Study of some almost complex and complex manifolds under certain conditions

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that thesis entitled "**Study of some almost complex and complex manifolds under certain conditions**" submitted by Mr. **Samser Alam** who got his name registered on 16th February, 2018 (Index no.: 60/18/Maths./25) for the award of Doctor of Philosophy (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Arindam Bhattacharyya** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated to
my mother
Salaher Begam
and
my beloved daughter
Sumedha Alam
for their patience, support and love.

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Preface

This doctoral thesis is devoted to study of some almost complex and complex manifolds under certain conditions. The thesis is divided into **six chapters**. The **first chapter** deals with some prerequisites that, in our opinion, are crucial for comprehending the core of the entire thesis.

In the **second chapter**, we investigate weakly symmetric Kähler manifolds that exhibit properties of being pseudo-projectively flat and quasi- conformally flat. Moreover, we examine weakly pseudo-projectively symmetric and quasi-conformally symmetric Kähler manifolds, which are further characterised as Einstein manifolds. Additionally, we establish the existence of pseudo-projectively flat weakly symmetric Kähler manifolds and quasi- conformally flat weakly symmetric Kähler manifolds, for which the Ricci tensor satisfies a certain relation.

In the **third chapter**, we conduct a comprehensive study and derive precise expressions for several curvature identities pertaining to a nearly Kähler manifold exhibiting con-circular and projective flatness. Furthermore, we attain intriguing findings concerning a 6-dimensional nearly Kähler manifold, and we present a detailed example to illustrate these results.

In **fourth chapter**, we examine different types of curvature identities present in Kähler-Norden manifolds, such as quasi-conformal flatness, pseudo-projective flatness, Weyl-conformal flatness, and Bochner flatness. We show that a Kähler-Norden manifold exhibits pseudo-projective symmetric if and only if it demonstrates local symmetric. Moreover, we explore semi-symmetric Kähler-Norden manifolds and prove that a Kähler-Norden manifold is semi-symmetric if and only if it possesses locally semi-symmetic.

In the **fifth chapter**, we study some curvature identities on a locally symmetric hyperKähler manifold. Next, we explore the concepts of conformal flatness and Bochner flatness of a hyperKähler manifold. Additionally, we prove that if M is a conformally flat hyperKähler manifold, then M is locally flat when the dimension of M is greater than 4 and M is locally symmetric with vanishing scalar curvature if the dimension of M is 4. Furthermore, we establish that a conformally flat and Bochner flat hyperKähler manifold of dimension $4n$ is an Einstein manifold. Later, we introduce the generalised W_2-curvature tensor and the study quasi-W_2-curvature tensor, and using these we develop that a generalised W_2-flat hyperKähler manifold and quasi-W_2-flat hyperKähler manifold are Ricci flat. Finally, we present some examples of hyperKähler manifolds.

In the **sixth chapter**, we carry out an investigation on various curvature properties in paraKähler manifolds that possess characteristics such as pseudo-quasi-conformal flatness, pseudo-projective flatness, W_2-flatness, and Bochner flatness. Moreover, we explore significant findings concerning the sectional curvature within the paraKähler manifolds. Additionally, we analyz the behavior of paraKähler space-time in the presence of a perfect fluid. Furthermore, we examine the behavior of weakly symmetric and weakly Ricci symmetric perfect fluids in the context of paraKähler space-time. Our study also encompasses the study of curvature identities in paraKähler space-time, specifically focusing on flatness properties related to the previously mentioned curvature tensors. Finally, we expand upon crucial properties associated with sectional curvature in paraKähler space-time.

The thesis contains the subject matter of the following papers whose titles, journal information and chapterwise distribution are given below:

Authors	Title of paper and journal information	Chapter
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Samser Alam and Arindam Bhattacharyya	"Properties of some curvature tensors on paraKähler space-time", Accepted, **Southeast Asian Bulletin of Mathematics**.	6

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1

Introduction

1.1 Introduction to almost complex manifold

Assume that M is a smooth manifold. An almost complex structure F on M is a linear complex structure or a linear map that squares to -1 on each tangent space of the manifold and varies smoothly on the manifold. In other words, we have a smooth tensor field F of degree $(1, 1)$ that is isomorphic to the tangent bundle when considered as a vector bundle. An almost complex manifold refers to a manifold that possesses an inherent structure known as an almost complex structure.

So, an almost complex structure refers to a smooth surface that exhibits a smooth linear complex structure on each of its tangent spaces. While it is true that every intricate manifold can be considered as an almost complex manifold, it should be noted that not all almost complex manifolds can be classified as complex manifolds. Applications in the symplectic geometry of almost complex structures are significant.

M must be even-dimensional if it ensures an almost complex structure. This can be understood as follows: Let us assume M is n-dimensional and $F\colon TM \to TM$ is an almost complex structure, where TM denotes the tangent bundle of the manifold. If $F^2 = -1$ then $(\det F)^2$ must also be $(-1)^n$. However, if M is a real manifold, then $\det F$ is a real number, and as a result, n must be even if M has an almost complex structure. Any even-dimensional vector space permits a linearly complex structure, as demonstrated by a simple exercise in linear algebra. As a result, a $(1, 1)$-rank tensor pointwise which is essentially a linear transformation on each tangent space is always admissible on an even-dimensional manifold such that $F_p^2 = -1$ at each point p. The pointwise linear complex structure only generates an almost complex structure, which is subsequently confirmed to be unique, when this local tensor can be connected to be described globally. It is analogous to changing the structure group of the tangent bundle from $GL(2n, \mathbb{R})$ to $GL(n, \mathbb{C})$ for this patching to be possible and consequently, for an almost complex structure to exist on the manifold M. Therefore, the existence query is exclusively algebraic-topological and is most well known.

Almost complex structures have important applications in symplectic geometry. The concept is due to Charles Ehresmann and Heinz Hopf [70] in the 1940.

Example 1.1.1. *Let M be an n-dimensional manifold. A structure on M given by a non-null tensor field f that satisfies $f^3 + f = 0$, is called an f-structure. If the rank of f (denoted as r) is a constant, i.e., $r = n$, then the f-structure provides an almost complex structure for the manifold M. In this case, n is even.*

Nijenhuis Tensor

Definition 1.1.1. *[18] (Nijenhuis Tensor) Let F be an almost complex structure in an almost complex manifolds M^n, where n is an even integer. Nijenhuis tensor in terms of F is a vector valued bilinear function N defined by $N_F(X_1, Y_1) = [FX_1, FY_1] - F[FX_1, Y_1] - F[X_1, FY_1] - [X_1, Y_1]$, where for X_1, $Y_1 \in \chi(M)$ and $[\ ,\ ]$ stand for Lie bracket.*

Theorem 1.1.1. *[18] In an even-dimensional almost complex manifold M^n*

$$(i)\ N(X_1, FY_1) = N(FX_1, Y_1) = -F(N(X_1, Y_1)) = F(N(FX_1, FY_1)),$$
$$(ii)\ N(FX_1, FY_1) = -N(X_1, Y_1) = -F(N(FX_1, Y_1)) = -F(N(X_1, FY_1)),$$

where $X_1, Y_1 \in \chi(M)$.

1.2 Introduction to complex manifold

An atlas of charts to the open unit disc and holomorphic transition maps are the characteristics of a complex manifold in differential geometry and complex geometry. Alternative definitions of the term "complex manifold" include an almost complex manifold and the complex manifold described above which can be expressed as an integrable complex manifold. The theories of smooth and complex manifolds have quite different flavors because holomorphic functions are substantially more rigid than smooth functions, compact complex manifolds are much more similar to algebraic varieties than to differentiable manifolds.

Definition 1.2.1. *[18] (Complex manifolds) A manifold that is almost complex and has a vanishing Nijenhuis tensor is referred to as a complex manifold.*

Example 1.2.1. *Consider $\mathbb{C}^n$ as the complex vector space that contains a collection of complex number sets, each consisting of n elements, with the notation $z = (z^1, z^2, ..., z^n)$. If we put $z^t = x^t + iy^t$, $x^t, y^t \in \mathbb{R}$, $t = 1, 2, ..., n$, then $\mathbb{C}^n$ can be associated with the real vector space $\mathbb{R}^{2n}$ containing $2n$-tuples of real numbers $(x^1, x^2, ..., x^n, y^1, y^2, ..., y^n)$. The identification of $\mathbb{C}^n$ with $\mathbb{R}^n$ will always be done through the correspondence $(z^1, z^2, ..., z^n) \to (x^1, x^2, ..., x^n, y^1, y^2, ..., y^n)$. The complex structure of $\mathbb{R}^{2n}$ induced from that of $\mathbb{C}^n$ maps $(x^1, x^2, ..., x^n, y^1, y^2, ..., y^n)$ into $(y^1, y^2, ..., y^n, -x^1, -x^2, ..., -x^n)$ and is known as the canonical complex structure of $\mathbb{R}^{2n}$. According to the natural basis of $\mathbb{R}^{2n}$, it is given by the matrix $F_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then $F_0^2 = -I$. As $N(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}) = 0$, this is an example of a complex manifold.*

1.2.1 Weakly symmetric manifold

In 1992, L. Tamassy and T. Q. Binh [69] introduced the concepts of the weakly symmetric and the weakly Ricci-symmetric manifolds. Aside from that, M. Prvanovic [52], U. C. De, and S. Bandyopdhayay [15] provided instances to clarify their points. Intriguing findings about the weakly symmetric and the weakly Ricci-symmetric Kähler manifolds were also discovered by L. Tamassy, U. C. De, and T. Q. Binh [68].

Definition 1.2.2. *[69] (Weakly symmetric manifold) A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition*

$$\begin{aligned}(\nabla_{X_1} R)(Y_1, Z_1, U_1, V_1) &= A(X_1)R(Y_1, Z_1, U_1, V_1) + B(Y_1)R(X_1, Z_1, U_1, V_1) \\ &\quad + C(Z_1)R(Y_1, X_1, U_1, V_1) + D(U_1)R(Y_1, Z_1, X_1, V_1) \\ &\quad + E(V_1)R(Y_1, Z_1, U_1, X_1), \qquad (1.2.1)\end{aligned}$$

and the manifold is called weakly Ricci symmetric if the Ricci tensor S *satisfies*

$$(\nabla_{X_1}S)(Y_1,Z_1)=A(X_1)S(Y_1,Z_1)+B(Y_1)S(X_1,Z_1)+C(Z_1)S(Y_1,X_1), \tag{1.2.2}$$

where A,B,C,D,E *are simultaneously non-vanishing* 1*-forms and* X_1,Y_1,Z_1,U_1,V_1 *are vector fields and* ∇ *be the covariant differentiation operator associated with the Riemannian metric* g.

The 1- forms are referred to as the associated 1-forms of the manifold, and an n-dimensional manifold with such properties is represented as $(WS)_n$.
M. Prvanovic [52] and P. Pandey [50] demonstrated that in a weakly symmetric manifold $B=C=D=E$ holds. If we consider $B=C=D=E=\omega$ (say) and then (1.2.1) and (1.2.2) becomes

$$\begin{aligned}(\nabla_{X_1}R)(Y_1,Z_1,U_1,V_1) &= A(X_1)R(Y_1,Z_1,U_1,V_1)+\omega(Y_1)R(X_1,Z_1,U_1,V_1)\\ &\quad +\omega(Z_1)R(Y_1,X_1,U_1,V_1)++\omega(U_1)R(Y_1,Z_1,X_1,V_1)\\ &\quad +\omega(V_1)R(Y_1,Z_1,U_1,X_1),\end{aligned} \tag{1.2.3}$$

and

$$(\nabla_{X_1}S)(Y_1,Z_1)=A(X_1)S(Y_1,Z_1)+\omega(Y_1)S(X_1,Z_1)+\omega(Z_1)S(Y_1,X_1), \tag{1.2.4}$$

where $g(X_1,\rho)=\omega(X_1)$ and $g(X_1,\alpha)=A(X_1)$. where ρ and α are vector fields.
In 2002, Prasad [53] defined and studied a tensor field $\overline{P}$ within the framework of a manifold possessing Riemannian geometry, with the dimension represented as $(n>2)$. This tensor field encompasses the projective curvature tensor $\overline{P}$, thereby exploring its properties and characteristics.

Definition 1.2.3. *[53] (Pseudo-projective curvature tensor) On a Riemannian manifold of dimension greater than* 2*, the pseudo-projective curvature tensor* $\overline{P}$ *is expressed by*

$$\begin{aligned}\overline{P}(X_1,Y_1)Z_1 &= aR(X_1,Y_1)Z_1+b[S(Y_1,Z_1)X_1-S(X_1,Z_1)Y_1]\\ &\quad -\frac{r}{n}\left[\frac{a}{n-1}+b\right][g(Y_1,Z_1)X_1-g(X_1,Z_1)Y_1],\end{aligned} \tag{1.2.5}$$

where a *and* b *are constants, that are non-zero. In this expression,* R *denotes the curvature tensor,* S *represents the Ricci tensor, and* r *stands the scalar curvature.*

A Riemannian manifold (M^n, g) with non-pseudo projective flatness, where the dimension n is greater than 2, is classified as weakly pseudo projectively symmetric manifold if certain conditions are satisfied. These conditions involve the pseudo-projective curvature tensor $\overline{P}$ of type $(0, 4)$ and can be expressed as follows:
When acting on vector fields X_1, Y_1, Z_1, U_1, V_1, the covariant derivative $(\nabla_{X_1}\overline{P})$ of the pseudo-projective curvature tensor satisfies the equation

$$\begin{aligned}(\nabla_{X_1}\overline{P})(Y_1, Z_1, U_1, V_1) &= A(X_1)\overline{P}(Y_1, Z_1, U_1, V_1) + B(Y_1)\overline{P}(X_1, Z_1, U_1, V_1) \\ &\quad +C(Z_1)\overline{P}(Y_1, X_1, U_1, V_1) + D(U_1)\overline{P}(Y_1, Z_1, X_1, V_1) \\ &\quad +E(V_1)\overline{P}(Y_1, Z_1, U_1, X_1), \qquad (1.2.6)\end{aligned}$$

where, A, B, C, D, E are 1-forms that do not vanish. This characterization is denoted as $(WPPS)_n$ for an n-dimensional manifold.
Yano and Sawaki [77] introduced the concept of the quasi-conformal curvature tensor, which can be described in the following manner:

Definition 1.2.4. *[77] (Quasi-conformal curvature tensor) The quasi-conformal curvature tensor is represented by this tensor field $\overline{C}$, which is given by*

$$\begin{aligned}\overline{C}(X_1, Y_1)Z_1 &= aR(X_1, Y_1)Z_1 + b[S(Y_1, Z_1)X_1 - S(X_1, Z_1)Y_1 \\ &\quad +g(Y_1, Z_1)QX_1 - g(X_1, Z_1)QY_1] \\ &\quad -\frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1], \qquad (1.2.7)\end{aligned}$$

where a and b are non-zero constants. If $a = 1$ and $b = -\frac{1}{n-2}$, then quasi-conformal curvature tensor is reduced to the Weyl-conformal curvature tensor [51], whose expression is given by:

$$\begin{aligned}W(X_1, Y_1)Z_1 &= R(X_1, Y_1)Z_1 - \frac{1}{n-2}[g(Y_1, Z_1)QX_1 - g(X_1, Z_1)QY_1 \\ &\quad +S(Y_1, Z_1)X_1 - S(X_1, Z_1)Y_1] \\ &\quad +\frac{r}{(n-1)(n-2)}[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1]. \qquad (1.2.8)\end{aligned}$$

So a manifold is Weyl-conformal flat if $\widetilde{W}(X_1, Y_1, Z_1, U_1) = g(W(X_1, Y_1)Z_1, U_1) = 0$.

