considered parameter values show that the equilibrium E^* is unique.

4.3.4 Stability of the equilibria

Under what parametric conditions an equilibrium state will be stable is essential for population persistence and sustainable yield. The stability of an equilibrium point means whether the system will return to the equilibrium point over time or not if the equilibrium point is perturbed. One way of determining such stability is the linearization technique of the system around the equilibrium point [258]. The Jacobian matrix of the system (4.6) at any arbitrary equilibrium point $\hat{E} = (\hat{S}, \hat{I}, \hat{H}, \hat{M})$ reads

$$J(\hat{S}, \hat{I}, \hat{H}, \hat{M}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \tag{4.9}$$

where $a_{11} = -f\hat{I} - \hat{H}q_1 - j\left(\frac{2(\hat{I}+\hat{S})}{L} - 1\right)$, $a_{12} = -\hat{S}f - j\left(\frac{2(\hat{I}+\hat{S})}{L} - 1\right)$, $a_{13} = -\hat{S}q_1$, $a_{21} = \hat{I}f$, $a_{22} = \hat{S}f - \mu - \hat{H}q_2$, $a_{23} = -\hat{I}q_2$, $a_{31} = \hat{H}\phi_1 q_1\left(\hat{M} - \tau\right)$, $a_{32} = \hat{H}\phi_1 q_2\left(\hat{M} - \tau\right)$, $a_{33} = \phi_1\left(\left(q_1\hat{S} + q_2\hat{I}\right)\left(\hat{M} - \tau\right) - c\right)$, $a_{34} = \phi_1\left(\hat{H}\hat{I}q_2 + \hat{H}\hat{S}q_1\right)$, $a_{41} = -\hat{H}\hat{M}\phi_2 q_1$, $a_{42} = -\hat{H}\hat{M}\phi_2 q_2$, $a_{43} = -\hat{M}\phi_2\left(\hat{I}q_2 + \hat{S}q_1\right)$, $a_{44} = -\phi_2(A_2\hat{M}^2 + A_1\hat{M} - A + \hat{H}\hat{I}q_2 + \hat{H}\hat{S}q_1) - \hat{M}\phi_2\left(A_1 + 2A_2\hat{M}\right)$. One can then prove the following stability theorem.

Theorem 4.2. *(i) The equilibrium points* $E_0 = (0,0,0,0)$, $E_4 = (S_4, I_4, 0, 0)$, *and* $E_5 = (L, 0, 0, 0)$ *are always unstable.*

(ii) The disease-free equilibrium $E_1 = (S_1, 0, H_1, M_1)$ *is locally asymptotically stable if the conditions* $\mathscr{R}_0 < 1$, $C_1 > 0$, $C_3 > 0$, *and* $C_1C_2 - C_3 > 0$ *are satisfied, otherwise it is unstable, where* C_1, C_2 *and* C_3 *are given in* (4.13).

(iii) If $c > (I_2q_2 + S_2q_1)(M_2 - \tau)$ *and* $2(S_2 + I_2) > L$, *then the harvesting-free equilibrium* $E_2 = (S_2, I_2, 0, M_2)$ *is locally asymptotically stable, and unstable otherwise.*

(iv) Whenever the conditions $\mu > Lf$, $c > Lq_1(M_3 - \tau)$ *hold, the harvesting-and disease-free equilibrium* $E_3 = (S_3, 0, 0, M_3)$ *remains locally asymptotically stable, and unstable otherwise.*

(v) A set of necessary and sufficient conditions for the stability of the coexisting equilibrium point $E^* = (S^*, I^*, H^*, M^*)$ *is* $\{C_4 > 0, C_6 > 0, C_7 > 0, C_4C_5C_6 - (C_6^2 + C_4^2C_7) > 0\}$*, where* C_4*,* C_5*,* C_6*, and* C_7 *are given in* (4.17).

Proof. (i) The variational matrix (4.9) at the trivial equilibrium point $E_0 = (0,0,0,0)$ reads

$$J_{E_0} = \begin{pmatrix} j & j & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & -c\phi_1 & 0 \\ 0 & 0 & 0 & A\phi_2 \end{pmatrix}. \tag{4.10}$$

Since two eigenvalues (j and $A\phi_2$) of the Jacobian matrix (4.10) are positive, the equilibrium point E_0 is always unstable. Similarly, a positive eigenvalue of the form $A\phi_2$ for both the equilibrium points $E_4 = (S_4, I_4, 0, 0)$ and $E_5 = (L, 0, 0, 0)$ makes them unstable.

(ii) At the disease-free equilibrium $E_1 = (S_1, 0, H_1, M_1)$, the variational matrix (4.9) reads

$$J_{E_1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 \\ 0 & b_{22} & 0 & 0 \\ b_{31} & b_{32} & 0 & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}, \tag{4.11}$$

where $b_{11} = -\frac{S_1 j}{L}$, $b_{12} = -S_1 f - j\left(\frac{2S_1}{L} - 1\right)$, $b_{13} = -S_1 q_1$, $b_{22} = S_1 f - \mu - H_1 q_2$, $b_{31} = H_1\phi_1 q_1(M_1 - \tau), b_{32} = H_1\phi_1 q_2(M_1 - \tau), b_{34} = H_1 S_1 \phi_1 q_1, b_{41} = -H_1 M_1 \phi_2 q_1, b_{42} = -H_1 M_1 \phi_2 q_2$, $b_{43} = -M_1 S_1 \phi_2 q_1$, $b_{44} = -M_1\phi_2(A_1 + 2A_2 M_1)$.

Its one eigenvalue is $S_1 f - \mu - H_1 q_2$, which is negative whenever the basic reproduction number $\mathscr{R}_0 < 1$. The other three eigenvalues are the roots of the equation

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0, \tag{4.12}$$

where

$$\begin{aligned} C_1 &= -(b_{11} + b_{44}),\ C_2 = -b_{13}b_{31} + b_{11}b_{44} - b_{34}b_{43}, \\ C_3 &= -b_{13}b_{34}b_{41} + b_{13}b_{31}b_{44} + b_{11}b_{34}b_{43}. \end{aligned} \tag{4.13}$$

Following Routh-Hurwitz criterion [259], the necessary and sufficient conditions for all roots of Eq. (4.12) to have negative real part are $C_1 > 0, C_3 > 0, C_1C_2 - C_3 > 0$. Therefore, the disease-free equilibrium $E_1 = (S_1, 0, H_1, M_1)$ is locally asymptotically stable under the condition $\mathscr{R}_0 < 1$, $C_1 > 0$, $C_3 > 0$, and $C_1C_2 - C_3 > 0$.

(iii) The Jacobian matrix at the harvesting-free equilibrium $E_2 = (S_2, I_2, 0, M_2)$ is

$$J_{E_2} = \begin{pmatrix} -I_2 f - j\left(\frac{2(I_2+S_2)}{L} - 1\right) & -S_2 f - j\left(\frac{2(I_2+S_2)}{L} - 1\right) & -S_2 q_1 & 0 \\ I_2 f & 0 & -I_2 q_2 & 0 \\ 0 & 0 & \begin{matrix}\phi_1((I_2 q_2 + S_2 q_1) \\ (M_2 - \tau) - c)\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}-M_2\phi_2(I_2 q_2 \\ +S_2 q_1)\end{matrix} & \begin{matrix}-M_2\phi_2(A_1 \\ +2A_2 M_2)\end{matrix} \end{pmatrix}. \quad (4.14)$$

Its two eigenvalues are $-M_2\phi_2(A_1 + 2A_2M_2) < 0$ and $\phi_1((I_2q_2 + S_2q_1)(M_2 - \tau) - c)$. The latter eigenvalue is negative provided $c > (I_2q_2 + S_2q_1)(M_2 - \tau)$, i.e., the cost per unit of fishing effort greater than the corresponding earnings. The other two eigenvalues are the roots of the equation

$$\lambda_1^2 + \left(I_2 f + j\left(\frac{2(I_2+S_2)}{L} - 1\right)\right)\lambda_1 + \left(S_2 f + j\left(\frac{2(I_2+S_2)}{L} - 1\right)\right) I_2 f = 0. \quad (4.15)$$

Clearly, the roots of Eq.(4.15) will have negative real parts whenever $2(S_2 + I_2) > L$. Thus, the equilibrium point $E_2 = (S_2, I_2, 0, M_2)$ is locally asymptotically stable under the conditions $c > (I_2q_2 + S_2q_1)(M_2 - \tau)$, $2(S_2 + I_2) > L$.

(iv) The characteristic equation corresponding to the Jacobian matrix (4.9) at the harvesting- and disease-free equilibrium $E_3(S_3, 0, 0, M_3)$ can be written as

$$(\lambda_3 + j)\{\lambda_3 - (Lf - \mu)\}\{\lambda_3 + \phi_1(c - Lq_1(M_3 - \tau))\}\{\lambda_3 + M_3\phi_2(A_1 + 2A_2M_3)\} = 0. \quad (4.16)$$

Therefore, the eigenvalues are $-j$, $Lf - \mu$, $-\phi_1(c - Lq_1(M_3 - \tau))$ and $-M_3\phi_2(A_1 + 2A_2M_3)$. Clearly, two eigenvalues $-j$ and $-M_3\phi_2(A_1 + 2A_2M_3)$ are always negative. The negativity of the remaining two is assured under the conditions $\mu > fL$ and $c > Lq_1(M_3 - \tau)$. Recall that the existence condition of equilibrium points E_2 and E_4 is $\mu < fL$. Therefore, whenever the equilibrium point E_2 or E_4 exists, the steady state E_3 cannot be stable. The other condition $c > Lq_1(M_3 - \tau)$ tells that the fishing cannot be profitable whenever E_3 is stable.

(v) Suppose an interior equilibrium $E^* = (S^*, I^*, H^*, M^*)$ of the system (4.6) exists. The Jacobian matrix in this case is evaluated as

$$J_{E^*} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{21} & 0 & c_{23} & 0 \\ c_{31} & c_{32} & 0 & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix},$$

where
$c_{11} = -\frac{2j(I^*+S^*)}{L}$, $c_{12} = -S^*f - j\left(\frac{2(I^*+S^*)}{L} - 1\right)$, $c_{13} = -S^*q_1$, $c_{21} = I^*f$, $c_{23} = -I^*q_2$, $c_{31} = H^*\phi_1 q_1(M^* - \tau)$, $c_{32} = H^*\phi_1 q_2(M^* - \tau)$, $c_{34} = \phi_1(H^*I^*q_2 + H^*S^*q_1)$, $c_{41} = -H^*M^*\phi_2 q_1$, c_{42}

$= -H^*M^*\phi_2 q_2, c_{43} = -M^*\phi_2\left(I^*q_2+S^*q_1\right),\ c_{44} = -M^*\phi_2(A_1+2A_2M^*)$.

The corresponding characteristic equation reads

$$\lambda_2{}^4 + C_4\lambda_2{}^3 + C_5\lambda_2{}^2 + C_6\lambda_2 + C_7 = 0,$$

where

$$\begin{aligned} C_4 &= (-c_{11}-c_{44}), \\ C_5 &= (c_{11}c_{44}-c_{13}c_{31}-c_{12}c_{21}-c_{23}c_{32}-c_{34}c_{43}), \\ C_6 &= \big(c_{11}c_{23}c_{32}-c_{12}c_{23}c_{31}-c_{13}c_{21}c_{32}+c_{12}c_{21}c_{44}+c_{11}c_{34}c_{43}+c_{13}c_{31}c_{44} \\ &\quad -c_{13}c_{34}c_{41}+c_{23}c_{32}c_{44}-c_{23}c_{34}c_{42}\big), \\ C_7 &= c_{11}c_{23}c_{34}c_{42}-c_{11}c_{23}c_{32}c_{44}+c_{12}c_{21}c_{34}c_{43}+c_{12}c_{23}c_{31}c_{44}-c_{12}c_{23}c_{34}c_{41} \\ &\quad +c_{13}c_{21}c_{32}c_{44}-c_{13}c_{21}c_{34}c_{42}. \end{aligned} \tag{4.17}$$

Following Routh-Hurwitz criterion [259], a set of necessary and sufficient conditions for the stability of the equilibrium point $E^* = (S^*, I^*, H^*, M^*)$ is

$$C_4 > 0, C_6 > 0, C_7 > 0, C_4C_5C_6 - (C_6^2 + C_4^2C_7) > 0.$$

This completes the proof of the theorem. □

4.3.5 Bifurcation analysis

Changes in the system dynamics for the variation of a system parameter may be well described through its bifurcation results. Considering the fishing tax τ as the control parameter, we investigate the occurrence of bifurcations in the system (4.6). One can prove the following theorem for the existence of bifurcations.

Theorem 4.3. *(i) The system* (4.6) *undergoes a transcritical bifurcation at the disease-free equilibrium point* $E_1(S_1, 0, H_1, P_1)$ *if* τ *reaches the critical value* τ_1^{TC}*, where* τ_1^{TC} *is the positive root of the equation*

$$fS_1(\tau) - \mu - q_2H_1(\tau) = 0,$$

and the transversality condition $f \neq \frac{q_2v_3}{v_1}$ *holds.*

(ii) The system (4.6) *undergoes a transcritical bifurcation at the harvesting-free equilibrium point* $E_2(S_2, I_2, 0, P_2)$ *if* τ *arrives the threshold level* τ_2^{TC}*, where*

$$\tau_2^{TC} = M_2 - \frac{c}{I_2q_2 + S_2q_1},$$

and the transversality condition $\tau_2^{TC} \neq M_2 + \frac{q_1 S_2 + q_2 I_2}{q_1 w_1 + q_2 w_2} w_4$ *holds.*

Proof. (i) From (4.11), one can observe that the Jacobian matrix leaves a zero eigenvalue if

$$fS_1(\tau) - \mu - q_2 H_1(\tau) = 0. \tag{4.18}$$

Let $\tau = \tau_1^{TC}$ be a positive root of the Eq. (4.18). Then, at τ_1^{TC}, the eigenvector corresponding to the zero eigenvalue of $J_{E_1}(S_1, 0, H_1, M_1)$ and $J_{E_1}(S_1, 0, H_1, M_1)^T$ are

$$\zeta = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $J_{E_1}(S_1, 0, H_1, M_1)^T$ is the transpose of $J_{E_1}(S_1, 0, H_1, M_1)$ and

$$\begin{aligned} v_1 &= -\frac{b_{32} v_2 + b_{34}}{b_{31}}, \\ v_2 &= \frac{b_{13} b_{34} (b_{31} b_{44} - b_{41} b_{34}) + b_{11} b_{34} b_{31} b_{43}}{(b_{12} b_{31} - b_{11} b_{32}) b_{31} b_{43} - b_{13} b_{31} (b_{42} b_{31} - b_{41} b_{32})}, \\ v_3 &= \frac{b_{11} b_{34} (b_{42} b_{31} - b_{41} b_{32}) - (b_{12} b_{31} - b_{11} b_{32})(b_{31} b_{44} - b_{41} b_{34})}{(b_{12} b_{31} - b_{11} b_{32}) b_{31} b_{43} - b_{13} b_{31} (b_{42} b_{31} - b_{41} b_{32})}. \end{aligned}$$

Now the three conditions of Sotomayor's theorem [119] for the existence of a degenerate transcritical bifurcation at $\tau = \tau_1^{TC}$ are

$$\begin{aligned} \eta^T R_\tau\Big(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big) &= 0, \\ \eta^T DR_\tau\Big(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big)\zeta &= 0, \\ \eta^T D^2 R\Big(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big)(\zeta, \zeta) &\neq 0. \end{aligned} \tag{4.19}$$

Here $R_\tau = \left(\frac{dF_1}{d\tau}, \frac{dF_2}{d\tau}, \frac{dF_3}{d\tau}, \frac{dF_4}{d\tau}\right)^T$ and $DR_\tau\Big(J_{E_1}(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big)\zeta$ is the linear transformation formed by the matrix of partial derivatives of the components of R_τ with respect to the state variables (S, I, H, M). Similarly, one can define the other linear transformation $D^2 R\Big(J_{E_1}(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big)(\zeta, \zeta)$. It is to be noted that the second condition of (4.19) needs to be non-zero for the appearance of non-degenerate transcritical bifurcation [119].

Now,

$$
\begin{aligned}
\eta^T R_\tau\Big(E_1;\tau=\tau_1^{TC}\Big) &= (0\;1\;0\;0)\begin{pmatrix} 0\\ 0\\ \phi_1 q_1 S_1 H_1\\ 0\end{pmatrix}_{\tau=\tau_1^{TC}}\\
&= 0,\\
\eta^T DR_\tau\Big(E_1;\tau=\tau_1^{TC}\Big)\zeta &= \eta^T\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \phi_1 q_1 H_1 & \phi_1 q_1 H_1 & \phi_1 q_1 S_1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}_{\tau=\tau_1^{TC}}\zeta\\
&= 0,\\
\eta^T D^2 R\Big(E_1;\tau=\tau_1^{TC}\Big)(\zeta,\zeta) &= \eta^T\begin{pmatrix} d_{11} & d_{12} & d_{13} & 0\\ d_{21} & d_{22} & d_{23} & 0\\ d_{31} & d_{32} & d_{33} & d_{34}\\ d_{41} & d_{42} & d_{43} & d_{44}\end{pmatrix}_{\tau=\tau_1^{TC}}\zeta\\
&= d_{21}v_1+d_{22}v_2+d_{23}v_3\\
&= 2(fv_1v_2-q_2v_2v_3),
\end{aligned}
$$

where
$d_{11}=-fv_2-\frac{2j}{L}(v_1+v_2)-q_1v_3$, $d_{12}=-fv_1-\frac{2j}{L}(v_1+v_2)$, $d_{13}=-q_1v_1$, $d_{21}=fv_2$, $d_{22}=fv_1-q_2v_3$, $d_{23}=-q_2v_2$, $d_{31}=\phi_1q_1((M_1-\tau_1^{TC})v_3+H_1)$, $d_{32}=\phi_2q_2((M_1-\tau_1^{TC})v_3+H_1)$, $d_{33}=\phi_1\{q_1S_1+(q_1v_1+q_2v_2)(M_1-\tau_1^{TC})\}$, $d_{34}=\phi_1\{q_1S_1v_3+q_1H_1(v_1+v_2)\}$, $d_{41}=-\phi_2q_1(H_1+M_1v_3)$, $d_{42}=-\phi_2q_2(H_1+M_1v_3)$, $d_{43}=-\phi_2\{M_1(q_1v_1+q_2v_2)+S_1q_1(v_3+1)\}$, $d_{44}=-\phi_2\{H_1(q_1v_1+q_2v_2)+S_1q_1+(2A_1+6A_2M_1)\}$.
Thus, following Sotomayars theorem [119], whenever the control parameter τ reaches the critical value $\tau=\tau_1^{TC}$, a degenerate transcritical bifurcation point occurs if the condition $f\neq\frac{q_2v_3}{v_1}$ holds.

(ii) Proceeding similarly, one can show that the variational matrix (4.14), corresponding to the harvesting effort-free equilibrium point $E_2(S_2,I_2,0,M_2)$, gives a zero eigenvalue at $\tau=M_2-\frac{c}{I_2q_2+S_2q_1}=\tau_2^{TC}$ (say). In this case, the eigenvectors of $J_{E_2}(S_2,I_2,0,M_2)$ and $J_{E_2}(S_2,I_2,0,M_2)^T$, corresponding to the zero eigenvalue at τ_2^{TC}, are

$$
\hat{\zeta}=\begin{pmatrix} w_1\\ w_2\\ w_3\\ w_4\end{pmatrix}\text{ and }\hat{\eta}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},
$$

where $w_1 = -\frac{q_2}{f}$, $w_2 = \frac{q_2\left(-fI_2 - j\left(\frac{2(S_2+I_2)}{L}-1\right)\right)(A_1+2A_2M_2) + f(q_1S_2+q_2I_2)\left(fS_2 + j\left(\frac{2(S_2+I_2)}{L}-1\right)\right)}{q_1 f \phi_2 S_2 I_2 M_2 (A_1+2A_2M_2)}$, $w_3 = 1$, $w_4 = \frac{q_1S_2+q_2I_2}{A_1+2A_2M_2}$.
Similar calculations show that there exists a degenerate transcritical bifurcation point at $\tau = \tau_2^{TC}$ if $\tau_2^{TC} \neq M_2 + \frac{q_1S_2+q_2I_2}{q_1w_1+q_2w_2}w_4$. □

4.3.6 Computational results

To visualize the previous bifurcations, and the in-between stabilities, we have presented a bifurcation diagram in Fig. 4.1 with the variations in τ. It shows three distinct dynamic behaviours of the system when the tax is varied in some stipulated range $0 < \tau < 11$.

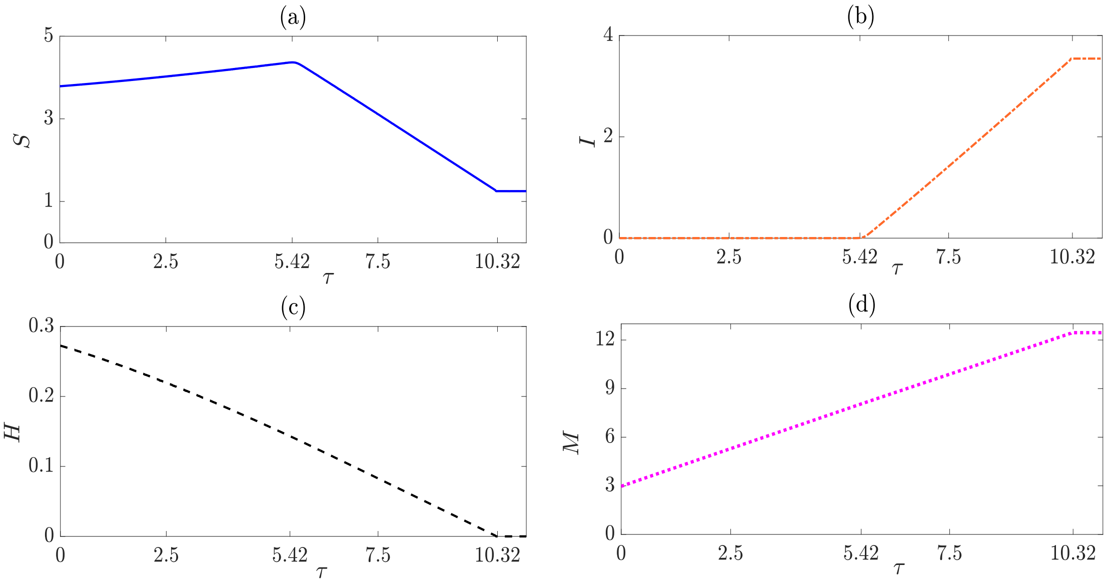

Figure 4.1: Bifurcation results of the system (4.6) when the tax, τ, is varied in the range $0 < \tau < 11$. We have plotted the maxima and minima of each state variables for each value of τ. This tax range is classified into three categories, low, intermediate and high, depending on the system's stabilities. The disease-free equilibrium (E_1) is stable when the tax is low $(0 < \tau < 5.42)$. The coexisting equilibrium (E^*) is stable in the intermediate tax, $5.42 < \tau < 10.32$. The harvesting-free equilibrium (E_2) is stable if the tax is high $(\tau > 10.32)$. Parameters are as in Table 4.1.

Solving Eq. (4.18), one gets the unique root as $\tau = \tau_1^{TC} = 5.42$. At this critical value, the eigenvector $\zeta = (v_1, v_2, v_3, 1)^T$ becomes $(26.30, -24.65, 0.01, 1)^T$ and therefore the transversality condition of Theorem 4.3(i) is satisfied as $f = 0.04 \neq \frac{qv_3}{v_1} = 0.0004$. Therefore, a transcritical bifurcation arises at $\tau = \tau_1^{TC} = 5.42$, following Theorem 4.3(i), where the disease-free equilibrium E_1 coalesces with the coexisting equilibrium E^* and exchanges their stability (see Fig. 4.1). At $\tau = \tau_2^{TC} = M_2 - \frac{c}{I_2q_2+S_2q_1} = 10.32$, one can obtain the eigenvector as $\hat{\zeta} = (w_1, w_2, w_3, w_4)^T = (-22.50, 0.83, 1, 31.12)^T$. Also the transversality condition of Theorem 4.3(ii) is satisfied as $\tau_2^{TC} = 10.32 \neq M_2 + \frac{q_1S_2+q_2I_2}{q_1w_1+q_2w_2}w_4 = 4.90$. Therefore, following Theorem 4.3(ii), another shift of stability through a transcritical bifurcation occurs at $\tau = 10.32$, where the coexisting equilibrium E^* and the harvesting-free equilibrium (E_2) met. Notice that the market price (M) increases as the tax increases, while the harvesting effort (H) steadily decreases in the same range $0 < \tau < 10.32$. The disease is established through the appearance of the I population as the imposed tax exceeds the first transcritical value $\tau_1^{TC} = 5.42$. The infected fish population increases rapidly for further increase in τ, while a gradual decline occurs in the healthy fish population. As the regulatory tax crosses the higher transcritical value $\tau_2^{TC} = 10.32$, harvesting effort declines to zero. Thus, there exist three different dynamic regimes for the variation in τ: (i) the system remains disease-free for low tax $(0 < \tau < 5.42)$, (ii) the disease persists when tax is intermediate $(5.42 < \tau < 10.32)$, and (iii) harvesting is not possible if the imposed tax is high $(\tau > 10.32)$. The harvesting-and disease-free equilibrium, E_3, does not appear in the bifurcation analysis results because it is always unstable whenever the equilibrium point E_2 or E_4 exists.

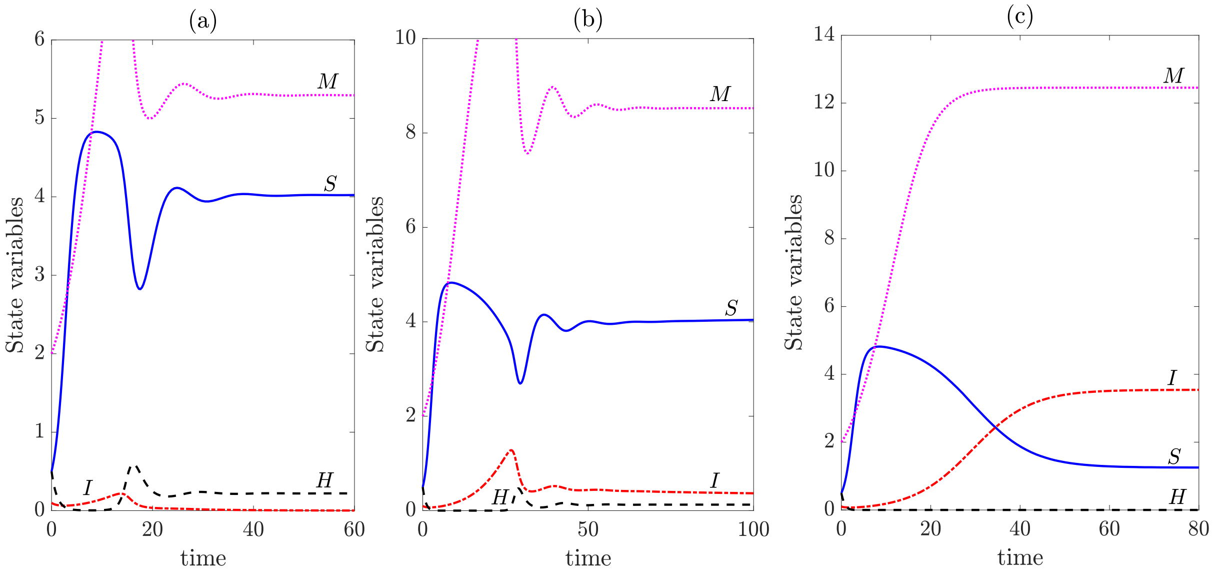

Figure 4.2: Time evolutions of the system (4.6) for some particular values of τ taken one from each region (see Fig. 4.1). (a) Stable behaviour of the infection-free equilibrium $E_1 = (4.02, 0, 0.21, 5.29)$ for $\tau = 2.5$. (b) The endemic equilibrium $E^* = (4.10, 0.31, 0.12, 8.52)$ is stable for $\tau = 6$. (c) For $\tau = 10.35$, harvesting effort becomes zero and the system stabilizes to the harvesting-free equilibrium $E_2 = (1.25, 3.54, 0, 12.45)$. In each case, the system started from the initial value $(0.5, 0.1, 0.5, 2)$. Parameters are as in Table 4.1.

The time series solutions (Fig. 4.2) of the system for three particular values of τ show the representative behaviour of the state variables for all τ in the considered range. At the lower value of the regulating tax (say $\tau = 2.5$) the required conditions of Theorem 4.2(ii) are satisfied as $C_1 = 0.91 > 0, C_3 = 0.16 > 0$, $C_1C_2 - C_3 = 0.28 > 0$. Here, the basic reproduction number is $\mathscr{R}_0 = 0.65 < 1$. Therefore, the system stabilizes to the disease-free equilibrium $E_1 = (4.02, 0, 0.21, 5.29)$ (Fig. 4.2a). Healthy fish stock in this state is high, at 4.02 units. Consequently, the price remains low ($M = 5.29$ units), and harvesting effort is high ($H = 0.21$ units) due to the availability of the fish stock. Intense harvesting reduces the infected fish, causing the elimination of infection from the system when the tax is low. If the imposed tax is moderate, say $\tau = 6$, the system converges to the endemic state E^*, by satisfying the set of necessary and sufficient conditions of Theorem 4.2(v) as $C_4 = 0.91 > 0, C_6 = 0.17 > 0, C_7 = 0.002 > 0, C_4C_5C_6 - (C_6^2 + C_4^2C_7) = 0.03 > 0$. This gives the stable solutions of all the state variables with equilibrium population levels $S^* = 4.10, I^* = 0.31, H^* = 0.12, M^* = 8.52$ (Fig. 4.2b). The infected fish can persist in the intermediate range of $5.42 < \tau < 10.32$. This is reasonable because increasing tax reduces harvesting and causes a compensatory increase in infected fish, which helps infection invade the host population. The total fish stock $(S^* + I^*)$ at E^* increases to 4.41 units from 4.02 units compared to the previous state. For higher tax, say $\tau = 10.35$ (> 10.32), the local stability condition given in Theorem 4.2(iii) becomes $c - (I_2q_2 + S_2q_1)(M - \tau) = 0.21 > 0$ and $2(S_2 + I_2) - L = 4 > 0$. Therefore, following Theorem 4.2(iii), the system converges to the harvesting-free equilibrium state $E_2 = (1.25, 3.54, 0, 12.45)$ (Fig. 4.2c), where each state variable has positive value except the fishing effort, which is zero. Observe that the fish market price in this state becomes too high ($M = 12.45$ units) for an imbalance in the demand and supply. Interestingly, even though the available fish stock is maximum ($S_2 + I_2 = 4.795$ units) in this case, the demand diminishes to zero due to the high market price (see Fig. 4.1d). Thus, there is a regime shift as τ crosses the upper transcritical value, where the system enters into a non-harvesting regime from the harvesting regime due to excessive fishing tax. Fishers opt out of fishing as harvesting is not economically viable at a higher tax ($\tau > 10.32$). Therefore, it is necessary to control the tax parameter sustainably, and the challenge for the regulating agency is to optimize this parameter for sustainable socio-economic benefits.