A curvature tensor, symbolised by W_2, was first suggested by G. P. Pokhariyal and R. S. Mishra [51] in 1970 and their relativistic implications were explored.

Definition 1.2.5. *[51] (W_2-curvature tensor) The formula for the W_2-curvature tensor on a manifold of dimension greater than 2 can be stated as follows:*

$$W_2(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 + \frac{1}{n-1}[g(X_1, Z_1)QY_1 - g(Y_1, Z_1)QX_1]. \tag{1.2.9}$$

When the condition $\widetilde{W}_2(X_1, Y_1, Z_1, U_1) = g(W_2(X_1, Y_1)Z_1, U_1) = 0$ is satisfied, it indicates that a manifold can be described as W_2-flat.

The Bochner curvature tensor, introduced in 1949 by Bochner [6, 76], plays a similar role in Kähler geometry to the Weyl curvature tensor on Riemannian manifolds.

Definition 1.2.6. *[8] (Bochner curvature tensor) The notion of Bochner curvature tensor is presented as:*

$$\begin{aligned} B(X_1, Y_1)Z_1 &= R(X_1, Y_1)Z_1 - \frac{1}{n+4}[g(Y_1, Z_1)QX_1 - g(X_1, Z_1)QY_1 \\ &\quad + S(Y_1, Z_1)X_1 - S(X_1, Z_1)Y_1 + g(FY_1, Z_1)QFX_1 \\ &\quad - g(FX_1, Z_1)QFY_1 + S(FY_1, Z_1)FX_1 - S(FX_1, Z_1)FY_1 \\ &\quad - 2S(FX_1, Y_1)FZ_1 - 2g(FX_1, Y_1)QFZ_1] \\ &\quad + \frac{r}{(n+2)(n+4)}[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1 \\ &\quad + g(FY_1, Z_1)FX_1 - g(FX_1, Z_1)FY_1 \\ &\quad - 2g(FX_1, Y_1)FZ_1], \end{aligned} \tag{1.2.10}$$

where Q represents the Ricci operator, defined by $g(QX_1, Y_1) = S(X_1, Y_1)$ and n denotes the dimension of the manifold. Additionally, a manifold is considered Bochner flat if $\widetilde{B}(X_1, Y_1, Z_1, U_1) = g(B(X_1, Y_1)Z_1, U_1) = 0$ is satisfied.

The concept of the pseudo-quasi-conformal curvature tensor $\widetilde{V}$ on a Riemannian manifold of dimension ≥ 3 was constructed and explored by the authors in [66], and it includes the projective, quasi-conformal, Weyl conformal, and concircular curvature tensors as special cases.

Definition 1.2.7. *[66] (Pseudo-quasi-conformal curvature tensor) A pseudo-quasi-conformal curvature tensor is described as:*

$$\begin{aligned} \widetilde{V}(X_1, Y_1)Z_1 &= (p+q)R(X_1, Y_1)Z_1 + (q - \frac{d}{n-1})[S(Y_1, Z_1)X_1 - S(X_1, Z_1)Y_1] \\ &\quad + q[g(Y_1, Z_1)QX_1 - g(X_1, Z_1)QY_1] - \frac{r\,[p + 2(n-1)q]}{n(n-1)}[g(Y_1, Z_1)X_1 \\ &\quad - g(X_1, Z_1)Y_1], \end{aligned} \tag{1.2.11}$$

where X_1, Y_1, and $Z_1 \in \chi(M)$, S is the Ricci tensor, r is the scalar curvature, Q is the Ricci operator corresponding to the Ricci tensor S, i.e. $g(QX_1, Y_1) = S(X_1, Y_1)$ and p, q, and d are real constants such that $p^2 + q^2 + d^2 > 0$ [16] and the manifold is n-dimensional.

Particularly, if

(1) $p = q = 0, d = 1$,

(2) $p \neq 0, q \neq 0, d = 0$,

(3) $p = 1, q = -\frac{1}{n-2}, d = 0$,

(4) $p = 1, q = d = 0$,

then $\widetilde{V}$ reduces to the projective curvature tensor, quasi-conformal-curvature tensor, conformal curvature tensor, and con-circular curvature tensor, respectively [20]. A manifold M^n, where $n > 3$, is called pseudo-quasi conformally flat if $\widetilde{V} = 0$.

A con-circular transformation, which was introduced by K. Yano [79] in 1940, is a type of transformation that maintains the shape of geodesic circles. This transformation is associated with a branch of geometry known as con-circular geometry. When a con-circular transformation is applied, the con-circular curvature tensor C remains unchanged.

Definition 1.2.8. *[79] (Con-circular curvature tensor) On an n-dimensional manifold M, the con-circular curvature tensor C is given by*

$$C(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 - \frac{r}{n(n-1)}[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1], \tag{1.2.12}$$

where X_1, Y_1, and Z_1 are arbitrady vector fields in $\chi(M)$.

Now, we introduce the concept of generalised W_2-curvature tensor ($n > 2$) as follows:

Definition 1.2.9 (Generalised W_2-curvature tensor)**.** *The generalised W_2-curvature tensor is defined by:*

$$\overline{W_2}(X_1, Y_1)Z_1 = aR(X_1, Y_1)Z_1 + \left(b + \frac{c}{n-1}\right)[g(X_1, Z_1)QY_1 - g(Y_1, Z_1)QX_1], \tag{1.2.13}$$

where a, b, and $c \neq 0$. In particular, if $a = 1, b = 0$, and $c = 1$, then it reduces to a W_2-curvature tensor. Again, if $b = 0$, we call the $\overline{W}_2$ tensor as a quasi-W_2 tensor and is denoted by $\widetilde{W_2}$. So a manifold is generalised W_2-flat if $g(\overline{W_2}(X_1, Y_1)Z_1, U_1) = 0$.

1.2.2 Weakly quasi-conformally symmetric manifold

Definition 1.2.10. *[62] (Weakly quasi-conformally symmetric manifold) A Riemannian manifold* $(M^n, g)(n > 2)$ *is said to be weakly quasi-conformally symmetric manifold, denoted by* $(WQCS)_n$*, if the quasi-conformally curvature tensor* $\overline{C}$ *of type* $(0,4)$ *satisfies the condition*

$$\begin{aligned}(\nabla_{X_1}\overline{C})(Y_1, Z_1, U_1, V_1) &= A(X_1)\overline{C}(Y_1, Z_1, U_1, V_1) + B(Y_1)\overline{C}(X_1, Z_1, U_1, V_1) \\ &\quad +C(Z_1)\overline{C}(Y_1, X_1, U_1, V_1) + D(U_1)\overline{C}(Y_1, Z_1, X_1, V_1) \\ &\quad +E(V_1)\overline{C}(Y_1, Z_1, U_1, X_1), \qquad (1.2.14)\end{aligned}$$

for all vectors fields X_1*,* Y_1*,* Z_1*,* U_1*,* V_1 *and* A*,* B*,* C*,* D*,* E *are non-vanishing* 1*-forms.*

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1.2.3 Almost Hermite manifold

Definition 1.2.11. *[18] (Almost Hermite manifold) An almost complex manifold endowed with a Riemannian metric* g *such that*

$$g(FX_1, FY_1) = g(X_1, Y_1),$$

is called an almost Hermite manifold, while (F, g) *is called an almost Hermite structure. A Hermite metric thus defined a Hermitian inner product on* TM *w.r.t. the complex structure* F*.*

1.3 Introduction to Kähler manifolds

In mathematics, particularly differential geometry, a Kähler manifold is a manifold that has three structures that are all mutually compatible: a complex structure, a Riemannian structure, and a symplectic structure. Erich Kähler [39] first presented the concept in 1933, while Jan Arnoldus Schouten and David van Dantzig first explored it in 1930. By André Weil, the terminology has been corrected. The study of Kähler manifolds, their geometry, and topology is referred to as Kähler geometry, as is the study of the structures and constructions that can be applied to Kähler manifolds, such as the existence of special connections like Hermitian Yang-Mills connections or special metrics like Kähler-Einstein metrics. Every smooth, complex projective variety is a Kähler manifold. Utilizing Kähler metrics, Hodge's theory is a fundamental concept in algebraic geometry.

Definition 1.3.1. *[10] (Kähler manifold) A Kähler manifold is an even-dimensional Riemannian manifold M with complex structure F on each tangent space of M that satisfies the following relations*

$$F^2(X_1) = -X_1, g(\overline{X_1}, \overline{Y_1}) = g(X_1, Y_1), (\nabla_{X_1} F)(Y_1) = 0,$$

where $F(X_1) = \overline{X_1}$, g is the Riemannian metric, and ∇ is the connection of covariant differentiation.

Theorem 1.3.1. *[18] An almost Hermitian manifold is a Kähler manifold if and only if $\nabla_{X_1} F(Y_1) = F(\nabla_{X_1} Y_1)$.*

Example 1.3.1. *In the field of mathematics, we define the n-dimensional complex coordinate space, also known as complex n-space, as the collection of all ordered n-tuples consisting of complex numbers. This space, symbolized as $\mathbb{C}^n$, corresponds to taking the Cartesian product of the complex plane $\mathbb{C}$ with itself n times. Symbolically,*

$$\mathbb{C}^n = \{(z_1, z_2, ..., z_n) \mid z_i \in \mathbb{C}\},$$

the variables z_i are the complex coordinates on the complex n-space. Complex coordinate space can be considered as a vector space over the complex numbers. Its addition and scalar multiplication operations are performed component-wise. By associating the real and imaginary parts of the coordinates, we can establish a one-to-one correspondence between $\mathbb{C}^n$ and the $2n$-dimensional real coordinate space, $\mathbb{R}^{2n}$. When equipped with the usual Euclidean topology, $\mathbb{C}^n$ becomes a topological vector space over the complex numbers. Therefore, the complex spaces $\mathbb{C}^n$ with standard Hermitian metric is a Kähler manifold.

1.3.1 Nearly Kähler manifold

Shun-ichi Tachibana [12] examined almost Tachibana manifolds, also referred to as nearly Kähler manifolds, in 1959, and Alfred Gray [28] further explored them from 1970 onwards.

Definition 1.3.2. *[18] (Nearly Kähler manifold) On an almost Hermite manifold M, if the almost complex structure F satisfies*

$$(\nabla_{X_1} F)(Y_1) + (\nabla_{Y_1} F)(X_1) = 0, \tag{1.3.15}$$

for arbitrady vector fields X_1 and $Y_1 \in \chi(M)$, then the manifold M is called a nearly Kähler manifold or an almost Tachibana manifold.

Putting X_1 for Y_1 in (1.3.15), we get

$$(\nabla_{X_1} F)(X_1) = 0.$$

If in an almost Tachibana manifold, Nijenhuis tensor vanishes, then it is called a Tachibana manifold.

Proposition 1.3.1. *[56] (i) For a nearly Kähler manifold*

$$N(X_1, Y_1) = 2\widetilde{M}(X_1, Y_1) = -4F((\nabla_{X_1} F)(Y_1)) = 4F((\nabla_{Y_1} F)(X_1)) = 4F((\nabla_{F(X_1)} F)F(Y_1)),$$

$$where \quad \widetilde{M}(X_1, Y_1) = \nabla_{F(X_1)} F(Y_1) - \nabla_{X_1} Y_1 - F(\nabla_{F(X_1)} Y_1) - F(\nabla_{X_1} F(Y_1)).$$

(ii) If M is nearly Kähler manifold then $N(X_1, Y_1) = F(\nabla_{X_1} F)Y_1$,

$$where \quad 4N(X_1, Y_1) = [X_1, Y_1] - [FX_1, FY_1] + F[FX_1, Y_1] + F[X_1, FY_1].$$

The nearly Kähler manifolds are well known in smaller dimensions. M is Kähler manifold if M is nearly Kähler with dim$M \leq 4$. The following is true if dim$M = 6$ (see [28], [27], [42], and [73]).

1.3.2 Kähler-Norden manifold

Definition 1.3.3. *[7] (Kähler-Norden manifold) A Kähler-Norden manifold is an even-dimensional connected differentiable manifold, denoted by M, with a dimension $n = 2m$, where m is greater than or equal to 2. It is equipped with a $(1,1)$-tensor field F and a pseudo-Riemannian metric g. The conditions that define a Kähler-Norden manifold are as follows:*

$$F^2 = -I, \quad g(FX_1, FY_1) = -g(X_1, Y_1), \quad \nabla F = 0,$$

these conditions holds for any vector fields X_1, $Y_1 \in TM$, which is the Lie algebra of vector fields on M, ∇ is the covariant differentiation operator of g. Additionally, in the given context, the symbol I represents the identity operator.

Then the metric g has necessarily a neutral signature (m, m), and we can find a holomorphic metric on the complex manifold M^n [21]. This fact gives us some topological obstructions to an anti-Kähler manifold, for instance, all its odd Chern numbers vanish because its holomorphic metric gives us a complex isomorphism between the complex tangent bundle and its dual and a compact simply connected Kähler manifold cannot be anti-Kähler because it does not admit a holomorphic metric. In a Kähler-Norden manifold (M,F,g), the Riemannian curvature operator R, the Riemannian curvature tensor $\widetilde{R}$, the Ricci tensor S, the scalar curvature r and the r^* curvature are defined by:

$$R(X_1, Y_1)Z_1 = [\nabla_{X_1}, \nabla_{Y_1}]Z_1 - \nabla_{[X_1,Y_1]}Z_1,$$

$$\widetilde{R}(X_1, Y_1, Z_1, W_1) = g(R(X_1, Y_1)Z_1, W_1),$$

$$S(X_1, Y_1) = trace \ \ of \ \ \{Z_1 \to R(Z_1, X_1)Y_1\},$$

$$r = trace \ \ S, \tag{1.3.16}$$

$$r^* = S(Fe_i, e_i). \tag{1.3.17}$$

1.3.3 Hyper Kähler manifold

A Riemannian $4n$-manifold called a hyperKähler manifold [67] if it has a family of almost complex structures that behave under composition like multiplication, purely imaginary, unit quaternions, and are covariantly constant w.r.t. the operator of the covariant differentiation. We acquire a quaternionic Kähler structure, at least if $n \geq 2$, if all that is required is for these almost complicated structures to exist locally and for the Levi-Civita connection to typically retain this family. Thus, quaternionic Kähler manifolds are a particular instance of hyperKähler manifolds. However, note that the quaternionic Kähler manifold is not required to be Kähler.

Definition 1.3.4. *[67] (HyperKähler manifold) For $n \in \mathbb{N}$ a natural number, a $4n$-dimensional Riemannian manifold is a hyperKähler manifold if its holonomy group is a subgroup of the quaternionic unitary group SP_n.*
Equivalently, a hyperKähler manifold is a Riemannian manifold with three complex structures which are Kähler with respect to the metric and satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -1$.

1.3.4 ParaKähler manifold

Definition 1.3.5. *[47] (ParaKähler manifold) A paraKähler manifold is an even dimensional connected differentiable manifold, denoted by M, with a dimension $n = 2m$, where m is greater than or equal to 2. It is equipped with a $(1,1)$-tensor field F and a pseudo-Riemannian metric g. The conditions that define a paraKähler manifold are as follows:*

$$F^2 = I, g(FX_1, FY_1) = -g(X_1, Y_1), \nabla F = 0 \tag{1.3.18}$$

these conditions holds for any vector fields X_1, $Y_1 \in TM$, which is the Lie algebra of vector fields on M, ∇ is the covariant differentiation operator of g. Additionally, in the given context, the symbol I represents the identity operator.

Example 1.3.2. *[13] An example of paraKähler structure is given in on $\mathbb{R}^n$ by $g = \begin{pmatrix} -I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and $F = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, where g is the pseudo-Euclidean metric, F is the almost complex structure and both the matrices are taken with respect to the canonical basis of $\mathbb{R}^n$.*

1.3.5 Kähler-Einstein manifold

Definition 1.3.6. *[4] (Kähler-Einstein manifold) A Kähler manifold is called Kähler-Einstein if the Ricci curvature tensor is equal to a constant λ times the metric tensor. i.e., $Ric = \lambda g$.*

1.3.6 Perfect fluid space-time

In the context of a perfect fluid within the framework of general relativity, the Einstein field equation [50], which incorporates the cosmological constant λ, can be expressed as follows:

$$S(X_1, Y_1) - \frac{r}{2} g(X_1, Y_1) + \lambda g(X_1, Y_1) = c[(\sigma + \tilde{p})\omega(X_1)\omega(Y_1) + \tilde{p} g(X_1, Y_1)], \tag{1.3.19}$$

where c represents the gravitational constant, σ corresponds the energy density, $\tilde{p}$ denotes the isotropic pressure of the fluid, and ω represents the 1-form determined by $\omega(X_1) = g(X_1, \rho)$, where ρ is the time-like vector field. The equation $g(\rho, \rho) = -1$ satisfies the fluid velocity connected to the time-like vector field ρ.

1.3.7 ParaKähler space-time

Spacetime is a conceptual framework in physics that merges the dimensions of space and time into a unified four-dimensional structure. It is represented mathematically as a manifold, enabling the visualization of relativistic phenomena, including the varying perceptions of events by different observers. In earlier times, it was commonly believed that the spatial geometry of the universe, defined by coordinates, distances, and directions, existed independently from the temporal dimension. However, the renowned physicist Albert Einstein played a fundamental role in introducing the notion of spacetime as an integral component of his theory of relativity. In 1983, V. R. Kaigorodov [40] conducted research on the curvature structure of space-time. Numerous differential geometers and mathematicians continued to develop these concepts of general relativity of space-time after that.

Definition 1.3.7. *[50] (ParaKähler space-time) A space-time with four-dimensions is classified as a paraKähler space-time if it meets the following criteria:*

$$F^2(X_1) = X_1, \tag{1.3.20}$$

$$g(FX_1, FY_1) = -g(X_1, Y_1), \tag{1.3.21}$$

$$(\nabla_{X_1} F) = 0, \tag{1.3.22}$$

where, F represents a tensor of type $(1,1)$, g denotes a Riemannian metric, and ∇ stands for the covariant differentiation operator.

Apart from the introductory chapter, this thesis consists of five chapters. A brief summary is given of these chapters as follows:

In the **second chapter**, in our investigation, we explore weakly symmetric Kähler manifolds that demonstrate characteristics of being both pseudo-projectively flat and quasi-conformally flat. Furthermore, we analyze weakly pseudo-projectively symmetric and quasi-conformally symmetric Kähler manifolds, which are also identified as Einstein manifolds. Additionally, we prove the presence of pseudo-projectively flat weakly symmetric Kähler manifolds and quasi-conformally flat weakly symmetric Kähler manifolds, where a specific relationship holds true for the Ricci tensor.

In the **third chapter**, we complete a thorough investigation and develop exact formulas for various curvature properties related to a nearly Kähler manifold that demonstrates both con-circular and projective flatness. Additionally, we obtain a fascinating result regarding a 6-dimensional nearly Kähler manifold, and we provide an example to demonstrate these outcomes.

In **fourth chapter**, we analyze various forms of curvature identities found in Kähler-Norden manifolds, including quasi-conformal flatness, pseudo-projective flatness, Weyl-conformal flatness, and Bochner flatness. Our findings indicate that a Kähler-Norden manifold displays pseudo-projective symmetric-ness if it exhibits local symmetric-ness and conversely. Additionally, we investigate semi-symmetric Kähler-Norden manifolds and establish that a Kähler-Norden manifold is semi-symmetric-ness if it possesses locally semi-symmetric-ness and conversely.

In the **fifth chapter**, we investigate certain curvature identities pertaining to a locally symmetric hyperKähler manifold. Additionally, we delve into the notions of conformal flatness and Bochner flatness in the context of hyperKähler manifolds. We successfully demonstrate that if the dimension of a conformally flat hyperKähler manifold exceeds 4, it is locally flat, whereas for a dimension of 4, it is locally symmetric with a scalar curvature of zero. Moreover, we establish that a conformally flat and Bochner flat hyperKähler manifold with a dimension of $4n$ qualifies as an Einstein manifold. Furthermore, we introduce a generalised W_2-flat hyperKähler manifold characterised by Ricci flatness, along with an equation relating the parameters a, b, and c as $a \neq \left(b + \frac{c}{4n-7}\right)$. We also investigate the scenario of a quasi-W_2-flat hyperKähler manifold, which is Ricci flat, under the condition that c is non-zero. Finally, we provide several examples illustrating the concepts discussed in hyperKähler manifolds.

In the **sixth chapter**, we conduct a study on several curvature identities in paraKähler manifolds that are pseudo-quasi-conformally flat, W_2-flat, pseudo-projectively flat, and Bochner flat. Furthermore, we investigate significant results concerning sectional curvature in paraKähler manifolds. Additionally, we examine the behavior of paraKähler space-time in the presence of a perfect fluid. Moreover, we explore the weakly symmetric perfect fluids and the weakly Ricci symmetric perfect fluids on paraKähler space-time. We also delve into the aforementioned curvature identities along with generalised W_2-

flat curvature identitity on paraKähler space-time. Lastly, we expand upon important properties associated with sectional curvature on paraKähler space-time.

2

Some Weakly Symmetric Kähler Manifolds

2.1 Introduction

After the introduction of the concepts of the weakly symmetric and weakly projective symmetric manifolds by L. Tamassy and T. Q. Binh [69] in 1989, M. Prvanovic [52], U. C. De, and S. Bandyopadhyay [15] etc. showed keen interests on those manifolds. The work of L. Tamassy, U. C. De, and T. Q. Binh [68] on the weakly symmetric and the weakly Ricci symmetric Kähler manifolds also produced a few curious results.
This chapter is divided into five sections. The first two sections consist of an introduction and preliminaries. In the third section, we show that a weakly pseudo-projectively symmetric Kähler manifold is an Einstein manifold with respect to vector field ρ satisfying the condition $g(X_1, \rho) = \omega(X_1)$. In the fourth section, we initiate in a pseudo-projectively flat weakly symmetric Kähler manifold, the Ricci tensor follows the relation $S(Z_1, \alpha) + S(Z_1, \rho) = -r\omega(Z_1)$, where $g(X_1, \alpha) = A(X_1)$. In the fifth segment, we illustrate that a weakly quasi-conformally symmetric Kähler manifold is again an Einstein manifold with respect to vector field ρ, which is defined earlier. In the last section, we show that in a quasi-conformally flat weakly symmetric Kähler manifold, the Ricci tensor satisfies the relation which is mentioned above.

2.2 Preliminaries

Definitions and some basic characteristics of the Kähler manifold, the weakly symmetric Kähler manifold, weakly pseudo-projectively symmetric Kähler and the weakly quasi-conformally symmetric Kähler manifold are provided in the introductory chapter. Here, we reflect on a few significant findings and apply them to our work.

In this context, we derive some formulae which will be essential to examine the behaviours of weakly pseudo-projectively symmetric manifold and weakly quasi-conformally symmetric manifold of dimension n. Let us consider an orthonormal basis $\{e_i\}_{i=1}^{n}$, of each tangent space of the manifold. Then from (1.2.5), we have the following:

$$\begin{aligned} (a)\sum_{i=1}^{n}\overline{P}(e_i,Y_1,Z_1,e_i) &= [a+(n-1)b][S(Y_1,Z_1)-\frac{r}{n}g(Y_1,Z_1)], \\ (b)\sum_{i=1}^{n}\overline{P}(X_1,Y_1,e_i,e_i) &= 0, \\ (c)\sum_{i=1}^{n}\overline{C}(e_i,Y_1,Z_1,e_i) &= [a+(n-2)b][S(Y_1,Z_1)-\frac{r}{n}g(Y_1,Z_1)], \\ (d)\sum_{i=1}^{n}\overline{C}(X_1,Y_1,e_i,e_i) &= 0. \end{aligned}$$

Now, we have proved the following proposition:

Proposition 2.2.1. *In a Riemannian manifold (M,g) with dimension greater than 2, the pseudo-projective curvature tensor and quasi-conformally curvature tensor satisfy the following relations:*

$$\begin{aligned} (I)\quad & \overline{P}(X_1,Y_1,Z_1,U_1)+\overline{P}(Y_1,Z_1,X_1,U_1)+\overline{P}(Z_1,X_1,Y_1,U_1)=0, \\ (II)\quad & \overline{P}(X_1,Y_1,U_1,Z_1)+\overline{P}(Y_1,Z_1,U_1,X_1)+\overline{P}(Z_1,X_1,U_1,Y_1)=0, \\ (III)\quad & \overline{C}(X_1,Y_1,Z_1,U_1)+\overline{C}(Y_1,Z_1,X_1,U_1)+\overline{C}(Z_1,X_1,Y_1,U_1)=0, \\ (IV)\quad & \overline{C}(X_1,Y_1,U_1,Z_1)+\overline{C}(Y_1,Z_1,U_1,X_1)+\overline{C}(Z_1,X_1,U_1,Y_1)=0. \end{aligned}$$

2.3 Weakly pseudo-projectively symmetric Kähler manifold

In this section, the following assertion is proved using the Kähler-Einstein metric of the weakly pseudo-projectively symmetric Kähler manifold. At the end of this section, we provide a corollary.