4.4 Optimal taxation policy

Here we explore the trade-off between the regulatory tax and the societal net benefit. The societal benefit (say, Θ) is defined here as the sum of net revenue from fish selling (say, Θ_1) and the income earned from the fishing tax (say, Θ_2), where

$$\begin{aligned} \Theta_1(S,I,H,M,\tau) &= \text{landed fish} \times \text{(market price minus fishing tax)} \\ &= H(q_1S + q_2I)(M - \tau), \end{aligned} \tag{4.20}$$

$$\begin{aligned} \Theta_2(S,I,H,M,\tau) &= \text{landed fish} \times \text{fishing tax} \\ &= H(q_1S + q_2I)\tau, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\Theta(S,I,H,M,\tau) &= \text{Revenue from fishing } (\Theta_1) + \text{Revenue from tax } (\Theta_2)\\ &= (q_1MS+q_2MI)H. \end{aligned} \tag{4.22}$$

We find whether there exists an optimal value of the imposed tax so that the societal benefit is maximum. To maximize the societal benefit, the optimal taxation problem may be defined as

$$\Im = \int_0^{\infty} \Theta(S,I,H,M,\tau)e^{-\delta t}dt, \tag{4.23}$$

where δ indicates the annual discount rate and Θ is defined in Eq. (4.22). The control variable τ is subject to the constraints $0 \leq \tau < \tau_{max}$, where τ_{max} denote the upper limits of the imposed tax. By virtue of the Pontryagin's maximum principle [292], one can write the Hamiltonian of the system as

$$\begin{aligned}\Upsilon(S,I,H,M,\tau) = {} & H(q_1MS+q_2MI)e^{-\delta t} + \xi_1\left[j(S+I)\left(1-\frac{S+I}{L}\right) - fSI - q_1SH\right] + \\ & \xi_2\left[fSI - \mu I - q_2IH\right] + \xi_3\left[\phi_1\big((q_1S+q_2I)(M-\tau)-c\big)H\right] + \\ & \xi_4\left[\phi_2M\big(A - A_1M - A_2M^2 - q_1SH - q_2IH\big)\right],\end{aligned} \tag{4.24}$$

subject to the system (4.6), where ξ_1, ξ_2, ξ_3 and ξ_4 are the adjoint variables. The optimal control variable τ has to satisfy the following conditions to maximize Υ [293]:

$$\frac{\partial \Upsilon}{\partial \tau} = 0,\ \frac{d\xi_1}{dt} = -\frac{\partial \Upsilon}{\partial S},\ \frac{d\xi_2}{dt} = -\frac{\partial \Upsilon}{\partial I},\ \frac{d\xi_3}{dt} = -\frac{\partial \Upsilon}{\partial H},\ \frac{d\xi_4}{dt} = -\frac{\partial \Upsilon}{\partial M}. \tag{4.25}$$

At any arbitrary equilibrium point $(\hat{S},\hat{I},\hat{H},\hat{M})$, $\frac{\partial \Upsilon}{\partial \tau} = 0$ gives $\xi_3\phi_1\big(-q_1\hat{S} - q_2\hat{I}\big)\hat{H} = 0$. For the nontrivial solution, one must have

$$\xi_3 = 0. \tag{4.26}$$

Again, $\frac{d\xi_4}{dt} = -\left[\frac{\partial \Upsilon}{\partial M}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$ gives

$$\frac{d\xi_4}{dt} = D_2e^{-\delta t} + D_1\xi_4, \tag{4.27}$$

where $D_1 = -\phi_2\big\{A - 2A_1\hat{M} - 3A_2\hat{M}^2 - q_1\hat{S}\hat{H} - q_2\hat{I}\hat{H}\big\}$ and $D_2 = -\big\{q_1\hat{S}\hat{H} + q_2\hat{I}\hat{H}\big\}$.
Solving (4.27), one gets

$$\xi_4 = -\frac{D_2}{D_1+\delta}e^{-\delta t}. \tag{4.28}$$

Also, $\frac{d\xi_3}{dt} = -\left[\frac{\partial \Upsilon}{\partial H}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$ provides

$$\xi_1 = F_1 e^{-\delta t} - F_2 \xi_2,$$

where $F_1 = \left(q_1\hat{S} + q_2\hat{I}\right)\left(\hat{M} + \frac{D_2\phi_2\hat{M}}{D_1+\delta}\right)\frac{1}{q_1\hat{S}}$ and $F_2 = \frac{q_2\hat{I}}{q_1\hat{S}}$.

Putting the value of ξ_1 in $\frac{d\xi_2}{dt} = -\left[\frac{\partial \Upsilon}{\partial I}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$, one gets

$$\begin{aligned} \xi_2 &= -\frac{D_4}{D_3+\delta} e^{-\delta t}, \text{ and consequently} \\ \xi_1 &= \left\{F_1 + \frac{D_4 F_2}{D_3+\delta}\right\} e^{-\delta t}, \end{aligned}$$

where $D_3 = F_2\left(j\left(1 - \frac{2(\hat{S}+\hat{I})}{L}\right) - f\hat{S}\right) - f\hat{S} + \mu + q_2\hat{H}$, and $D_4 = -q_2\hat{H}\hat{M} - F_1\left(j\left(1 - \frac{2(\hat{S}+\hat{I})}{L}\right) - f\hat{S}\right) - \frac{q_2\phi_2 D_2\hat{H}\hat{M}}{(D_1+\delta)}$.

Observe that each of these adjoint variables (ξ_1, ξ_2, ξ_3, ξ_4) is bounded. Substituting the values of these adjoint variables in $\frac{d\xi_1}{dt} = -\left[\frac{\partial \Upsilon}{\partial S}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$, one gets the optimal tax equation as

$$\begin{aligned} \Gamma(\tau) &= q_1\hat{M}\hat{H} + \left(j\left(1 - \frac{2(\hat{S}+\hat{I})}{L}\right) - f\hat{I} - q_1\hat{H} - \delta\right)\left(F_1 + \frac{D_4F_2}{D_3+\delta}\right) - \frac{D_4 f\hat{I}}{D_3+\delta} + \\ & \frac{D_2\phi_2 q_1\hat{H}\hat{M}}{(D_1+\delta)} = 0 \end{aligned} \tag{4.29}$$

for a suitable choice of the annual discount rate, δ. The positive values of τ for which $\Gamma(\tau) = 0$ are the possible optimal candidates. The optimal value $\tau = \tau^c$ is the value for which Θ is maximum. If there are i number of equilibrium points with non-zero harvesting value, we will obtain i number of critical τ^c's. Then the optimal societal revenue, $\Theta^{\max}$, is given by

$$\Theta^{\max} = \max_i \Theta(S, I, H, M, \tau_i^c). \tag{4.30}$$

To compute the optimum tax level and the corresponding societal revenue $\Theta(S, I, H, M, \tau)$ for the parameter values considered in Table 4.1 with an annual discount rate $\delta = 0.001$, we solve Eq. (4.29) at the disease-free and endemic equilibrium points, where harvesting has non-zero equilibrium value. We obtain two optimal values of τ, namely, $\tau_1^c = 4.44$M\$/metric ton at the infection-free equilibrium state, and $\tau_2^c = 9.22$ M\$/metric ton at the endemic equilibrium state (See Fig. 4.3a). The societal benefit or the net revenue at these two optimal tax values are computed from (4.22) as $\Theta(\tau_1^c) = 4.096$ M\$/year and $\Theta(\tau_2^c) = 1.478$M\$/year. Thus, the maximum net revenue is $\Theta^{\max} = \max(4.09, 1.478) = 4.09$M\$/year and the optimal tax is $\tau_1^c = 4.44$M\$/metric ton, which is obtained at the disease-free equilibrium state, E_1. Following similar calculations, one can get the optimal equation for the fishing tax revenue Θ_2 as

$$\begin{aligned}\Gamma_1(\tau) \;=\;& \frac{1}{q_1\hat{S}}\left\{-\delta + j\left\{1-\frac{2(\hat{S}+\hat{I})}{L}\right\} - f\hat{I} - q_1\hat{H}\right\}\left\{\frac{D_6 q_2\hat{I}}{D_5+\delta} + (q_1\hat{S}+q_2\hat{I})\hat{M} - c - \frac{\delta}{\phi_1} + \right.\\ & \left.\frac{D_2}{D_1+\delta}\right\} + q_1\tau\hat{H} - \frac{D_6 f\hat{I}}{D_5+\delta} + q_1(\hat{M}-\tau)\hat{H} + \frac{D_2\tau\hat{H}}{D_1+\delta} = 0, \qquad (4.31)\end{aligned}$$

where

$$\begin{aligned} D_5 \;=\;& \frac{q_2\hat{I}}{q_1\hat{S}}\left\{j\left(1-\frac{2(\hat{S}+\hat{I})}{L}\right) - f\hat{S}\right\} + \mu - f\hat{S} + q_2\hat{H},\\ D_6 \;=\;& -q_2\tau\hat{H} - \left\{(q_1\hat{S}+q_2\hat{I})\hat{M} - c - \frac{\delta}{\phi_1} + \frac{D_2}{D_1+\delta}\right\}\left\{j\left(1-\frac{2(\hat{S}+\hat{I})}{L}\right) - f\hat{S}\right\} -\\ & \frac{D_2\phi_2 q_2\hat{M}\hat{H}}{D_1+\delta}.\end{aligned}$$

The solution of Eq. (4.31) provides the optimal value of τ as $\tau^{c1}_{\Theta_2} = 1$ M\$/metric ton and $\tau^{c2}_{\Theta_2} = 5.68$ M\$/metric ton (See Fig. 4.3b). The earnings from fishing tax at these two optimal tax values are computed from Eq. (4.21) as $\Theta_2(\tau^{c1}_{\Theta_2}) = 0.78$ M\$/year and $\Theta_2(\tau^{c2}_{\Theta_2}) = 2.717$ M\$/year. Thus, the maximum fishing tax revenue is $\Theta_2^{\max} = 2.717$ M\$/year and the optimal tax is $\tau^{c2}_{\Theta_2} = 5.68$M\$/metric ton, which is obtained at the endemic equilibrium state, E^*. It is worth mentioning that the fisherman revenue (Θ_1) is a decreasing tax function, and it is maximum

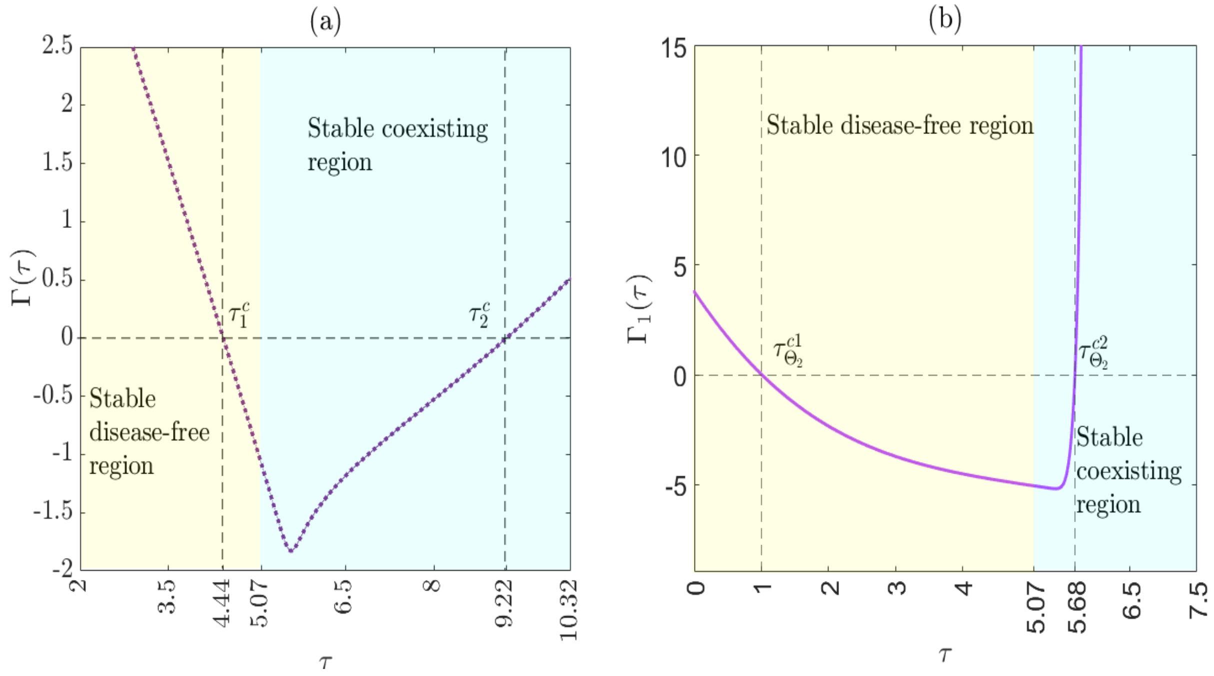

Figure 4.3: (a) Plot of the optimal tax equation (4.29) for Θ with respect to τ. It shows that there are two optimal values of τ, viz., $\tau_1^c = 4.44$ and $\tau_2^c = 9.22$, for which $\Gamma(\tau) = 0$. (b) Similar plot of (4.31) for Θ_2 shows that there exists two optimal values of τ, viz., $\tau^{c1}_{\Theta_2} = 1$ and $\tau^{c2}_{\Theta_2} = 5.68$. Here the annual discount rate is $\delta = 0.001$, and the other parameters are as in Table 4.1.

when $\tau = 0$. In Table 4.2, we have presented the equilibrium values of the state variables and the revenues at equilibrium points E_1 and E^* for some particular discounts of τ. It shows that the societal income is maximum ($\Theta = 4.096$) when $\tau = 4.44$. Fishermen's earnings from selling fish are maximum ($\Theta_1 = 2.453$) when $\tau = 0$ and the revenue from the fishing tax is maximum ($\Theta_2 = 2.717$) when $\tau = 5.68$. It is interesting to note that the total equilibrium fish stock ($\hat{S}+\hat{I}$) is maximum (4.795) in the endemic state; however, the maximum societal revenue (4.096) is generated at the disease-free equilibrium state for the optimal tax $\tau = 4.44$.

Table 4.2: This table evaluates the societal revenue $\Theta(\hat{S},\hat{I},\hat{H},\hat{M})$, fisherman's revenue $\Theta_1(\hat{S},\hat{I},\hat{H},\hat{M})$, and tax revenue $\Theta_2(\hat{S},\hat{I},\hat{H},\hat{M})$ at the equilibrium states E_1 and E^*, where harvesting is possible, for some particular values of fishing tax with an annual discount rate $\delta = 0.001$. Observe that societal revenue is maximum (4.096 M$/year) in the disease-free state (where $\hat{I} = 0$) for $\tau = 4.44$ M$/metric ton. Tax revenue is maximum (2.717 M$/year) in the endemic state for $\tau = 5.68$ M$/metric ton. Fishers' revenue is maximum (2.453 M$/year) when there is no fishing tax and gradually declines with increasing τ. The optimum values are written in boldface. The parameters are as in Fig. 4.3.

τ (M$/MT*)	$\hat{S}$ (MT)	$\hat{I}$ (MT)	$\hat{S}+\hat{I}$ (MT)	$\hat{H}$ (SFU)	$\hat{M}$ (M$/MT)	Θ_1 (M$/year)	Θ_2 (M$/year)	$\Theta = \Theta_1 + \Theta_2$ (M$/year)
0	3.789	0	3.789	0.273	2.970	**2.453**	0	2.453
2.5	4.024	0	4.024	0.2195	5.296	1.976	1.767	3.743
4.44	4.244	0	4.244	0.1701	7.091	1.531	2.565	**4.096**
5	4.3123	0	4.3123	0.1549	7.609	1.394	2.672	4.066
5.07	4.320	0	4.320	0.153	7.674	1.377	2.681	4.058
5.68	4.305	0.0839	4.3889	0.1359	8.237	1.22	**2.717**	3.941
6	4.101	0.3118	4.4128	0.1267	8.527	1.140	2.707	3.847
8	2.780	1.791	4.571	0.0685	10.345	0.6132	2.0918	2.705
9.22	1.971	2.715	4.686	0.0321	11.459	0.289	1.189	1.478
10.32	1.250	3.545	4.795	0	12.454	0	0	0

*MT stands for metric ton.

The equilibrium revenue curves for varying taxes are plotted in Fig. 4.4. It shows that the fisherman's revenue (Θ_1) is maximum when there is no fishing tax. Here the societal benefit (Θ) coincided with the fishers' earnings. Otherwise, societal benefits exceed the fishers' incomes for the feasible range of τ. The societal benefit is always higher from the generated revenue from the fishing tax (Θ_1) in the same range. It is observable that the societal revenue gradually increases with τ and becomes maximum in the disease-free state for $\tau = 4.44$, and after that, it decreases to zero. Whereas the tax revenue increases till $\tau = 5.68$ and then declines to zero. The maximum tax earned at $\tau = 5.68$ when disease persists in the system. These results show the existence of a trade-off between the revenue earnings and the imposed tax.

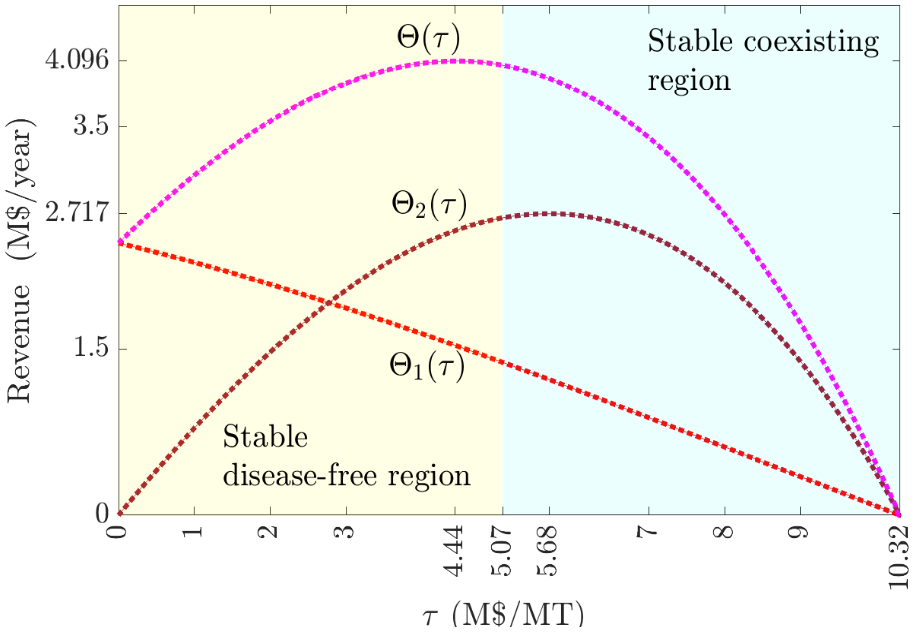

Figure 4.4: Equilibrium revenue curves are plotted against the tax. The societal revenue ($\Theta(\tau)$) is maximum at $\tau_1^c = 4.44$ M\$/metric ton, and the corresponding maximum revenue is 4.096 M\$/year. The maximum revenue generated from the imposed tax ($\Theta_2(\tau)$) is obtained at the optimal tax $\tau_{\Theta_2}^c = 5.68$ M\$/metric ton, and the corresponding tax revenue is 2.717 M\$/year. At $\tau = 0$, the fishers' revenue is maximum, and the corresponding earning is 2.453 M\$/year. Here the annual discount rate is $\delta = 0.001$, and the other parameters are as in Table 4.1.

4.4.1 Sensitivity analysis

We estimated the changes (see Table 4.3) in the optimal societal revenue due to the changes in the parameter values. Table 4.3 shows only those parameters out of 13 parameters in the Table 4.1 which bring significant change in the result. While determining the sensitivity of a parameter, all other parameters remain fixed as in Table 4.1 with $\tau = 4.44$ M\$/metric ton at which societal revenue is maximum (4.096 M\$/year). This table shows that the maximal demand A is the most sensitive parameter. If the parameter A is enhanced by 50% or 25% from its default value 0.9 (see Table 4.1), then the optimal societal revenue will be increased by 83.86% or 40.18%, respectively. On the contrary, if it is decreased by 50% or 25%, the societal revenue decreases by 76.07% or 37.95%, respectively. It is observable that the stability region interchanges between disease-free and endemic states with the variation of most of the parameters. However, the scenario is completely different with the variation of j and A_1, where the stability region always remains disease-free. It is interesting to observe that the optimal societal revenue always decreases from its default value with any increment or decrement of the parameter τ. This implies that the value of τ (4.44 M\$/metric ton) is optimal and the corresponding societal revenue (4.096 M\$/year) is also optimal.

Table 4.3: Effect on optimal societal revenue due to the change in the key parameters and the corresponding changes in the stability state. The seven parameters are varied 25 or 50 per cent upside or downside from their default values mentioned in Table 4.1, and the corresponding changes in the optimal societal revenue are tabulated. Here a "$+$" sign indicates a shift in the upside, and a "$-$" sign suggests a change in the downside.

Parameters	Changes in parameters (%)	Changes in Θ (%)	Stability region
A	+50	83.86	Disease-free
	−50	−76.07	Endemic
	+25	40.18	Disease-free
	−25	−37.95	Endemic
A_2	+50	−20.81	Endemic
	−50	22.85	Disease-free
	+25	−10.65	Disease-free
	−25	11.19	Disease-free
L	+50	−2.91	Disease-free
	−50	−24.00	Endemic
	+25	−1.07	Disease-free
	−25	−2.84	Endemic
j	+50	−0.08	Disease-free
	−50	−1.79	Disease-free
	+25	−0.13	Disease-free
	−25	−0.03	Disease-free
A_1	+50	−6.09	Disease-free
	−50	6.21	Disease-free
	+25	−3.06	Disease-free
	−25	3.09	Disease-free
c	+50	−4.001	Endemic
	−50	−4.47	Disease-free
	+25	−0.999	Disease-free
	−25	−1.13	Disease-free
τ	+50	−12.59	Endemic
	−50	−11.11	Disease-free
	+25	−3.03	Endemic
	−25	−2.96	Disease-free

4.5 Discussion

The fishery has become one of the significant subsistences across the globe. According to the 2021 report of the Food and Agriculture Organization (FAO), about 38.98 million people are engaged in fisheries [294], justifying why most fisheries are under stress. Some governing agencies try to restrict harvesting by imposing a tax per unit of biomass of landed fish. Although taxation controls overfishing, an irrational tax policy may negatively affect fishery dynamics and revenue generation. It may help increase fishery-related infection and drastically reduce the amount of landed fish, causing a significant difference between the demand and supply of this globally accepted food item. A pronounced effect of this imbalance is the price hike of the fish stock, which may directly impact the fishery & related industries and employability. So the question is – how much taxation benefits a fishery in the presence of infection? Does there exist any trade-off? How does the intricacy of demand, tax, and disease play a role in fishery dynamics and revenue generation? We proposed a nonlinear bioeconomic harvesting model of a single-species fishery with infection, variable market price, and nonlinear demand to answer these questions and explore taxation's ecological and economic effects. To our knowledge, such a theoretical investigation is rare in the literature. We have considered a nonlinear quadratic market demand to represent the demand-price relation. Such a quadratic demand may be a more suitable demand function, compared to constant [295], linear [142], and saturated [223] types functions, when the demand of a particular commodity decreases sharply if its price is high.

Our system has seven equilibrium points, of which three are always unstable, and the remaining four may be stable or unstable depending on the parametric conditions. The bifurcation analysis for the tax parameter classified the system stability into three distinct dynamic regimes. It is revealed that the system remains disease-free if the regulatory tax is low, which promotes intensive harvesting. Such intense harvesting reduces the infected fish, causing the elimination of infection from the system. A reduction in the harvesting efforts due to increased tax helps the infection spread, and the disease can invade the fish population for an extended range of intermediate tax. Healthy fish density gradually decreases in this case with a complementary increase in the infected fish density. Since fish harvesting is relatively low in the medium range of tax, its supply reduces significantly, increasing the difference between demand and supply with the growing tax. Therefore, the fish price steadily increases following the open market theory. As the market price becomes too high, the demand gradually diminishes to zero (see Fig. 4.5).

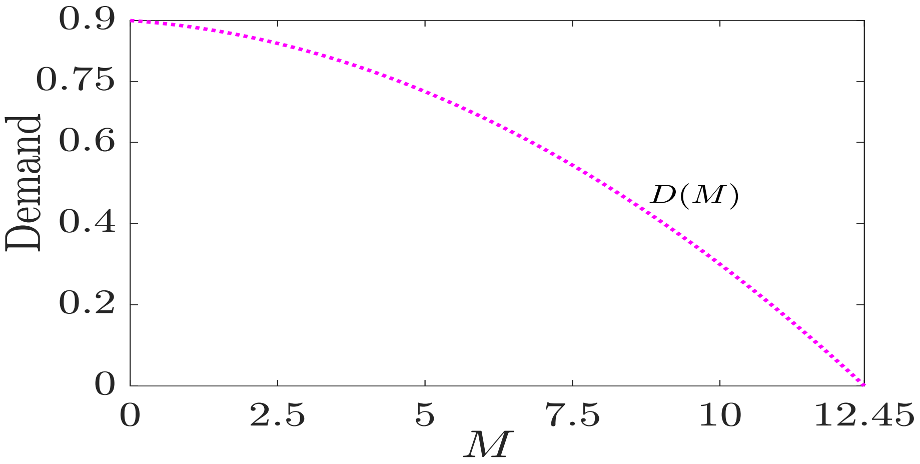

Figure 4.5: Quadratic demand curve $D(M) = A - A_1 - A_2M^2$ is plotted as a function of price, M, in the range $0 \leq M \leq 12.45$. The upper value of M is fixed from Fig. 4.2(c), where the harvesting-free equilibrium E_2 is stable. It shows that demand decreases from its maximum when the price is zero to its minimum when it is high. The parameters are as in Table 4.1.

Thus, the fisheries experience a tax-induced functioning instability at the higher level of fishing tax. In such a case, fishing is no more economically viable, and the fishers opt out of fisheries due to a lack of demand and high fishing tax. The ecological and economic effect of such a non-harvesting regime shift is immense. Such a shift from a harvesting regime to a non-harvesting regime is not due to the scarcity of harvested stock but the need for better governance. Therefore, it is necessary to control the tax parameter sustainably, and the challenge for the regulatory agency is to optimize this parameter for maximal socio-economic benefits.

It is worth mentioning that the fisherman's income will be maximum if they do not pay any fishing tax. Indeed, their earnings will gradually decrease with the increasing tax. On the other hand, the regulatory authority earns more revenue by charging a higher fishing tax. Imposing a tax is beneficial because it controls harvesting and saves fishery from overexploitation. Secondly, the regulatory authority may use the tax revenue for various welfare measures for the people associated with the fishery, marine ecosystem, coastal management, and related value chains for sustainable development and economic prosperity. Therefore, an effective regulatory taxation policy may play a crucial role in the sustainable use of fisheries through a win-win solution. A low tax may help make the system infection-free, while infection may persist if the tax is high. A higher regulatory tax, however, may put an end to harvesting. It implies that there exists a trade-off. Consequently, an optimal taxation policy is necessary to make a balance among the harvesting intensity, infection spreading, market demand & supply, and revenue earnings.

It is revealed that some optimum tax exists, where the societal income is maximum and occurs at the disease-free state for some lower optimal tax. However, the regulatory authority earns the maximum revenue for some higher optimal tax in the disease state. Fishers' income is maximized with no tax and steadily decreases to zero with increasing tax. Noticeably, the gap between demand and supply of fish widens with the increasing tax, causing a steady price increase in this globally accepted renewable food item. Thus, the higher regulatory tax causes an imbalance in the fish supply and price, which may severely impact fishery, fishery-related industries, and employability. Therefore, there should be an optimal tax policy for which the fishery sustains and maximizes societal revenue. The future of fishing thus depends on many interconnected factors, including infection control, ecosystem management, maintaining the demand-supply chain, and implementing a justifiable regulatory taxation policy through good governance. Indeed, this will help put a step forward in achieving the sustainable development goals by 2030 as set by the United Nations.

In recent decades, climate change has triggered significant regime shifts in various oceanic indices [296, 297]. While the predominant cause of this shift in the fishery is attributed to overfishing [135, 136], there is growing evidence linking it to climate change as well [137, 138]. In the forthcoming chapter, our investigation aims to explore the possibility of additional factors contributing to the regime shift in the fishery, which have not been addressed in previous studies.

5

Demand-induced regime shift in fishery: A mathematical perspective[1]

5.1 Introduction

A regime shift is a change in the average value of a particular data series within a year to a decadal scale [131]. A more recent definition of regime shift is defined as a large, abrupt, and persistent change in the system behaviour that causes significant impacts on human well-being [132, 133]. In the case of fisheries, regime shift means a change in the harvested species due to a change in the non-harvested species [134]. For example, different driving forces, like eutrophication, pollution, climate change, etc., may alter the distribution of biotic and abiotic factors, which may cause a difference in the spatial distribution of planktonic invertebrates. Therefore, a regime shift due to the change in plankton distribution may occur in the economic species [134]. A regime shift in many oceanic indices due to climate change has been identified in the last three decades [296, 297]. Though overfishing is the primary reason for a regime shift in the fishery [135, 136], climate change has a close association with it [137, 138]. In this chapter, we demonstrate a different reason for the regime shift in fishery, not reported earlier to the best of our knowledge. We show that high demand for fish may cause a regime shift in a fishery in a shorter time. Therefore, economic theory should be a part of the current fishery management policy.

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[1] *The bulk of this chapter has been published in Mathematical Biosciences, DOI:https://doi.org/10.1016/j.mbs.2023.109008, (2023).*

Modern fishery management faces multi-factorial challenges. For example, on the one hand, they plan for long-term sustainable fish production while maintaining the aquatic ecosystem's health. On the other hand, plans for the amount of harvested fish biomass to be harvested to meet the market demand. The per capita consumption rose to 20.2 kg compared to 9.9 kg in the 1960s, and the target is to increase it to 21.4 kg by 2030 [95]. The fish demand to meet the protein and amino-acids supply for the world's growing population is expected further to aggravate the existing over-harvesting problems of fishery and aquaculture. Therefore, a new challenge for the fishery management policy is to monitor the market demand for fishery items and make the production target accordingly. Keeping the price of fish and fishery products affordable is challenging for the management authorities. Price control is more critical in the case of developing and underdeveloped countries to ensure the food supply. Different studies consider that there exists an inverse relation between fish price and available fish stocks [96, 97]. Remarkably, the cost of high-valued fish in the UK market varies significantly with the landed fish quantity [98]. There is a significant difference in the market price of highly demanded Hilsa fish (*Tenualosa ilisha*) in South Asian countries due to the variation in the availability of the fish [298]. Therefore, dynamic price monitoring depending on the instantaneous supply and demand of fish may provide insights for the better benefit of society.

There are many pieces of evidence of fish extinction due to overharvesting [15, 16, 17]. A fifteen years tenure of commercial harvesting reduces the community biomass of fishes by over 80% [299]. Undoubtedly, the disappearance of commercial fish has a profound direct impact on livelihood and national income. However, their indirect effect is more severe and rarely accounted for [25]. Fish species are not isolated. Instead, they are interconnected with the other species of the aquatic ecosystem. A gradual reduction of fish species may alter the energy flow of the food chain. Many harvested fishes are intermediate predators. They feed on zooplankton but are predated by larger fish. Thus, both the top-down and bottom-up cascading effects are observed due to overfishing [25, 26], causing reduced ecosystem services. Therefore, the regulatory authority should take appropriate strategies to reduce the harvesting effort and minimize its impact on the ecosystem services. Some popular and practical approaches to reduce harvesting pressure are to levy a tax on the landed fish [30], put a tax on the fishing vessels [31], fix a fishing quota [32] and fishing days [33]. In this chapter, we only consider the tax on landed fish as the controlling measure.

Another concern of fishery management is the infectious disease caused by a virus, bacteria, protists, and metazoans [55, 149]. Such contagious diseases cause a significant economic loss in fishery either by reducing the biological productivity of the diseased fish and/or by lowering the commercial value of the infected fish [149, 160]. White spot syndrome virus (WSSV) is a predominant infectious disease in shrimp. Since 1992, this disease has devastated shrimp production and related industries in many countries, like Thailand [161], Ecuador [162], India [163], Iran [164] and USA [165]. This virus spreads rapidly from one infected shrimp to another susceptible shrimp and can kill them within seven to ten days [166, 167]. Thus, the infected shrimps have no chance to reproduce but to die. The economic loss due to WSSV in the last two-three decades has been reported to be huge. During 2010-2016, Thailand reported a financial loss of 11.58 B$ and working days loss of 0.1 million [168]. India reported 250 M$ loss due to WSSV in 2006-2008, along with 2.15 million man-days employment loss [163]. Shrimp production in Bangladesh dropped to 18,630 tonnes in 1998 from 25,742 tonnes in 1997, causing a significant economic loss [169]. The income from shrimp export drastically reduced in Iran due to this viral infection. The frozen shrimp export of Iran declined to 2,290 tonnes in 2007 compared to 7,680 tonnes in 2004, causing a reduction in income from 32.8 M$ in 2004 to 8.7 M$ in 2007 [164].

Many bioeconomic models consider that the harvesting effort follows a catch per unit effort (CPUE) hypothesizes [139, 181, 140]. However, the CPUE type harvesting has several abridgments, such as the harvesting rate becoming infinite as the harvested stock is infinite or the effort is infinite. This is quite unrealistic from the applicability and theoretical viewpoints [179, 142, 182]. Recently, Moussaoui and Auger [142] proposed a bioeconomic fishery model considering a nonlinear saturated type harvesting effort. According to classical economic theory [99], an essential commercial factor in any bioeconomic study is maintaining the balance between the landed biomass and market demand. Although most studies considered linear demand function [142, 175, 208], it is well-known that nonlinear demand provides a better market scenario than the former [184, 185]. Here, we consider such nonlinear saturated type demand to integrate the dynamic price of fish with its instantaneous demand and supply. We analyze a four-dimensional bioeconomic model taking into consideration all these aspects. The objective is to decipher the consequences of a single-species fishery model in the presence of infection, nonlinear saturated harvesting rate, fishing tax, and market demand. How increasing demand may cause a regime shift in the fish and fishery is the most crucial objective of this work.

The subsequent chapter consists of the following sections in sequential order. The immediate next section describes the bioeconomic model formulation for a single-species fishery. Analytical results of the model, like well-posedness, basic reproduction number, and equilibrium points & their stability analysis, are presented in Section 5.3. The optimal fishing tax analysis is given in Section 5.4. Section 5.5 contains the simulation results, including one and two-parameter bifurcation results. Furthermore, the price-induced regime shift of the system is discussed here. Section 5.6 sums up the entire chapter.

5.2 The model

In Chapter 3, we explored the dynamical behaviour of the following single-species bioeconomic fishery model in the presence of infection:

$$\begin{aligned}
\frac{dX}{dt} &= rX\left(1-\frac{X+Y}{K}\right)-\lambda XY-Q_1(X,H),\\
\frac{dY}{dt} &= \lambda XY-\mu Y-Q_2(Y,H),\\
\frac{dH}{dt} &= \phi_1\Big\{P(q_1X+q_2Y)H-cH\Big\},\\
\frac{dP}{dt} &= \phi_2\Big\{D(P)-(q_1X+q_2Y)H\Big\},
\end{aligned} \tag{5.1}$$

where X & Y are the densities of susceptible and infected fishes, and H & P are the harvesting effort and market price of fish at time t. This model says that fish species are classified into two classes, susceptible fish and infected fish when the disease invades the fish population. Disease spreads through contact following mass action law with λ as the transmission rate. Infected fish cannot give birth and does not recover. The death (natural plus virulence) of infected fish occurs at an exponential rate with a rate constant μ. Under the same effort, H, the harvesting rate of healthy fish ($Q_1(X,H)$) and infected fish ($Q_2(Y,H)$) are

$$Q_1(X,H)=q_1XH,\;\; Q_2(Y,H)=q_2YH,$$

where q_1 and q_2 are the catchability coefficients of susceptible and infected fishes, respectively. Harvesting effort at any time is proportional to the profit margin (selling price − cost of fishing), and the corresponding instantaneous price change is proportional to the difference between the demand and supply of fish. The demand decreases linearly with the price as $D(P)=A-\alpha P$, where A is the maximum demand and α is the demand decreasing rate. The parameters ϕ_1,ϕ_2 are proportionality constant.

We mentioned its several drawbacks that should be addressed in the future study. For example, this model considers that fish harvesting follows the CPUE hypothesis, which states that the harvesting rate at any time is proportional to the product of the fish stock and harvesting effort. Such harvesting has some serious loopholes. For example, the harvesting rate becomes unbounded when either effort or fish stocks become large. To remove such unrealistic features of the harvesting rate with a finite effort, we consider

$$Q_1(X,H)=\frac{q_1XH}{X+D_1},\;\; Q_2(Y,H)=\frac{q_2YH}{Y+D_2},$$

where D_1,D_2 are the half-saturation constants of susceptible and infected fishes. Secondly, the demand function considered in the model (5.1) varies linearly with the price. It is, however, pointed out that the demand for many commodities in a real market scenario is nonlinear and saturates as price becomes high [184, 185]. We, therefore, consider here a saturated demand of the form [184]

$$D(P) = \frac{A}{1+BP},$$

where A is the maximum demand as before, and B is the demand sensitivity parameter. Third, the cost of the healthy and infected fishes in model (5.1) was considered the same. However, it is certainly not true in the real market scenario. The insalubrious fish is sold at a much lower price than its healthy counterpart. In many cases, such fish are sold at a fixed lower price [300]. The demand for salubrious fish varies with its price; however, it is not the case for insalubrious fish. We further assume that the regulatory agency imposes a tax $\tau(>0)$ on the per unit biomass of the landed fish to control the overexploitation of fish [271, 267]. Taking into account that fishes are harvested with a saturated catch, a fishing tax, different prices for healthy and infected fish, and saturated demand, we propose the following bioeconomic fishery model:

$$\begin{aligned}
\frac{dX}{dt} &= X\left\{ r\left(1-\frac{X+Y}{K}\right) - \lambda Y - \frac{q_1 H}{X+D_1} \right\}, \\
\frac{dY}{dt} &= Y\left\{ \lambda X - \mu - \frac{q_2 H}{Y+D_2} \right\}, \\
\frac{dH}{dt} &= \phi_1 H\left\{ \left(\frac{q_1(P-\tau)X}{X+D_1} + \frac{q_2(p-\tau)Y}{Y+D_2}\right) - c \right\}, \qquad (5.2) \\
\frac{dP}{dt} &= \phi_2 P\left(\frac{A}{1+BP} - \frac{q_1 X H}{X+D_1}\right),
\end{aligned}$$

where $p(<< P)$ is the constant price per unit biomass of the infected fish. It is to be noted that the epidemiological characteristics of shrimp due to WSSV match the assumptions of our model. All parameters are nonnegative from biological viewpoints. The variables and parameters are described in Table 5.1.

Several studies have considered the harvesting of ecological species. Gakkhar and Singh [301] studied a food web model where the top predator is harvested following the CPUE type rate. They examined the local stability of the equilibrium points and optimized the net revenue. It is also demonstrated that complex dynamics like chaos may be suppressed through harvesting. Ang and Safuan [283] considered CPUE harvesting in an intraguild predator-prey model with variable carrying capacity and studied the bionomic equilibrium and optimal harvesting policy. CPUE harvesting was considered in a predator-prey model with imprecise biological parameters in [302]. The optimal harvest policy and the bionomic equilibrium were studied there in detail. Chakraborty et al. [153] considered a ratio-dependent predator-prey model with CPUE predator harvesting. They explained the coexistence of the species under harvesting. A harvested predator-prey model with infection and the ratio-dependent functional response was proposed and analyzed in [155]. They observed the effect of harvesting on the qualitative behavior of the system and prescribed the conditions for which the system would not show periodic solutions. The impact of fishing tax has been considered in many bioeconomic models [303, 268, 270, 90]. Ghosh and Kar [90] considered a predator-prey model with prey harvesting and observed that the optimal taxation policy is bang-bang control. Kar [269] executed a similar study in a ratio-dependent predator-prey harvesting model, where predator species are subject to harvesting. Saturated harvesting effort was considered in several fishery models [179, 183, 304, 305]. However, these studies did not consider price as a dynamic variable following open market theory and did not examine the effect of disease on the harvested stock. This chapter explicitly combines all these essential ecological and epidemiological ingredients and fulfills the shortcomings of many studies.

Table 5.1: State variables and parameters with their descriptions and default values.

Variable	Description	Unit	
$X(t)$	Healthy fish biomass at time t	metric tonnes (MT)	
$Y(t)$	Infected fish biomass at time t	metric tonnes	
$H(t)$	Fishing effort at time t	SFU*	
$P(t)$	Market price per unit of biomass at time t	M**/metric tonne	
Parameter	Description	Default Value	Reference
r	Intrinsic growth rate of healthy fish	0.9 /year	[208]
K	Environmental carrying capacity	metric tonnes	Variable
D_1	Half saturation level of susceptible fish	4 metric tonnes	This chapter
D_2	Half saturation level of infected fish	4.8 metric tonnes	This chapter
λ	Transmission rate	metric tonnes/year	Variable
q_1	Catchability coefficient of susceptible fish	0.8 metric tonnes/SFU/year	[287]
μ	Total death (natural + virulence) rate of infected fish	0.05 /year	[208]
q_2	Catchability coefficient of infected fish	0.9 metric tonnes/SFU/year	[287]
c	Cost per unit of fishing effort	0.05 $/SFU/year	This chapter
A	Maximum demand	metric tonnes/year	Variable
B	Demand sensitivity parameter	5 metric tonnes/M$	This chapter
ϕ_1	Stiffness parameter	0.1 SFU/M$	[208]
ϕ_2	Proportionality constant	0.15 /metric tonne	[208]
p	Fixed market price of infected fish	0.05 $/metric tonne	This chapter
τ	Tax per unit biomass of harvesting	M$/metric tonne	Variable

* SFU stands for Standardized Fishing Unit [242, 243] and ** M$ indicates million USD.

5.3 Analytical results

In any population model, it is vital to show that model solutions are positively invariant and bounded. The region where every solution of the system (5.2) is positive and uniformly bounded is given in the following lemma. The proof is similar to lemma 2.1 in Chapter 2.

Lemma 5.1. *Every solution of the system* (5.2)*, having a starting point* $(X_0, Y_0, H_0, P_0) \in \mathbb{R}_+^{4,0}$*, is positive and uniformly bounded in*

$$\mathscr{M} = \{(X,Y,H,P) : 0 < X < \zeta + \hat{\varepsilon},\ 0 < \mathscr{S}(X,Y,H,P) < \frac{s_2}{s_1} + \varepsilon, \textit{for any positive } \ \hat{\varepsilon}, \varepsilon\},$$

where $s_1 = \min\{\frac{r}{K}, \phi_2 q_1 D_1, \phi_1 q_1\}$, $s_2 = r + \phi_1 q_2 p + \phi_2 A$, *and* $\zeta = \max\{X_0, K\}$.

5.3.1 Equilibrium points

Our model system (5.2) has five equilibrium points. The trivial equilibrium point $\xi_0 = (0,0,0,0)$ and the only healthy fish equilibrium $\xi_1 = (K,0,0,0)$ always exist. The healthy and infected fishes equilibrium $\xi_2 = (\frac{\mu}{\lambda}, \frac{r(\lambda K-\mu)}{\lambda(\lambda K+r)}, 0, 0)$ exists whenever the condition $\lambda K > \mu$ is hold. The infection-free equilibrium point $\xi_3 = (X_3, 0, H_3, P_3)$ has the equilibrium components

$$H_3 = \frac{r}{q_1}\big(1 - \frac{X_3}{K}\big)(X_3 + D_1), \;\; P_3 = \left(\frac{A}{rX_3\big(1 - \frac{X_3}{K}\big)} - 1\right)\frac{1}{B},$$

where X_3 is the positive root of the equation

$$C_1 X_3^2 + C_2 X_3 + C_3 = 0, \tag{5.3}$$

having coefficients

$$C_1 = q_1 rB\big(\tau + \frac{1}{B}\big) + rBc, \; C_2 = rBc(D_1 - K) - q_1 rBK(\tau + \frac{1}{B}), \; C_3 = q_1 AK - rBcD_1 K.$$

The Eq. (5.3) has exactly one positive real root if $C_1 C_3 < 0$, and consequently, H_3, P_3 are unique. Note that H_3 is always positive as $X_3 < K$. Since $0 < (1 - \frac{X_3}{K}) < 1$, the positivity of P_3 is assured if $A > rX_3\big(1 - \frac{X_3}{K}\big)$. Thus, the disease-free equilibrium point ξ_3 uniquely exists if $A > rX_3\big(1 - \frac{X_3}{K}\big)$ and $C_1 C_3 < 0$. There may exist two disease-free equilibria under some parametric conditions. Observe that Eq. (5.3) will have two real roots if $C_2^2 - 4C_1C_3 > 0$. These roots will be positive if the signs of C_1, C_2, C_3 are, respectively, $+, -, +$ or $-, +, -$. In any case, the signs of C_1 and C_2 are opposite, and the signs of C_1 and C_3 are the same (either both positive or both negative). Thus, Eq. (5.3) will have two real positive roots and hence two disease-free equilibria, if $C_2^2 - 4C_1C_3 > 0, C_1 > 0, C_2 < 0, C_1 > 0$ or $C_2^2 - 4C_1C_3 > 0, C_1 > 0, C_2 < 0, C_1 > 0$.

The endemic equilibrium point is represented by $\xi_4 = (\bar{X}, \bar{Y}, \bar{H}, \bar{P})$, where

$$\bar{Y} = \frac{q_2 \bar{H}}{(\lambda \bar{X} - \mu)} - D_2, \; \bar{P} = \left(\frac{A(\bar{X} + D_1)}{q_1 \bar{X} \bar{H}} - 1\right)\frac{1}{B}. \tag{5.4}$$

Observe that $\bar{Y} > 0$ if $\lambda < \frac{D_2\mu + q_2\bar{H}}{D_2\bar{X}}$ and $\bar{P} > 0$ if $D_1 > (q_1\bar{H} - A)\bar{X}$. The other two equilibrium components $\bar{X}, \bar{Y}$ are the positive solutions of the equations

$$\begin{aligned} \frac{A}{B\bar{H}} - \frac{q_1\bar{X}}{B(\bar{X} + D_1)} - \frac{q_1\tau\bar{X}}{\bar{X} + D_1} - c + (p - \tau)\left(1 - \frac{D_2(\lambda\bar{X} - \mu)}{\bar{H}}\right) &= 0, \\ r(K - \bar{X}) - (r + \lambda K)\left(\frac{q_2\bar{H}}{\lambda\bar{X} - \mu} - D_2\right) - \frac{q_1\bar{H}}{\bar{X} + D_1} &= 0. \end{aligned} \tag{5.5}$$

The complexity of Eq. (5.5) restricts its analytical solution, however, the unique equilibrium will be numerically computed in the simulation section.

5.3.2 Basic reproduction number

Whether a disease will be successful to invade a host population is determined by the basic reproduction number (BRN) of the disease, usually denoted by $\mathscr{R}_0$. If $\mathscr{R}_0 < 1$ then there will be no epidemic. Following [290], one can determine the basic reproduction number of the system (5.2) as follows.

Note that the system (5.2) has one infection state, Y. Consider that $\mathscr{F}$ and $\mathscr{V}$, respectively, indicate the rate of appearance of new infection and the rate of other transitions. Then

$$\mathscr{F} = \big(\lambda XY\big)_{1\times 1} \text{ and } \mathscr{V} = \left(\mu Y + \frac{q_2 YH}{Y+D_2}\right)_{1\times 1}.$$

At the infection-free equilibrium point $\xi_3 = (X_3, 0, H_3, P_3)$, the transmission matrix $\hat{F}$ and the transition matrix $\hat{V}$ associated with system (5.2) are given by

$$F = \left[\frac{\partial \mathscr{F}}{\partial Y}\right]_{\xi_3=(X_3,0,H_3,P_3)} = \lambda X_3, \text{ and } V = \left[\frac{\partial \mathscr{V}}{\partial Y}\right]_{\xi_3=(X_3,0,H_3,P_3)} = \mu + \frac{q_2 H_3}{D_2}.$$

The corresponding next generation matrix becomes

$$\mathscr{K} = FV^{-1} = \left(\frac{\lambda X_3 D_2}{\mu D_2 + q_2 H_3}\right)_{1\times 1}.$$

Hence, the basic reproduction number of the system (5.2), which is the spectral radius of the next generation matrix $\mathscr{K}$ [291], is given by

$$\mathscr{R}_0 = \frac{\lambda X_3 D_2}{\mu D_2 + q_2 H_3}.$$

5.3.3 Stability of the equilibrium points

We use the following terminologies to facilitate the stability analysis. The pure growth rate (Ξ, say) of the healthy fish, having biomass X, is measured by $\Xi = rX$ [183]. This type of growth is also known as Malthusian growth. At the disease-free and endemic equilibria, this value becomes $\Xi_D = rX_3$ and $\Xi_E = r\bar{X}$, respectively.

The maximum harvesting (Π) when healthy fish population is too large is

$$\Pi = \lim_{X\to\infty} \frac{q_1 XH}{X+D_1} = q_1 H.$$

Thus, the maximum harvesting of healthy fish at the disease-free and endemic equilibrium states are given by $\Pi_D = q_1 H_3$ and $\Pi_E = q_1 \bar{H}$, respectively.
The price elasticity of demand measures how the fish quantity demanded changes in response to fish price variation. If Υ is the price elasticity of demand for healthy fish, then

$$\Upsilon = -\frac{P}{D}\frac{dD}{dP} = -\frac{BP}{1+BP} \text{ [306].}$$

At the infection-free and endemic equilibrium states, this quantity is given by $\Upsilon_D = -\frac{BP_3}{1+BP_3}$ and $\Upsilon_E = -\frac{B\bar{P}}{1+B\bar{P}}$, respectively.

The variational matrix of the system (5.2) at an arbitrary equilibrium point $\hat{\xi} = (\hat{X}, \hat{Y}, \hat{H}, \hat{P})$ is given by

$$J = \begin{pmatrix} r\left(1-\frac{2X+Y}{K}\right) - \lambda Y - \frac{q_1 H}{X+D_1} + \frac{q_1 XH}{(X+D_1)^2} & -\left(\frac{r}{K}+\lambda\right)X & -\frac{q_1 X}{X+D_1} & 0 \\ \lambda Y & \lambda X - \mu - \frac{q_2 H}{Y+D_2} + \frac{q_2 YH}{(Y+D_2)^2} & -\frac{q_2 Y}{Y+D_2} & 0 \\ \frac{\phi_1 q_1 (P-\tau)H}{X+D_1} - \frac{\phi_1 q_1 (P-\tau)XH}{(X+D_1)^2} & \frac{\phi_1 q_2 (p-\tau)H}{Y+D_2} - \frac{\phi_1 q_2 (p-\tau)YH}{(Y+D_2)^2} & \phi_1\left[\frac{q_1(P-\tau)X}{X+D_1} + \frac{q_2(p-\tau)Y}{Y+D_2} - c\right] & \frac{\phi_1 q_1 XH}{X+D_1} \\ \frac{\phi_2 q_1 HP}{X+D_1} - \frac{\phi_2 q_1 XHP}{(X+D_1)^2} & 0 & \frac{\phi_2 q_1 XP}{X+D_1} & \phi_2\left(\frac{A}{1+BP} - \frac{ABP}{(1+BP)^2} - \frac{q_1 XH}{X+D_1}\right) \end{pmatrix}_{(\hat{X},\hat{Y},\hat{H},\hat{P})}. \tag{5.6}$$

It is easy to show that the variational matrices evaluated at the trivial equilibrium point ξ_0 have two positive eigenvalues, namely r and $\phi_2 A$, making it always unstable. Similarly, the equilibrium points $\xi_1(K,0,0,0)$ and $\xi_2(\frac{\mu}{\lambda}, \frac{r(\lambda K-\mu)}{\lambda(\lambda K+r)}, 0, 0)$ are also unstable as the corresponding Jacobian matrices always have a positive eigenvalue of the form $\phi_2 A$ (> 0). Hence, the following theorem is true.

Theorem 5.2. *The trivial equilibrium point* $\xi_0(0,0,0,0)$*, the only healthy fish equilibrium* $\xi_1(K,0,0,0)$*, and the healthy and infected fishes equilibrium* $\xi_2 = (\frac{\mu}{\lambda}, \frac{r(\lambda K-\mu)}{\lambda(\lambda K+r)}, 0, 0)$ *are always unstable.*

Theorem 5.3. *The disease-free equilibrium point* $\xi_3 = (X_3, 0, H_3, P_3)$*, whenever it exists, is locally asymptotically stable if* $\frac{\lambda_3 (X_3+D_1)^2}{b_{44} b_{13} \phi_1 (P_3-\tau) D_1} < \Pi_D < \left[\frac{\Xi_D}{K} + \frac{\mu D_2 + q_2 H_3}{D_2}(1-\mathscr{R}_0) - \phi_2 D(P_3)\Upsilon_D\right] \frac{(X_3+D_1)^2}{X_3}$, $\Upsilon_D > \frac{(b_{13} b_{41} - b_{11} b_{43}) b_{34}}{\phi_2 D(P_3) b_{31} b_{13}}$, $\mathscr{R}_0 < 1$, *and* $\Xi_D > K\left[\frac{b_{22}(b_{11}+b_{44}) - b_{43} b_{34} - b_{13} b_{31} - B_2^2 - B_0 B_3}{\phi_2 D(P_3)\Upsilon_D B_0 B_2} + \frac{\Pi_D X_3}{(X_3+D_1)^2}\right]$ *hold simultaneously, where* B_0*,* B_1*,* B_2 *and* B_3 *are defined in* (5.9).

Proof. At the disease-free equilibrium point $\xi_3 = (X_3, 0, H_3, P_3)$, one can evaluate the variational matrix as

$$J(X_3, 0, H_3, P_3) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 \\ 0 & b_{22} & 0 & 0 \\ b_{31} & b_{32} & 0 & b_{34} \\ b_{41} & 0 & b_{43} & b_{44} \end{pmatrix}, \tag{5.7}$$

where
$b_{11} = -\frac{rX_3}{K} + \frac{q_1X_3H_3}{(X_3+D_1)^2}$, $b_{12} = -\frac{rX_3}{K} - \lambda X_3$, $b_{13} = -\frac{q_1X_3}{X_3+D_1}$, $b_{22} = \lambda X_3 - \mu - \frac{q_2H_3}{D_2}$, $b_{31} = \frac{\phi_1q_1(P_3-\tau)H_3}{X_3+D_1} - \frac{\phi_1q_1(P_3-\tau)X_3H_3}{(X_3+D_1)^2}$, $b_{32} = \frac{\phi_1q_2(p-\tau)H_3}{D_2}$, $b_{34} = \frac{\phi_1q_1X_3H_3}{X_3+D_1}$, $b_{41} = \frac{\phi_2q_1H_3P_3}{X_3+D_1} - \frac{\phi_2q_1X_3H_3P_3}{(X_3+D_1)^2}$, $b_{43} = \frac{\phi_2q_1X_3P_3}{X_3+D_1}$, $b_{44} = -\frac{\phi_2ABP_3}{(1+BP_3)^2}$.
The corresponding characteristic equation reads

$$\beta^4 + B_0\beta^3 + B_1\beta^2 + B_2\beta + B_3 = 0, \tag{5.8}$$

where

$$\begin{aligned}
B_0 &= -b_{11} - b_{22} - b_{44},\ B_1 = b_{11}b_{22} + b_{22}b_{44} + b_{11}b_{44} - b_{43}b_{34} - b_{13}b_{31}, \\
B_2 &= b_{22}b_{43}b_{34} + b_{13}b_{22}b_{31} - b_{11}b_{22}b_{44} + b_{11}b_{44}b_{34} + b_{31}b_{44}b_{13} - b_{13}b_{34}b_{41}, \qquad (5.9) \\
B_3 &= b_{13}b_{34}b_{41}b_{22} - b_{31}b_{44}b_{13}b_{22} - b_{11}b_{22}b_{34}b_{43}.
\end{aligned}$$

The characteristic equation (5.8) will have roots with negative real parts iff [259] $B_0 > 0, B_2 > 0, B_3 > 0$ and $B_0B_1B_2 > (B_2^2 + B_0^2B_3)$. Now $B_0 > 0, B_2 > 0, B_3 > 0$ and $B_0B_1B_2 > (B_2^2 + B_0^2B_3)$ respectively gives

$$\begin{aligned}
\Pi_D &< \left[\frac{\Xi_D}{K} + \frac{\mu D_2 + q_2H_3}{D_2}(1-\mathscr{R}_0) - \phi_2 D(P_3)\Upsilon_D\right]\frac{(X_3+D_1)^2}{X_3}, \\
\Pi_D &> \frac{\lambda_3(X_3+D_1)^2}{b_{44}b_{13}\phi_1(P_3-\tau)D_1}, \\
\Upsilon_D &> \frac{(b_{13}b_{41} - b_{11}b_{43})b_{34}}{\phi_2 D(P_3)b_{31}b_{13}} \ \&\ \mathscr{R}_0 < 1\ ,\ \text{and} \\
\Xi_D &> K\left[\frac{b_{22}(b_{11}+b_{44}) - b_{43}b_{34} - b_{13}b_{31} - B_2^2 - B_0B_3}{\phi_2 D(P_3)\Upsilon_D B_0 B_2} + \frac{\Pi_D X_3}{(X_3+D_1)^2}\right],
\end{aligned}$$

where $\lambda_3 = b_{13}(b_{34}b_{41} - b_{22}b_{31}) - b_{22}(b_{43}b_{34} + b_{11}b_{44}) - b_{11}b_{44}b_{34}$. Hence the theorem. □

Theorem 5.4. *The disease-free equilibrium ξ_3 of system* (5.2)*, if it exists, is globally asymptotically stable if the basic reproduction number is less than unity (i.e., $\mathscr{R}_0 < 1$).*

Proof. To prove the global stability of ξ_3, we consider a Lyapunov function

$$\hat{\Theta}(X,Y,H,P) = Y.$$

It is zero at ξ_3, and positive for all feasible values of $(X,Y,H,P) \in \mathbb{R}_4^+$ Its time derivatives along the solution of (5.2) gives

$$\frac{d\hat{\Theta}}{dt} = \left[\lambda X - \mu - \frac{q_2H}{Y+D_2}\right]Y.$$

Therefore,

$$\left[\frac{d\hat{\Theta}}{dt}\right]_{\xi_3} = \left[\lambda\bar{X} - \mu - \frac{q_2\bar{H}}{D_2}\right]\hat{\Theta} = \left[\left\{\mu + \frac{q_2\bar{H}}{D_2}\right\}(\mathscr{R}_0 - 1)\right]\hat{\Theta}. \tag{5.10}$$

Clearly, if $\mathscr{R}_0 < 1$ then $\frac{d\hat{\Theta}}{dt} < 0$. Therefore, following Lyapunov stability theorem [307], ξ_3 is globally asymptotically stable if $\mathscr{R}_0 < 1$. □

Theorem 5.5. *The endemic equilibrium point* $\xi_4 = (\bar{X}, \bar{Y}, \bar{H}, \bar{P})$ *of system* (5.2)*, if it exists, is locally asymptotically stable under the conditions* $\Xi_E > K\left[\frac{q_1\bar{X}\bar{H}}{(\bar{X}+D_1)^2} + \frac{q_2\bar{Y}\bar{H}}{(\bar{Y}+D_2)^2} + \phi_2 D(\bar{P})\Upsilon_E\right]$, $\frac{\gamma_2(\bar{X}+D_1)}{\phi_2 q_1\bar{X}a_{22}a_{31}D(\bar{P})} < \Upsilon_E < \frac{\gamma_1(\bar{X}+D_1)}{\phi_2 q_1\bar{X}D(\bar{P})a_{31}}$, *and* $\Pi_E < \frac{(\bar{X}+D)^2}{\bar{X}}\left(\frac{r\bar{X}}{K} - a_{22} - a_{44} - \frac{A_2^2 + A_0^2 A_3}{A_1 A_2}\right)$, *where* A_0, A_1, A_2 *and* A_3 *are given in* (5.12)*.*

Proof. The Jacobian matrix of system (5.2) at the co-existing equilibrium $\xi_4 = (\bar{X}, \bar{Y}, \bar{H}, \bar{P})$ is evaluated as

$$J(\bar{X}, \bar{Y}, \bar{H}, \bar{P}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix},$$

where
$a_{11} = -\frac{r\bar{X}}{K} + \frac{q_1\bar{X}\bar{H}}{(\bar{X}+D_1)^2}$, $a_{12} = -\frac{r\bar{X}}{K} - \lambda\bar{X}$, $a_{13} = -\frac{q_1\bar{X}}{\bar{X}+D_1}$, $a_{21} = \lambda\bar{Y}$, $a_{22} = \frac{q_2\bar{Y}\bar{H}}{(\bar{Y}+D_2)^2}$, $a_{23} = -\frac{q_2\bar{Y}}{\bar{Y}+D_2}$, $a_{31} = \frac{\phi_1 q_1(\bar{P}-\tau)\bar{H}}{\bar{X}+D_1} - \frac{\phi_1 q_1(\bar{P}-\tau)\bar{X}\bar{H}}{(\bar{X}+D_1)^2}$, $a_{32} = \frac{\phi_1 q_2(p-\tau)\bar{H}}{\bar{Y}+D_2} - \frac{\phi_1 q_2(p-\tau)\bar{Y}\bar{H}}{(\bar{Y}+D_2)^2}$, $a_{34} = \frac{\phi_1 q_1\bar{X}\bar{H}}{\bar{X}+D_1}$, $a_{41} = \frac{\phi_2 q_1\bar{H}\bar{P}}{\bar{X}+D_1} - \frac{\phi_2 q_1\bar{X}\bar{H}\bar{P}}{(\bar{X}+D_1)^2}$, $a_{43} = \frac{\phi_2 q_1\bar{X}\bar{P}}{\bar{X}+D_1}$, $a_{44} = -\frac{\phi_2 AB\bar{P}}{(1+B\bar{P})^2}$.
The corresponding characteristic equation is represented by

$$\beta^4 + A_0\beta^3 + A_1\beta^2 + A_2\beta + A_3 = 0, \tag{5.11}$$

where

$$\begin{aligned}
A_0 &= -a_{11} - a_{22} - a_{44}, \\
A_1 &= a_{11}a_{22} + a_{22}a_{44} + a_{11}a_{44} - a_{43}a_{34} - a_{13}a_{31} - a_{32}a_{23} - a_{12}a_{21}, \\
A_2 &= a_{22}a_{43}a_{34} + a_{13}a_{22}a_{31} - a_{11}a_{22}a_{44} + a_{11}a_{43}a_{34} + a_{31}a_{44}a_{13} - a_{13}a_{32}a_{21} + \\
&\quad a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{23}a_{44}a_{32} - a_{11}a_{22}a_{44} - a_{13}a_{34}, \qquad (5.12) \\
A_3 &= a_{13}a_{34}a_{41}a_{22} - a_{31}a_{44}a_{13}a_{22} - a_{11}a_{22}a_{34}a_{43} + a_{13}a_{32}a_{44}a_{21} - a_{32}a_{44}a_{11}a_{23} - \\
&\quad a_{12}a_{23}a_{31}a_{44} + a_{12}a_{21}a_{34}a_{43} - a_{12}a_{34}a_{41}.
\end{aligned}$$

Observe that $A_0 > 0$ whenever the condition

$$\Xi_E > K\left[\frac{q_1\bar{X}\bar{H}}{(\bar{X}+D_1)^2} + \frac{q_2\bar{Y}\bar{H}}{(\bar{Y}+D_2)^2} + \phi_2 D(\bar{P})\Upsilon_E\right] \tag{5.13}$$

is satisfied. $A_2 > 0$ implies

$$\Upsilon_E < \frac{\gamma_1(\bar{X}+D_1)}{\phi_2 q_1\bar{X}D(\bar{P})a_{31}}, \tag{5.14}$$

where
$\gamma_1 = -a_{22}a_{43}a_{34} - a_{13}a_{22}a_{31} + a_{11}a_{22}a_{44} - a_{11}a_{43}a_{34} + a_{13}a_{32}a_{21} - a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31} - a_{23}a_{44}a_{32} + a_{11}a_{22}a_{44} + a_{13}a_{34}$.
Next, $A_3 > 0$ gives

$$\Upsilon_E > \frac{\gamma_2(\bar{X}+D_1)}{\phi_2 q_1\bar{X}a_{22}a_{31}D(\bar{P})}, \tag{5.15}$$

where
$\gamma_2 = a_{13}a_{34}a_{41}a_{22} - a_{11}a_{22}a_{34}a_{43} + a_{13}a_{32}a_{44}a_{21} - a_{32}a_{44}a_{11}a_{23} - a_{12}a_{23}a_{31}a_{44} + a_{12}a_{21}a_{34}a_{43} - a_{12}a_{34}a_{41}$.
Again, the relation $A_0A_1A_2 > A_2^2 + A_0^2A_3$ provides

$$\Pi_E < \frac{(\bar{X}+D)^2}{\bar{X}}\left(\frac{r\bar{X}}{K} - a_{22} - a_{44} - \frac{A_2^2+A_0^2A_3}{A_1A_2}\right). \tag{5.16}$$