Theorem 2.3.1. *A weakly pseudo-projectively symmetric Kähler manifold is an Einstein manifold with respect to the vector field ρ satisfies $g(X_1,\rho)=\omega(X_1)$.*

Proof. If the manifold M is a weakly pseudo-projectively symmetric Kähler manifold, then we have proved

$$\overline{P}(\overline{Y_1}, \overline{Z_1}, U_1, V_1) = \overline{P}(Y_1, Z_1, U_1, V_1). \tag{2.3.1}$$

Considering the covariant derivative along with an arbitrady vector X_1, we have

$$(\nabla_{X_1}\overline{P})(\overline{Y_1}, \overline{Z_1}, U_1, V_1) = (\nabla_{X_1}\overline{P})(Y_1, Z_1, U_1, V_1). \tag{2.3.2}$$

Applying (1.2.3) and (1.2.6) in (2.3.2), we obtain

$$\begin{aligned} &\omega(Y_1)\overline{P}(X_1, Z_1, U_1, V_1) + \omega(Z_1)\overline{P}(Y_1, X_1, U_1, V_1) \\ &= \omega(\overline{Y_1})\overline{P}(X_1, \overline{Z_1}, U_1, V_1) + \omega(\overline{Z_1})\overline{P}(\overline{Y_1}, X_1, U_1, V_1). \end{aligned} \tag{2.3.3}$$

Setting $Z_1 = U_1 = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\begin{aligned} &(a-b)\omega(Y_1)S(X_1, V_1) - \frac{(a-1)br}{n}\omega(Y_1)g(X_1, V_1) - aR(Y_1, X_1, V_1, \rho) \\ &+bg(Y_1, V_1)S(X_1, \rho) - 2bg(X_1, V_1)S(Y_1, \rho) + \frac{2r}{n}\left[\frac{a}{n-1} + b\right] g(X_1, V_1)g(Y_1, \rho) \\ &-\frac{r}{n}\left[\frac{a}{n-1} + b\right] g(Y_1, V_1)g(X_1, \rho) \\ &= (a+b)\omega(\overline{Y_1})S(X_1, \overline{V_1}) - \frac{r}{n}\left[\frac{a}{n-1} + b\right] g(X_1, \overline{V_1})\omega(\overline{Y_1}) + aR(\overline{Y_1}, X_1, V_1, \overline{\rho}) \\ &+\frac{r}{n}\left[\frac{a}{n-1} + \frac{(r-n)b}{r}\right] g(\overline{Y_1}, V_1)S(X_1, \overline{\rho}). \end{aligned} \tag{2.3.4}$$

Putting $X_1 = V_1 = e_i$, $1 \leq i \leq n$ and summing over i, we acquire

$$\begin{aligned} &rg(Y_1, \rho)\left[a(1-b) + \left(2 - \frac{1}{n}\right)\left(\frac{a}{n-1} + b\right)\right] \\ &= S(Y_1, \rho)\left[2a + 2b(n-1) + \frac{ar}{n(n-1)} + \frac{br}{n}\right]. \end{aligned} \tag{2.3.5}$$

We achieve

$$S(Y_1, \rho) = fg(Y_1, \rho).$$

This is an Einstein manifold for every vector field ρ.
Hence, the proof. □

The above theorem leads to the following corollary:

Corollary 2.3.1. *For a weakly pseudo-projectively symmetric Kähler manifold if* ρ *is a unit vector field, then the expression for scalar curvature is,* $r = \frac{2nh[a+(n-1)b]}{2na+(n-h-1)(a+b)}$, *provided* $2na + (n-h-1)(a+b) \neq 0$, *where* $h = S(\rho, \rho)$. *In addition, if* $a + (n-1)b = 0$, *then the scalar curvature vanishes.*

Proof. Setting $Y_1 = \rho$, we obtain our desired result. □

2.4 Pseudo-projectively flat weakly symmetric Kähler manifold

In this section, the following result is proved by applying the flat curvature tensor property of the weakly pseudo-projectively symmetric Kähler manifold as the Ricci tensor.

Theorem 2.4.1. *In a pseudo-projectively flat weakly symmetric Kähler manifold, the Ricci tensor obeys the relation* $S(Z_1, \alpha) + S(Z_1, \rho) = -r\omega(Z_1)$.

Proof. For pseudo-projectively flat curvature tensor, $\overline{P}(Y_1, Z_1, U_1, V_1) = 0$, then

$$\begin{aligned}&aR(Y_1, Z_1, U_1, V_1) + bS(Z_1, U_1)g(Y_1, V_1) - bS(Y_1, U_1)g(Z_1, V_1)\\ &-\frac{r}{n}\left[\frac{a}{n-1} + b\right] g(Z_1, U_1)g(Y_1, V_1) + \frac{r}{n}\left[\frac{a}{n-1} + b\right] g(Y_1, U_1)g(Z_1, V_1) = 0.\end{aligned}$$

Then

$$\begin{aligned}R(Y_1, Z_1, U_1, V_1) = &-\frac{b}{a}[S(Z_1, U_1)g(Y_1, V_1) - S(Y_1, U_1)g(Z_1, V_1)]\\ &+ \frac{r}{an}\left[\frac{a}{n-1} + b\right] [g(Z_1, U_1)g(Y_1, V_1) - g(Y_1, U_1)g(Z_1, V_1)]. \qquad (2.4.1)\end{aligned}$$

Taking covariant differentiation w.r.t. X_1, we get

$$(\nabla_{X_1}R)(Y_1, Z_1, U_1, V_1) = -\frac{b}{a}[g(Y_1, V_1)(\nabla_{X_1}S)(Z_1, U_1) - g(Z_1, V_1)(\nabla_{X_1}S)(Y_1, U_1)], \qquad (2.4.2)$$

then (2.4.2) reduces to

$$\begin{aligned}&A(X_1)R(Y_1, Z_1, U_1, V_1) + \omega(Y_1)R(X_1, Z_1, U_1, V_1) + \omega(Z_1)R(Y_1, X_1, U_1, V_1)\\ &+\omega(U_1)R(Y_1, Z_1, X_1, V_1) + \omega(V_1)R(Y_1, Z_1, U_1, X_1) = -\frac{b}{a}[g(Y_1, V_1)A(X_1)S(Z_1, U_1)\\ &+\omega(Z_1)S(X_1, U_1) + \omega(U_1)S(Z_1, X_1) - g(Z_1, V_1)A(X_1)S(Y_1, U_1) + \omega(Y_1)S(X_1, U_1)\\ &+\omega(U_1)S(Y_1, X_1)]. \qquad (2.4.3)\end{aligned}$$

Putting $Y_1 = V_1 = e_i$, $1 \leq i \leq n$ and summing over i, we obtain

$$\left[1 + \frac{b}{a}(n-1)\right] [A(X_1, U_1) + \omega(Z_1)S(X_1, U_1) + \omega(U_1)S(Z_1, X_1)] = 0. \tag{2.4.4}$$

Taking $X_1 = U_1 = e_i$, $1 \leq i \leq n$ and summing over i, we acquire

$$\left[1 + \frac{b}{a}(n-1)\right] [S(Z_1, \alpha) + r\omega(Z_1) + S(Z_1, \rho)] = 0, \tag{2.4.5}$$

for any vector field ρ defined by $g(X_1, \rho) = \omega(X_1)$ and $g(X_1, \alpha) = A(X_1)$, then we have

$$S(Z_1, \alpha) + S(Z_1, \rho) = -r\omega(Z_1).$$

This completes the theorem. □

2.5 Weakly quasi-conformally symmetric Kähler manifold

In this section, the Kähler-Einstein metric of the weakly quasi-conformally symmetric Kähler manifold is used to demonstrate the claim that follows. This section's conclusion is followed by a corollary.

Theorem 2.5.1. *A weakly quasi-conformally symmetric Kähler manifold is an Einstein manifold with respect to the vector field ρ which satisfies $g(X_1, \rho) = \omega(X_1)$.*

Proof. If the manifold M is a weakly quasi-conformally symmetric Kähler manifold, then we have proved

$$\overline{C}(\overline{Y_1}, \overline{Z_1}, U_1, V_1) = \overline{C}(Y_1, Z_1, U_1, V_1). \tag{2.5.1}$$

Considering the covariant derivative along with an arbitrady vector X_1, we have

$$(\nabla_{X_1}\overline{C})(\overline{Y_1}, \overline{Z_1}, U_1, V_1) = (\nabla_{X_1}\overline{C})(Y_1, Z_1, U_1, V_1). \tag{2.5.2}$$

Using (1.2.7) in (2.5.2), we obtain

$$\begin{aligned}&\omega(Y_1)\overline{C}(X_1, Z_1, U_1, V_1) + \omega(Z_1)\overline{C}(Y_1, X_1, U_1, V_1)\\ &= \omega(\overline{Y_1})\overline{C}(X_1, \overline{Z_1}, U_1, V_1) + \omega(\overline{Z_1})\overline{C}(\overline{Y_1}, X_1, U_1, V_1).\end{aligned} \tag{2.5.3}$$

By putting $Z_1 = U_1 = e_i$, $1 \leq i \leq n$ and summing over i, we acquire

$$\begin{aligned}
&[a+(n-4)b]\omega(Y_1)S(X_1,V_1) - \frac{r}{n}[a+(n-2)b]\omega(Y_1)g(X_1,V_1) - aR(Y_1,X_1,V_1,\rho)\\
&+bg(Y_1,V_1)S(X_1,\rho) - 2bg(X_1,V_1)S(Y_1,\rho) + bg(X_1,\rho)S(Y_1,V_1)\\
&= (a+2b)\omega(\overline{Y_1})S(X_1,\overline{V_1}) - \frac{r}{n}\left[\frac{a}{n-1}+2b\right]\omega(\overline{Y_1})g(X_1,\overline{V_1})\\
&+aR(\overline{Y_1},X_1,V_1,\overline{\rho}) - bg(Y_1,\overline{V_1})S(X_1,\overline{\rho})\\
&+\frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(\overline{Y_1},V_1)g(X_1,\overline{\rho}) - bg(X_1,\overline{\rho})S(\overline{Y_1},V_1). \qquad (2.5.4)
\end{aligned}$$

Again, putting $X_1 = V_1 = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$-rg(Y_1,\rho)\left[2b+\frac{a}{n(n-1)}+\frac{2b}{n}\right] = S(Y_1,\rho)[2a-3b+2bn]. \qquad (2.5.5)$$

We achieve

$$S(Y_1,\rho) = fg(Y_1,\rho).$$

This is again an Einstein manifold for every vector field ρ.
Therefore, the proof. ☐

Theorem 2.5.1 gives raise to the following corollary:

Corollary 2.5.1. *For a weakly quasi-conformally symmetric Kähler manifold if ρ is a unit vector field, then the expression for scalar curvature is, $r = -\frac{n(n-1)h[2a+(2n-3)b]}{a+(n^2-1)2b}$, provided $a+(n^2-1)2b \neq 0$, where $h = S(\rho,\rho)$. In addition, if $2a+(2n-3)b = 0$, then the scalar curvature vanishes.*

Proof. Putting $Y_1 = \rho$, we get our desired result. ☐

2.6 Quasi-conformally flat weakly symmetric Kähler manifold

This section uses the weakly quasi-conformally symmetric Kähler manifold's flat curvature tensor property as the Ricci tensor to present the following claim.

Theorem 2.6.1. *In a quasi-conformally flat weakly symmetric Kähler manifold, the Ricci tensor follows the relation $S(Z_1,\alpha) + S(Z_1,\rho) = -r\omega(Z_1)$.*

Proof. For quasi-conformally flat curvature tensor, $\overline{C}(Y_1, Z_1, U_1, V_1) = 0$, then

$$\begin{aligned} &aR(Y_1, Z_1, U_1, V_1) + bS(Z_1, U_1)g(Y_1, V_1) - bS(Y_1, U_1)g(Z_1, V_1) + bg(Z_1, U_1)g(QY_1, V_1) \\ &-bg(Y_1, U_1)g(QZ_1, V_1) - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right] g(Z_1, U_1)g(Y_1, V_1) \\ &+\frac{r}{n}\left[\frac{a}{n-1} + 2b\right] g(Y_1, U_1)g(Z_1, V_1) = 0. \qquad (2.6.1) \end{aligned}$$

Then

$$\begin{aligned} R(Y_1, Z_1, U_1, V_1) &= -\frac{b}{a}[S(Z_1, U_1)g(Y_1, V_1) - S(Y_1, U_1)g(Z_1, V_1) \\ &+g(Z_1, U_1)g(QY_1, V_1) - g(Y_1, U_1)g(QZ_1, V_1)] \\ &+\frac{r}{an}\left[\frac{a}{n-1} + 2b\right][g(Z_1, U_1)g(Y_1, V_1) \\ &-g(Y_1, U_1)g(Z_1, V_1)]. \qquad (2.6.2) \end{aligned}$$

Taking covariant differentiation w.r.t. X_1, we get

$$(\nabla_{X_1}R)(Y_1, Z_1, U_1, V_1) = -\frac{b}{a}[g(Y_1, V_1)(\nabla_{X_1}S)(Z_1, U_1) - g(Z_1, V_1)(\nabla_{X_1}S)(Y_1, U_1)], \qquad (2.6.3)$$

then

$$\begin{aligned} &A(X_1)R(Y_1, Z_1, U_1, V_1) + \omega(Y_1)R(X_1, Z_1, U_1, V_1) + \omega(Z_1)R(Y_1, X_1, U_1, V_1) \\ &+\omega(U_1)R(Y_1, Z_1, X_1, V_1) + \omega(V_1)R(Y_1, Z_1, U_1, X_1) = -\frac{b}{a}[g(Y_1, V_1)A(X_1)S(Z_1, U_1) \\ &+\omega(Z_1)S(X_1, U_1) + \omega(U_1)S(Z_1, X_1) - g(Z_1, V_1)A(X_1)S(Y_1, U_1) \\ &+\omega(Y_1)S(X_1, U_1) + \omega(U_1)S(Y_1, X_1)]. \qquad (2.6.4) \end{aligned}$$

By putting $Y_1 = V_1 = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\left[1 + \frac{b}{a}(n-1)\right][A(X_1)S(Z_1, U_1) + \omega(Z_1)S(X_1, U_1) + \omega(U_1)S(Z_1, X_1)] = 0. \qquad (2.6.5)$$

Again, taking $X_1 = U_1 = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\left[1 + \frac{b}{a}(n-1)\right][S(Z_1, \alpha) + r\omega(Z_1) + S(Z_1, \rho)] = 0, \qquad (2.6.6)$$

for any vector field ρ defined by $g(X_1, \rho) = \omega(X_1)$ and $g(X_1, \alpha) = A(X_1)$, then we have

$$S(Z_1, \alpha) + S(Z_1, \rho) = -r\omega(Z_1). \qquad (2.6.7)$$

Hence the proof. □

3

Some Curvature Identities on Nearly Kähler Manifolds

3.1 Introduction

Gray [28] identified nearly Kähler manifolds while studying weak holonomy, whose Riemannian curvature operators fulfil certain identities. These identities are only somewhat more difficult than and similar to the related formula for the Riemannian curvature operator of Kähler manifolds. These manifolds are referred to as nearly Kähler manifolds by Gray, who was able to demonstrate that many findings regarding the topology and geometry of Kähler manifolds generalise to nearly Kähler manifolds as well as identify several new topological and geometric properties. Gray termed them almost Kähler manifolds, and he was able to demonstrate that many topological and geometric conclusions on Kähler manifolds generalised to nearly Kähler manifolds, as well as discover new topological and geometric aspects of these manifold.