Following Routh-Hurwitz criterion [259] and applying Eq. (5.13), (5.14), (5.15), and (5.16), one must have the required stability condition as $\Xi_E > K\left[\frac{q_1\bar{X}\bar{H}}{(\bar{X}+D_1)^2} + \frac{q_2\bar{Y}\bar{H}}{(\bar{Y}+D_2)^2} + \phi_2 D(\bar{P})\Upsilon_E\right]$, $\frac{\gamma_2(\bar{X}+D_1)}{\phi_2 q_1\bar{X}a_{22}a_{31}D(\bar{P})} < \Upsilon_E < \frac{\gamma_1(\bar{X}+D_1)}{\phi_2 q_1\bar{X}D(\bar{P})a_{31}}$ and $\Pi_E < \frac{(\bar{X}+D)^2}{\bar{X}}\left(\frac{r\bar{X}}{K} - a_{22} - a_{44} - \frac{A_2^2+A_0^2A_3}{A_1A_2}\right)$. Hence the theorem. ☐

Theorem 5.6. *The interior equilibrium point* $\xi_4 = (\bar{X},\bar{Y},\bar{H},\bar{P})$ *of the system* (5.2)*, whenever exists, is globally asymptotically stable if* $\Xi_E > \frac{1}{c_{23}^2}[K\bar{X}\{2(c_{13}c_{14}c_{34} + c_{12}c_{13}c_{23}) - c_{11}c_{34}^2\}]$*, and* $0 < \Pi_E^2 < \frac{1}{c_{23}^2F_1^2}[4(\bar{X}+D_1)^2(F_1F_2F_3 - F_3^2 - F_1^2c_{12}^2c_{34}^2)]$ *hold simultaneously, where* $F_1,\ F_2,\ F_3,$ *and* F_4 *are in* (5.19)*.*

Proof. To prove the global stability of ξ_4, we consider the Lyapunov function

$$\begin{aligned}\Theta(X,Y,H,P) \;=\;& \left[(X-\bar{X}) - \bar{X}\ln\left(\frac{X}{\bar{X}}\right)\right] + h\left[(Y-\bar{Y}) - \bar{Y}\ln\left(\frac{Y}{\bar{Y}}\right)\right] + i\left[(H-\bar{H}) - \bar{H}\ln\left(\frac{H}{\bar{H}}\right)\right] \\ & + j\left[(P-\bar{P}) - \bar{P}\ln\left(\frac{P}{\bar{P}}\right)\right],\end{aligned}$$

for some suitable choice of the positive constants $h,\ i,$ and j. At the interior equilibrium point

$\xi_4 = (\bar{X}, \bar{Y}, \bar{H}, \bar{P})$, Θ is zero and positive for all feasible values of $(X, Y, H, P) \in \mathbb{R}_4^+$. One can write the time derivative of Θ along the solution of (5.2) as

$$\begin{aligned}
\dot{\Theta} &= \left(\frac{X-\bar{X}}{X}\right)\dot{X} + h\left(\frac{Y-\bar{Y}}{Y}\right)\dot{Y} + i\left(\frac{H-\bar{H}}{H}\right)\dot{H} + j\left(\frac{P-\bar{P}}{P}\right)\dot{P} \\
&= (X-\bar{X})\left(r\left(1-\frac{X+Y}{K}\right) - \lambda Y - \frac{q_1 H}{X+D_1}\right) + (Y-\bar{Y})\left(\lambda X - \mu - \frac{q_2 H}{Y+D_2}\right) + \phi_1(H-\bar{H}) \\
&\quad \left(\frac{q_1(P-\tau)X}{X+D_1} + \frac{q_2(p-\tau)Y}{Y+D_2} - c\right) + \phi_2(P-\bar{P})\left(\frac{A}{1+BP} - \frac{q_1 X H}{X+D_1}\right) \\
&\quad \text{(where } h = i = j = 1\text{)} \\
&= (X-\bar{X}, Y-\bar{Y}, H-\bar{H}, P-\bar{P})^{\intercal}\, Q\, (X-\bar{X}, Y-\bar{Y}, H-\bar{H}, P-\bar{P}),
\end{aligned}$$

where

$$Q = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & 0 & c_{23} & 0 \\ c_{13} & c_{23} & 0 & c_{34} \\ c_{14} & 0 & c_{34} & 0 \end{pmatrix}. \tag{5.17}$$

The matrix Q is symmetric, having coefficients $c_{11} = -\frac{r}{K}$, $c_{12} = -\frac{1}{2}\frac{r}{K}$, $c_{13} = \frac{1}{2}\left(-\frac{q_1}{\bar{X}+D_1} + \frac{\phi_1 q_1(\bar{P}-\tau)}{\bar{X}+D_1}\right)$, $c_{14} = -\frac{q_1\bar{H}}{2(\bar{X}+D_1)}$, $c_{23} = \frac{1}{2}\left(-\frac{q_2}{\bar{Y}+D_2} + \frac{\phi_1 q_2(p-\tau)}{\bar{Y}+D_2}\right)$, $c_{34} = \frac{1}{2}\left(\frac{\phi_1 q_1\bar{X}}{\bar{X}+D_1} + \frac{\phi_2 q_1\bar{X}}{\bar{X}+D_1}\right)$, and therefore, its roots are real. To show $\dot{\Theta} < 0$ for all $(X, Y, H, P) \neq (\bar{X}, \bar{Y}, \bar{H}, \bar{P})$, the matrix Q should be negative definite [308]. Since the matrix Q is symmetric, it will be negative definite if all roots are negative [309]. The characteristic equation of the matrix Q is expressible in the form

$$\eta^4 + F_1\eta^3 + F_2\eta^2 + F_3\eta + F_4 = 0, \tag{5.18}$$

where

$$\begin{aligned}
F_1 &= \frac{r}{K},\ F_2 = -(c_{14}^2 + c_{34}^2 + c_{13}^2 + c_{23}^2 + c_{12}^2), \\
F_3 &= c_{11}c_{23}^2 - 2(c_{13}c_{14}c_{34} + c_{12}c_{13}c_{23}) + c_{11}c_{34}^2 \\
F_4 &= c_{14}^2c_{23}^2 + c_{12}^2c_{34}^2.
\end{aligned} \tag{5.19}$$

Observe that F_1 is always positive, $F_3 > 0$ under the condition $\Xi_E > \frac{K\bar{X}[2(c_{13}c_{14}c_{34}+c_{12}c_{13}c_{23})-c_{11}c_{34}^2]}{c_{23}^2}$, $F_4 > 0$ implies

$\Pi_E^2 > \dfrac{-4(\bar{X}+D_1)^2c_{12}^2c_{34}^2}{c_{23}^2}$ and $F_1F_2F_3 > (F_3^2 + F_1^2F_4)$ gives $\Pi_E^2 < \dfrac{4(X+D_1)^2(F_1F_2F_3 - F_3^2 - F_1^2c_{12}^2c_{34}^2)}{c_{23}^2F_1^2}$.

So, the roots of (5.18) will be real negative if pure growth rate at endemic equilibrium point (ξ_4) satisfies

$$\Xi_E > \frac{K\bar{X}[2(c_{13}c_{14}c_{34} + c_{12}c_{13}c_{23}) - c_{11}c_{34}^2]}{c_{23}^2},$$

and maximum harvested healthy species at the endemic equilibrium state Π_E satisfies

$$0 < \Pi_E^2 < \frac{4(\bar{X} + D_1)^2(F_1F_2F_3 - F_3^2 - F_1^2c_{12}^2c_{34}^2)}{c_{23}^2F_1^2}.$$

Thus, the theorem is proven. □

5.3.4 Bifurcations analysis

In this section by using Sotomayor's theorem [119], we will obtain the existence condition of saddle-node bifurcation at the disease-free equilibrium point $\xi_3 = (X_3, 0, H_3, P_3)$.

Theorem 5.7. *Whenever the set of condition* $\left\{ b_{41} - b_{11} - b_{31} = 0,\ -\frac{b_{11}b_{43}}{b_{13}} - \frac{b_{31}b_{44}}{b_{34}} + b_{41} \left(\frac{b_{12}b_{43}b_{34} + b_{13}b_{32}b_{44}}{b_{13}b_{22}b_{34}}\right) = 0,\ \textit{and}\ \hat{w}_1\tilde{f}_1 + \hat{w}_2\tilde{f}_2 + \hat{w}_3\tilde{f}_3 + \tilde{f}_4 \neq 0 \right\}$ *is satisfied, the system* (5.2) *exhibits a saddle-node bifurcation at* $A = A^{SN}$, *where* A^{SN} *is given in Eq.* (5.21).

Proof. Observe that the characteristic equation (5.8) can be rewritten in the form

$$(\beta - b_{22})\left(\beta^3 + J_1\beta^2 + J_2\beta + J_3\right) = 0, \tag{5.20}$$

where $J_1 = -b_{11} - b_{44}$, $J_2 = b_{11}b_{44} + b_{13}b_{31} - b_{34}b_{43}$, $J_3 = b_{13}b_{31}b_{44} + b_{11}b_{34}b_{43} - b_{13}b_{34}b_{41}$. Observe that the Eq. (5.20) will have a simple zero root, i.e., the Jocobian matrix (5.7) contains a simple zero eigenvalue if

$$\begin{aligned} & J_3 = b_{13}b_{31}b_{44} + b_{11}b_{34}b_{43} - b_{13}b_{34}b_{41} = 0 \\ \text{or,} \quad & A = \frac{(1 + BP_3)^2}{\phi_2 b_{13}b_{31}P_3}(b_{11}b_{34}b_{43} - b_{13}b_{34}b_{41}) = A^{SN}\ \text{(say)}. \end{aligned} \tag{5.21}$$

At $A = A^{SN}$, the other three roots of the Eq. (5.20) can be given by $\frac{1}{2}(-J_1 \pm \sqrt{J_1^2 - 4J_2})$, and $\lambda X_3 - \mu - \frac{q_2H_3}{D_2}$. Clearly, $\lambda X_3 - \mu - \frac{q_2H_3}{D_2} < 0$ as at $\xi_3 = (X_3, 0, H_3, P_3)$, $\mathscr{R}_0 < 1$, and the negativity conditions of other two roots will be $J_1 < 0$ and $J_2 > 0$. Presume that the right and left eigenvector of the matrix $J(X_3, 0, H_3, P_3)$ are $\hat{v}$ and $\hat{w}$, respectively. Then those vectors can be expressed as follows:

$$\hat{v} = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \\ \hat{v}_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -\frac{b_{11}}{b_{13}} \\ -\frac{b_{31}}{b_{34}} \end{pmatrix}, \text{ and } \hat{w}^t = \begin{pmatrix} \hat{w}_1 \\ \hat{w}_3 \\ \hat{w}_2 \\ \hat{w}_4 \end{pmatrix} = \begin{pmatrix} -\frac{b_{43}}{b_{13}} \\ \frac{b_{12}b_{43}b_{34} + b_{13}b_{32}b_{44}}{b_{13}b_{22}b_{34}} \\ -\frac{b_{44}}{b_{34}} \\ 1 \end{pmatrix},$$

under the conditions $b_{41} - b_{11} - b_{31} = 0$ and $-\frac{b_{11}b_{43}}{b_{13}} - \frac{b_{31}b_{44}}{b_{34}} + b_{41}\left(\frac{b_{12}b_{43}b_{34}+b_{13}b_{32}b_{44}}{b_{13}b_{22}b_{34}}\right) = 0$, respectively. Now assume the set of differential equation $\left(\frac{dX}{dt}, \frac{dY}{dt}, \frac{dH}{dt}, \frac{dP}{dt}\right) = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4) = \mathscr{A}$ and define

$$\mathscr{B}_1 = \hat{w}\frac{d\mathscr{A}}{dA}\bigg|_{(\xi_3, A^{SN})} = -\frac{\phi_2 A^{SN} B P_3}{(1+BP_3)^2} \neq 0.$$

Also,

$$\begin{aligned} \mathscr{B}_2 &= \hat{w}\left[D^2\mathscr{A}(\hat{v},\hat{v})\right]|_{(\xi_3, A^{SN})} \\ &= \hat{w}(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4)^t|_{(\xi_3, A^{SN})} \\ &= \hat{w}_1\tilde{f}_1 + \hat{w}_2\tilde{f}_2 + \hat{w}_3\tilde{f}_3 + \tilde{f}_4, \end{aligned} \tag{5.22}$$

where $D\mathscr{A}$ indicates the Jacobian of $\mathscr{A}$ and $\tilde{f}_1 = \left(-\frac{2r}{K} + \frac{q_1 D_1 H_3}{(X_3+D_1)^3}\right)\hat{v}_1{}^2 - 2\left(\frac{r}{K} + \lambda\right)v_1 v_2 - \frac{2q_1\hat{v}_1\hat{v}_3 D_1}{(X_3+D_1)^2}$, $\tilde{f}_2 = 2\lambda\hat{v}_1\hat{v}_2 + \frac{q_2 D_2 H_3 \hat{v}_2{}^2}{(D_2)^3} - \frac{2q_2 D_2 H_3 \hat{v}_2 \hat{v}_3}{(D_2)^2}$, $\tilde{f}_3 = -\frac{\phi_1 q_1 (P_3-\tau)\hat{v}_1{}^2 H_3 D_1}{(X_3+D_1)^3} + \frac{2\phi_1 q_1 (P_3-\tau)\hat{v}_1\hat{v}_3 D_1}{(X_3+D_1)^2} + \frac{2\phi_1 q_1 \hat{v}_1 \hat{v}_4 H_3 D_1}{(X_3+D_1)^2} - \frac{\phi_1 q_2 (p-\tau)\hat{v}_2{}^2 H_3 D_2}{(D_2)^3} + \frac{2\phi_1 q_2 (p-\tau)\hat{v}_2\hat{v}_3 D_2}{(D_2)^2} + \frac{2\phi_1 q_1 X_3 v_3 v_4}{(X_3+D_1)}$, $\tilde{f}_4 = \frac{2\phi_2 q_1 P_3 D_1 v_1 v_3}{(X_3+D_1)^2} - \frac{\phi_2 q_1 P_3 H_3 D_1 v_1^2}{(X_3+D_1)^3} - \frac{2\phi_2 q_1 H_3 D_1 v_1 v_4}{(X_3+D_1)^2} - \frac{2\phi_2 q_1 H_3 v_3 v_4}{(X_3+D_1)} - \frac{\phi_2 A^{SN} B \hat{v}_4{}^2}{(1+BP_3)^3}$.
Thus following Perko [119], one can state that the model system (5.2) manifests saddle-node bifurcation when $A = A^{SN}$ iff the set of condition $\left\{b_{41} - b_{11} - b_{31} = 0,\ -\frac{b_{11}b_{43}}{b_{13}} - \frac{b_{31}b_{44}}{b_{34}} + b_{41}\left(\frac{b_{12}b_{43}b_{34}+b_{13}b_{32}b_{44}}{b_{13}b_{22}b_{34}}\right) = 0, \text{ and } \hat{w}_1\tilde{f}_1 + \hat{w}_2\tilde{f}_2 + \hat{w}_3\tilde{f}_3 + \tilde{f}_4 \neq 0\right\}$ is satisfied. Hence the proof.

□

Following the line of [119], one can similarly obtain the Hopf and transcritical bifurcation points with respect to several important system parameters. However, we ignore the analytical computation, but numerically show the existence of all local bifurcations.

5.4 Optimal taxation policy

As we mentioned earlier that taxation has both the positive and negative effects on the fishery. In the one hand, it helps to control overexploitation and generate revenue. On the other hand, higher taxation may negatively affect the fishing. Fishermen may lost interest if the imposed tax is significantly high, which, in turn, may increase the infected fish population due to low harvesting. The system might also become unstable, causing a non-constant yield, if the fishing tax is high. So, a balanced tax should be implemented by the regulatory agency to maximize the societal benefit. The societal benefit is defined here as the sum of revenues generated by the fishermen by selling fish, and the revenue earned by the agency from the levied tax. If Γ_1 and Γ_2 are, respectively, the earnings from fishing excluding tax and the earning of the regulatory agency form tax revenue, then the societal revenue $(\Gamma(X,Y,H,P,\tau))$ is

$$\begin{aligned}\Gamma(X,Y,H,P,\tau) &= \Gamma_1(X,Y,H,P,\tau)+\Gamma_2(X,Y,H,P,\tau)\\ &= \left(\frac{q_1(P-\tau)X}{X+D_1}+\frac{q_2(p-\tau)Y}{Y+D_2}\right)H+\left(\frac{q_1\tau X}{X+D_1}+\frac{q_2\tau Y}{Y+D_2}\right)H\\ &= \left(\frac{q_1PX}{X+D_1}+\frac{q_2pY}{Y+D_2}\right)H. \qquad (5.23)\end{aligned}$$

If δ is the annual discount rate, then the optimal taxation problem may be defined as

$$\Upsilon=\int_0^\infty e^{-\delta t}\Gamma(X,Y,H,P,\tau)dt, \qquad (5.24)$$

subject to the state variables (5.2), and Γ is the economic rent defined in (5.23). The control variable τ is subject to the constraints $0<\tau<\tau_{max}$, where τ_{max} is the upper cap of the imposed tax. Our objective is to find the optimal value of τ which will maximize Γ. By virtue of the Pontryagin's maximum principle, one can write the Hamiltonian

$$\begin{aligned}\Im(X,Y,H,P,\tau) &= e^{-\delta t}\left(\frac{q_1PX}{X+D_1}+\frac{q_2pY}{Y+D_2}\right)H+\varepsilon_1\left[rX\left(1-\frac{X+Y}{K}\right)-\lambda XY-\frac{q_1XH}{X+D_1}\right]+\\ &\quad \varepsilon_2\left[\lambda XY-\mu Y-\frac{q_2YH}{Y+D_2}\right]+\varepsilon_3\phi_1\left(\left(\frac{q_1(P-\tau)X}{X+D_1}+\frac{q_2(p-\tau)Y}{Y+D_2}\right)H-cH\right)\\ &\quad +\varepsilon_4\phi_2P\left(\frac{A}{1+BP}-\frac{q_1XH}{X+D_1}\right). \qquad (5.25)\end{aligned}$$

Here $\varepsilon_1,\varepsilon_2,\varepsilon_3$ and ε_4 are the adjoint variables to be determined. To maximize $\Im$, the optimal control variable τ has to satisfy the following conditions:

$$\frac{\partial\Im}{\partial\tau}=0,\ \frac{d\varepsilon_1}{dt}=-\frac{\partial\Im}{\partial X},\ \frac{d\varepsilon_2}{dt}=-\frac{\partial\Im}{\partial Y},\ \frac{d\varepsilon_3}{dt}=-\frac{\partial\Im}{\partial H},\ \frac{d\varepsilon_4}{dt}=-\frac{\partial\Im}{\partial P}. \qquad (5.26)$$

At any arbitrary equilibrium point $\xi(\hat{X},\hat{Y},\hat{H},\hat{P})$, $\frac{\partial\Im}{\partial\tau}=0$ gives $\varepsilon_3\phi_1\big(-\frac{q_1\hat{X}}{\hat{X}+D_1}-\frac{q_2\hat{Y}}{\hat{Y}+D_2}\big)\hat{H}=0$. For non trivial solution, one needs

$$\varepsilon_3=0. \qquad (5.27)$$

Again, $\frac{d\varepsilon_4}{dt}=-\big[\frac{\partial\Im}{\partial P}\big]_{(\hat{X},\hat{Y},\hat{H},\hat{P})}$ gives

$$\frac{d\varepsilon_4}{dt}=G_1e^{-\delta t}+G_2\varepsilon_4,$$

where

$$G_1=-\frac{q_1\hat{X}\hat{H}}{\hat{X}+D_1},\ G_2=-\phi_2\left(\frac{A}{1+B\hat{P}}-\frac{q_1\hat{X}\hat{H}}{\hat{X}+D_1}\right)+\frac{\phi_2AB\hat{P}}{(1+B\hat{P})^2}.$$

Solving this first order differential equation in ε_4, one gets

$$\varepsilon_4 = -\frac{G_1}{G_2+\delta}e^{-\delta t}. \tag{5.28}$$

Now, $\frac{d\varepsilon_3}{dt} = -\left[\frac{\partial \Im}{\partial H}\right]_{(\hat{X},\hat{Y},\hat{H},\hat{P})}$ provides

$$\varepsilon_1 = \frac{G_3(\hat{X}+D_1)}{q_1\hat{X}}e^{-\delta t} - \frac{q_2\hat{Y}(\hat{X}+D_1)}{q_1\hat{X}(\hat{Y}+D_2)}\varepsilon_2. \tag{5.29}$$

Putting this value of ε_1 in $\frac{d\varepsilon_2}{dt} = -\left[\frac{\partial \Im}{\partial Y}\right]_{(\hat{X},\hat{Y},\hat{H},\hat{P})}$, one gets

$$\varepsilon_2 = -\frac{G_4}{G_5+\delta}e^{-\delta t}, \tag{5.30}$$

where $G_3 = \frac{q_1\hat{P}\hat{X}}{\hat{X}+D_1} + \frac{q_2p\hat{Y}}{\hat{Y}+D_2} + \frac{q_1\phi_2G_1\hat{X}\hat{P}}{(\hat{X}+D_1)(G_2+\delta)}$, $G_4 = \left(-\frac{q_2p}{\hat{Y}+D_2} + \frac{q_2p\hat{Y}}{(\hat{Y}+D_2)^2}\right)\hat{H} + \frac{G_3(\hat{X}+D_1)\left(\frac{r}{K}+\lambda\right)}{q_1}$ and $G_5 = -\lambda\hat{X} + \mu + \frac{q_2\hat{H}}{\hat{Y}+D_2} - \frac{q_2\hat{Y}\hat{H}}{(\hat{Y}+D_2)^2} - \frac{q_2\hat{Y}\hat{H}}{(\hat{Y}+D_2)^2} - \frac{q_2\hat{Y}(\hat{X}+D_1)\left(\frac{r}{K}+\lambda\right)}{q_1(\hat{Y}+D_2)}$.
The following optimal tax equation can be obtained by substituting the values of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 in $\frac{d\varepsilon_1}{dt} = -\left[\frac{\partial \Im}{\partial X}\right]_{(\hat{X},\hat{Y},\hat{H},\hat{P})}$ for a suitable choice of annual discount rate (δ) at any equilibrium point:

$$\begin{aligned}&\frac{q_1D_1\hat{H}\hat{P}\hat{X}}{(\hat{X}+D_1)^2} + \left(r\left(1-\frac{2\hat{X}+\hat{Y}}{K}\right) - \lambda\hat{Y} - \frac{q_1D_1\hat{H}}{(\hat{X}+D_1)^2}\right)\left(\frac{G_3(\hat{X}+D_1)}{q_1\hat{X}} + \frac{G_4q_2\hat{Y}(\hat{X}+D_1)}{q_1\hat{X}(G_5+\delta)(\hat{Y}+D_2)}\right)\\ &-\frac{\lambda\hat{Y}G_4}{G_5+\delta} - \frac{G_1D_1\phi_2q_1\hat{H}\hat{P}}{(G_2+\delta)(\hat{X}+D_1)^2} - \delta\left(\frac{G_3(\hat{X}+D_1)}{q_1\hat{X}} + \frac{G_4q_2\hat{Y}(\hat{X}+D_1)}{q_1\hat{X}(G_5+\delta)(\hat{Y}+D_2)}\right) = 0.\end{aligned} \tag{5.31}$$

The positive solution of Eq. (5.31) provides the optimal tax, $\hat{\tau}$, of τ. The optimal societal revenue $\hat{\Gamma}$ evaluated at the equilibrium $(\hat{X},\hat{Y},\hat{H},\hat{P})$ is then obtained by substituting $\tau = \hat{\tau}$ in (5.23). A similar method has to be applied if one wants to find the optimal societal revenue for an another equilibrium point.

5.5 Simulation results

For an illustration of the analytical results, the parameter values mentioned in Table 5.1 will be considered as the baseline values and will remain fixed unless it is mentioned. It is to be mentioned that these parameters follow some rules/criteria to be meaningful for ecological systems, as prescribed in [310]. We carefully selected four system parameters K, λ, A and τ to observe their variational effects on the system dynamics. The parameter K is selected from the ecological point of view. A population model can show the well-known *paradox of enrichment* phenomenon and can make a system unstable with large amplitude oscillations from its stable state if the system is enriched by increasing its carrying capacity [264]. The disease transmission rate parameter, λ, which plays a vital role in disease persistence, is chosen from the epidemic point of view. The spread and control of infectious diseases depend on this parameter [252]. The parameter A is vital from the point of view of open market theory. It gives an upper cap in the market demand even when the price is low. The parameter τ has two types of importance. First, it regulates the overfishing of renewable resources and ensures fish persistence for sustainable use. Secondly, the revenue generated from the imposed tax can be used for the welfare of fishermen and the aquatic environment. We first presented the dynamic changes observed due to the change in a single parameter while keeping other parameter values unchanged and then showed their combined effects.

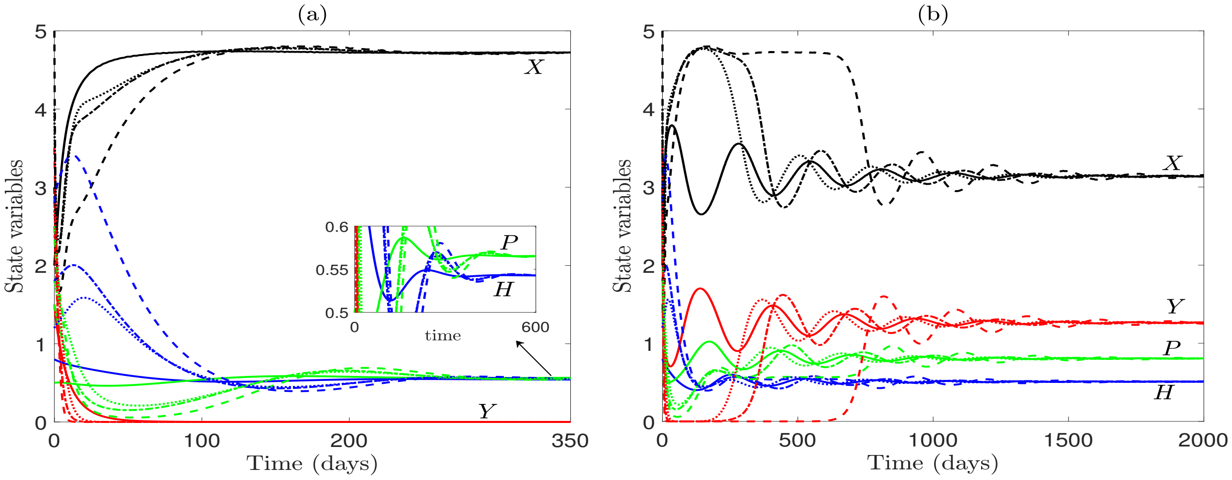

Figure 5.1: Global stability of the (a) disease-free equilibrium $\xi_3 = (4.72, 0, 0.54, 0.57)$ and (b) the endemic equilibrium $\xi_4 = (3.14, 1.26, 0.51, 0.81)$. In each case, the system started from the initial values $(2, 1.5, 0.8, 0.5)$, $(3, 2.5, 1.8, 1.5)$, $(5, 3.5, 2.8, 2.5)$, $(4, 2.7, 1.2, 1.8)$ and converged to the respective equilibrium points after initial fluctuations, depicting the global stability of the equilibrium points ξ_3 (Fig. (a)) and ξ_4 (Fig. (b)). The black, red, blue, and green curves indicate X, Y, H, and P. Here $\lambda = 0.02$ and 0.04 for the figures (a) and (b), respectively. Other parameters are as in Table 5.1 with $K = 5, A = 0.9, \tau = 0.45$.

At the lower value of the transmission rate ($\lambda = 0.02$), the basic reproduction number $\mathscr{R}_0$ goes below 1 ($\mathscr{R}_0 = 0.62$) and the system stabilizes globally to the disease-free equilibrium $\xi_3 = (4.72, 0, 0.54, 0.57)$ (Fig. 5.1a), following Theorem 5.4. Healthy fish stock in this state is high, 4.72 metric tonnes, causing a compensatory decrease in the fish price. The harvesting effort is also high because of the availability of fish stock. On the contrary, if the transmission rate is high, say 0.04, the system converges to the endemic state ξ_4 (Fig. 5.1b) with equilibrium population levels $\bar{X} = 3.14, \bar{Y} = 1.26, \bar{H} = 0.51, \bar{P} = 0.81$. It is to be mentioned that parameter values satisfy the global stability conditions of Theorem 5.6. At the equilibrium state ξ_4, the net fish stock $(\bar{X} + \bar{Y})$ is around 4.40 units, in which 3.14 units are salubrious and the other 1.26 units are insalubrious. Total fish stock in this state decreases to 4.40 units from 4.72 units in the infection-free state. Due to low fish production in the stable endemic state, supply cannot completely fulfill the demand, and the equilibrium market price goes high (0.81 per unit biomass of fish from its previous price 0.57). The harvesting effort, however, is comparatively low here (around 0.51 units) because of reduced stock level. It is important to note that the income from infected fish is negative in this case as the tax (τ) to be paid per unit biomass of infected fish is higher than its fixed selling price (p). Still, the fishers may be interested in fishing and profit is earned because of the higher price of healthy fish.

5.5.1 One parameter bifurcation results

We here explored the switching phenomena of the system under the variation of the parameters A, λ, and K. We varied the demand parameter, A, in some stipulated range, $0 < A < 5$, and observed four distinct dynamic regions, viz. R_1, R_2, R_3 and R_4, as shown in Fig. 5.2.

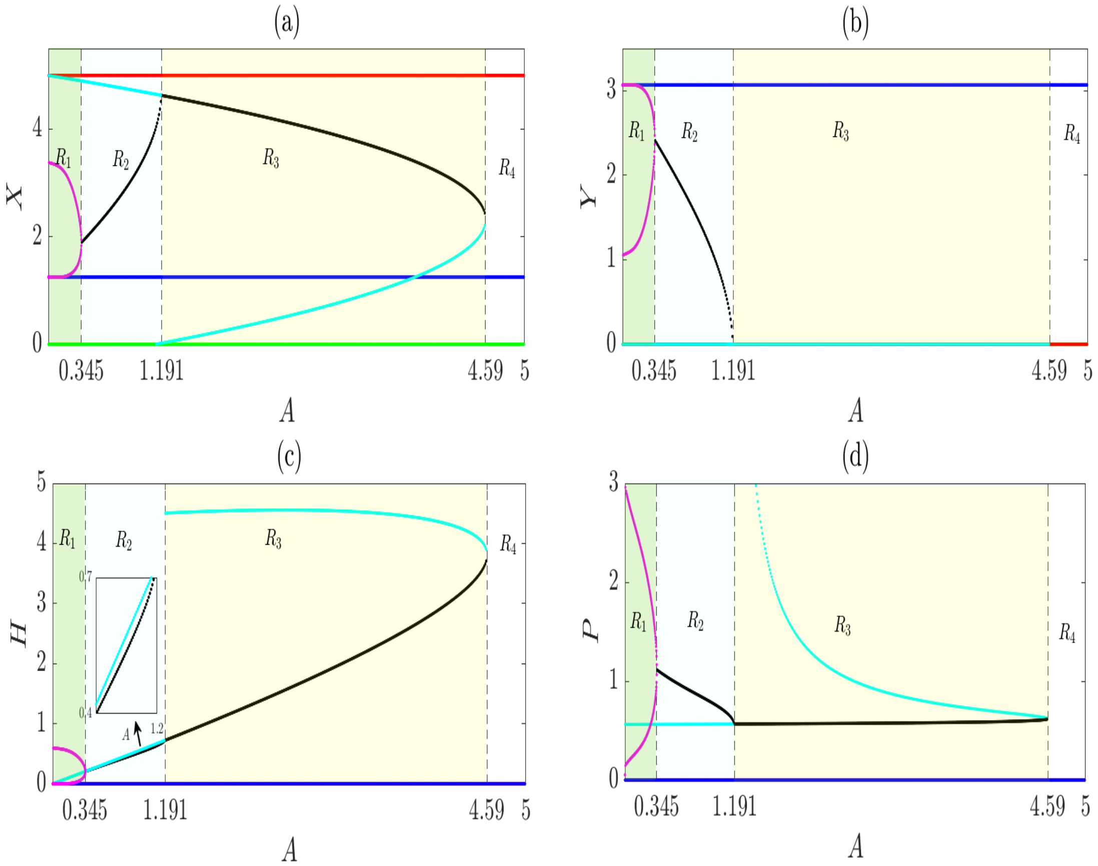

Figure 5.2: Bifurcation results when the maximum demand A varies in the ranges $0 < A < 5$. Black lines are the stable equilibrium points of the system (5.2). Green, red, blue and cyan dotted lines indicate the unstable equilibrium points ξ_0, ξ_1, ξ_2, and ξ_3. Magenta dotted lines give the maxima and minima of the stable limit cycle. The endemic equilibrium ξ_4 is unstable in $0 < A < 0.345$ and stable in $0.345 < A < 1.191$. One disease-free equilibrium ξ_3 is stable in $1.191 < A < 4.59$, while the other is unstable. The stable and unstable disease-free equilibria are annihilated through a saddle-node bifurcation at $A = 4.59$. The price becomes unbounded when $A > 4.59$ due to the scarcity of fish. High harvesting effort persists as fish demand and price are high. Parameters are as in Table 5.1 with $K = 5, \lambda = 0.04, \tau = 0.45$.

A transition from the unstable oscillatory state at the lower value of A to the stable coexistence state occurs through a Hopf bifurcation in the backward direction at $A = 0.345$. Then a transcritical bifurcation arises at $A = 1.191$, where the endemic equilibrium ξ_4 coalesces with the disease-free equilibrium ξ_3 and exchanges their stabilities. Two disease-free equilibrium points exist in the region R_3, one stable and the other unstable. These two equilibria approach each other with increasing A and are annihilated at $A = 4.59$ through a saddle-node bifurcation by satisfying the conditions of Theorem 5.7.

Following the open market theory, the fish price grows as its market demand grows. The increased price causes harvesting efforts to increase, which causes a steady decline in the fish biomass. As demand exceeds and crosses the critical threshold $A = 4.59$, the fish cost becomes unbounded in the region R_4, causing excessive harvesting pressure and eventual regime shift. A number of equilibrium points exist in this region (ξ_0, ξ_1 and ξ_2), but all of them are unstable.

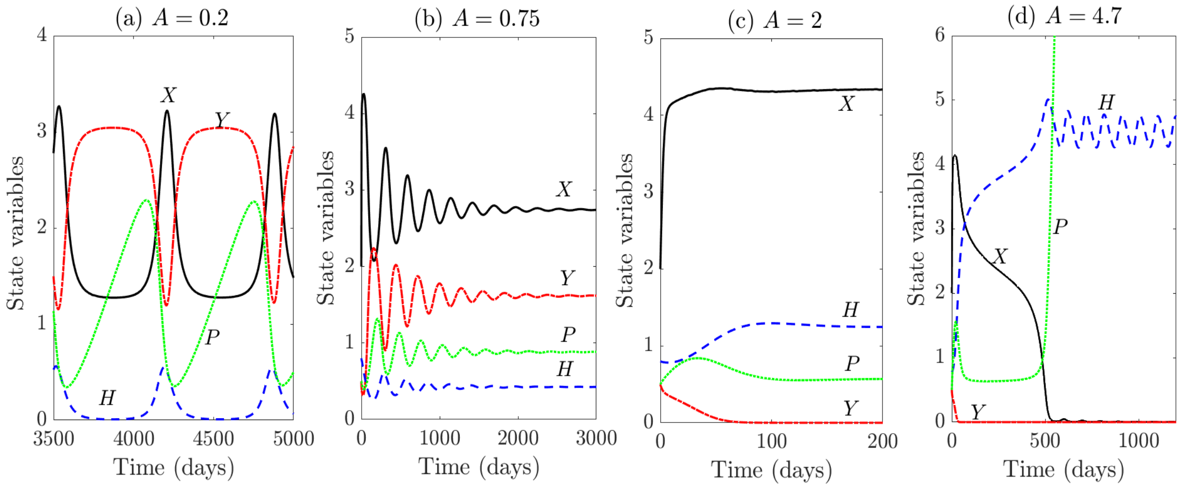

Figure 5.3: Time evolutions of the system (5.2) for some fixed values of A taken one from each zone of Fig. 5.2. (a) The endemic equilibrium ξ_4 is unstable for $A = 0.2$. (b) ξ_4 is stable for $A = 0.75$. (c) The disease-free equilibrium ξ_3 is stable for $A = 2$. (d) The price is unbounded for $A = 4.7$. Other parameters are as in Fig. 5.2.

The four distinct behaviours of the system for four fixed values taken one each from the four regions of Fig. 5.2 are represented by time series solutions in Fig. 5.3. It is observable that the harvesting effort continuously grows with the increasing demand. The fishers put a higher effort into fishing to meet the growing demand, which causes a steady decline in the healthy fish stock. A catastrophic shift of the system's state occurs when the market demand crosses some upper threshold value, $A = 4.59$, due to the excessive gap between demand and supply. The fish population sharply declines to near extinction as the harvesting effort grows to meet the high demand for fish (see Fig. 5.3d). The existence of unbounded fish price is also visible if one looks at the last equation of the model system (5.2). For the existence of equilibrium, one obtains from Eq. (5.2)d $\frac{A}{1+BP} - \frac{q_1XH}{X+D_1} = 0$ for nonzero P. If any of X or H is zero, then $\frac{A}{1+BP}$ has to be zero to hold the equality. This would be possible only when P becomes very large, as A and B are positive finite. Such demand-induced regime shift from a harvesting to a non-harvesting state was unreported earlier, as far as our knowledge goes.

Similar bifurcation analysis with respect to the parameters λ (the transmission rate) and K (the environmental carrying capacity) reveal that the system remains disease-free for the lower values of these parameters (Fig. 5.4). A stable endemic equilibrium appears in each case through a transcritical bifurcation when λ and K cross their respective critical values, $\lambda = 0.032$ and $K = 4.25$. In the endemic stable region of λ, healthy fish stock steadily decreases as the infection rate increases (Fig. 5.4a). The infected fish stock initially increases with increasing λ but then decreases slowly for further increases in λ. This is due to the lack of sufficient numbers of susceptible fishes to be newly infected at the higher value of λ. The price increases very slowly as healthy fish becomes scarce. Fishermen, therefore, put extra effort into harvesting, and a gradual increment is observed in the harvesting effort, H. On the other hand, a moderate system enrichment supports the increased production of the harvested stock. However, a significant increment in the equilibrium infected fish stock may occur if the system is too enriched (Fig. 5.4b). As the price of infected fish is low, fishing may not be profitable even though the gross fish production is high at a higher value of K. So a steady decline in the harvesting effort may occur with increasing K, causing a gradual increase in fish price.

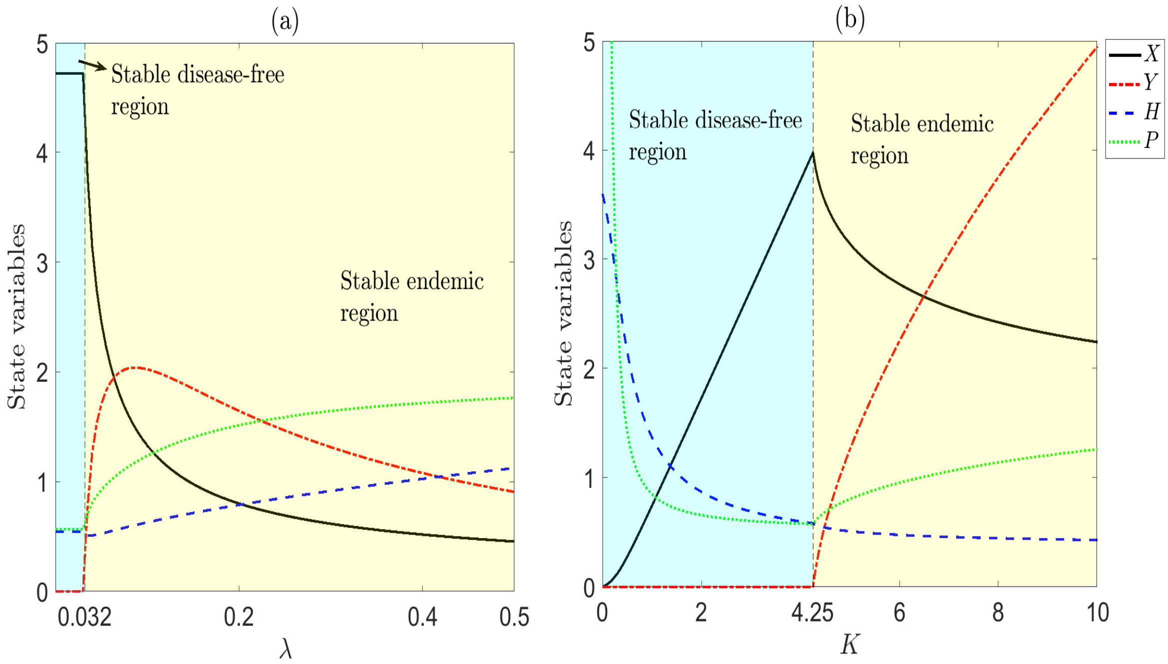

Figure 5.4: Bifurcation diagrams with respect to (a) the transmission rate, λ, and (b) environmental carrying capacity, K. Other parameters are as in Table 5.1 with (a) $A = 0.9, K = 5, \tau = 0.45$, and (b) $A = 0.9, \lambda = 0.04, \tau = 0.45$.

Tax (τ) per unit biomass of the landed fish is a regulatory measure imposed by the controlling authority to restrict over-harvesting. Figure 5.5 illustrates three changes in the dynamical states due to the variation in τ. The disease-free equilibrium is stable in the region $0 < \tau < 0.26$, and then the endemic equilibrium (ξ_4) becomes stable in the range $0.26 < \tau < 0.64$ through a transcritical bifurcation at $\tau = 0.26$. For further increment in the tax $(\tau > 0.64)$, the endemic equilibrium becomes unstable through a forward Hopf bifurcation. It reveals that a disease can invade the fish population if the imposed tax crosses some threshold value. The system even can lose its stable state and exhibit unstable periodic behavior if the levied tax exceeds some upper threshold value.

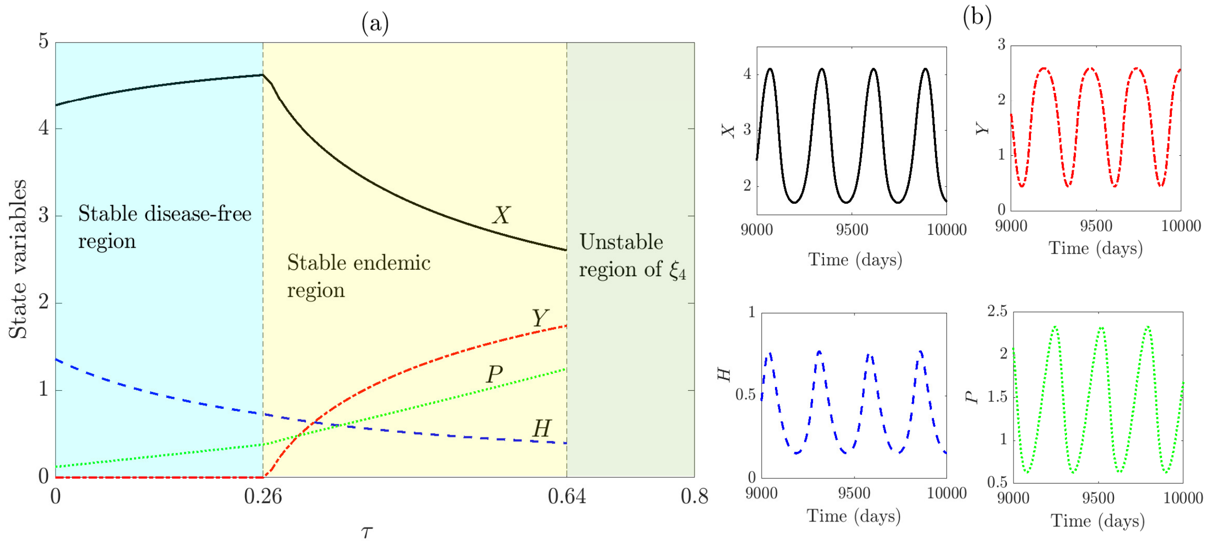

Figure 5.5: Effect of taxation on the system dynamics: (a) The infection-free equilibrium is stable in the lower range of τ, $(0 < \tau < 0.26)$. In the intermediate range $2.6 < \tau < 0.64$, the endemic equilibrium ξ_4 is stable and unstable for $\tau > 0.64$. A Hopf bifurcation is realized at $\tau = 0.64$. (b) The time evaluations of the state variables X, Y, H, and P show limit cycle oscillations for $\tau = 0.7 (> 0.64)$. Other parameters are as in Table 5.1 with $A = 0.9, K = 5, \lambda = 0.04$.

5.5.2 Two parameter bifurcation results

We present two-parameter bifurcation results to demonstrate the more extensive dynamical behavior of the system. Fig. 5.6 delineates the system's dynamic state when two parameters vary simultaneously. Figure 5.6a shows that there is a large stable parametric domain in the $A - \lambda$ plane, where the endemic equilibrium ξ_4 is stable. The stable disease-free equilibrium ξ_3 dominates the region where the transmission rate is low. The system can tolerate a higher transmission rate and maintains a disease-free state if the demand is also high. However, for too higher values of A, the system enters into the price-unbounded region. In the latter area, the fish price overgrows due to high demand and the nonavailability of the fish stock. For a more extended range of λ, the endemic equilibrium ξ_4 is unstable (oscillatory), provided the demand is low.

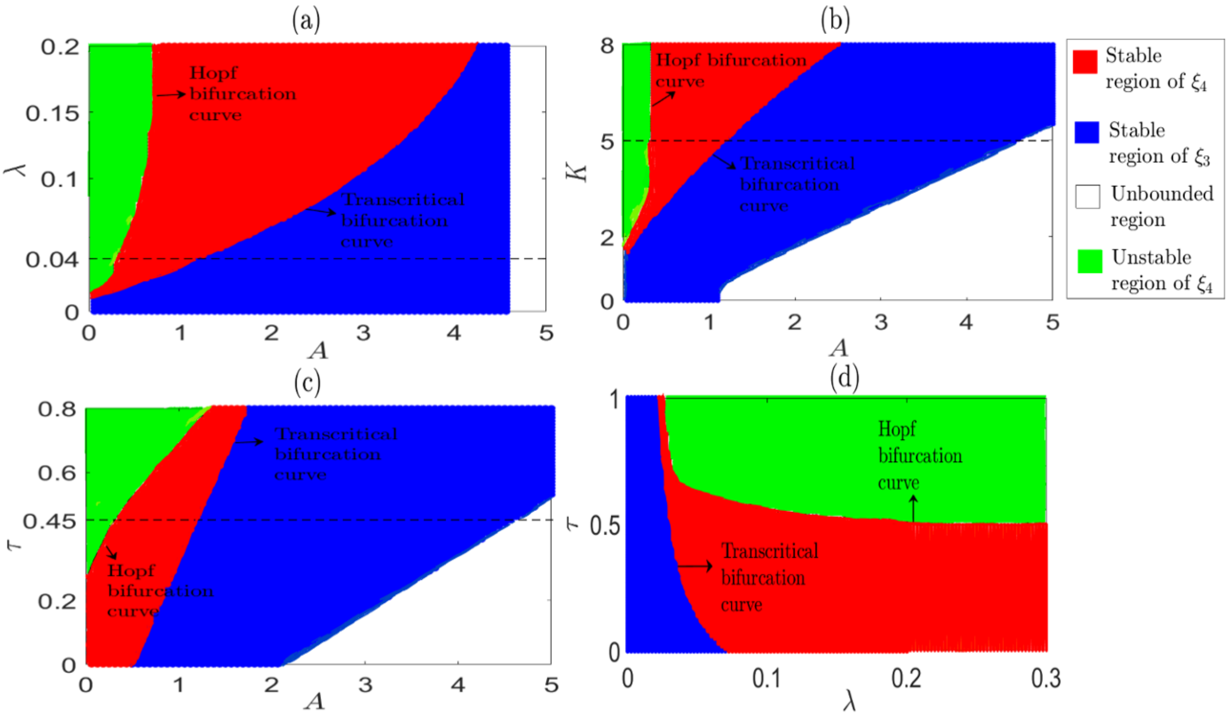

Figure 5.6: System dynamics when two parameters are varied simultaneously. (a) Four different dynamic regimes may be observed in the $A-\lambda$ parametric plane. (b, c) In the $A-K$ and $A-\tau$ planes, there are also four different dynamic behaviors of the system. (d) Three dynamic behaviors are observed when λ and τ simultaneously vary. The red and blue colors represent the stable regions for the endemic equilibrium ξ_4 and the disease-free equilibrium ξ_3, respectively. The green region is the unstable (oscillatory) region of ξ_4, whereas the white area represents the region of the disease-free state with the unbounded price hike. Parameters are as in Table 5.1 with $K=5, \tau=0.45$ for Fig. (a), $\lambda=0.04, \tau=0.45$ for Fig. (b), $\lambda=0.04, K=5$ for Fig. (c) and $K=5, A=0.9$ for Fig. (d).

There are four different dynamic regimes for the simultaneous variation of demand (A) and environmental carrying capacity (K) (Figure 5.6b). The stable disease-free region separates the stable endemic region from the price-unbounded area. The endemic state is unstable for the extended range of K if the demand is deficient. Such instability, which causes large amplitude oscillations in the system populations due to eutrophication, is driven through the enrichment of the system and known as the *paradox of enrichment* in the ecological literature [264, 311, 312]. Thus, a bioeconomic system may also show the paradox of enrichment only when the demand is deficient. However, these oscillations are suppressed, and the system backs to a stable state through a backward Hopf bifurcation if demand increases. Further increase in demand may push the system into a price-unbounded region. More or less, similar dynamics are observed for simultaneous variations in the parameters A and τ (Figure 5.6c). It is clear from this diagram that for a given maximum demand A, it is possible to choose a level of taxation that stabilizes the fishery and eliminates infection. The dynamics are relatively more straightforward in the $\lambda-\tau$ plane. Here the system shows three types of dynamic behaviors. Infection cannot persist, and the system becomes disease-free for any tax if the transmission rate is low (Figure 5.6d). The disease, however, persists if the transmission rate increases. In this case, the endemic equilibrium is either stable or unstable, depending on the fishing tax. The coexistence equilibrium is stable for low to intermediate tax and unstable if the tax is high. A similar behavior is observed in the system dynamics when K and τ are varied simultaneously (figure not shown). Thus, tax may have both stabilizing and destabilizing effects on the system dynamics. Thus, Fig. 5.6 demonstrates that an additional dynamic regime, where the price is unbounded, may appear only in the presence of the demand parameter. Fish stock rapidly goes extinct due to intensive harvesting led by the high demand. The fishery system may collapse due to the non-availability of the fish, making the price unbounded in response to the fish demand.

We here compare the maximum societal revenue at different equilibrium states where fishing is possible. For this, we search for the optimal tax from Eq. (5.23) for the parameter values of Fig. 5.5. The optimal value of τ is determined as $\tau_{opt} = 0.64$ at the endemic equilibrium ξ_4 with an annual discount rate $\delta = 0.08$, and the same at the disease-free equilibrium ξ_3 is obtained as $\tau_{opt} = 0.26$. It is to be recalled that the societal benefit (Γ) is the sum of the revenues earned by the fishermen from the harvested fish after paying the tax (Γ_1), and the income earned by the regulatory authority from the taxation imposed on the harvested fish (Γ_2). The fishermen's income (Γ_1) comes from the selling of healthy and infected fish, making $\Gamma_1 = \Gamma_1^{'} + \Gamma_1^{''}$. The societal benefit is therefore computed as

$$\Gamma = \Gamma_1 + \Gamma_2 = \Gamma_1^{'} + \Gamma_1^{''} + \Gamma_2.$$

It is worth mentioning that the societal revenue (Γ) coincides with the fishermen's earnings (Γ_1) in the absence of fishing tax (i.e., $\Gamma = \Gamma_1$ when $\tau = 0$), otherwise they are different.

Earlier, we demonstrated (see Fig. 5.5) how the stability of the equilibrium points ξ_3 and ξ_4 change with the variation of τ. It shows that steady harvesting is possible in $0 < \tau < 0.64$, where the system is stable around the equilibrium points ξ_3 and ξ_4. Figure 5.7 shows all types of revenues evaluated at these stable equilibrium points. Observe that $\Gamma = \Gamma_1$ at $\tau = 0$, $\Gamma_1 = \Gamma_1^{'}$ in the disease-free range, $0 < \tau < 0.26$, and $\Gamma_1 = \Gamma_1^{'} + \Gamma_1^{''}$ for $0.26 < \tau > 0.64$. These revenue curves show that fishermen's earnings are maximum and the societal benefit is minimum when there is no fishing tax. The fishermen's income (Γ_1) decreases with the increasing tax. The revenue (Γ_2) earned by the regulating agency as fishing tax is an increasing function of τ. In the range $0.26 < \tau < 0.64$, the fishermen's revenue comes from selling susceptible and infected fish. Though the fishermen incur a loss (see the red dashed line $\Gamma_1^{''}$) by selling the infected fish at a lower fixed price than the tax paid, the net income (Γ_1) is positive because of higher income from the healthy fish (red dotted line, $\Gamma_1^{'}$). Note that the healthy fish stock gradually declines (see Fig. 5.5) in the range $0.26 < \tau < 0.64$, but the revenue earned by selling them still increases due to increasing fish price in this stable range of the endemic equilibrium point, ξ_4. Thus, the total revenue becomes positive by compensating for the loss incurred from the infected fish. The societal revenue is maximum ($\Gamma = 0.167$ M\$/year) at $\tau = 0.64$ M\$/metric tonne. Furthermore, the revenue earned by selling the healthy fish and the maximum loss incurred from the infected fish occurs at the same value of τ. Stability is lost if τ exceeds this optimal value. Noticeably, there is no trade-off between the fishing tax and revenue earnings. A comparative study showing equilibrium points and revenues under various heads is presented in Table 5.2.

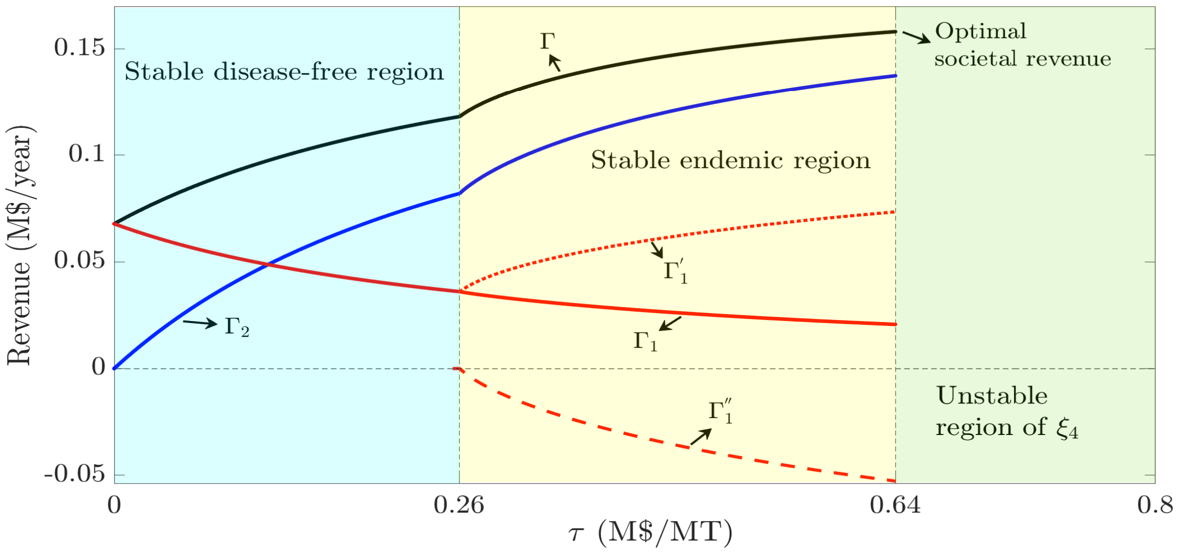

Figure 5.7: Plots of the societal revenue Γ (solid black line), revenue from fishing Γ_1 (solid red line), and revenue earned from the fishing tax Γ_2 (solid blue line) for different values of τ. Red dotted, and red dashed lines indicate the fishermen's earnings from the susceptible fish ($\Gamma_1^{'}$) and infected fish ($\Gamma_1^{''}$), respectively. Parameters are as in Fig. 5.5.

Table 5.2: This table compares various earnings from the fishery at the equilibrium states ξ_3 and ξ_4 for some particular fishing taxes with annual discount rate $\delta = 0.08$. Other parameters are as in Fig. 5.7. Observe that maximum societal benefit occurs in the endemic state at $\tau = 0.64$ M\$/MT. For the disease-free equilibrium, the societal benefit is maximum at $\tau = 0.26$ M\$/MT, and the same for the endemic equilibrium occurs at $\tau = 0.64$ M\$/MT.

τ M\$/MT	$\hat{X}$ MT	$\hat{Y}$ MT	$\hat{H}$ SFU	$\hat{P}$ M\$/MT	$\Gamma_1^{'}$ M\$/year	$\Gamma_1^{''}$ M\$/year	$\Gamma_1 = \Gamma_1^{'} + \Gamma_1^{''}$ M\$/year	Γ_2 M\$/year	$\Gamma = \Gamma_1 + \Gamma_2$ M\$/year
0	4.27	0	1.36	0.12	0.068	0	**0.068**	0	0.068
0.02	4.32	0	1.27	0.14	0.063	0	0.063	0.011	0.074
0.15	4.53	0	0.91	0.27	0.045	0	0.045	0.058	0.103
0.26	4.63	0	0.72	0.38	0.036	0	0.036	**0.082**	0.118
0.35	3.66	0.81	0.60	0.58	0.054	-0.023	0.031	0.108	0.139
0.5	2.96	1.42	0.47	0.92	0.068	-0.044	0.024	0.129	0.153
0.64	2.61	1.74	0.39	1.25	0.076	-0.056	0.020	**0.147**	**0.167**

5.6 Discussion

Fish harvesting is a social-ecological interaction where human and nonhuman species (including the harvested and non-harvested species) interact. Such social-ecological systems are dynamic and complex [204]. The complexity may multiply if the social-ecological phenomenon of interest is connected with the economic ingredients. In this work, we have studied such an extended social-ecological-economic interaction by coupling the social-ecological interaction of fish harvesting in the presence of infection with the fish market price, where the demand and supply of the fish determine the latter. The objective is deciphering how market demand controls fish, fisheries and infection. Whether there is any emerging dynamic due to the extension of the social-ecological system to the social-ecological-economic system is another important objective of this chapter.

For this, we have considered a bioeconomic fishery model, where an infectious disease of SI type circulates. Various assumptions are made to make the model realistic and to reduce the knowledge gap. For example, we have considered nonlinear saturated harvesting, a nonlinear price-dependent saturated demand function, different prices for healthy and infected fish from the actual market scenario, and fishing tax. A fishing tax levied on landed fish is an effective control measure against overfishing. The regulatory tax helps to reduce harvesting pressure and can help maintain a sustainable fishery with healthy fish. Our one-parameter bifurcation result shows (see Fig. 5.5) that increasing the fishing tax may help disease spreading by reducing the harvesting effort. An increased fish price may also cause periodic oscillations in the fish stock and its price. Such periodic solutions are undesirable because they provoke periods of minimal fish density close to extinction and weak fishing activity.

The two-parameter bifurcation results of the tax, τ, plotted with the maximum demand A and the transmission rate λ, demonstrate the broader dynamics of the system. It reveals that for a given maximum demand A or transmission rate λ, it would be possible to choose a level of taxation that stabilizes the fishery and eliminates infection. The bifurcation results thus provide insights into how to deal with the multi-difficulties of fisheries and how economic dynamics can significantly affect maintaining a sustainable fishery with healthy fish.

Furthermore, an increased fishing tax may negatively affect the fishery if it is excessively high. Income from fishery may reduce significantly, and the fishers may opt out of the fishery due to overfishing tax. This may aggravate the poor socioeconomic condition of the local people, particularly in low-income countries, where an alternative livelihood is challenging to find. Therefore, the regulatory authority should levy a tax more scientifically rather than imposing it abruptly. An optimal tax calculation may be helpful in such a decision-making process. Some studies demonstrate that there exists a trade-off between fishing tax and revenue generation [91]. However, this study shows that there is no such trade-off. The tax revenue levied by the regulating authority and the societal revenue increase with increasing tax. Fisher's income, however, decreases with increasing fishing tax. It is minimum at the disease-free state when there is no tax, but the former two earnings are maximum at the endemic state before the system shifts to the unstable state. The nonexistence of a trade-off between fishing tax and revenue generation is attributed to the saturated type demand.

Overharvesting is the primary cause of uncontrolled fishing. It is reported that such overharvesting may cause a regime shift in many fisheries [136]. Some studies have shown anthropogenic causes as the driver of the marine regime shifts [138]. These studies, however, do not consider the economic concept of demand and supply. Some recent papers [182, 232] have reported catastrophic changes in fish stock when the price is considered a time-varying state equation. In these studies, the price is considered in a fast time scale, allowing the price to quickly adjusted and settle at the equilibrium value. Such a slow-fast time scale reduces the system dimension. One can perform a similar slow-fast study for our model. In this case, however, the system may arrive at an equilibrium where fish biomass is zero, but the effort is nonzero. And the reduced system will not have an equilibrium where fish biomass is nonzero, but the effort is zero. This happens due to the system's transformation to a slow time scale treating price as a constant. The equilibrium price will then grow unboundedly as the fish biomass tends to zero. However, one may find it challenging to think of a natural fishery where an equilibrium of the form $(X = 0, H \neq 0)$ exists and becomes stable, but an equilibrium of the form $(X \neq 0, H = 0)$ never exists. Our study is a generalization of these studies [182, 232] and does not allow such equilibria.

The demand parameter A plays a vital role in the dynamics of social-ecological-economic interaction. Demand can make the system stable from its unstable state. It also plays a role in removing the infection from the system. Increasing demand corresponds to increased harvesting effort, which helps eliminate the disease. On the contrary, the negative side of high demand could be severe. There may be a drastic change in the system's behaviour. A regime shift from a harvested state to a non-harvested state may occur in the system, causing an imbalance between demand and supply and the people's socioeconomic condition associated with the fishery. The non-harvesting regime, where the price is unbounded, is not observed in any other bifurcation results where A is not one of the bifurcation parameters. Overexploitation is known to cause a regime shift in a fishery. But demand may also contribute to such regime shift was unreported earlier. One reason is that earlier harvesting models were not coupled with the economic concept of demand and supply. Thus, reducing harvesting pressure may not be sufficient to protect fisheries from collapse if other mechanisms also contribute to the process. Therefore, understanding and managing regime shift is a global challenge [313]. The controlling agency should take appropriate measures to minimize the socioeconomic imbalance caused by excessive demand. Such a demand-induced regime shift could be avoided if the gap between demand and supply is minimized, which is a fundamental economic problem. Understanding various causes or mechanisms of sudden shifts in the system's state will help us protect and sustain our renewable marine resources. However, translating the regime shift theory for its application may be an arduous task [314].

There is a major drawback to the chapter since it only examines a single species of fishery, but the majority of fisheries contain predator and prey species. Predator-prey fisheries are important for ecological significance, promoting biodiversity, maintaining ecological balance, and triggering trophic cascades. It also has economic value, sustains fisheries, and provides essential ecosystem services. Understanding and managing these interactions are vital for marine ecosystem health and human well-being. Therefore, in Chapter 6, we present and analyze a harvesting model that integrates ecological interactions of predators and prey fish with ecotourism and open market economies.

6

An ecological-economic fishery model: Maximizing the societal benefit through an integrated approach of fishing and ecotourism[1]

6.1 Introduction

Ecotourism is one of the leading and potential branches of the tourism industry because of its significant role in sustainable development [194, 195, 196]. Ecotourism was advocated in 1968 when Hetzer [197] integrated culture, education, and tourism in a string, later becoming a pillar of conservation and sustainable development. According to The International Ecotourism Society (TIES), ecotourism is a liable excursion into the natural environment that must conserve the area's ecology, maintain the territorial inhabitant's prosperity, and the consciousness to preserve the ecosystem [198]. Recreational fishing and non-extractive recreational activities may be an integral part of ecotourism. Coral reef ecotourism and fishing are one of the most important nature-based tourism having potential ecological and economic value [199, 200]. Marine wildlife, including marine mammals, may be another potential player in the modern fishery that could play a significant role in achieving the goal of fishery-based ecotourism [201, 202, 203].