In 2002, Nagy [44, 45] characterised nearly Kähler manifolds as an almost Hermitian manifolds whose canonical Hermitian connection has parallel and totally skew-symmetric torsion and displayed that any complete strict nearly Kähler manifold is finitely covered by a product of homogeneous 3-symmetric manifolds, twistor spaces over quaternionic Kähler manifolds with their canonical nearly Kähler structure and 6-dimensional strict nearly Kähler manifolds.

We want to investigate various characteristics of curvature identities on nearly Kähler manifolds, which is inspired by these publications ([28], [72], [44], and [45]).

This chapter consists of six sections. The fist two sections contain an introduction and preliminaries, respectively. In the third section, we discuss some results on the nearly Kähler manifold. In the fourth section, we study and obtain expressions of some curvature identities on nearly Kähler manifold that is con-circularly flat and projectively flat. In the later section, we present interesting results on a 6-dimensional nearly Kähler manifold. Lastly, we provide an example of a nearly Kähler manifold towards the results.

3.2 Preliminaries

In the introduction chapter, definition of the nearly Kähler manifold, conditions and properties of nearly Kähler manifold, are all discussed. We would like to take this occasion to provide a few helpful findings that serve as evidence for the findings.

In this section, we explain our notation and write down some important curvature identities. For a connected almost Hermitian manifold (M,g,F), we have $g(FX_1, FY_1) = g(X_1, Y_1)$ for all X_1 and Y_1 in TM. Throughout this chapter we shall assume that (M,g,F) is nearly Kähler, that is $(\nabla_{X_1}F)(X_1) = 0$ for all $X_1 \in TM$. Let R denote the Riemannian curvature tensor. Then we have the following identities [72], [28], and [27]:

$$(\nabla_{X_1}F)(Y_1) + (\nabla_{FX_1}F)(F) = 0, \tag{3.2.1}$$

$$(\nabla_{X_1}F)(FY_1) + F((\nabla_{X_1}F)(Y_1)) = 0, \tag{3.2.2}$$

$$R(W_1, X_1, Y_1, Z_1) - R(W_1, X_1, FY_1, FZ_1) = g((\nabla_{W_1}F)(X_1), (\nabla_{Y_1}F)(Z_1)), \tag{3.2.3}$$

$$\textit{and} \;\; R(W_1, X_1, Y_1, Z_1) = R(FW_1, FX_1, FY_1, FZ_1). \tag{3.2.4}$$

We now define linear transformations R_1 and R_1^* by

$$Ric(X_1, Y_1) = g(R_1(X_1), Y_1) = \sum_{i=1}^{n} R(X_1, e_i, Y_1, e_i) \;\; \textit{and}$$

$$Ric^*(X_1, Y_1) = g(R_1^*(X_1), Y_1) = \frac{1}{2}\sum_{i=1}^{n} R(X_1, FY_1, e_i, Fe_i)$$

respectively, where $\{e_1, ..., e_n\}$ denotes a local orthonormal basis. We shall call *Ric* the *Ricci* tensor of the metric and *Ric** the *Ricci** tensor respectively. Now note that $Ric - Ric^*$ is given by the formula

$$(Ric - Ric^*)(X_1, Y_1) = \sum_{i=1}^{n} g((\nabla_{X_1}F)e_i, (\nabla_{Y_1}F)e_i),$$

for all vector fields X_1 and Y_1 on M [56]. Furthermore, Gray [27] proved that

$$\sum_{i,j=1}^{n}(Ric - Ric^*)(e_i, e_j)(R(X_1, e_i, Y_1, e_j) - 5R(X_1, e_i, FY_1, Fe_j)) = 0.$$

Proposition 3.2.1. *[26] For a strict nearly Kähler manifold (M,g,F) of dimension* 6, *we have for an arbitrady* X_1, $Y_1 \in \chi(M)$
(i) ∇F *has a constant type, that is*

$$g((\nabla_{X_1}F)(Y_1), (\nabla_{X_1}F)(Y_1)) = \frac{r}{30}(g(X_1, X_1)g(Y_1, Y_1) - g(X_1, Y_1)^2 - g(FX_1, Y_1)^2)$$

for any vector fields X_1 *and* Y_1,
(ii) the first Chern class of (M, F) *vanishes, and*
(iii) M *is Einstein manifold,*

$$Ric = \frac{r}{6}g, Ric^* = \frac{r}{30}g.$$

Furthermore, from this proposition, we have the following lemma (see [28], [27], and [73]).

Lemma 3.2.1. *[72] For any vector fields* W_1, X_1, Y_1, *and* Z_1, *we have*

$$\begin{aligned} g((\nabla_{W_1}F)(X_1), (\nabla_{Y_1}F)(Z_1)) &= \frac{r}{30}[g(W_1, Y_1)g(X_1, Z_1) - g(W_1, Z_1)g(X_1, Y_1) \\ &\quad -g(W_1, FY_1)g(X_1, FZ_1) + g(W_1, FZ_1)g(X_1, FY_1)] \end{aligned}$$

and

$$g((\nabla_{W_1}\nabla_{Z_1}X_1), Y_1) = \frac{r}{30}[g(W_1, Z_1)g(FX_1, Y_1) - g(W_1, X_1)g(FZ_1, Y_1) + g(W_1, Y_1)g(FZ_1, X_1)].$$

The above results are useful to prove in the next sections.

3.3 Some results on nearly Kähler manifold

Theorem 3.3.1. *A necessary and sufficient condition for an almost Hermite manifold to be an almost nearly Kähler manifold is*

$$\nabla_{X_1}F(Y_1) + \nabla_{Y_1}F(X_1) = F(\nabla_{X_1}Y_1) + F(\nabla_{Y_1}X_1).$$

Proof. First, we suppose that an almost Hermite manifold is an almost nearly Kähler manifold. Then

$$\begin{aligned}
&(\nabla_{X_1}F)(Y_1) + (\nabla_{Y_1}F)(X_1) = 0,\\
or,\ &\nabla_{X_1}F(Y_1) - F(\nabla_{X_1}Y_1) + \nabla_{Y_1}F(X_1) - F(\nabla_{Y_1}X_1) = 0,\\
or,\ &\nabla_{X_1}F(Y_1) + \nabla_{Y_1}F(X_1) = F(\nabla_{X_1}Y_1) + F(\nabla_{Y_1}X_1).
\end{aligned}$$

Conversely, we suppose that

$$\begin{aligned}
&\nabla_{X_1}F(Y_1) + \nabla_{Y_1}F(X_1) = F(\nabla_{X_1}Y_1) + F(\nabla_{Y_1}X_1),\\
or,\ &\nabla_{X_1}F(Y_1) - F(\nabla_{X_1}Y_1) + \nabla_{Y_1}F(X_1) - F(\nabla_{Y_1}X_1) = 0,\\
or,\ &(\nabla_{X_1}F)(Y_1) + (\nabla_{Y_1}F)(X_1) = 0.
\end{aligned}$$

Hence, the manifold becomes an almost nearly Kähler manifold. □

Theorem 3.3.2. *If the Nijenhuis tensor vanishes on a nearly Kähler manifold, then the manifold becomes a Kähler manifold.*

Proof. From Proposition 1.3.1, we have

$$N(X_1, Y_1) = -4F((\nabla_{X_1}F)(Y_1)).$$

If $N(X_1, Y_1) = 0$, then $F((\nabla_{X_1}F)(Y_1)) = 0$. That is, $F^2(\nabla_{X_1}F)Y_1 = 0$.
Hence, $(\nabla_{X_1}F)(Y_1) = 0$.
Therefore, the manifold is a Kähler manifold. □

Theorem 3.3.3. *On a nearly Kähler manifold* $divF = 0$.

Proof. On a nearly Kähler manifold, we obtain

$$(\nabla_{X_1}F)(Y_1) + (\nabla_{Y_1}F)(X_1) = 0.$$

Now contracting X_1 and Y_1, we have

$$(\nabla_{X_1}F)(X_1) = 0.$$

That is, $divF = 0$. □

3.4 Curvature identities on nearly Kähler manifold

Here, we prove some properties of curvature identities on nearly Kähler manifold.

Theorem 3.4.1. *For a con-circularly flat nearly Kähler manifold the following relation holds*

$$\begin{aligned}&2g(F(R(X_1,Y_1)Z_1,W_1))+g[(\nabla_{X_1}F)(\nabla_{Y_1}Z_1),W_1]-g[(\nabla_{Y_1}F)(\nabla_{X_1}Z_1),W_1]\\&=\frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1)].\end{aligned}$$

Proof. Now, (1.2.12) can be written as

$$\widetilde{C}(X_1,Y_1,Z_1,W_1)=\widetilde{R}(X_1,Y_1,Z_1,W_1)-\frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1)], \tag{3.4.1}$$

where,

$$\widetilde{C}(X_1,Y_1,Z_1,W_1)=g(C(X_1,Y_1)Z_1,W_1),\widetilde{R}(X_1,Y_1,Z_1,W_1)=g(R(X_1,Y_1)Z_1,W_1)$$

and r is the scalar curvature.

Now for con-circularly flat manifold, we have $\widetilde{C}(X_1,Y_1,Z_1,W_1)=0$. Hence from (3.4.1), we get

$$\widetilde{R}(X_1,Y_1,Z_1,W_1)=\frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1)]. \tag{3.4.2}$$

Now putting $Z_1=F(Z_1)$ in (3.4.2), we get

$$\begin{aligned}&g(\nabla_{X_1}\nabla_{Y_1}F(Z_1),W_1)-g(\nabla_{Y_1}\nabla_{X_1}F(Z_1),W_1)-g(\nabla_{[X_1,Y_1]}F(Z_1),W_1)\\&=\frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1)].\end{aligned} \tag{3.4.3}$$

By using

$$\nabla_{X_1}F(Y_1)=(\nabla_{X_1}F)Y_1+F(\nabla_{X_1}Y_1),$$

and nearly Kähler condition

$$(\nabla_{X_1}F)(Y_1)+(\nabla_{Y_1}F)(X_1)=0,$$

we have

$$\begin{aligned}&-g[\nabla_{X_1}(\nabla_{Z_1}F)Y_1,W_1]+g[(\nabla_{X_1}F)(\nabla_{Y_1}Z_1),W_1]+g(F(\nabla_{X_1}\nabla_{Y_1}Z_1),W_1)\\&+g[\nabla_{Y_1}(\nabla_{Z_1}F)X_1,W_1]-g[(\nabla_{Y_1}F)(\nabla_{X_1}Z_1),W_1]-g(F(\nabla_{Y_1}\nabla_{X_1}Z_1),W_1)\\&-g[(\nabla_{[X_1,Y_1]}F)Z_1,W_1]-g(F(\nabla_{[X_1,Y_1]}Z_1),W_1)\\&=\frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1)],\end{aligned}$$

this implies

$$\begin{aligned}&2g(F(R(X_1,Y_1)Z_1,W_1))+g[(\nabla_{X_1}F)(\nabla_{Y_1}Z_1),W_1]-g[(\nabla_{Y_1}F)(\nabla_{X_1}Z_1),W_1]\\&=\frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1)].\end{aligned}$$

Hence the proof. □

Theorem 3.4.2. *For a con-circularly flat nearly Kähler manifold the following expression holds*

$$\sum_{i=1}^{n} g((\nabla_{e_i}F)(e_i),(\nabla_{e_i}F)(e_i))=0.$$

Proof. In a nearly Kähler manifold, the curvature tensor $\widetilde{R}$ follows the following relations [18]

$$\widetilde{R}(X_1,Y_1,X_1,Y_1)=\widetilde{R}(X_1,Y_1,F(X_1),F(Y_1))+g((\nabla_{X_1}F)(Y_1),(\nabla_{X_1}F)(Y_1)),$$

where $\widetilde{R}(X_1,Y_1,X_1,Y_1)=g(R(X_1,Y_1)X_1,Y_1)$.
Also, for a con-circularly flat manifold, we have $\widetilde{C}(X_1,Y_1,Z_1,W_1)=0$. So

$$\widetilde{R}(X_1,Y_1,Z_1,W_1)=\frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1)]. \tag{3.4.4}$$

Now from (3.4.4) and putting $X_1=Y_1=e_i$, $1\leq i\leq n$ and summing over i, we obtain

$$\sum_{i=1}^{n} g((\nabla_{e_i}F)(e_i),(\nabla_{e_i}F)(e_i))=0.$$

This completes the proof. □

Note 3.4.1. *For a conformally flat, projectively flat, con-harmonic flat, and Bochner flat nearly Kähler manifold the following relations hold*

$$\sum_{i=1}^{n} g((\nabla_{e_i}F)(e_i),(\nabla_{e_i}F)(e_i))=0.$$

Theorem 3.4.3. *If a nearly Kähler manifold with constant holomorphic sectional curvature c at every point P is con-circularly flat, then*

$$\sum_{i=1}^{n} g((\nabla_{X_1}F)(Y_1),(\nabla_{e_i}F)(e_i))=0.$$

Proof. We know that in a nearly Kähler manifold M with a constant holomorphic sectional curvature c at every point P in M, the Riemannian curvature tensor of M takes the following form [18]

$$\begin{aligned}\widetilde{R}(X_1,Y_1,Z_1,W_1) &= \frac{c}{4}[g(X_1,W_1)g(Y_1,Z_1) - g(X_1,Z_1)g(Y_1,W_1)\\ &+g(X_1,F(W_1))g(Y_1,F(Z_1)) - g(X_1,F(Z_1))g(Y_1,F(W_1))\\ &-2g(X_1,F(Y_1))g(Z_1,F(W_1))]\\ &+\frac{1}{4}[g((\nabla_{X_1}F)W_1,(\nabla_{Y_1}F)Z_1) - g((\nabla_{X_1}F)Z_1,(\nabla_{Y_1}F)W_1)\\ &-2g((\nabla_{X_1}F)Y_1,(\nabla_{Z_1}F)W_1)]. \qquad (3.4.5)\end{aligned}$$