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One of the ecotourism principles is providing direct financial benefits for conservation. For this, tourist entry fees at the ecotourism spot may be one step forward to maintaining the preservation of the tourist spot and the economic development of the people associated with such program [8].

Fishing is a major and sustainable livelihood. According to the recent report of the Food and Agriculture Organization (FAO) [294], almost 38.98 million people make a living by fishing, and the 7% source of global protein comes from seafood. Over the last fifty years, the production of fish has become quadruplicated, and the average fish consumption reaches almost twice that compared to half a century ago. About 38 fish species with 90 fish stock faced rapid collapse in the last five years of the 20th century [128]. International Ocean Management (IOM) was established by the United Nations Sustainable Development Goals (UNSDG) to ensure a healthy and wealthy sea for the sustainable use of marine resources [315]. A prime objective of today's global fishery is to support the blue economy, which advocates the use of marine ecosystems sustainably for economic development, improvement of livelihood of local people, and overall national development while preserving the marine health [316, 317, 318, 319]. The fishermen usually exhibit consolidated interest in economic incentives rather than taking care of the species depletion. It is, therefore, essential for the controlling agency to impose some kinds of restrictions for regulated harvesting. For instance, many developed countries applied to catch share management systems (CSMS) to bring flexibility and accountability in fisheries involving the stakeholders [320, 321, 322]. Enforcement of a fishing tax is a significant footstep toward CSMS [189]. Such policy is effectively implemented in Iceland for both the inland and marine fishes [192].

6.1.1 Empirical example

Chilika, located in Odisha state of eastern India between the latitude $19^{\circ}20'13.06'' - 19^{\circ}54'47.02''$ to longitude $85^{\circ}06'49.15'' - 85^{\circ}35'32.87''$, is the largest lagoon in Asia. This lagoon is fed with freshwater by three rivers and connected with the salty water of the Bay of Bengal [323]. Depending on the season, this brackish water lagoon extends over an area between 906 and 1165 km^2 and contains a large variety of 318 fish species and a home for migratory birds [323]. Chilika lagoon provides the livelihood of 200000 fishers [324]. The actual fin fish catch from 2011 to 2015 is 7456.02, 7114.3, 7699.71, and 7146.77 tonnes, giving an average fish catch of 7354.2 tons per year [325]. However, the total fish production has decreased from 12714.95 tonnes in 2016-17 to 9406.0 tonnes in 2020-21 [326]. Irrawaddy dolphin (*Orcaella brevirostris*), enlisted as endangered species globally, is a resident of this lagoon and is a top predator of this species-enriched ecosystem. According to the 2007 census, there are 138 dolphins in this lagoon. Out of them, 115 were adults, 17 adolescents, and six calves [327]. Chilika lagoon is only 37 km away from Puri, one of the holiest places of Hindu pilgrimages, and the most attractive sea beaches in India [113]. Due to its geographical advantage, the Chilika lagoon is a hotspot for ecotourism. Dolphin watching in a boat is an attraction for tourists in Chilika, and many visitors take a boat ride to watch dolphins. Local cooperatives regulate this tourism [328]. According to a survey report conducted during 2010-2011, the average charge for dolphin-watching was US$7 per person for an average watching time of 2 hr 55 minutes [329]. The average annual tourist number was 154,036 in the survey period [329], making a revenue of US$1,078,000 annually. Similar empirical phenomena are also observed in many fishery-related ecotourism spots [330, 114]. However, such combined activities always lack proper management for resource sustainability and revenue generation [115]. Therefore, a scientific evaluation of the integrated ecological-economic issues is vital for the maximum economic benefit while preserving aquatic resources.

Different theoretical models are studied to explore the effect of tourism on an open-access fishery [110, 331, 332, 91]. Some other models [268, 267, 270, 90, 282] consider the optimal fishing tax to restrict the over-harvesting of renewable resources. Coupling fishing-based tourism with the optimal fishing tax may be more challenging and essential from a socioeconomic viewpoint. Here we integrated a harvested predator-prey model with prey harvesting and ecotourism activity involving the predatory fish. From a realistic viewpoint, we assume that fish harvesting is nonlinear and saturates at higher fish density. The variation in the fishing effort and fish price depends on the market's demand and supply of the harvested fish. A fishing tax regulates overfishing. The tourist entry depends on the recreational fish abundance and the entrance fee. Our stability analysis shows that three equilibrium states are ecologically and economically important. We provide their local stability and prove the global stability of the most crucial interior equilibrium point. Using Pontryagin's maximum principle, the tax was optimized for maximizing the equilibrium societal revenue. It is shown that there exists an optimal tax level where every stakeholder will be in a win-win situation.

The rest of the chapter follows the following order. A two-species ecological-economic predator-prey harvesting model is proposed in the next Section 6.2. Mathematical results like positivity, boundedness of the solutions, and local & global stability of the equilibrium points are presented in Section 6.3. It also contains the optimal fishing tax determination result. The simulation results are given in Section 6.4. The chapter ends with a discussion in Section 6.5.

6.2 Model construction

Consider the interaction between prey and its generalist predator, where the predators are the marine mammals (dolphins, sharks, whales) and prey is the smaller fish (like krill, shrimp, squid, mackerel, catfish, etc.). The growth equation of such interacting species in the absence of harvesting and ecotourism can be represented by [91]

$$\begin{aligned} \frac{dX}{dt} &= \rho_1 X\left(1-\frac{X}{K_1}\right)-\eta_1 XY, \\ \frac{dY}{dt} &= \rho_2 Y\left(1-\frac{Y}{K_2}\right)+\eta_2 XY, \end{aligned} \tag{6.1}$$

where (X,Y), respectively, indicate the biomass of prey fish and predatory fish at time t, (ρ_1,ρ_2) are their respective intrinsic growth rates and (K_1,K_2) are the levels of carrying capacities supported by the environment. The marine predator is usually of generalist type [91], who feeds on other prey, including the focal prey. In the absence of focal prey, the predator maintains its growth logistically. The focal prey's consumption contributes to the predator's additional growth, as represented in the model system (6.1).

Per unit effort harvesting may not be infinite even when there is plenty of fish. So we assume that the prey fish harvest reaches a finite value when fish stock is large and, therefore, consider a nonlinear saturated catch function. In such a case, the amount of harvested stock per unit of time can be represented by [142]

$$\mathscr{H}(X,H)=\frac{\xi XH}{X+D_1},$$

where ξ is the catchability coefficient and D_1 as the half-saturation level.

The tourist number $\mathscr{N}$ is presumed to depend on the level of predatory fish Y and tourist entry fee, b. It is usually assumed that the number of tourists exponentially decays with the entry fee [333]. Then the number of tourists $\mathscr{N}$ at any time t can be expressed as [110]

$$\mathscr{N}(Y)=N_0 Y^a e^{-bg},$$

where N_0 is a proportionality constant and g is a sensitivity parameter, which measures the tourist's sensitivity to the entry fee. Here a is a controlling parameter, which determines how the tourist number depends on the predatory fish level. For example, $a=0,1,>1,<1$ imply, respectively, no dependency on the predatory fish level, depends linearly on the predatory fish level, Y, or at a faster rate than Y, or a slower rate than the level Y. Then the growth equation of each species in the presence of harvesting and ecotourism can be written as

$$\begin{aligned} \frac{dX}{dt} &= \rho_1 X\left(1-\frac{X}{K_1}\right)-\eta_1 XY-\mathscr{H}(X,H), \\ \frac{dY}{dt} &= \rho_2 Y\left(1-\frac{Y}{K_2}\right)+\eta_2 XY-\beta\mathscr{N}(Y)Y. \end{aligned} \tag{6.2}$$

We now further assume that the harvesting effort H applied to catch prey fish at any time t varies with the profit margin, which is defined by the income from fish selling minus the incurred cost for making use of an effort H. If M is the per unit market price of fish and τ is the fixed fishing tax levied on the per unit of landed fish, then the per unit net selling price of fish is $(M - \tau)$. In an open market, the per capita fish price at any time t is balanced by the difference between demand and supply of fish at that time. If Φ is the maximum demand and r is the rate at which demand decreases linearly with increasing price, then the demand ($\mathscr{D}$) has a mathematical expression [139]

$$\mathscr{D}(M) = \Phi - rM.$$

Based on these assumptions, one can formulate the following predator-prey fishery model in the presence of ecotourism and prey fish harvesting with a fishing tax:

$$\begin{aligned}
\frac{dX}{dt} &= \rho_1 X\left(1-\frac{X}{K_1}\right) - \eta_1 XY - \mathscr{H}(X,H),\\
\frac{dY}{dt} &= \rho_2 Y\left(1-\frac{Y}{K_2}\right) + \eta_2 XY - \beta\mathscr{N}(Y)Y,\\
\frac{dH}{dt} &= \alpha_1\Big(\mathscr{H}(X,H)(M-\tau) - \kappa H\Big),\\
\frac{dM}{dt} &= \alpha_2 M\Big(\mathscr{D}(M) - \mathscr{H}(X,H)\Big).
\end{aligned} \tag{6.3}$$

All parameters are positive from ecological and economic viewpoints. The assumptions of this model well-fit the ecological and economic interface of Chilika lagoon. The system (6.3) involving only the first two equations with catch per unit effort (CPUE) harvesting $\mathscr{H}(X,H) = qXH$ was considered in [91] to describe the importance of optimal tourist entrance fee for sustainable development. Predator-prey model with CPUE harvesting has been considered in other studies [334]. Krishna et al. [183] studied a similar version of the above model involving the first three equations with the underline modified Lotka-Volterra type predator-prey model to maximize the income from fishing through optimal taxation. They did not consider the effect of ecotourism and variation in the market price following open market policy. Using the method of aggregation of variables, Auger et al. [139] studied the effects of market price on the dynamics of a reduced dimension fishery model with a price equation without considering the tax policy and ecotourism. Bioeconomic models of a fishery with saturated catch and variable price were considered in many other studies [142, 184, 208]. These papers, however, ignored the concept of ecotourism and its effect on socioeconomic development. Here we integrate the idea of taxation, ecotourism, and market-based price variation of the harvested fish and then analyze it to demonstrate the ecological and economic effects of fishing tax and some other parameters.

6.3 Mathematical results

6.3.1 Positivity and boundedness

The following positivity and boundedness results of the system (6.3) can be proved as in lemma 2.1.

Lemma 6.1. *Under the initial value* $(X_0, Y_0, H_0, M_0) \in \mathbb{R}_+^{4,0}$, *every solution* $(X(t), Y(t), H(t), M(t))$ *of the system* (6.3) *is positive and uniformly bounded in* $\mathscr{G}_{\mathscr{L}}$, *where*
$\mathscr{G}_{\mathscr{L}} = \{(X,Y,H,M) : 0 < X < \zeta + \hat{\varepsilon},\ 0 < M < \hat{\zeta} + \tilde{\varepsilon},\ 0 < S(X,Y,H,M) < \frac{s_2}{s_1} + \varepsilon$, *for any positive* $\hat{\varepsilon}, \tilde{\varepsilon}, \varepsilon\}$, *where* $s_1 = \min\{\frac{\rho_1}{K_1}, (\frac{\rho_2}{K_2} + \eta_1), \alpha_2\xi, \alpha_2 r\}$, $s_2 = \rho_1 + \rho_2 + \alpha_2\Phi + \eta_2\zeta + \xi\hat{\zeta}$ *and* $\zeta = \max\{X_0, K_1\}$, $\hat{\zeta} = \max\{M_0, \hat{K}_1\}$.

6.3.2 Equilibrium points and their local stabilities

System (6.3) contains nine equilibrium points:

1. The zero equilibrium $E_0 = (0,0,0,0)$.

2. The only market price equilibrium $E_1 = (0,0,0,\frac{\Phi}{r})$.

3. The only predatory fish equilibrium $E_2 = (0, Y_2, 0, 0)$, where Y_2 is the positive root of the equation

$$\beta N_0 Y^{(a+1)} e^{-bg} + \frac{Y}{K_2} - 1 = 0. \tag{6.4}$$

4. Predatory fish and price equilibrium $E_3 = (0, Y_3, 0, \frac{\Phi}{r})$, where Y_3 is again the positive root of (6.4).

5. The only prey equilibrium $E_4 = (K_1, 0, 0, 0)$.

6. The prey species and its market price equilibrium $E_5 = (K_1, 0, 0, \frac{\Phi}{r})$.

7. The predatory fish-free equilibrium $E_6 = (X_6, 0, H_6, M_6)$, where

$$H_6 = \frac{\rho_1(X_6 + D_1)(K_1 - X_6)}{\xi K_1}, M_6 = \frac{1}{r}\left(\Phi - \rho_1 X_6\big(1 - \frac{X_6}{K_1}\big)\right),$$

   and X_6 is the positive root of the equation

$$\xi\rho X_6^3 - \xi\rho X_6^2 + (\xi\Phi - \kappa K_1 r - \xi r\tau K_1)X_6 - \kappa r K_1 D_1 = 0.$$

   Note that there is a unique positive root of X_6 if $\xi\Phi - \kappa K_1 r - \xi r\tau K_1 < 0$.

8. The harvesting effort-free equilibrium $E_7 = (X_7, Y_7, 0, \frac{\Phi}{r})$, where

$$X_7 = \left(1 + \frac{Y_7 \eta_1}{\rho_1}\right) K_1,$$

and Y_7 is the positive root of the equation

$$\rho_1 N_0 \beta K_2 e^{-bg} Y_7^a - (\rho_1 \rho_2 + \eta_1 \eta_2 K_1 K_2) Y_7 + \rho_1 K_2 (\rho_2 - \eta_2 K_1) = 0.$$

9. The coexisting equilibrium point $E^* = (X^*, Y^*, H^*, M^*)$, where

$$\begin{aligned} X^* &= \frac{1}{\eta_2}\left\{\rho_2\left(\frac{Y^*}{K_2} - 1\right) + N_0 Y^{*a} \beta \, \mathrm{e}^{-bg}\right\}, \\ M^* &= \tau + \frac{\kappa}{\xi}\left\{\frac{D_1 \eta_2}{\rho_2\left(\frac{Y^*}{K_2} - 1\right) + N_0 Y^{*a} \beta \, \mathrm{e}^{-bg}} + 1\right\}, \\ H^* &= \frac{(M^* - \tau)}{\kappa}\left[\Phi - r\left(\tau + \frac{\kappa}{\xi}\left\{\frac{D_1 \eta_2}{\rho_2\left(\frac{Y^*}{K_2} - 1\right) + N_0 Y^{*a} \beta \, \mathrm{e}^{-bg}} + 1\right\}\right)\right], \end{aligned}$$

and Y^* is the positive root of the equation

$$\begin{aligned} & C_1 Y^{*(3a)} - C_2 Y^{*(2a+1)} - C_3 Y^{*(2a)} + C_4 Y^{*(a+2)} - C_5 Y^{*(a+1)} + C_6 Y^{*(a)} + C_7 Y^{*3} \\ & -C_8 Y^{*2} - C_9 Y^* - C_{10} = 0, \end{aligned} \tag{6.5}$$

where
$C_1 = {K_2}^3 {N_0}^3 \beta^3 \rho_1 \xi$, $C_2 = {K_2}^2 {N_0}^2 \beta^2 \xi \, \mathrm{e}^{bg} (3 \rho_1 \rho_2 + K_1 K_2 \eta_1 \eta_2)$, $C_3 = \rho_1 \beta^2 N_0^2 \xi K_2^3 e^{bg}$ $(K_1 \eta_2 + 3 \rho_2)$, $C_4 = K_2 N_0 \beta \rho_2 \xi \, \mathrm{e}^{2bg} (3 \rho_1 \rho_2 + 2 K_1 K_2 \eta_1 \eta_2)$, $C_5 = 2 {K_2}^2 N_0 \beta \rho_2 \xi \, \mathrm{e}^{2bg}$ $(3 \rho_1 \rho_2 + K_1 K_2 \eta_1 \rho_2 + K_1 \eta_2 \rho_1)$, $C_6 = {K_2}^3 N_0 \beta e^{2bg} (3 \rho_1 {\rho_2}^2 \xi + K_1 \Phi {\eta_2}^2 \xi - K_1 {\eta_2}^2 \kappa r +$ $2 K_1 \eta_2 \rho_1 \rho_2 \xi - K_1 {\eta_2}^2 r \tau \xi)$, $C_7 = \xi \, \mathrm{e}^{3bg} {\rho_2}^2 (\rho_1 \rho_2 K_1 K_2 \eta_1 \eta_2)$, $C_8 = K_2 \rho_2^2 \xi e^{3bg} (3 \rho_1 \rho_2 +$ $K_1 \eta_2 \rho_1 + 2 K_1 \eta_1 \eta_2)$, $C_9 = K_2^2 \rho_2 e^{3bg} + K_1 {\eta_2}^2 \kappa r - 2 K_1 \eta_2 \rho_1 \rho_2 \xi - K_1 \Phi {\eta_2}^2 \xi - 3 \rho_1 {\rho_2}^2 \xi -$ $K_1 K_2 \eta_1 \eta_2 \rho_2 \xi + K_1 {\eta_2}^2 r \tau \xi$, $C_{10} = {K_2}^3 \mathrm{e}^{3bg} + \rho_1 {\rho_2}^3 \xi + K_1 \eta_2 \rho_1 {\rho_2}^2 \xi + D_1 K_1 {\eta_2}^3 \kappa r +$ $K_1 \Phi {\eta_2}^2 \rho_2 \xi - K_1 {\eta_2}^2 \kappa r \rho_2 - K_1 {\eta_2}^2 r \rho_2 \tau \xi$.

Acknowledging the analytical complexities for the nontrivial solutions of the equations (6.4) and (6.5), we will go for the numerical computations of these equations in the simulation section. Note that the equilibrium components of E^* contain the fishing tax parameter τ. Once we determine the optimal tax value $\tau = \tau^*$, then the equilibrium $E^*(\tau^*)$ is said to be an optimal equilibrium [183].

The equilibrium point should be locally stable for obtaining a viable optimal policy with respect to some equilibrium. We use the linearization technique to determine the local stability of these equilibrium points. The Jacobian matrix of the nonlinear system (6.3) evaluated at an arbitrary equilibrium point $\hat{E} = (\hat{X}, \hat{Y}, \hat{H}, \hat{M})$ is evaluated as

$$J = \begin{pmatrix} -\frac{\xi D_1 H}{(D_1+X)^2} - \rho_1 \left(\frac{2X}{K_1} - 1\right) - Y\eta_1 & -X\eta_1 & -\frac{X\xi}{D_1+X} & 0 \\ Y\eta_2 & X\eta_2 - \rho_2\left(\frac{2Y}{K_2} - 1\right) - N_0 Y^a \beta\, \mathrm{e}^{-bg}(a+1) & 0 & 0 \\ \alpha_1\left(\frac{H D_1 \xi (M-\tau)}{(D_1+X)^2}\right) & 0 & -\alpha_1\left(\kappa - \frac{X\xi(M-\tau)}{D_1+X}\right) & \frac{HX\alpha_1\xi}{D_1+X} \\ -\frac{\xi\alpha_2 D_1 HM}{(D_1+X)^2} & 0 & -\frac{MX\alpha_2\xi}{D_1+X} & -\alpha_2\left(Mr - \Phi + \frac{\xi XH}{D_1+X}\right) - \alpha_2 rM \end{pmatrix}_{(\hat{X},\hat{Y},\hat{H},\hat{M})} . \quad (6.6)$$

From the eigenvalues of this matrix, one can easily prove the following theorem.

Theorem 6.2. *(i) The equilibrium points* $E_0 = (0,0,0,0)$, $E_1 = (0,0,0,\frac{\Phi}{r})$, $E_2 = (0,Y_2,0,0)$, $E_4 = (K_1,0,0,0)$, $E_5 = (K_1,0,0,\frac{\Phi}{r})$, *and* $E_6 = (X_6,0,H_6,M_6)$ *are always unstable.*
(ii) The equilibrium $E_3 = (0,Y_3,0,\frac{\Phi}{r})$ *is locally asymptotically stable if the relation* $\rho_1 < Y_3\,\eta_1$ *is satisfied, otherwise it is unstable.*
(iii) The equilibrium $E_7 = (X_7,Y_7,0,\frac{\Phi}{r})$ *is locally asymptotically stable if the condition* $\kappa > \frac{X_7\xi\left(\frac{\Phi}{r}-\tau\right)}{D_1+X_7}$ *holds.*
(iv) Whenever the equilibrium point $E^* = (X^*,Y^*,H^*,M^*)$ *exists, it is locally asymptotically stable if* $\mathscr{B}_1 > 0, \mathscr{B}_3 > 0, \mathscr{B}_4 > 0$ *and* $\mathscr{B}_1\mathscr{B}_2\mathscr{B}_3 > (\mathscr{B}_3^2 + \mathscr{B}_1^2\mathscr{B}_4)$, *where* $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3$ *and* $\mathscr{B}_4$ *are given in* (6.11).

Proof. (i) At the zero equilibrium point $E_0 = (0,0,0,0)$, the Jacobian matrix (6.6) becomes

$$J(0,0,0,0) = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & -\alpha_1\kappa & 0 \\ 0 & 0 & 0 & \Phi\alpha_2 \end{pmatrix}. \quad (6.7)$$

There exists three positive eigenvalues corresponding to the Jacobian matrix (6.7), namely, ρ_1, ρ_2 and $\Phi\alpha_2$ and hence, the equilibrium point $E_0 = (0,0,0,0)$ is always unstable. Similarly, there exists two positive eigenvalues ρ_1, ρ_2 for the equilibrium point $E_1 = (0,0,0,\frac{\Phi}{r})$, one positive eigenvalue $\alpha_2\Phi$ for both the equilibrium points $E_2 = (0,Y_2,0,0)$ and $E_4 = (K_1,0,0,0)$, making these equilibrium points unstable. Each of the equilibrium point $E_5 = (K_1,0,0,\frac{\Phi}{r})$ and $E_6 = (X_6,0,H_6,M_6)$ has one positive eigenvalue $\rho_2 + X_6$ and $\rho_2 + X_7$, respectively. Hence, each of these equilibrium points is always unstable.

(ii) The variational matrix at the equilibrium $E_3 = (0,Y_3,0,\frac{\Phi}{r})$ is

$$J(0,Y_3,0,\frac{\Phi}{r}) = \begin{pmatrix} \rho_1 - Y_3\eta_1 & 0 & 0 & 0 \\ Y_3\eta_2 & -\frac{\rho_2 Y_3}{K_2} - aN_0 Y_3^a \beta\, \mathrm{e}^{-bg} & 0 & 0 \\ 0 & 0 & -\alpha_1\kappa & 0 \\ 0 & 0 & 0 & -\Phi\alpha_2 \end{pmatrix}.$$

Here eigenvalues are $-\Phi\,\alpha_2(<0)$, $-\alpha_1\,\kappa(<0)$, $-\frac{\rho_2 Y_3}{K_2} - aN_0\,Y_3^a\,\beta\,\mathrm{e}^{-bg}(<0)$, and $\rho_1 - Y_3\,\eta_1$. Thus, the equilibrium point $E_3 = (0, Y_3, 0, \frac{\Phi}{r})$ is locally asymptotically stable if $\rho_1 < Y_3\,\eta_1$, otherwise it is unstable.

(iii) At the equilibrium $E_7 = (X_7, Y_7, 0, \frac{\Phi}{r})$, one can obtain the Jacobian matrix as

$$J(X_7, Y_7, 0, \frac{\Phi}{r}) = \begin{pmatrix} -\frac{\rho_1 X_7}{K_1} & -X_7\,\eta_1 & -\frac{X_7\,\xi}{D_1+X_7} & 0 \\ Y_7\,\eta_2 & -\frac{\rho_2 Y_7}{K_2} - aN_0\,Y_7^a\,\beta\,\mathrm{e}^{-bg} & 0 & 0 \\ 0 & 0 & -\alpha_1\left(\kappa - \frac{X_7\,\xi\,(\frac{\Phi}{r}-\tau)}{D_1+X_7}\right) & 0 \\ 0 & 0 & -\frac{\Phi X_7\,\alpha_2\,\xi}{r(D_1+X_7)} & -\Phi\,\alpha_2 \end{pmatrix}.$$

Two eigenvalues of the above matrix are $\lambda_1 = -\Phi\,\alpha_2$ and $\lambda_2 = -\alpha_1\left(\kappa - \frac{X_7\,\xi\,(\frac{\Phi}{r}-\tau)}{D_1+X_7}\right)$. Here λ_1 is always negative, and $\lambda_2 < 0$ if $\kappa > \frac{X_7\,\xi\,(\frac{\Phi}{r}-\tau)}{D_1+X_7}$. The other two eigenvalues are the roots of the equation

$$\lambda^2 + \mathscr{A}_1\lambda + \mathscr{A}_2 = 0, \tag{6.8}$$

where

$$\begin{aligned} \mathscr{A}_1 &= \frac{\mathrm{e}^{-bg}\left(K_2 X_7\,\rho_1\,\mathrm{e}^{bg} + K_1\,Y_7\,\rho_2\,\mathrm{e}^{bg} + K_1\,K_2 N_0\,Y_7^a\,a\,\beta\right)}{K_1\,K_2}, \\ \mathscr{A}_2 &= \frac{X_7\,\mathrm{e}^{-bg}\left(Y_7\,\rho_1\,\rho_2\,\mathrm{e}^{bg} + K_2 N_0\,Y_7^a\,a\,\beta\,\rho_1 + K_1\,K_2\,Y_7\,\eta_1\,\eta_2\,\mathrm{e}^{bg}\right)}{K_1\,K_2}. \end{aligned}$$

Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are positive, both roots of the Eq. (6.8) are either real negative or complex conjugates with negative real part. Therefore, the equilibrium point $E_7 = (X_7, Y_7, 0, \frac{\Phi}{r})$ is locally asymptotically stable under the condition $\kappa > \frac{X_7\,\xi\,(\frac{\Phi}{r}-\tau)}{D_1+X_7}$.

(iv) Suppose an interior equilibrium $E^* = (X^*, Y^*, H^*, M^*)$ of the system (6.3) exists. The corresponding Jacobian matrix is evaluated as

$$J(X^*, Y^*, H^*, M^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & 0 & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix}, \tag{6.9}$$

where
$a_{11} = -\frac{\xi X^* H^*}{(D_1+X^*)} - \frac{\rho_1 X^*}{K_1}, a_{12} = -X^*\,\eta_1, a_{13} = -\frac{X^*\,\xi}{D_1+X^*}, a_{21} = Y^*\,\eta_2, a_{22} = -\frac{\rho_2 Y^*}{K_2} - aN_0\,Y^{*a}\,\beta\,\mathrm{e}^{-bg}$,
$a_{31} = \alpha_1\left(\frac{H^*\,D_1\,\xi\,(M^*-\tau)}{(D_1+X^*)^2}\right), a_{34} = \frac{H^* X^* \alpha_1\,\xi}{D_1+X^*}, a_{41} = -\frac{\xi\,\alpha_2 D_1 H^* M^*}{(D_1+X^*)^2}, a_{43} = -\frac{M^*\,X^*\,\alpha_2\,\xi}{D_1+X^*}, a_{44} = -\alpha_2 r M^*$.

The characteristic equation then reads

$$\lambda^4+\mathscr{B}_1\lambda^3+\mathscr{B}_2\lambda^2+\mathscr{B}_3\lambda+\mathscr{B}_4=0, \tag{6.10}$$

where

$$\begin{aligned}
\mathscr{B}_1 &= -(a_{11}+a_{22}+a_{44}),\\
\mathscr{B}_2 &= a_{11}a_{22}-a_{12}a_{21}-a_{13}a_{31}+a_{11}a_{44}+a_{22}a_{44}-a_{34}a_{43},\\
\mathscr{B}_3 &= a_{13}a_{22}a_{31}-a_{11}a_{22}a_{44}+a_{12}a_{21}a_{44}+a_{13}a_{31}a_{44}-a_{13}a_{34}a_{41}+a_{11}a_{34}a_{43}+a_{22}a_{34}a_{43},\\
\mathscr{B}_4 &= -a_{13}a_{22}a_{31}a_{44}+a_{13}a_{22}a_{34}a_{41}-a_{11}a_{22}a_{34}a_{43}+a_{12}a_{21}a_{34}a_{43}.
\end{aligned} \tag{6.11}$$

All roots of (6.10) have negative real parts [259] and the locally asymptotically stability of E^* is assured if $\mathscr{B}_1>0, \mathscr{B}_3>0, \mathscr{B}_4>0$ and $\mathscr{B}_1\mathscr{B}_2\mathscr{B}_3>(\mathscr{B}_3^2+\mathscr{B}_1^2\mathscr{B}_4)$. ☐

6.3.3 Global stability of E^*

In this section, we want to observe the basin of attraction of the most significant equilibrium point E^*, where all state variables coexist. If $\mathbb{R}_+^{4,0}$ is the basin of attraction of E^*, it is said to be globally asymptotically stable. The method of Lyapunov function [260] is the most well-known scheme for the global study of an equilibrium point. But there are several constraints to constructing a Lyapunov function and its subsequent analysis to show that the considered function is negative definite for a complex system, like (6.3). The high-dimensional Bendixson criterion [261] may be an alternative method for such a study. If the conditions of the following theorem hold, then the equilibrium E^* is globally asymptotically stable.