Also, for a con-circularly flat manifold, we have $\widetilde{C}(X_1,Y_1,Z_1,W_1) = 0$. So

$$\widetilde{R}(X_1,Y_1,Z_1,W_1) = \frac{r}{n(n-1)}[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)]. \qquad (3.4.6)$$

Now, from the equation (3.4.5) and putting $Z_1 = W_1 = e_i$, $1 \leq i \leq n$ and summing over i, we have

$$\sum_{i=1}^{n} g((\nabla_{X_1}F)(Y_1),(\nabla_{e_i}F)(e_i)) = 0.$$

Hence the proof. □

Note 3.4.2. *For a conformally flat, projectively flat, con-harmonic flat, and Bochner flat nearly Kähler manifold M with constant holomorphic sectional curvature c at every point P in M, the following expression holds*

$$\sum_{i=1}^{n} g((\nabla_{X_1}F)(Y_1),(\nabla_{e_i}F)(e_i)) = 0.$$

3.5 Curvature identities in 6-dimensional nearly Kähler manifold

For a 6-dimensional nearly Kähler manifold the con-circular curvature tensor represents the form

$$\widetilde{C}(X_1,Y_1,Z_1,W_1) = \widetilde{R}(X_1,Y_1,Z_1,W_1) - \frac{r}{30}[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)].$$

So, we deduce the following result:

Result 3.5.1. *For a con-circularly flat* 6*-dimensional nearly Kähler manifold, the expression of the Ricci curvature tensor is* $S(X_1, Y_1) = \frac{r}{6} g(X_1, Y_1)$*. So the manifold is an Einstein manifold.*

3.6 Example of nearly Kähler manifold

A 6-dimensional unit sphere S^6 has an almost complex structure F defined by the vector cross product in the space of purely imaginary Cayley numbers. This almost complex structure is not integrable and satisfies $(\nabla_{X_1} F)(X_1) = 0$, for every vector field X_1 on S^6. Hence, S^6 is a nearly Kähler manifold which is not Kähler.

The results can be verified in the above example.

4

Some Curvature Identities on Kähler-Norden Manifolds

4.1 Introduction

Norden [46] was the first to study almost complex manifolds with his metric. In order to classify an almost complex manifold with respect to the covariant derivative of the almost complex structure, Ganchev et al. [21] used the Norden metric. Ganchev et al. [22] classified the almost-contact manifolds with Norden-metric in 1993 and introduced the geometry of these manifolds.

The criteria of the pseudosymmetry and semisymmetry types for the Riemann, Ricci, and Weyl curvature tensors of Kählerian and paraKählerian manifolds were investigated in the publications [35, 36, 48] and several others. Using a Kähler-Norden manifold, we expand Sluka's [64] result in this chapter. Sluka [65] created some illustrations of semisymmetric and locally symmetric Kähler-Norden manifolds, as well as ones that are holomorphically projectively flat.

This chapter contains five sections of which the first two sections are the introduction and preliminaries. In the third section, we study some curvature identities on Kähler-Norden manifolds, specifically focusing on quasi-conformally flat, pseudo-projectively flat, Weyl-conformally flat, and Bochner flat. In the next section, we show that a Kähler-Norden manifold is pseudo-projectively symmetric if and only if it is locally symmetric and proved that Kähler-Norden manifolds are quasi-conformally symmetric, Weyl-conformally symmetric, and Bochner symmetric if and only if these are all locally symmetric. In the last section, we also conduct a study on semi-symmetric Kähler-Norden manifold and proved that Kähler-Norden manifolds are pseudo-projectively semi-symmetric, quasi-conformally semi-symmetric, Weyl-conformally semi-symmetric, and Bochner semi-symmetric if and only if these are all semi-symmetric.

4.2 Preliminaries

Definitions and some basic characteristics of the Kähler-Norden manifold, various types of curvature tensors, and r^* curvature tensors are provided in the introductory chapter. Here, we review several important findings from the previous and apply these to our current work.

Now, within a Kähler-Norden manifold [14], the subsequent properties are fulfilled:

$$R(FX_1, FY_1)Z_1 = -R(X_1, Y_1)Z_1, \tag{4.2.1}$$

$$R(FX_1, Y_1)Z_1 = R(X_1, FY_1)Z_1, \tag{4.2.2}$$

$$S(FX_1, Y_1) = S(FY_1, X_1), \tag{4.2.3}$$

$$S(FX_1, FY_1) = -S(X_1, Y_1). \tag{4.2.4}$$

If we take Q as the Ricci operator then the Ricci tensor S in terms of Q is expressed as

$$S(X_1, Y_1) = g(QX_1, Y_1), \tag{4.2.5}$$

where

$$rQY_1 = -\sum_i \epsilon_i R(e_i, Y_1)e_i,$$

and $\{e_i\}$, $1 \leq i \leq n$ is an orthonormal basis and ϵ_i are the indicators of e_i. The Riemannian metric g in terms of e_i and ϵ_i is given by

$$\epsilon_i = g(e_i, e_i) = \pm 1, \tag{4.2.6}$$

$$g(Fe_i, e_i) = 0. \tag{4.2.7}$$

Definition 4.2.1. *[14] A Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of the manifold.*

Definition 4.2.2. *[14] A pseudo-projectively curvature tensor is said to be parallel if the covariant derivative of pseudo-projective curvature tensor vanishes i.e.* $\nabla \overline{P} = 0$*, and this type of manifold is called a pseudo-projectively symmetric manifold.*

Definition 4.2.3. *[58, 30] Let (M,g) be a Riemannian or pseudo-Riemannian manifold is called semi-symmetric if* $R(X_1, Y_1).R = 0$*, Ricci semi-symmetric if* $R(X_1, Y_1).S = 0$*, where* $R(X_1, Y_1)$ *denotes the derivation in the tensor algebra at each point of the manifold.*

4.3 Some results on curvature identities on Kähler-Norden manifold

In the following part, we take the manifold into consideration as a even-dimensional Kähler-Norden manifold where the corresponding Ricci tensors fulfill the r^* curvature tensors.

Theorem 4.3.1. *In a quasi-conformally flat Kähler-Norden manifold, the Ricci tensor follows the relation* $S(Y_1, W_1) = \frac{br^*}{a-2b} g(FY_1, W_1)$*, provided* $a \neq 2b$*.*

Proof. In an Kähler-Norden manifold of dimension n, the Ricci tensor S is expressed by

$$S(X_1, Y_1) = \sum_{i=1}^{n} \epsilon_i \widetilde{R}(F(e_i), F(Y_1), e_i, W_1). \tag{4.3.1}$$

Considering the inner product of (1.2.7) with W_1, we obtain

$$\begin{aligned} g(\overline{C}(X_1, Y_1)Z_1, W_1) \;=\;& a\widetilde{R}(X_1, Y_1, Z_1, W_1) + b[S(Y_1, Z_1)g(X_1, W_1) \\ & -S(X_1, Z_1)g(Y_1, W_1) \\ & +g(Y_1, Z_1)S(X_1, W_1) - g(X_1, Z_1)S(Y_1, W_1)] \\ & -\frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y_1, Z_1)g(X_1, W_1) \\ & -g(X_1, Z_1)g(Y_1, W_1)]. \end{aligned} \tag{4.3.2}$$

Now, as the manifold is quasi-conformally flat, from (4.3.2) we get

$$\begin{aligned} & a\widetilde{R}(X_1, Y_1, Z_1, W_1) + b[S(Y_1, Z_1)g(X_1, W_1) - S(X_1, Z_1)g(Y_1, W_1) \\ & +g(Y_1, Z_1)S(X_1, W_1) - g(X_1, Z_1)S(Y_1, W_1)] \\ & -\frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1)] = 0. \end{aligned} \tag{4.3.3}$$

Setting $X_1 = Fe_i, Y_1 = FY_1, Z_1 = e_i$ in (4.3.3) and summing over $i = 1, 2, ..., n$, and applying (4.3.1), (1.3.17), (4.2.3), (4.2.4) and (4.2.7), we have

$$(a-2b)S(Y_1, W_1) - br^* g(FY_1, W_1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right) g(Y_1, W_1) = 0. \tag{4.3.4}$$

Taking $Y_1 = W_1 = e_i$ in (4.3.4) and summing over $i = 1, 2, ..., n$, and applying (1.3.16), we obtain

$$anr = 0.$$

This implies

$$r = 0, \quad \textit{provided} \;\; a \neq 0.$$

Then (4.3.4) becomes

$$(a-2b)S(Y_1, W_1) - br^* g(FY_1, W_1) = 0.$$

This implies

$$S(Y_1, W_1) = \frac{br^*}{a-2b} g(FY_1, W_1), \quad \textit{provided} \;\; a \neq 2b.$$

This completes the proof. □

Theorem 4.3.2. *In a pseudo-projectively flat Kähler-Norden manifold, the Ricci tensor obeys the relation* $S(Y_1, W_1) = \frac{br^*}{a-b} g(FY_1, W_1)$*, provided* $a \neq b$.

Proof. Taking scalar product of (1.2.5) with W_1 leads to

$$\begin{aligned} g(\overline{P}(X_1, Y_1)Z_1, W_1) &= a\widetilde{R}(X_1, Y_1, Z_1, W_1) + b[S(Y_1, Z_1)g(X_1, W_1) - S(X_1, Z_1)g(Y_1, W_1)] \\ &\quad - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1)]. \end{aligned} \tag{4.3.5}$$

Now, as the manifold is pseudo-projectively flat, from (4.3.5) we get

$$\begin{aligned} &a\widetilde{R}(X_1, Y_1, Z_1, W_1) + b[S(Y_1, Z_1)g(X_1, W_1) - S(X_1, Z_1)g(Y_1, W_1)] \\ &-\frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1)] = 0. \end{aligned} \tag{4.3.6}$$

Setting $X_1 = Fe_i, Y_1 = FY_1, Z_1 = e_i$ in (4.3.6) and summing over $i = 1, 2, ..., n$, and applying (4.3.1), (1.3.17), (4.2.3), (4.2.4) and (4.2.7), we have

$$(a-b)S(Y_1, W_1) - br^* g(FY_1, W_1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right) g(Y_1, W_1) = 0. \tag{4.3.7}$$

Taking $Y_1 = W_1 = e_i$ in (4.3.7) and summing over $i = 1, 2, ..., n$, and applying (1.3.16), we obtain

$$anr = 0.$$

This implies

$$r = 0, \quad \textit{provided} \;\; a \neq 0.$$

Then (4.3.7) becomes

$$(a - b)S(Y_1, W_1) - br^* g(FY_1, W_1) = 0.$$

This implies

$$S(Y_1, W_1) = \frac{br^*}{a - b} g(FY_1, W_1), \quad \textit{provided} \;\; a \neq b.$$

Hence the proof. □

Theorem 4.3.3. *In a Weyl-conformally flat Kähler-Norden manifold, the Ricci tensor follows the property* $S(Y_1, U_1) = -\frac{r^*}{n} g(FY_1, U_1)$*, provided* $n \neq 0$.

Proof. Considering the inner product of (1.2.8) with U_1, we acquire

$$\begin{aligned} g(W(X_1, Y_1)Z_1, U_1) = \widetilde{R}(X_1, Y_1, Z_1, U_1) &- \frac{1}{n - 2}[g(Y_1, Z_1)S(X_1, U_1) - g(X_1, Z_1)S(Y_1, U_1) \\ &+ S(Y_1, Z_1)g(X_1, U_1) - S(X_1, Z_1)g(Y_1, U_1)] \\ &+ \frac{r}{(n - 1)(n - 2)}[g(Y_1, Z_1)g(X_1, U_1) - g(X_1, Z_1)g(Y_1, U_1)]. \end{aligned} \tag{4.3.8}$$

Now, as the manifold is Weyl-conformally flat, from (4.3.8) we obtain

$$\begin{aligned} &\widetilde{R}(X_1, Y_1, Z_1, U_1) - \frac{1}{n - 2}[g(Y_1, Z_1)S(X_1, U_1) - g(X_1, Z_1)S(Y_1, U_1) \\ &+ S(Y_1, Z_1)g(X_1, U_1) - S(X_1, Z_1)g(Y_1, U_1)] \\ &+ \frac{r}{(n - 1)(n - 2)}[g(Y_1, Z_1)g(X_1, U_1) - g(X_1, Z_1)g(Y_1, U_1)] = 0. \end{aligned} \tag{4.3.9}$$

Setting $X_1 = Fe_i, Y_1 = FY_1, Z_1 = e_i$ in (4.3.9) and summing over $i = 1, 2, ..., n$, and applying (4.3.1), (1.3.17), (4.2.3), (4.2.4) and (4.2.7), we get

$$\frac{n}{n - 2} S(Y_1, U_1) + \frac{r^*}{n - 2} g(FY_1, U_1) - \frac{r}{(n - 1)(n - 2)} g(Y_1, U_1) = 0. \tag{4.3.10}$$

Taking $Y_1 = U_1 = e_i$ in (4.3.10) and summing over $i = 1, 2, ..., n$, and also applying (1.3.16), we have

$$nr = 0.$$

This implies

$$r = 0, \quad \textit{provided} \;\; n \neq 0$$

Then (4.3.10) becomes

$$\frac{n}{n-2}S(Y_1, U_1) + \frac{r^*}{n-2}g(FY_1, U_1) = 0.$$

This implies

$$S(Y_1, U_1) = -\frac{r^*}{n}g(FY_1, U_1), \quad \textit{provided} \;\; n \neq 0.$$

This completes the proof. □

Theorem 4.3.4. *In a Bochner flat Kähler-Norden manifold, the Ricci tensor obeys the relation* $S(Y_1, W_1) = -\frac{r^*}{2(n+4)}g(FY_1, W_1)$, *provided* $n + 4 \neq 0$.

Proof. Considering the scalar product of (1.2.10) with W_1, we achieve

$$\begin{aligned} g(B(X_1, Y_1)Z_1, W_1) \;\; = \;\; & \widetilde{R}(X_1, Y_1, Z_1, W_1) - \frac{1}{n+4}[g(Y_1, Z_1)S(X_1, W_1) \\ & -g(X_1, Z_1)S(Y_1, W_1) + S(Y_1, Z_1)g(X_1, W_1) \\ & -S(X_1, Z_1)g(Y_1, W_1) + g(FY_1, Z_1)S(FX_1, W_1) \\ & -g(FX_1, Z_1)S(FY_1, W_1) + S(FY_1, Z_1)g(FX_1, W_1) \\ & -S(FX_1, Z_1)g(FY_1, W_1) - 2S(FX_1, Y_1)g(FZ_1, W_1) \\ & -2g(FX_1, Y_1)S(FZ_1, W_1)] \\ & +\frac{r}{(n+2)(n+4)}[g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1) \\ & +g(FY_1, Z_1)g(FX_1, W_1) - g(FX_1, Z_1)g(FY_1, W_1) \\ & -2g(FX_1, Y_1)g(FZ_1, W_1)]. \end{aligned} \tag{4.3.11}$$

Now, as the manifold is Bochner flat, from (4.3.11) we get

$$\begin{aligned} \widetilde{R}(X_1, Y_1, Z_1, W_1) \;\; = \;\; & \frac{1}{n+4}[g(Y_1, Z_1)S(X_1, W_1) - g(X_1, Z_1)S(Y_1, W_1) \\ & +S(Y_1, Z_1)g(X_1, W_1) - S(X_1, Z_1)g(Y_1, W_1) \\ & +g(FY_1, Z_1)S(FX_1, W_1) - g(FX_1, Z_1)S(FY_1, W_1) \\ & +S(FY_1, Z_1)g(FX_1, W_1) - S(FX_1, Z_1)g(FY_1, W_1) \\ & -2S(FX_1, Y_1)g(FZ_1, W_1) - 2g(FX_1, Y_1)S(FZ_1, W_1)] \\ & -\frac{r}{(n+2)(n+4)}[g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1) \\ & +g(FY_1, Z_1)g(FX_1, W_1) - g(FX_1, Z_1)g(FY_1, W_1) \\ & -2g(FX_1, Y_1)g(FZ_1, W_1)]. \end{aligned} \tag{4.3.12}$$

Putting $X_1 = Fe_i, Y_1 = FY_1, Z_1 = e_i$ in (4.3.12) and summing over $i = 1, 2, ..., n$, and applying (4.3.1), (1.3.16), (1.3.17), (4.2.3), (4.2.4) and (4.2.7), we have

$$S(Y_1, W_1) = -\frac{r^*}{2(n+4)} g(FY_1, W_1), \quad \textit{provided} \ \ n + 4 \neq 0.$$

Hence the proof. □

Corollary 4.3.1. *In a Bochner Kähler-Norden manifold, the scalar curvature vanishes.*

Proof. Setting $Y_1 = W_1 = e_i$ in the above equation, and taking the summation over $i = 1, 2, ..., n$, we obtain $r = 0$.
Therefore, the proof is complete. □

4.4 Symmetric Kähler-Norden manifold

Theorem 4.4.1. *A Kähler-Norden manifold is pseudo-projectively symmetric if it is locally symmetric and conversely.*

Proof. Taking the covariant derivative of equation (1.2.5) and putting $X_1 = Fe_i, Y_1 = FY_1, Z_1 = e_i, W_1 = W_1$, and also using $\nabla \overline{P} = 0$, we acquire

$$(a-b)(\nabla_{X_1} S)(Y_1, W_1) - b dr^*(X_1) g(FY_1, W_1) + \frac{dr(X_1)}{n}\left(\frac{a}{n-1} + b\right) g(Y_1, W_1) = 0. \tag{4.4.1}$$

Now, putting $Y_1 = W_1 = e_i$ in (4.4.1), we have

$$an(dr(X_1)) = 0. \tag{4.4.2}$$

Since $a \neq 0$, which implies

$$dr(X_1) = 0. \tag{4.4.3}$$

Again, using (4.4.3) in (4.4.1), we obtain

$$(\nabla_{X_1} S)(Y_1, W_1) = \frac{b}{a-b} dr^*(X_1) g(FY_1, W_1). \tag{4.4.4}$$

Putting $Y_1 = FY_1$ in (4.4.4), we get

$$(\nabla_{X_1} S)(FY_1, W_1) = -\frac{b}{a-b} dr^*(X_1) g(Y_1, W_1). \tag{4.4.5}$$

Once again, replacing Y_1 and W_1 in equation (4.4.5) by e_i, we have

$$\left(1 + \frac{bn}{a-b}\right) dr^*(X_1) = 0, \tag{4.4.6}$$

this implies

$$dr^*(X_1) = 0. \tag{4.4.7}$$

Applying (4.4.7) in (4.4.4), we get

$$(\nabla_{X_1} S)(Y_1, W_1) = 0. \tag{4.4.8}$$

Now, taking the covariant derivative of (1.2.5) and using (4.4.3) and (4.4.8), we obtain

$$(\nabla_{X_1}\overline{P})(Y_1, Z_1, U_1, V_1) = a(\nabla_{X_1}R)(Y_1, Z_1, U_1, V_1), where \;\; a \neq 0.$$

This proves the theorem. □

From theorem 4.4.1, we get the following corollary:

Corollary 4.4.1. *Kähler-Norden manifolds are quasi-conformally symmetric, Weyl-con formally symmetric and Bochner symmetric if and only if these are all locally symmetric.*

4.5 Semi-symmetric Kähler-Norden manifold

Theorem 4.5.1. *A Kähler-Norden manifold is pseudo-projectively semi-symmetric if it is semi-symmetric and conversely.*

Proof. From equation (1.2.5) and putting $X_1 = Fe_i, Y_1 = FY_1, Z_1 = e_i, W_1 = W_1$, we obtain

$$\sum_{i=1}^{n} \epsilon_i \overline{P}(Fe_i, FY_1)e_i = (a-b)QY_1 - br^*FY_1 + \frac{r}{n}\left(\frac{a}{n-1} + b\right)Y_1, \tag{4.5.1}$$

where r^* is the trace of FQ and is known as $*$-scalar curvature. If pseudo-projectively curvature tensor in Kähler-Norden manifold satisfies $R.\overline{P} = 0$, then from equation (4.5.1), $R.Q = 0$ and hence $R.S = 0$. Since we know that the Ricci tensors are expressed by $S(X_1, Y_1) = g(QX_1, Y_1)$ and $S(FX_1, Y_1) = g(QFX_1, Y_1)$, then from equation (1.2.5), if $R.\overline{P} = 0$ and $R.S = 0$, then we obtain $R.R = 0$. Conversely if

$$R.R = 0 \Rightarrow R.S = 0 \Rightarrow R.Q = 0, \tag{4.5.2}$$

then from (4.5.1), we have $R.\overline{P} = 0$. Hence the proof. □

From theorem 4.5.1, we get the following corollary:

Corollary 4.5.1. *Kähler-Norden manifolds are quasi-conformally semi-symmetric, Weyl-conformally semi-symmetric and Bochner semi-symmetric if and only if these are all semi-symmetric.*

5

Some Curvature Identities on hyperKähler Manifolds

5.1 Introduction

Riemannian manifolds with only one such automorphism are referred to as Kähler manifolds. Even while the term "hyperKähler" recalls Grassmann's "hypercomplex numbers" rather than Hamilton's quaternions, it was established with E. Calabi [34] and is a correct description—the metric is Kählerian for multiple complex structures. But there is a crucial distinction between hyperKähler and Kähler manifolds. Simply by including a hermitian form $\partial\bar{\partial}f$ for every sufficiently small C^∞ function f, one can change the Kähler metric on a given complex manifold to another. Kähler metrics' space is infinitely dimensional as a result. Examples of Kähler manifolds are also widely available. Since a Kähler metric is inherited by every complex submanifold of CP_n, merely setting down the algebraic equations for a projective variety provides a huge number of examples.

HyperKähler metrics, in comparison, are far more strict. If one such metric exists on a compact manifold, then up to isometry there is only a finite-dimensional space of them. Finding examples is also difficult. They are obviously impossible to locate as quaternionic submanifolds of the quaternionic projective space $\mathbb{H}P_n$ [28].

Considering M. Berger's [33] description of the holonomy groups of Riemannian manifolds in 1955, the concept of a hyperKähler manifold first emerged. Since I, J, and K are covariant constants on a hyperKähler manifold, parallel translation preserves them. As a result, the holonomy group is contained in both the orthogonal group O_{4n} and the group $GL(n,\mathbb{H})$ of quaternionic invertible matrices (i.e., those linear transformations that commute with right multiplication by i, j and k). The group of $n \times n$ quaternionic unitary matrices is the maximum such intersection in SP_n. In Berger's list, this group performed. The linear transformations of C^{2n} that preserve a non-degenerate skew form, U_{2n} and $SP(2n, C)$, intersect to produce the group SP_n. Thus, a hyperKähler manifold is a naturally complex manifold with a holomorphic symplectic form. By using the three Kähler two-forms, $\omega_1(X_1, Y_1) = g(IX_1, Y_1)$, $\omega_2(X_1, Y_1) = g(JX_1, Y_1)$, $\omega_3(X_1, Y_1) = g(KX_1, Y_1)$ for X_1, $Y_1 \in TM$, defined for the complex structures I, J and K, one can clearly understand this. In terms of complex structures, I, J and K, the complex form $\omega_1 = \omega_2 + i\omega_3$ is non-degenerate and covariant constant, making it closed and holomorphic.

This chapter contains seven sections of which the first two sections are the introduction and preliminaries, respectively. In the third section, we study some curvature identities on hyperKähler manifold that is locally symmetric. In the next section, we study conformal flatness of a hyperKähler manifold. Also, for a conformally flat hyperKähler manifold of dimension ≥ 4, we prove that the manifold is locally symmetric. Particularly, if the dimension of the manifold is equal to 4 then its scalar curvature vanishes identically. Next, we investigate a conformally flat hyperKähler manifold of dimension $4n$ which becomes an Einstein manifold. In the later section, we discuss the Bochner flatness of a hyperKähler manifold and prove that this manifold is an Einstein manifold. Later, we establish a generalised W_2-flat hyperKähler manifold and prove that this manifold is an Einstein manifold. Also it is Ricci flat, provided $a \neq \left(b + \frac{c}{4n-7}\right)$. Now, we examine a quasi-W_2 flat hyperKähler manifold is Ricci flat, provided $c \neq 0$. In the last section, we give some examples of a hyperKähler manifold to support our results.

5.2 Preliminaries

In the introductory chapter, the almost hypercomplex manifold, hypercomplex manifold, and hyperKähler manifold are discussed. We want to take this opportunity to mention a few helpful findings that are used to obtain some results.

Let M be a Riemannian manifold with I, J, and K compatible almost complex structures parallel with respect to the operator of the covariant differentiation satisfying $IJ = K = -JI$. Consequently, (a) I, J, and K are integrable, and (b) $\omega_1 = g(I.,.)$, etc. are

symplectic forms. Let $\mathbb{H} = \mathbb{R}^4$ with basis $\{1, i, j, k\}$, $i^2 = -1 = j^2 = k^2$, quaternion division algebra. In $\mathbb{H}^n$, $Iq = -qi$ holds, with a standard inner product. We also know $SP_1 = SU_2 = \{ai + bj + ck : a^2 + b^2 + c^2 = 1\}$ acts on the right. $SP_n = \{A \in M_n(\mathbb{H}) \mid \overline{A}^T A = I_n\}$ is the centraliser in SO_{4n} of SP_1.
Now, we have the following propositions:

Proposition 5.2.1. *[37, 57] A hyperKähler manifold M is defined as a complex manifold that possesses a holomorphic symplectic form. Conversely, any compact Kähler manifold with a holomorphic symplectic form is hyperKähler.*

Proposition 5.2.2. *[57] A hyperKähler manifold is a C^∞ Riemannian manifold together with three covariantly constant orthogonal endomorphisms I, J and K of the tangent bundle which satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$.*

Note that I, J, and K induce quaternionic vector space structure on each tangent space. So, dimension of a hyperKähler manifold is divisible by 4. Since I, J, and K are covariantly constant, a parallel transport commutes with the quaternionic multiplication, and so the holonomy group is contained in $O_{4n} \bigcap GL_n(\mathbb{H}) \cong SP_n$, the group of quaternionic unitary $n \times n$ matrices. In particular, since $SP_n \subseteq SU_{2n}$ for every hyperKähler manifold is Calabi-Yau [34].

5.3 Results of Some Curvature Identities on hyper Kähler Manifold

We investigated some properties of curvature tensors and Ricci tensors of the hyperKähler manifold.

Theorem 5.3.1. *On a hyperKähler manifold M, the Riemannian curvature tensor R*

satisfies

$$\begin{aligned}
&(i) R(X_1,Y_1)IZ_1 = IR(X_1,Y_1)Z_1,\\
&(ii) R(IX_1,IY_1)Z_1 = R(X_1,Y_1)Z_1,\\
&(iii) R(IX_1,Y_1)Z_1 + R(X_1,IY_1)Z_1 = 0,\\
&(iv) \widetilde{R}(IX_1,IY_1,IZ_1,IW_1) = \widetilde{R}(X_1,Y_1,Z_1,W_1),\\
&(v) \widetilde{R}(IX_1,Y_1,IZ_1,W_1) = \widetilde{R}(X_1,IY_1,Z_1,IW_1),\\
&(vi) \widetilde{R}(X_1,Y_1,IZ_1,JW_1) = -\widetilde{R}(IX_1,IY_1,Z_1,IJW_1),\\
&(vii) \widetilde{R}(IX_1,IY_1,JZ_1,JW_1) = \widetilde{R}(X_1,Y_1,IJZ_1,IJW_1),
\end{aligned}$$

where $\widetilde{R}(X_1,Y_1,Z_1,W_1) = g(R(X_1,Y_1)Z_1,W_1)$.