Theorem 6.3. *Suppose the equilibrium E^* exists. If*

$$\begin{aligned}
\nu \;=\; \max\Bigg\{&\beta_{11}+\frac{\beta_{14}\omega_1}{\omega_4},\ \beta_{22}+\beta_{23}\omega_2+\frac{\beta_{24}\omega_2}{\omega_4},\ \frac{\beta_{32}}{\omega_2}+\beta_{33}+\frac{\beta_{35}}{\omega_4}+\frac{\beta_{36}}{\omega_5},\ \frac{\beta_{41}\omega_3}{\omega_1}+\frac{\beta_{42}\omega_3}{\omega_2}+\\
&\frac{\beta_{44}\omega_2}{\omega_3}+\frac{\beta_{45}\omega_3}{\omega_4},\ \frac{\beta_{51}\omega_4}{\omega_1}+\beta_{53}\omega_4+\frac{\beta_{54}\omega_4}{\omega_3}+\beta_{55},\ \frac{\beta_{62}\omega_5}{\omega_2}+\beta_{63}\omega_5+\beta_{66}\Bigg\}<0,
\end{aligned} \tag{6.12}$$

then the system (6.3) *can not exhibit any non-trivial periodic solution. Furthermore, the coexisting equilibrium point E^* is globally asymptotically stable in $\mathbb{R}_+^{4,0}$.*

Proof. Let us assume O be an open subset of $\mathbb{R}_+^{4,0}$ and consider the function $\mathscr{F}:\mathscr{X}\to\mathscr{F}(\mathscr{X})\in\mathbb{R}_+^{4,0}$ which is a differentiable and have continuous gradient on $\mathscr{X}\in O$. Presume the system (6.3) as

$$\frac{d\mathscr{X}}{dt}=\mathscr{F}(\mathscr{X}), \tag{6.13}$$

where $\mathscr{X}=(X,Y,H,M)^T$ and $\mathscr{F}(\mathscr{X})=\Bigg[\rho_1 X\left(1-\frac{X}{K_1}\right)-\eta_1 XY-\mathscr{H}(X,H),\ \rho_2 Y\left(1-\frac{Y}{K_2}\right)+$ $\eta_2 XY-\beta\mathscr{N}(Y)Y,\ \alpha_1\Big(\mathscr{H}(X,H)(M-\tau)-\kappa H\Big),\ \alpha_2 M\Big(\mathscr{D}(M)-\mathscr{H}(X,H)\Big)\Bigg]^T.$

One can utilize the high-dimensional Bendixson criterion by showing that the second additive compound matrix

$$\frac{d\mathscr{Z}}{dt}=\frac{\partial\mathscr{F}}{\partial\mathscr{X}}^{[2]}(\mathscr{X}(t,\mathscr{X}_0))(\mathscr{Z}(t)) \tag{6.14}$$

at a solution $\mathscr{X}(t,\mathscr{X}_0)\in O$ of Eq. (6.13) is equi-uniformly asymptotically stable [262, 263]. The second additive compound matrix for the variational matrix

$$\frac{\partial\mathscr{F}}{\partial\mathscr{X}}=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & 0\\ \alpha_{21} & \alpha_{22} & 0 & 0\\ \alpha_{31} & 0 & \alpha_{33} & \alpha_{34}\\ \alpha_{41} & 0 & \alpha_{43} & \alpha_{44}\end{pmatrix},$$

where
$\alpha_{11}=-\frac{\xi D_1 H}{(D_1+X)^2}-\rho_1\left(\frac{2X}{K_1}-1\right)-Y\eta_1,\ \alpha_{12}=-X\eta_1,\ \alpha_{13}=-\frac{X\xi}{D_1+X},\ \alpha_{21}=Y\eta_2,\ \alpha_{22}=X\eta_2-\rho_2\left(\frac{2Y}{K_2}-1\right)-N_0 Y^a\beta\,\mathrm{e}^{-bg}(a+1),\ \alpha_{31}=\alpha_1\left(\frac{HD_1\xi(M-\tau)}{(D_1+X)^2}\right),\ \alpha_{33}=-\alpha_1\left(\kappa-\frac{X\xi(M-\tau)}{D_1+X}\right),\ \alpha_{34}=\frac{HX\alpha_1\xi}{D_1+X},\ \alpha_{41}=-\frac{\xi\alpha_2 D_1 HM}{(D_1+X)^2},\ \alpha_{43}=-\frac{MX\alpha_2\xi}{D_1+X},\ \alpha_{44}=-\alpha_2\left(Mr-\Phi+\frac{\xi XH}{D_1+X}\right)-\alpha_2 rM,$
will be a $\left({}^4C_2\times{}^4C_2\right)=(6\times 6)$ matrix and that can be represented as [208]

$$\frac{\partial\mathscr{F}}{\partial\mathscr{X}}^{[2]}=\begin{pmatrix}\alpha_{11}+\alpha_{22} & 0 & 0 & -\alpha_{13} & 0 & 0\\ 0 & \alpha_{11}+\alpha_{33} & \alpha_{34} & \alpha_{12} & 0 & 0\\ 0 & \alpha_{43} & \alpha_{11}+\alpha_{44} & 0 & \alpha_{12} & \alpha_{13}\\ -\alpha_{31} & \alpha_{21} & 0 & \alpha_{22}+\alpha_{33} & \alpha_{34} & 0\\ -\alpha_{41} & 0 & \alpha_{21} & \alpha_{43} & \alpha_{22}+\alpha_{44} & 0\\ 0 & -\alpha_{41} & \alpha_{31} & 0 & 0 & \alpha_{33}+\alpha_{44}\end{pmatrix}.$$

The region contained in any compact two-dimensional space in O falls exponentially if the system (6.13) exhibits equi-uniform asymptotic stability nature. However, there can't exist any invariant simple closed rectifiable curve within O, including periodic orbits, if O is simply connected. Thus, one can obtain the following proposition.

Proposition 6.4. *Consider a simply connected region $O\subset\mathbb{R}_+^{4,0}$. Assume that the family of nonlinear system* (6.13) *is equi-uniformly asymptotically stable. Then, if O is positively invariant and contains a unique coexisting equilibrium point E^*, then E^* is globally asymptotically stable in O.*

One can write the second compound system (6.14) as

$$(\dot{\zeta}_1, \dot{\zeta}_2, \dot{\zeta}_3, \dot{\zeta}_4, \dot{\zeta}_5, \dot{\zeta}_6)^T = \frac{\partial \mathscr{F}^{[2]}}{\partial \mathscr{X}}(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)^T,$$

where

$$\begin{aligned}
\dot{\zeta}_1 &= \Bigg[\rho_1\left(1-\frac{2X}{K_1}\right) - \frac{\xi D_1 H}{(D_1+X)^2} - Y\,\eta_1 + X\,\eta_2 + \rho_2\left(1-\frac{2Y}{K_2}\right) - N_0 Y^a \beta\, \mathrm{e}^{-bg}(a+1)\Bigg]\zeta_1 - \\
&\quad \left[\frac{X\,\xi}{D_1+X}\right]\zeta_4, \\
\dot{\zeta}_2 &= \Bigg[\rho_1\left(1-\frac{2X}{K_1}\right) - \frac{\xi D_1 H}{(D_1+X)^2} - Y\,\eta_1 - \alpha_1\left(\kappa - \frac{X\,\xi\,(M-\tau)}{D_1+X}\right)\Bigg]\zeta_2 + \left[\frac{H X\,\alpha_1\,\xi}{D_1+X}\right]\zeta_3 - [X\,\eta_1]\zeta_4, \\
\dot{\zeta}_3 &= \left[-\frac{M X\,\alpha_2\,\xi}{D_1+X}\right]\zeta_2 + \Bigg[\rho_1\left(1-\frac{2X}{K_1}\right) - \frac{\xi D_1 H}{(D_1+X)^2} - Y\,\eta_1 - \alpha_2\left(Mr - \Phi + \frac{\xi X H}{D_1+X}\right) - \alpha_2 rM\Bigg] \\
&\quad \zeta_3 - [\eta_1 X]\,\zeta_5 - \left[\frac{\xi X}{X+D_1}\right]\zeta_6, \\
\dot{\zeta}_4 &= -\left[\alpha_1\left(\frac{H D_1\,\xi\,(M-\tau)}{(D_1+X)^2}\right)\right]\zeta_1 + [Y\,\eta_2]\,\zeta_2 - \Bigg[X\,\eta_2 - \rho_2\left(\frac{2Y}{K_2}-1\right) - N_0 Y^a \beta\,\mathrm{e}^{-bg}(a+1) \qquad (6.15)\\
&\quad -\alpha_1\left(\kappa - \frac{X\,\xi\,(M-\tau)}{D_1+X}\right)\Bigg]\zeta_4 + \left[\frac{H X\,\alpha_1\,\xi}{D_1+X}\right]\zeta_5, \\
\dot{\zeta}_5 &= \left[\frac{\xi\,\alpha_2 D_1 H M}{(D_1+X)^2}\right]\zeta_1 + [Y\eta_2]\,\zeta_3 - \left[\frac{M X\,\alpha_2\,\xi}{D_1+X}\right]\zeta_4 + \Bigg[X\,\eta_2 - \rho_2\left(\frac{2Y}{K_2}-1\right) - N_0 Y^a \beta\,\mathrm{e}^{-bg}(a+1) - \\
&\quad \alpha_2\left(Mr - \Phi + \frac{\xi X H}{D_1+X}\right) - \alpha_2 rM\Bigg]\zeta_5, \\
\dot{\zeta}_6 &= \left[\frac{\xi\,\alpha_2 D_1 H M}{(D_1+X)^2}\right]\zeta_2 + \left[\alpha_1\left(\frac{H D_1\,\xi\,(M-\tau)}{(D_1+X)^2}\right)\right]\zeta_3 + \Bigg[-\alpha_1\left(\kappa - \frac{X\,\xi\,(M-\tau)}{D_1+X}\right) - \alpha_2\Bigg(Mr - \Phi + \\
&\quad \frac{\xi X H}{D_1+X}\Bigg) - \alpha_2 rM\Bigg]\zeta_6.
\end{aligned}$$

Construct the set

$$\mathscr{W}(\mathscr{X}) = \max\{\omega_1|\zeta_1|, \omega_2|\zeta_2|, |\zeta_3|, \omega_3|\zeta_4|, \omega_4|\zeta_5|, \omega_5|\zeta_6|\}.$$

Then at the coexisting equilibrium point $E^*(X^*, Y^*, H^*, M^*)$, the succession of system (6.15) can be written in the following inequality forms

$$\begin{aligned}
&\frac{d^+}{dt}\omega_1|\zeta_1| \le \beta_{11}\omega_1|\zeta_1| + \frac{\beta_{14}\omega_1}{\omega_4}\omega_4|\zeta_4|, \\
&\frac{d^+}{dt}\omega_2|\zeta_2| \le \beta_{22}\omega_2|\zeta_2| + \beta_{23}\omega_2|\zeta_3| + \frac{\beta_{24}\omega_2}{\omega_4}\omega_4|\zeta_4|, \\
&\frac{d^+}{dt}|\zeta_3| \le \frac{\beta_{32}}{\omega_2}\omega_2|\zeta_2| + \beta_{33}|\zeta_3| + \frac{\beta_{35}}{\omega_4}\omega_4|\zeta_5| + \frac{\beta_{36}}{\omega_5}\omega_5|\zeta_6|, \qquad (6.16)
\end{aligned}$$

$$\frac{d^+}{dt}\omega_3|\zeta_4| \leq \frac{\beta_{41}\omega_3}{\omega_1}\omega_1|\zeta_1| + \frac{\beta_{42}\omega_3}{\omega_2}\omega_2|\zeta_2| + \frac{\beta_{44}\omega_2}{\omega_3}\omega_3|\zeta_4| + \frac{\beta_{45}\omega_3}{\omega_4}\omega_4|\zeta_5|,$$
$$\frac{d^+}{dt}\omega_4|\zeta_5| \leq \frac{\beta_{51}\omega_4}{\omega_1}\omega_1|\zeta_1| + \beta_{53}\omega_4|\zeta_3| + \frac{\beta_{54}\omega_4}{\omega_3}\omega_3|\zeta_4| + \beta_{55}\omega_4|\zeta_5|,$$
$$\frac{d^+}{dt}\omega_5|\zeta_6| \leq \frac{\beta_{62}\omega_5}{\omega_2}\omega_2|\zeta_2| + \beta_{63}\omega_5|\zeta_3| + \beta_{66}\omega_5|\zeta_6|,$$

where $\frac{d^+}{dt}$ indicates the right-hand derivative, and
$\beta_{11} = -\frac{\xi X^* H^*}{(D_1+X^*)} - \frac{\rho_1 X^*}{K_1} + -\frac{\rho_2 Y^*}{K_2} - aN_0 Y^{*a}\beta\,\mathrm{e}^{-bg}, \beta_{14} = \frac{X^*\xi}{D_1+X^*}, \beta_{22} = -\frac{\xi X^* H^*}{(D_1+X^*)} - \frac{\rho_1 X^*}{K_1}$,
$\beta_{23} = \frac{H^* X^* \alpha_1 \xi}{D_1+X^*}, \beta_{24} = -X^*\eta_1,\ \beta_{32} = -\frac{M^* X^* \alpha_2 \xi}{D_1+X^*}, \beta_{33} = -\frac{\xi X^* H^*}{(D_1+X^*)} - \frac{\rho_1 X^*}{K_1} + -\alpha_2 r M^*$,
$\beta_{35} = -X^*\eta_1, \beta_{36} = -\frac{X^*\xi}{D_1+X^*},\ a_{21} = Y^*\eta_2, \beta_{41} = -\alpha_1\left(\frac{H^* D_1 \xi\,(M^*-\tau)}{(D_1+X^*)^2}\right), \beta_{42} = Y^*\eta_2, \beta_{44}$
$= -\frac{\rho_2 Y^*}{K_2} - aN_0 Y^{*a}\beta\,\mathrm{e}^{-bg}, \beta_{45} = \frac{H^* X^* \alpha_1 \xi}{D_1+X^*},\ \beta_{51} = \frac{\xi\alpha_2 D_1 H^* M^*}{(D_1+X^*)^2},\ \beta_{53} = Y^*\eta_2,\ \beta_{54} = -\frac{M^* X^* \alpha_2 \xi}{D_1+X^*}$,
$\beta_{55} = -\frac{\rho_2 Y^*}{K_2} - aN_0 Y^{*a}\beta\,\mathrm{e}^{-bg} - \alpha_2 r M^*,\ \beta_{61} = 0,\ \beta_{62} = \frac{\xi\alpha_2 D_1 H^* M^*}{(D_1+X^*)^2},\ \beta_{63} = \alpha_1\left(\frac{H^* D_1 \xi\,(M^*-\tau)}{(D_1+X^*)^2}\right)$,
$\beta_{66} = -\alpha_2 r M^*$.
It implies

$$\frac{d^+}{dt}\mathscr{W}(\mathscr{Z}(t)) \leq \nu\mathscr{W}(\mathscr{Z}(t)),$$

where

$$\begin{aligned}\nu &= \max\left\{\beta_{11} + \frac{\beta_{14}\omega_1}{\omega_4},\ \beta_{22} + \beta_{23}\omega_2 + \frac{\beta_{24}\omega_2}{\omega_4},\ \frac{\beta_{32}}{\omega_2} + \beta_{33} + \frac{\beta_{35}}{\omega_4} + \frac{\beta_{36}}{\omega_5},\ \frac{\beta_{41}\omega_3}{\omega_1} + \frac{\beta_{42}\omega_3}{\omega_2} + \right.\\ &\quad \left.\frac{\beta_{44}\omega_2}{\omega_3} + \frac{\beta_{45}\omega_3}{\omega_4},\ \frac{\beta_{51}\omega_4}{\omega_1} + \beta_{53}\omega_4 + \frac{\beta_{54}\omega_4}{\omega_3} + \beta_{55},\ \frac{\beta_{62}\omega_5}{\omega_2} + \beta_{63}\omega_5 + \beta_{66}\right\}.\end{aligned}$$

We have already shown that the system (6.3) is bounded. Hence, whenever the parametric condition $\nu < 0$ is satisfied, one can get a positive constant $\hat{\nu}$ such that $\nu \leq -\hat{\nu} < 0$. Which gives

$$\mathscr{W}(\mathscr{Z}(t)) \leq \mathscr{W}(\mathscr{Z}((\hat{t}))\exp(-\hat{\nu}(t-\hat{t})),\ t \geq \hat{t} > 0.$$

This guarantees the equi-uniform asymptotic stability of the second compound system (6.14) and hence, following Proposition (6.4), the coexisting equilibrium point of the system (6.3) is globally asymptotically stable. This completes the proof. □

6.3.4 Optimum tax determination

Imposing a fishing tax helps to control drastic harvesting, protect renewable resources from extinction, and support the long-term availability of this renewable resource. On the other hand, an abundance of prey fish provides the required food for the predatory fish to maintain their growth. These predatory fish, in turn, attract recreation tourists and generate revenue for society. However, the fishers may lose interest and abstain from fishing if the imposed tax on the landed fish is too high. So, it is essential to follow an optimal tax policy to help maintain a balance between fishing tax and societal benefit.

The societal revenue corresponding to the system (6.3) can be obtained by summing the fisher's income from selling the fish (Ω_1), earnings from ecotourism (Ω_2), and the earnings from fishing tax (Ω_3). These three earning components are expressed as

$$\Omega_1(X,Y,H,M,\tau) = (M-\tau)\left(\frac{\xi XH}{X+D_1}\right), \tag{6.17}$$

$$\Omega_2(X,Y,H,M,\tau) = bN_0Y^ae^{-bg}, \tag{6.18}$$

$$\Omega_3(X,Y,H,M,\tau) = \frac{\tau\xi XH}{X+D_1}. \tag{6.19}$$

Then the societal revenue ($\Omega(X,Y,H,M,\tau)$) is the sum of above three and is given by

$$\Omega(X,Y,H,M,\tau) = \frac{\xi XHM}{X+D_1}+bN_0Y^ae^{-bg}. \tag{6.20}$$

We intend to find the optimum fishing tax to provide optimum societal revenue. For this, the objective function is defined as

$$\Xi = \int_0^\infty e^{-\iota t}\Omega(X,Y,H,M,\tau)dt, \tag{6.21}$$

subject to the constraints (6.3). Here ι is the discount rate that includes inflation and time-value of money. The control variable τ is bounded by 0 and $\tau^{\max}$, where $\tau^{\max}$ indicates the upper limits of the imposed tax.

Following Pontryagin's maximum principle, one gets the Hamiltonian of the system as

$$\begin{aligned}\Psi(X,Y,H,M,\tau) = {} & e^{-\iota t}\left(\frac{\xi XHM}{X+D_1}+bN_0Y^ae^{-bg}\right)+\varepsilon_1\left(\rho_1X\left(1-\frac{X}{K_1}\right)-\eta_1XY-\frac{\xi XH}{X+D_1}\right)+ \\ & \varepsilon_2\left(\rho_2Y\left(1-\frac{Y}{K_2}\right)+\eta_2XY-\beta N_0Y^{a+1}e^{-bg}\right)+\varepsilon_3\Bigg(\alpha_1\Bigg(\frac{\xi XH}{X+D_1}(M-\tau) \\ & -\kappa H\Bigg)\Bigg)+\varepsilon_4\left(\alpha_2M\left(\Phi-rM-\frac{\xi XH}{X+D_1}\right)\right),\end{aligned} \tag{6.22}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 are the adjoint variables. To optimize the societal revenue Ω, the following conditions on the Hamiltonian have to be satisfied:

$$\frac{\partial\Psi}{\partial\tau}=0,\ \frac{d\varepsilon_1}{dt}=-\frac{\partial\Psi}{\partial X},\ \frac{d\varepsilon_2}{dt}=-\frac{\partial\Psi}{\partial Y},\ \frac{d\varepsilon_3}{dt}=-\frac{\partial\Psi}{\partial H},\ \frac{d\varepsilon_4}{dt}=-\frac{\partial\Psi}{\partial M}. \tag{6.23}$$

Recall that fish harvesting with the existence of both species is possible in the coexisting state only, so we are interested in examining the optimality at this equilibrium. The first condition of (6.23) evaluated at the equilibrium point $E^*(X^*,Y^*,H^*,M^*)$ gives $\varepsilon_3\alpha_1\left(-\frac{\xi X^*H^*}{X^*+D_1}\right)=0$. For the non-trivial solution, one have

$$\varepsilon_3 = 0.$$

Now, $\frac{d\varepsilon_4}{dt} = -\left[\frac{\partial \Psi}{\partial M}\right]_{(X^*,Y^*,H^*,M^*)}$ gives

$$\frac{d\varepsilon_4}{dt} = G_1 e^{-\iota t} + G_2 \varepsilon_4,$$

where

$$G_1 = -\frac{H^* X^* \xi}{D_1 + X^*} \text{ and } G_2 = \alpha_2 \left(2M^* r - \Phi + \frac{H^* X^* \xi}{D_1 + X^*}\right).$$

Solving this ordinary differential equation, one gets

$$\varepsilon_4 = -\frac{G_1}{G_2 + \iota} e^{-\iota t}.$$

Substituting the value of ε_3 and ε_4 in $\frac{d\varepsilon_3}{dt} = -\left[\frac{\partial \Psi}{\partial H}\right]_{(X^*,Y^*,H^*,M^*)}$, we have

$$\varepsilon_1 = M^* \left(1 + \frac{G_1}{G_2 + \iota}\right) e^{-\iota t}.$$

Similarly, substituting the value of ε_1 in $\frac{d\varepsilon_2}{dt} = -\left[\frac{\partial \Psi}{\partial Y}\right]_{(X^*,Y^*,H^*,M^*)}$, one have

$$\varepsilon_2 = -\frac{G_3}{G_4 + \iota} e^{-\iota t}, \text{ where}$$

$$\begin{aligned} G_3 &= -M^* X^* \eta_1 \left(\frac{H^* X^* \xi}{\left(\delta + \alpha_2 \left(2M^* r - \Phi + \frac{H^* X^* \xi}{D_1 + X^*}\right)\right)(D_1 + X^*)} - 1\right) - N_0 Y^{*(a-1)} ab \mathrm{e}^{-bg}, \\ G_4 &= \rho_2 \left(\frac{2Y^*}{K_2} - 1\right) - X^* \eta_2 + N_0 Y^{*a} \beta \mathrm{e}^{-bg} (a+1). \end{aligned}$$

Then, putting the values of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 in $\frac{d\varepsilon_1}{dt} = -\left[\frac{\partial \Psi}{\partial X}\right]_{(X^*,Y^*,H^*,M^*)}$, one can reach the following optimal tax equation for a fixed choice of annual discount rate (ι) at the coexisting equilibrium point:

$$\Upsilon(\tau) = \frac{G_1 G_7}{G_2 + \delta} - G_5 - M^* (G_6 - \delta) \left(\frac{G_1}{G_2 + \delta} + 1\right) + \frac{G_3 Y^* \eta_2}{G_4 + \delta} = 0, \tag{6.24}$$

where

$$G_5 = \frac{D_1 H^* M^* \xi}{(D_1 + X^*)^2},\ G_6 = -Y^* \eta_1 - \rho_1 \left(\frac{2X^*}{K_1} - 1\right) - \frac{D_1 H^* \xi}{(D_1 + X^*)^2},\ G_7 = -\frac{D_1 H^* M^* \alpha_2 \xi}{(D_1 + X^*)^2}.$$

Note that (X^*, Y^*, H^*, M^*) contains τ and the optimum fishing tax τ^* is the positive solution of Eq. (6.24). Then the corresponding optimal societal revenue may be obtained from (6.20) at the coexisting equilibrium point for the optimum tax $\tau = \tau^*$.

6.4 Simulation results

For the visual paradigm, the parameter values given in Table 6.1 will be considered as the baseline values and will remain fixed unless it is mentioned. Considering $K_1 = 30,\ \Phi = 2.5,\ \tau = 0.5$ and $K_2 = 4$, the conditions given in Theorem 6.2(iv) are satisfied, and therefore all the state variables coexist in a stable state at their equilibrium values $X^* = 2.26, Y^* = 4.39, H^* = 13.27, M^* = 1.82$ (See Fig. 6.1a). Note that all the state variables have considerable levels and may be ideal for fishing and ecotourism. Observe that the harvesting effort is around 13.27 units in this state which is high enough. Since the harvesting effort is proportional to the profit margin, fishing will be profitable in this state. The predatory fish biomass (around 4.39 units) also has a reasonable level to encourage tourists. However, at the higher values of the predatory fish carrying capacity ($K_2 = 13$), the coexisting state loses its stability, and the system stabilizes to the harvesting effort-free equilibrium $E_7(0.21, 5.66, 0, 4.16)$. A higher value of K_2 allows the predatory fish to grow. This higher level of predatory fish consumes most prey fish, and the remaining fish is insufficient to support harvesting. Consequently, the market price of fish becomes high ($M = 4.16$ units) due to its unavailability. Suppose the predatory fish is further allowed to increase by enhancing its carrying capacity. In that case, the prey fish also goes to extinction due to predation pressure (see Fig. 6.1c), causing the system to converge

Table 6.1: Variables and parameters with their descriptions

State Variables	Descriptions	Units	
$X(t)$	Prey fish biomass at time *t*	MT*	
$Y(t)$	Predator fish biomass at time *t*	MT	
$H(t)$	Fishing effort for prey species at time *t*	SFU$^{\odot}$	
$M(t)$	Market price per unit biomass of prey species at time *t*	M\$$^{\otimes}$/MT	
Dependent Functions	**Descriptions**	**Units**	
$\mathscr{H}(X,H)$	Harvested prey fish biomass at time *t* (a function of prey fish biomass and fishing effort)	MT/year	
$\mathscr{N}(Y)$	Number of visiting tourist at time *t* (a function of predatory fish)	million/year	
$\mathscr{D}(M)$	Per capita demand rate at time *t* (a function of prey fish price)	MT/year	
Parameters	**Descriptions**	**Default Values**	**Reference**
(ρ_1, ρ_2)	Intrinsic growth rate of prey fish and its predators	(4,2)/year	[91]
(K_1, K_2)	Environmental carrying capacity of prey fish and predatory	variables MT	Estimated
η_1	Predation rate of predatory fish	0.7 /MT/year	[91]
η_2	Biomass transformation rate	0.6 /MT/year	[335]
ξ	Catchability coefficient of prey fish	0.14 MT/SFU/year	[336]
D_1	Half saturation level of prey fish	2 MT	Assumed
β	Predator mortality rate due to tourism activities	0.3 /million	Assumed
N_0	Baseline number of tourists	2.5 million/year/(MT)^a	[91]
a	Tourist sensitivity parameter	0.3 unit-less	Assumed
b	Tourist entrance fee	0.4 M$/million tourist	[337]
g	Cost sensitivity parameter	0.02 million tourist/M$	Assumed
κ	Cost per unit of fishing effort	0.14 M$/SFU/year	Assumed
Φ	Maximal demand	variable MT/year	Estimated
r	The rate at which the demand decrease	0.6 MT^2/M\$/year	[208]
τ	Regulatory tax imposed on per unit of landed fish	variable M$/MT	Estimated
α_1	Stiffness parameter	0.6 SFU/M$	Assumed
α_2	Proportionality constant	0.5 /MT	Assumed

*MT represents metric tons, $^{\odot}$ SFU stands for Standardized Fishing Unit [242, 243] and $^{\otimes}$ M$ indicates million USD.

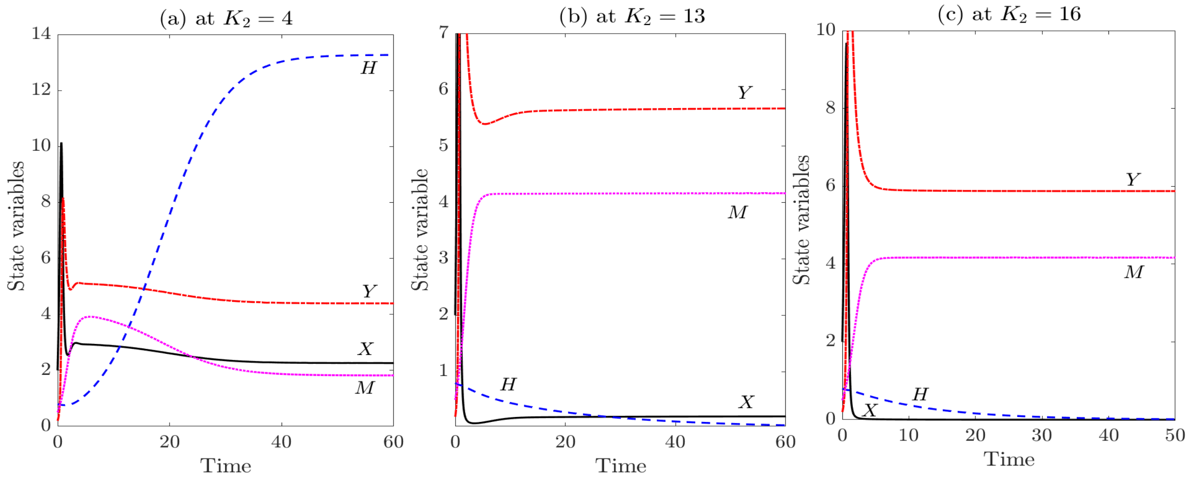

Figure 6.1: Time series solution of the model (6.3) with initial value $(2, 0.2, 0.8, 0.5)$. (a) The interior equilibrium $E^*(2.26, 4.39, 13.27, 1.82)$ is locally asymptotically stable for $K_2 = 4$. (b) The equilibrium $E_7(0.21, 5.66, 0, 4.16)$ is locally asymptotically stable for $K_2 = 13$. (c) The equilibrium $E_3(0, 5.88, 0, 4.17)$ is locally asymptotically stable for $K_2 = 16$. Parameter values are as in Table 6.1 with $K_1 = 30$, $\Phi = 2.5$, $\tau = 0.5$.

at the equilibrium $E_3(0, 5.88, 0, 4.17)$, where prey fish and its harvesting is not possible. If we compute the societal benefit at these equilibrium points for the fixed fishing tax $\tau = 0.5$, one obtains from Eq. (6.20), $\Omega(E^*(2.26, 4.39, 13.27, 1.82)) = 4.11$, $\Omega(E_7(0.21, 5.66, 0, 4.16)) = 1.66$ and $\Omega(E_3(0, 5.88, 0, 4.17)) = 1.69$. Note that the societal benefit is highest at the coexisting equilibrium E^*, where prey fish harvesting is possible. Therefore, the governing agency will try to maintain the circumstances so that the system remains stable at E^*. One way to preserve the stability of E^* is to keep the predatory fish level low, and this would be further ascertained in the bifurcation analysis.

Considering the parameter values as in Fig. 6.1a and selecting $\omega_1 = 3$, $\omega_2 = 2$, $\omega_3 = 4$, $\omega_4 = 0.5$, and $\omega_5 = 0.1$, one can verify the global stability condition given in (6.12) as $\nu = \max\{-1.9839, -5.0319, -4.3718, -2.0309, -4.3871, -0.5202\} = -0.5202 < 0$. Hence, following Theorem 6.3, the interior equilibrium E^* is globally asymptotically stable in $\mathbb{R}^4_+$. Figure 6.2 shows that the solutions of the system (6.3) with different initial populations converge asymptotically to the coexisting equilibrium point $E^*(2.26, 4.39, 13.27, 1.82)$, implying the independency of the solution's starting value.

6.4.1 Single parameter bifurcation

Four important system parameters K_1, Φ, K_2, and τ are selected to observe their variational impacts on the system behavior. The parameter Φ has economic significance and is considered for appraising the market sensitivity of prey fish. Fish harvesting and price depend very much on its demand, ϕ. The parameters K_1 and K_2 have their ecological significance. The enhanced value of the system's carrying capacity can make a stable predator-prey interaction unstable

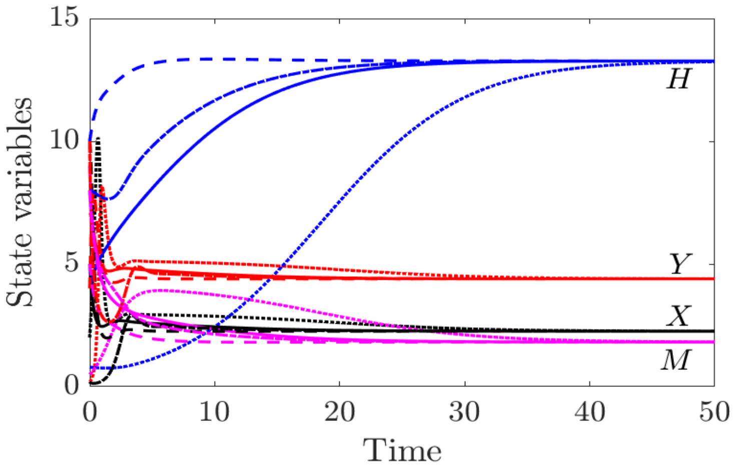

Figure 6.2: Global stability of the interior equilibrium point E^*. Solutions starting from the initial points $(2,0.2,0.8,0.5),(5,7,4,9),(10,4,10,5)$ and $(0.2,10,8,8)$ converge to $E^*(2.26,4.39,13.27,1.82)$.

through large amplitude oscillations, the phenomenon known as the *paradox of enrichment* [264]. Secondly, these parameters maintain how many prey or predator species are essential to maintain the equipoise between harvesting and ecotourism. The fishing tax parameter τ also has two significant importance. First, it regulates the overfishing of renewable resources and ensures their sustainable use [338, 339]. Secondly, the revenue generated from the imposed tax can be used for the welfare of fishermen, and the aquatic environment [192, 340]. The importance of these parameters may be further justified by the sensitivity analysis of the system parameters by calculating their PRCC (partial ranked correlation coefficients) values (see Fig. 6.3).

We first look at the switching phenomena of the system due to the solo variation in the parameters mentioned above. The one-parameter bifurcation diagram with respect to the demand parameter ϕ (Fig. 6.4) shows three distinct dynamical switchings of the system due to demand variation. If the demand is low ($0 < \Phi < 0.99$), there is no harvesting, and the harvesting-free equilibrium E_7 becomes stable. As the demand increases, fish harvesting is initiated, and the interior equilibrium becomes stable for $0.99 < \Phi < 3.7$. However, this equilibrium becomes unstable, and the oscillations appear as ϕ exceeds the value 3.7. A transcritical bifurcation and a Hopf bifurcation occur at the critical values $\Phi = 0.99$ and $\Phi = 3.7$, respectively. Interestingly, the predatory fish population declines once harvesting is initiated and maintains the downtrend with increasing demand. This is reasonable because increasing demand causes increased harvesting, which forces the prey fish population to reduce. This lower prey fish population can only support a lower predatory fish. However, for much higher demand ($\Phi > 3.70$), the system losses its stability, and there are regular ups and downs in all the state variables.

Fig. 6.5 provides similar kind of bifurcation diagrams when the environmental carrying capacity of predatory fish K_2 (Fig 6.5a), fishing tax τ (Fig 6.5b) and the environmental carrying capacity of prey fish K_1 (Fig 6.5c) are varied. The fish harvesting is possible for $0 \leq \tau < 3$ (see

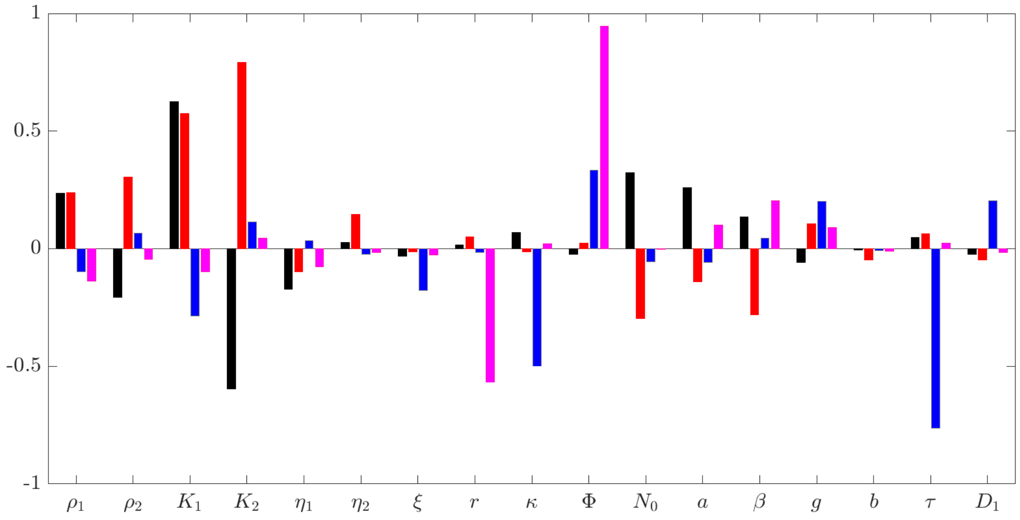

Figure 6.3: Global sensitivity analysis (GSA) of model parameters using the partially ranked correlation coefficients (PRCC) technique. All parameters mentioned in Table 6.1 with $K_1 = 30$, $\Phi = 2.5$, $\tau = 0.5$ have been varied two times up and down. It shows that K_1, Φ, K_2, and τ are the most sensitive parameters with p value less than 0.05. The stiffness parameter α_1 and the proportionality constant α_2 were kept outside this study as they should be small enough and cannot be enhanced two times.

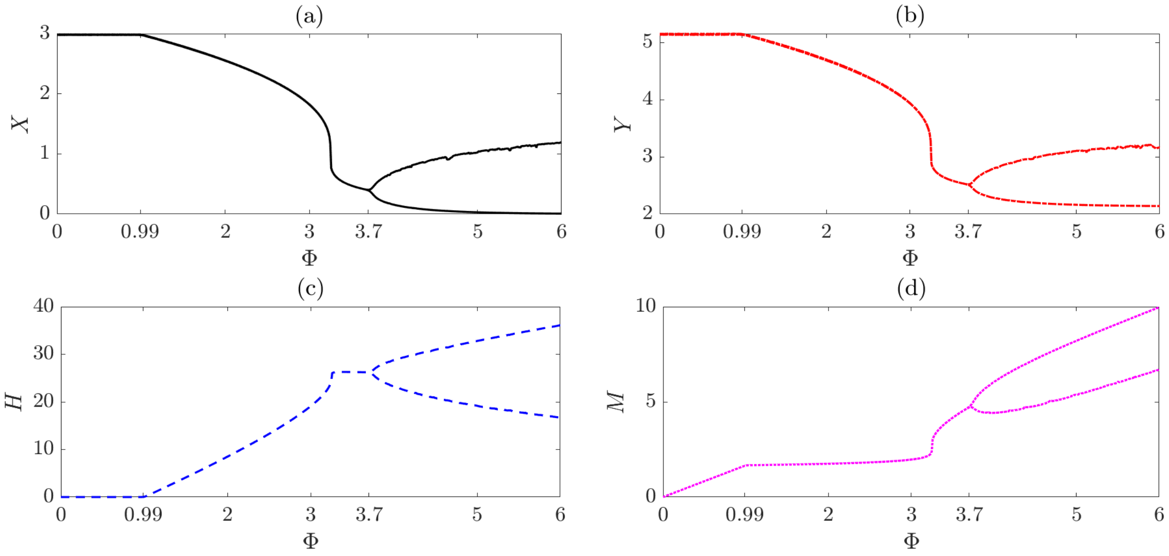

Figure 6.4: Bifurcation diagram of the state variables when the demand parameter Φ varies in the ranges $0 < \Phi < 6$. The fishing effort-free equilibrium E_7 is stable in $0 < \Phi < 0.99$. The coexisting equilibrium E^* is stable in the $0.99 < \Phi < 3.70$ and unstable for $\Phi > 3.70$. Other parameters remain the same as in Fig. 6.1a.

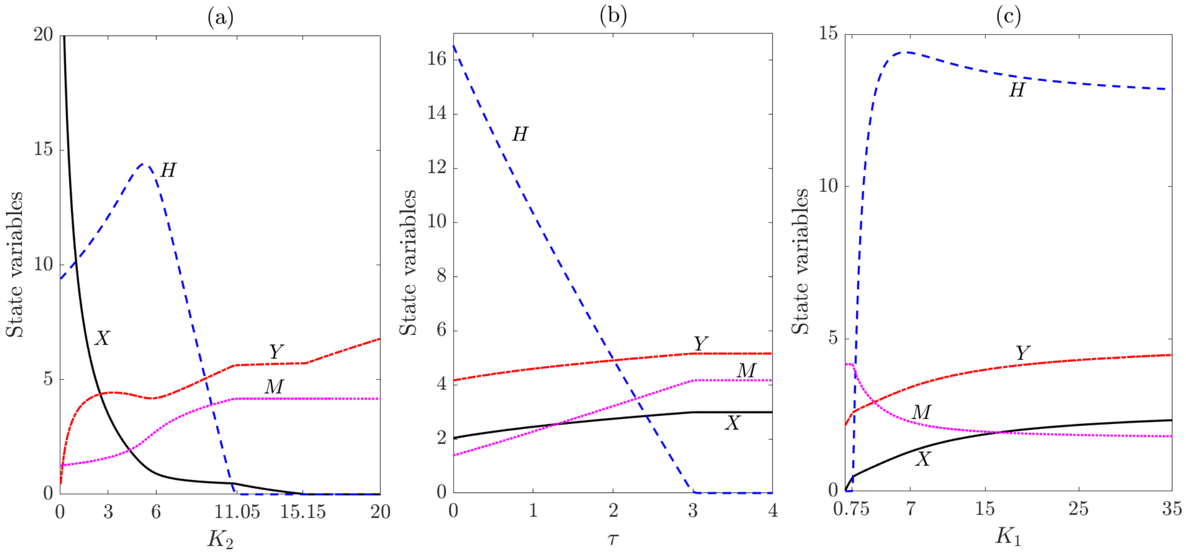

Figure 6.5: Bifurcation diagrams with respect to (a) the environmental carrying capacity of the predatory fish population, K_2, (b) fishing tax, τ, and (c) the environmental carrying capacity of prey fish population, K_1. Other parameters are as in Table 6.1 with $\Phi = 2.5,\ \tau = 0.5,\ K_1 = 30$ for Fig. (a); $\Phi = 2.5,\ K_1 = 30,\ K_2 = 4$ for Fig. (b); and $\Phi = 2.5,\ \tau = 0.5,\ K_2 = 4$ for Fig. (c).

Fig. 6.5b), and therefore, the upper bound of τ is set to $\tau^{\max} = 3$. Three different dynamical states arise when the parameter K_2 varies. The coexisting state E^* is stable at the lower values of K_2 ($0 < K_2 < 11.05$). It is noticeable that harvesting effort initially increases but then diminishes to zero as K_2 is high. The growing population of predatory fish sends the prey fish to extinction, and the harvesting stops much before the fish become extinct. The increasing trend of H is observed for the variation of K_1, and its higher value supports populations to coexist in a stable state. Interestingly, no *paradox of enrichment* is observed here due to the enrichment of the system by increasing the carrying capacities.

6.4.2 Two parameter bifurcation

A broader dynamic behavior of the system is unveiled through the two-parameter bifurcation analysis, presented in Fig. 6.6. Its upper row shows that the parametric plane $\tau - K_1$ is differentiated by the stability regions of the equilibrium points E^* and E_7, whereas the parametric planes $K_2 - K_1$ and $\tau - K_2$ are delineated by the stability domains of E^*, E_7 and E_3. No oscillatory existence of the interior equilibrium is observed in these cases. The lower row of this figure indicates that there are two stability regions and one oscillatory coexistence region in the $\tau - \phi$ and $K_1 - \phi$ planes. The dynamics are richer in the $K_2 - \phi$ plane, where three stable regimes and one oscillatory regime are noticed. In each parametric plane, the coexisting equilibrium E^* contains the largest stability domain, and whenever this equilibrium oscillates, the corresponding domain of attraction occupies the least space. The stable coexisting state E^* and the harvesting-free equilibrium E_7 are the only states in every bifurcation diagram. Observe

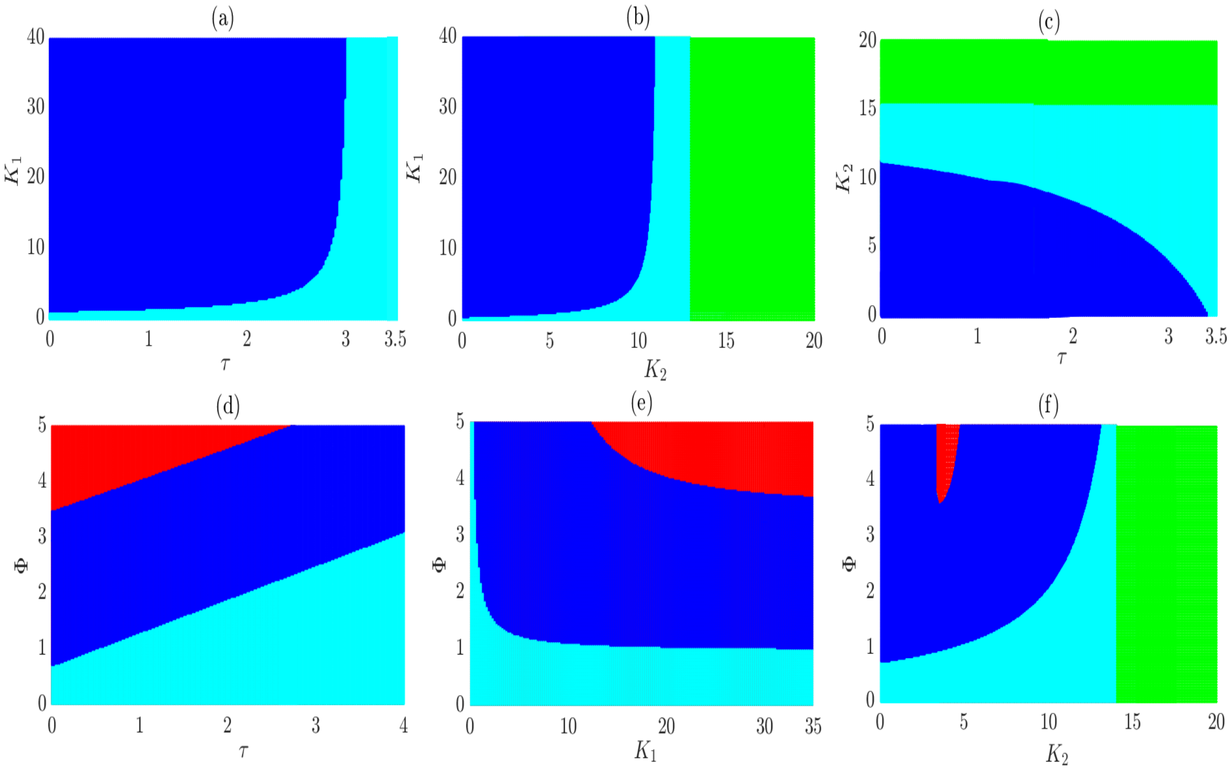

Figure 6.6: Stability and instability regimes of different equilibrium points of the system (6.3) due to the variation of a pair of parameters. These figures show that there may exist two, three, and four isolated dynamic regimes for the simultaneous variations of two parameters. The blue, cyan, and green colors indicate, respectively, the stable regions for the coexisting equilibrium (E^*), the harvesting-free equilibrium (E_7), and the predatory fish and price equilibrium (E_3). The oscillatory (unstable) state of the steady state E^* has been shown in red color. Here (a) $\Phi = 2.5, K_2 = 4$, (b) $\Phi = 2.5, \tau = 0.5$, (c) $\Phi = 2.5, K_1 = 30$, (d) $K_1 = 30, K_2 = 4$, (e) $\tau = 0.5, K_2 = 4$ and (f) $K_1 = 4, \tau = 0.5$. Other parameters are as in Table 6.1

that the equilibrium E_3 appears in the parametric plane only when there is a variation in the parameter K_2. It follows that if predator species carrying capacity is not allowed to increase significantly, then the equilibrium point $E_3(0, Y, 0, \frac{\phi}{r})$ will not be stable. On the other hand, the oscillatory coexistence state never appears if the demand is not high.

For the numerical computation of the optimal tax, we consider the parameter values given in Table 6.1 with $K_1 = 30$, $\Phi = 2.5$, $K_2 = 4$, $\iota = 0.7$ and solve the equation (6.24) for τ. Figure 6.7a shows that the optimal tax value evaluated at the coexisting equilibrium point E^* is $\tau^* = 0.83$ M\$/MT. The corresponding societal revenue at this optimal equilibrium $E^*(\tau^*)$, computed from (6.20), is $\Omega(\tau^*) = 4.167$ M\$/year, which is maximum (see Fig. 6.7b, top magenta colour curve). If one wants to find the optimal value of τ that will maximize the incomes from fishing tax or ecotourism, then $\Omega(X, Y, H, M)$ of Eq. (6.21) has to be replaced by $\Omega_3(X, Y, H, M)$ or $\Omega_2(X, Y, H, M)$, respectively, and then perform the subsequent analysis to obtain an equation similar to Eq. (6.24). Figure 6.7b indicates that the income from fishing tax, Ω_3, is maximum (1.29 M\$/year) for the optimal tax value $\tau^* = 1.53$ M\$/MT. The income from ecotourism (Ω_2) shows a little change due to the variation in fishing tax because its change

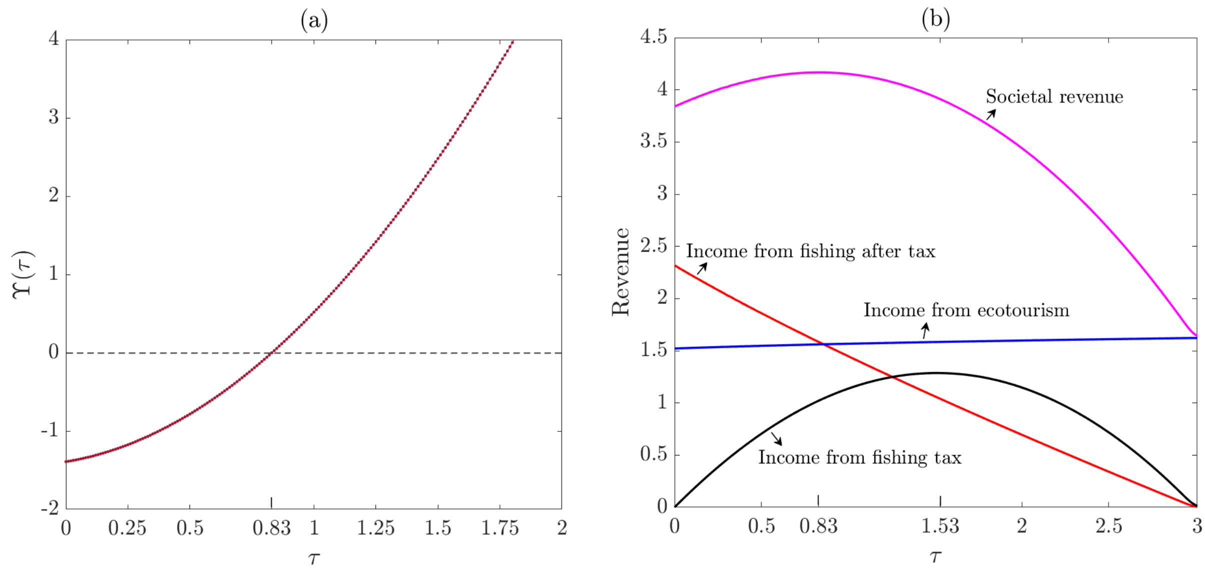

Figure 6.7: (a) Plot of $\Upsilon(\tau)$ under the variation of the parameter τ. It shows that the Eq. (6.24) has only one solution, i.e., there is a single optimal value of τ, viz., $\tau^* = 0.83$, for which $\Upsilon(\tau) = 0$. (b) Magenta, red, blue and black lines, respectively, indicate the societal revenue, income from fishing after tax, earning from ecotourism and income from fishing tax. The optimal societal revenue ($\Omega(\tau^*) = 4.167$ M\$/year) is attained at $\tau^* = 0.83$, M\$/MT and the optimal revenue earned from the imposed tax ($\Omega_3 = 1.29$ M\$/year) is obtained at the optimal tax $\tau^*_{\Theta_3} = 1.53$ M\$/MT. The parameters are as in Table 6.1 with $K_1 = 30$, $\Phi = 2.5$, $K_2 = 4$ and the annual discount rate $\iota = 0.7$.

occurs through the change in the equilibrium level of Y^* only and does not depend directly on τ. It is worth mentioning that the income from fishing after tax (Ω_1) is a decreasing function of tax, and it is maximum when $\tau = 0$. Whereas, the earning from tourism after tax (Ω_2) is an increasing function of τ, and attains its optimal value at its highest value. One can observe that each income component attains its maximum value at different tax levels. Interestingly, the maximum societal revenue is earned at the lowest non-zero optimal fishing tax $\tau = 0.83$ M\$/MT. Further comparison in income components can be made at different values of τ (see Table 6.2). It shows that the societal revenue is maximum ($\Omega = 4.167$ M\$/year) when $\tau = 0.83$ M\$/MT. The income from fishing after tax ($\Omega_1 = 2.32$ M\$/year) is maximum at $\tau = 0$, and the earning from fishing tax ($\Omega_3 = 1.29$ M\$/year) is maximum at $\tau = 1.53$ M\$/MT.

6.5 Discussion

One of the goals of the blue economy is to protect marine species for sustainable use and improve the local people's livelihood and economy while preserving the marine ecosystem's health or blue resources. A scientific integration of fishery and ecotourism may be one way to reduce pressure on fish and fisheries as well as improve the economic and social development of coastal people on a long-term basis. Intermingling a market-based fishing strategy in

Table 6.2: This table reports the societal revenue (Ω), income from fishing after tax (Ω_1), income from ecotourism (Ω_2), tax from fishing (Ω_3) at some fixed values of fishing tax with an annual discount rate $\iota = 0.7$. The optimum tax and the corresponding optimal revenue are marked in boldface. The parameters are as in Fig. 6.7.

τ (M$/MT)	X^* (MT)	Y^* (MT)	H^* (SFU)	M^* (M$/MT)	Ω_1 (M$/year)	Ω_2 (M$/year)	Ω_3 (M$/year)	$\Omega = \Omega_1 + \Omega_2 + \Omega_3$ (M$/year)
0	2.03	4.16	16.54	1.39	**2.32**	1.522	0	3.84
0.25	2.16	4.29	14.85	1.60	2.08	1.535	0.39	4
0.5	2.26	4.40	13.27	1.82	1.86	1.547	0.70	4.11
0.75	2.36	4.50	11.77	2.04	1.65	1.558	0.96	4.15
0.83	2.39	4.53	11.31	2.12	1.58	1.561	1.02	**4.167**
1.25	2.53	4.67	8.94	2.51	1.25	1.576	1.25	4.07
1.53	2.61	4.76	7.42	2.77	1.04	1.584	**1.29**	3.91
1.75	2.68	4.83	6.26	2.97	0.88	1.591	1.25	3.72
2	2.74	4.90	4.96	3.21	0.69	1.598	1.15	3.44
2.25	2.81	4.97	3.69	3.45	0.52	1.605	0.97	3.09
2.75	2.93	5.09	1.20	3.93	0.17	1.617	0.39	2.18
3	2.98	5.15	0.06	4.15	0.001	**1.622**	0.02	1.64

a multi-species fishery with a fair taxation policy may provide long-term sustainable growth. Fishery-based ecotourism is another potential component that may contribute significantly to the financial improvement of the local people by preserving the aquatic ecosystem. An entry tax may be levied on tourists to develop the ecotourism locality. Here we have proposed and analyzed a harvesting model that integrates the ecological interaction of a predator and prey fish with ecotourism and the open market economy theory. In the ecological interaction, the prey fish is harvested commercially, whose market price is determined by the demand-supply relationship. The harvesting effort employed on the prey fish is proportional to the profit margin. The generalist predatory fish (dolphin, shark, whale) is restricted from commercial harvesting but used for recreation purposes for the visitors as a part of the ecotourism. The mentioned social-ecological-economic interaction may match various commercial and fishery-based ecotourism sites, including the Chilika lagoon.

The regulatory authority imposes a fishing tax on landed fish as a controlling measure to restrict overfishing. Tax levied on per unit of landed fish sometimes may negatively affect the fishery. If the charged tax is high, fishers may avoid fishing and go for other occupations. This may have serious socioeconomic consequences, particularly for the people of underdeveloped and developing countries, where fewer options exist for alternative livelihood. Another apparent negative effect of the increasing fishing tax is an imbalance between supply and demand in this global food item, which may cause a hike in the market price as a cascading effect of the nonavailability of fish even when there is high demand. Of course, the harvested species will get relief from overexploitation if there is no harvesting. It is worth mentioning that such a situation is not desirable for a fishery. On the contrary, the fishers' income will be maximum when there is no fishing tax but will steadily decline with increasing tax. However, none of the above cases is worthwhile for sustainable development. A balance between harvesting and fishing tax may be a more suitable solution. Therefore, defining an optimal fishing tax is imperative to benefit fishers, fish species, and regulatory authorities for a win-win situation.

Our model analysis revealed that the system has nine equilibrium points. Out of which, six equilibrium points are always unstable, and the rest three are conditionally stable. The interior equilibrium point is ecologically and economically more important. We have proven the global stability of this equilibrium using the high-dimension Bendixson criterion. To illustrate the dynamic features of the system, we selected four important system parameters based on their ecological & economic importance and PRCC analysis. One-and-two-parameter bifurcation results of these parameters revealed that the system might switch among the equilibrium points E^*, E_7, E_3 due to the variation of the identified parameters. Oscillations in the interior equilibrium E^* may occur if the bifurcation involves the demand parameter, Φ. Even the two carrying capacities (K_1, K_2) cannot produce sustained oscillations unless the fish demand is significantly high. Thus, there is a mere chance of occurring the paradox of enrichment in this integrated system due to enrichment.

The societal revenue is an ensemble income, including the fishers' income from fish selling after paying the tax, income from the entrance fee of ecotourism, and the fishing tax. Each individual earning may be optimal for different optimal taxes, and the optimal tax that maximizes the societal revenue may also be unlike the others. So, we searched for an optimum tax level to maximize the overall earnings for maximal social benefit. Our analysis revealed that the optimal tax corresponding to the maximum societal revenue is much lower than in the other two cases. Effective management for fishery-based ecotourism is more crucial because fish & fishery industries overlap the tourism industry. One cannot overlook the influence of one over the other. This chapter shows that a justified taxation policy may significantly improve the overall health of fish & fishery and may contribute to the blue economy, as mentioned in the different sustainable development goals (SDG-3, SDG-8, and SDG-14-17). However, there are further avenues for the extension of this chapter. For example, we have here optimized the fishing tax only. One can optimize the fishing tax and tourist entrance fee so that the societal benefit can be further maximized. Secondly, harvesting in a marine environment is a stochastic process; therefore, one can go for the stochastic optimization of various incomes for more realistic outcomes.

7

Conclusions and future work

World fisheries have increased tremendously in the last fifty years due to the high demand for fishery products, the use of sophisticated fishing gear & vessel technology, and growing trade [92]. Millions of people live on fishing, and many fisheries are stressed due to intense harvesting and emerging diseases. A proper management policy with a scientific understanding of species interaction is a footstep in a long-term sustainable fishery. In this thesis, we present a theoretical study that clubs species interaction in the presence and absence of infection and the demand-supply theory based on open market policy.

In conclusion, this thesis has addressed several critical issues that have not been extensively explored in previous bioeconomic modelling. Listed below are some key findings associated with each chapter of this thesis.

- Chapter 2 reveals a two-dimensional bioeconomic model with a variable market price, which depends on supply and demand. Notably, this chapter identified previously overlooked aspects in bioeconomic modelling and addressed several vital socio-ecological-economic issues. Furthermore, the analysis considered various demand-supply interactions, observing that certain equilibria rely on market demand to exist. For instance, the fish-free equilibrium only occurs when demand functions follow polynomial or logarithmic types. We also identified several demand-supply relationships that play a role in catastrophic regime shifts, while the occurrence of an economic trade-off is solely dependent on the demand function. This study further reveals that for the traditional CPUE harvesting function, the MEY can indeed be achieved with the least amount of imposed fishing effort whenever an economic trade-off exists. This finding underscores the significance of understanding the complexities of demand-supply interactions in achieving sustainable fishery management practices and optimizing economic outcomes while maintaining fishery health.

- Chapter 3 delves into the critical investigation of the environmental carrying capacity (K) required to sustain harvesting. This chapter explores the potential for harvesting the maximum number of fish and generating maximum revenue across different enrichment levels. The one-parameter bifurcation analysis uncovers that harvesting becomes feasible only when the environmental carrying capacity surpasses a specific threshold, allowing for a higher fish population. Moreover, it is observed that infection diminishes once harvesting is initiated, leading to an infection-free system over an extended range of environmental carrying capacities. The ecological parameter exhibits multiple equilibrium point switches as it transitions from lower to higher values. Additionally, the equilibrium harvest demonstrates an upward trend with increased carrying capacity. Notably, the harvested stock consistently exceeds the infection-free equilibrium at lower values of K while remaining lower than the interior equilibrium at higher values of K.

- Chapter 4 uncovers significant implications related to fish disease and taxation in the system dynamics, with potential impacts on revenue generation. While taxation can help control overfishing, it may inadvertently contribute to disease spread and price increases if set at high levels. A critical finding is the potential for a regime shift, wherein an excessively high regulatory tax leads to a transition from a harvesting regime to a non-harvesting one. Moreover, we identify existing trade-offs between revenue generation and regulatory tax. The overall societal revenue reaches its peak at the optimal tax level. However, it is crucial to note that individual earnings in these aspects vary across different tax levels. This underscores the complex interplay between taxation policies, revenue streams, and overall system dynamics, highlighting the importance of carefully calibrating regulatory measures to achieve sustainable outcomes.

- Chapter 5 presents intriguing findings from various socio-economic viewpoints. It reveals that by appropriately selecting a level of taxation, the fishery can be stabilized, and infection can be eliminated, providing valuable insights into addressing the complex challenges of fisheries and their sustainability amidst economic dynamics. However, an excessively high fishing tax could negatively impact the fishery, reducing fishers' income and potential opt-outs from the fishery, especially in low-income countries. To avoid such consequences, the regulatory authority should levy taxes more judiciously, and optimal tax calculations can aid in informed decision-making. This chapter further unveils that there is no trade-off between fishing tax and revenue generation, contrary to previous assumptions. Tax and societal revenues increase with rising taxes, while fisher's income decreases. Notably, the saturated type of demand plays a crucial role in negating the trade-off. Additionally, the chapter investigates the contribution of the open market philosophy to regime shifts in the fishing industry, revealing a novel reason for such shifts, i.e., high demand for fish can hasten regime shifts when considering the interplay between aquatic ecology and market dynamics.

- Chapter 6 explores the integration of commercially harvested fishery and recreational fishery to alleviate fisheries pressure and promote economic and social development in coastal communities. The analysis reveals that the optimal tax for maximum societal revenue is considerably lower than the individual revenues of fishers, tourism organizations, and tax authorities. Effective management of fishery-based ecotourism is crucial due to the overlap of fish & fishery industries with tourism. This thesis emphasizes that a well-justified taxation policy can significantly enhance the overall health of fish & fishery and contribute to the blue economy, aligning with sustainable development goals (SDG-3, SDG-8, and SDG-14-17).

These findings highlight the importance of considering various ecological, economic, and social factors when modelling and managing fisheries. By addressing these issues, the thesis contributes a more comprehensive understanding of fishery dynamics and provides valuable insights for sustainable fishery management and conservation efforts.

Despite the many exciting results in this thesis, a few limitations will require further exploration, which are some areas of future research. A noteworthy limitation in each chapter of this thesis is the deterministic nature of the model systems, where all parameters are assumed to be constant. However, this does not accurately represent natural environments, as birth, death, growth, infection, and tax fluctuate around average values. To address this crucial aspect, it is essential to incorporate environmental stochasticity into the model systems. By studying the corresponding stochastic models for each considered model, we can capture natural systems' inherent randomness and variability. The extension to stochastic modelling holds significant promise in providing a more realistic representation of the complexities and uncertainties present in marine environments. Accounting for environmental fluctuations and stochastic processes will lead to more robust and reliable predictions, allowing us to understand better the resilience and dynamics of fish populations, ecosystems, socio-economic interactions, and sustainable fishery management strategies.

In Chapters 2, 4, 5, and 6, our focus was on optimizing a single parameter, either the fishing effort or the fishing tax. However, a promising avenue for further exploration involves optimizing coupled parameters, such as the fishing tax and tourist entrance fee, to maximize societal benefits further. Considering the interplay between these variables, we can unlock additional opportunities to enhance overall outcomes.

This thesis has focused on the non-selective harvesting of certain commercial species. However, in the future, we aim to delve into sustainable fishery management, focusing on age-specific harvesting, driven by two critical reasons. Firstly, we have witnessed numerous species extinction instances resulting from excessive juvenile fish harvesting. Addressing this issue is crucial for ensuring fish populations' long-term health and viability. Secondly, harvesting fish stock beyond a certain age allows species to reach their maximum body size, leading to optimal economic returns from the harvested stock. With these objectives in mind, each chapter of this thesis can be expanded to explore the intricacies of age-specific harvesting and its implications on sustainable fishery management. By understanding the age-specific dynamics and incorporating age-related factors into our analysis, we can develop more comprehensive and practical strategies to ensure the preservation and sustainable use of fish resources for the benefit of both ecosystems and coastal communities. Despite such limitations, this thesis may give valuable insights into managing socio-ecological-economic interactions towards sustainability.

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List of publications related to thesis

Published

1. Nandadulal Bairagi, **Santanu Bhattacharya**, Pierre Auger, and Biswajit Sarkar. "Bioeconomics fishery model in presence of infection: Sustainability and demand-price perspectives", *Applied Mathematics and Computation*, 2021, doi.org/10.1016/j.amc.2021.126225 (Elsevier Journal, Q1, Impact Factor 4.397).
2. Nandadulal Bairagi, **Santanu Bhattacharya**, and Biswajit Sarkar., "Demand-induced regime shift in fishery: A mathematical perspective", *Mathematical Biosciences*, 2023, doi.org/10.1016/j.mbs.2023.109008 (Elsevier Journal, Q1, Impact Factor 3.935).
3. Biswajit Sarkar, **Santanu Bhattacharya**, and Nandadulal Bairagi, "An ecological-economic fishery model: Maximizing the societal benefit through an integrated approach of fishing and ecotourism", *Mathematical Methods in the Applied Sciences*, 2023, doi.org/10.1002/mma.9356 (Wiley Journal, Q1, Impact Factor 3.007).
4. Biswajit Sarkar, **Santanu Bhattacharya**, Nandadulal Bairagi. "Dynamic behaviour of a single-species nonlinear fishery model with infection: The role of fishing tax and time-dependent market price", *Journal of Nonlinrar Science and Applications*, 2023"

Communicated

1. **Santanu Bhattacharya**, Biswajit Sarkar, Nandadulal Bairagi. "Dynamic consequence of an ecological-economic model with different harvesting strategies and demand functions"