Proof. (i) Since I is parallel, i.e., $(\nabla_{X_1}I)(Y_1) = 0$, we get

$$\nabla_{X_1}I(Y_1) = I(\nabla_{X_1}Y_1).$$

Now

$$\begin{aligned}
R(X_1,Y_1)I(Z_1) &= \nabla_{X_1}\nabla_{Y_1}I(Z_1) - \nabla_{Y_1}\nabla_{X_1}I(Z_1) - \nabla_{[X_1,Y_1]}I(Z_1)\\
&= \nabla_{X_1}I(\nabla_{Y_1}Z_1) - \nabla_{Y_1}I(\nabla_{X_1}Z_1) - I(\nabla_{[X_1,Y_1](Z_1)})\\
&= I(\nabla_{X_1}\nabla_{Y_1}Z_1) - I(\nabla_{Y_1}\nabla_{X_1}Z_1) - I(\nabla_{[X_1,Y_1]}Z_1)\\
&= I(R(X_1,Y_1)Z_1).
\end{aligned}$$

(ii) Since $g(R(X_1,Y_1)V_1,U_1) = g(R(U_1,V_1)Y_1,X_1)$, we have

$$\begin{aligned}
g(R(IX_1,IY_1)V_1,U_1) &= g(R(U_1,V_1)IY_1,IX_1)\\
&= g(I(R)(U_1,V_1)Y_1,IX_1)\\
&= -g(R(U_1,V_1)Y_1,I^2(X_1)), \quad [\mathit{since}\ \ g(IX_1,Y_1) = -g(X_1,IY_1)]\\
&= g(R(U_1,V_1)Y_1,X_1), \quad [\mathit{since}\ \ I^2 = J^2 = K^2 = -1\\
&\qquad\qquad \mathit{and}\ \ IJ = -K = JI]\\
&= g(R(X_1,Y_1)V_1,U_1).
\end{aligned}$$

Hence, $R(IX_1,IY_1)V_1 = R(X_1,Y_1)V_1$.

(iii) Putting $X_1 = IX_1$ in *(ii)*, we obtain *(iii)*.

(iv) Now

$$\begin{aligned} g(R(IX_1, IY_1)IZ_1, IW_1) &= -g(I(R)(IX_1, IY_1)IZ_1, W_1), \\ &\qquad\qquad [since \;\; g(IX_1, Y_1) = -g(X_1, IY_1)] \\ &= -g(R(IX_1, IY_1)Z_1, W_1) \\ &= g(R(X_1, Y_1)Z_1, W_1), \\ &\qquad [using \;\; g(R(IX_1, IY_1)V_1, U_1) = g(R(X_1, Y_1)V_1, U_1)] \end{aligned}$$

Therefore, $\widetilde{R}(IX_1, IY_1, IZ_1, IW_1) = \widetilde{R}(X_1, Y_1, Z_1, W_1)$.

(v) Setting $Y_1 = IY_1$, $W_1 = IW_1$ in *(iv)*, we get *(v)*.

(vi) Putting $X_1 = IX_1$, $Y_1 = IY_1$, $W_1 = KW_1$, where $IJ = K = -JI$ in equation *(iv)*, then we obtain

$$\widetilde{R}(X_1, Y_1, IZ_1, JW_1) = -\widetilde{R}(IX_1, IY_1, Z_1, IJW_1).$$

(vii) Again putting $Z_1 = KZ_1$, $W_1 = KW_1$, where $IJ = K = -JI$ in equation *(iv)*, then we have $\widetilde{R}(IX_1, IY_1, JZ_1, JW_1) = \widetilde{R}(X_1, Y_1, IJZ_1, IJW_1)$. □

Remark 5.3.1. *Accordingly, theorem 5.3.1 holds for operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above curvature identities also hold for the operators IJ and JI.*

Theorem 5.3.2. *The Ricci tensor of a hyperKähler manifold follows the following relations*

$$\begin{aligned} &(i) S(IX_1, IY_1) = S(X_1, Y_1), \\ &(ii) S(IX_1, Y_1) + S(X_1, IY_1) = 0. \end{aligned}$$

Proof.

$$\begin{aligned}
S(IX_1, IY_1) &= trace\{Z_1 \to R(Z_1, IX_1)IY_1\} \\
&= trace\{IZ_1 \to R(IZ_1, IX_1)IY_1\} \\
&= trace\{IZ_1 \to R(Z_1, X_1)IY_1\}, \quad [by\ (ii)\ of\ theorem\ 5.3.1] \\
&= trace\{IZ_1 \to IR(Z_1, X_1)Y_1\}, \quad [sinceIR = RI] \\
&= trace\{Z_1 \to R(Z_1, X_1)Y_1\} \\
&= S(X_1, Y_1),
\end{aligned}$$

which proves (i).
Now setting $X_1 = IY_1$ in (i), we obtain (ii). □

Remark 5.3.2. *In parallel, theorem 5.3.2 holds for the operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above Ricci tensor of a hyperKähler manifold also satisfies the operators IJ and JI.*

Theorem 5.3.3. *For a hyperKähler manifold of dimension $4n$ the following relation holds, i.e., $\sum\limits_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), X_1, I(Y_1)) = 0$.*

Proof. We have

$$\begin{aligned}
S(X_1, Y_1) &= \sum_{i=1}^{4n} \epsilon_i g(R(e_i, X_1)Y_1, e_i) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(I(e_i), I(X_1))Y_1, e_i) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(e_i, Y_1)I(X_1), I(e_i)) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(Y_1, e_i)I(e_i), I(X_1)) \\
&= \sum_{i=1}^{4n} \epsilon_i g(R(Y_1, e_i)I(X_1), I(e_i))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{4n} \epsilon_i g(R(I(X_1), e_i)Y_1, I(e_i)) + \sum_{i=1}^{4n} \epsilon_i g(R(Y_1, I(X_1))e_i, I(e_i)), \\
&\qquad\qquad\qquad\qquad\qquad\qquad [using\ Bianchi's\ identities] \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(e_i, I(X_1))Y_1, I(e_i)) - \sum_{i=1}^{4n} \epsilon_i g(R(I(X_1), Y_1)e_i, I(e_i)) \\
&= \sum_{i=1}^{4n} \epsilon_i g(I(R)(e_i, I(X_1))Y_1, e_i) + \sum_{i=1}^{4n} \epsilon_i g(I(R)(I(X_1), Y_1)e_i, e_i) \\
&= \sum_{i=1}^{4n} \epsilon_i g(R(e_i, I(X_1))I(Y_1), e_i) + \sum_{i=1}^{4n} \epsilon_i g(R(I(X_1), Y_1)I(e_i), e_i) \\
&= S(IX_1, IY_1) - \sum_{i=1}^{4n} \epsilon_i g(R(Y_1, I(X_1))I(e_i), e_i) \\
&= S(X_1, Y_1) + \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), X_1, I(Y_1)).
\end{aligned}
$$

So, this implies $\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), X_1, I(Y_1)) = 0$. □

Remark 5.3.3. *Comparably, theorem 5.3.3 holds for the operators* J, K*. Since* $I^2 = J^2 = K^2 = -1$ *and* $IJ = -K = JI$*, so the above global form of curvature tensors of a hyperKähler manifold also satisfy for the operators* IJ *and* JI*, i.e.*

$$
\begin{aligned}
(i)\ &\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, J(e_i), X_1, J(Y_1)) = 0, \\
(ii)\ &\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, IJ(e_i), X_1, IJ(Y_1)) = 0, \\
(iii)\ &\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, JI(e_i), X_1, JI(Y_1)) = 0.
\end{aligned}
$$

5.4 Conformal flatness of hyperKähler manifold

We concentrate on dimension $4n \geq 4$ because every 3-dimensional Riemannian or pseudo-Riemannian manifold is conformally flat. We demonstrate the following theorem using the identities from the preceding section.

Theorem 5.4.1. *Let* M *be a conformally flat hyperKähler manifold. Then*
(i) M *is locally flat if dim* $M \geq 4$*,*
(ii) M *is locally symmetric, and its scalar curvature vanishes identically if dim* $M = 4$*.*

Proof. By the vanishing of the conformal curvature tensor, we have

$$\begin{aligned}\widetilde{R}(X_1, Y_1, Z_1, W_1) = &\frac{1}{4n-2}[-g(X_1, Z_1)S(Y_1, W_1) - g(Y_1, W_1)S(X_1, Z_1)\\ &+ g(X_1, W_1)S(Y_1, Z_1) + g(Y_1, Z_1)S(X_1, W_1)]\\ &+ \frac{r}{(4n-1)(4n-2)}[g(X_1, Z_1)g(Y_1, W_1) - g(X_1, W_1)g(Y_1, Z_1)], \quad (5.4.1)\end{aligned}$$

r being the scalar curvature. From the above equation with the help of the theorem 5.3.2 and theorem 5.3.1, we get

$$\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), Z_1, I(W_1)) = -\frac{2}{4n-2}S(Z_1, W_1) + \frac{r}{(4n-1)(4n-2)}g(Z_1, W_1).$$

Then using the result of theorem 5.3.3, we obtain

$$S(Z_1, W_1) = \frac{r}{2(4n-1)}g(Z_1, W_1). \quad (5.4.2)$$

Now, setting $Z_1 = W_1 = e_i, 1 \leq i \leq 4n$ and summing over i, then from equation (5.4.2), we have $(4n-1)r = 0$, where $r = \sum\limits_{i=1}^{4n} \epsilon_i S(e_i, e_i)$. Then this implies $r = 0$, when $4n \geq 4$. Now, putting the value of $r = 0$ in the equation (5.4.2), we get $S = 0$.

So from equation (5.4.1), it follows that the manifold is locally flat. Now, also if $4n = 4$ then $r = 0$, then its scalar curvature vanishes identically. Next, we prove the manifold is locally symmetric. Now, we assume that conformally flatness implies

$$(\nabla_{X_1} S)(Y_1, Z_1) - (\nabla_{Y_1} S)(X_1, Z_1) = \frac{1}{6}[(X_1 r)g(Y_1, Z_1) - (Y_1 r)g(X_1, Z_1)]. \quad (5.4.3)$$

Putting $r = 0$ in the equation (5.4.3), we acquire

$$(\nabla_{X_1} S)(Y_1, Z_1) = (\nabla_{Y_1} S)(X_1, Z_1). \quad (5.4.4)$$

On the other hand, from theorem 5.3.2 it follows that

$$(\nabla_{X_1} S)(Y_1, I(Z_1)) + (\nabla_{X_1} S)(Z_1, I(Y_1)) = 0. \quad (5.4.5)$$

Using the equality and the equation (5.4.5), we get

$$(\nabla_{X_1} S)(Y_1, I(Z_1)) + (\nabla_{Y_1} S)(Z_1, I(X_1)) = 0. \quad (5.4.6)$$

Applying the result of theorem 5.3.2 to the above equation, we obtain $\nabla S = 0$. Now from the equation (5.4.1) and $\nabla S = 0$, implies that $\nabla R = 0$, i.e., the manifold is locally symmetric. □

The subsequent corollary follows from theorem 5.4.1.

Corollary 5.4.1. *A conformally flat hyperKähler manifold of dimension* $4n$ *is an Einstein manifold.*

5.5 Bochner flatness of hyperKähler manifold

Theorem 5.5.1. *A Bochner flat hyperKähler manifold is an Einstein manifold.*

Proof. Taking inner product in the equation (1.2.10) by W_1, we get

$$\begin{aligned} g(B(X_1,Y_1)Z_1,W_1) &= \widetilde{R}(X_1,Y_1,Z_1,W_1)-\frac{1}{4n+4}[g(Y_1,Z_1)S(X_1,W_1) \\ &\quad -g(X_1,Z_1)S(Y_1,W_1)+S(Y_1,Z_1)g(X_1,W_1) \\ &\quad -S(X_1,Z_1)g(Y_1,W_1)+g(IY_1,Z_1)S(IX_1,W_1) \\ &\quad -g(IX_1,Z_1)S(IY_1,W_1)+S(IY_1,Z_1)g(IX_1,W_1) \\ &\quad -S(IX_1,Z_1)g(IY_1,W_1)-2S(IX_1,Y_1)g(IZ_1,W_1) \\ &\quad -2g(IX_1,Y_1)S(IZ_1,W_1)] \\ &\quad +\frac{r}{(4n+2)(4n+4)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1) \\ &\quad +g(IY_1,Z_1)g(IX_1,W_1)-g(IX_1,Z_1)g(IY_1,W_1) \\ &\quad -2g(IX_1,Y_1)g(IZ_1,W_1)]. \end{aligned} \tag{5.5.1}$$

As the manifold is Bochner flat, we can rewrite the foregoing equation as

$$\begin{aligned} \widetilde{R}(X_1,Y_1,Z_1,W_1) &= \frac{1}{4n+4}[g(Y_1,Z_1)S(X_1,W_1)-g(X_1,Z_1)S(Y_1,W_1) \\ &\quad +S(Y_1,Z_1)g(X_1,W_1)-S(X_1,Z_1)g(Y_1,W_1) \\ &\quad +g(IY_1,Z_1)S(IX_1,W_1)-g(IX_1,Z_1)S(IY_1,W_1) \\ &\quad +S(IY_1,Z_1)g(IX_1,W_1)-S(IX_1,Z_1)g(IY_1,W_1) \\ &\quad -2S(IX_1,Y_1)g(IZ_1,W_1)-2g(IX_1,Y_1)S(IZ_1,W_1)] \\ &\quad -\frac{r}{(4n+2)(4n+4)}[g(Y_1,Z_1)g(X_1,W_1)-g(X_1,Z_1)g(Y_1,W_1) \\ &\quad +g(IY_1,Z_1)g(IX_1,W_1)-g(IX_1,Z_1)g(IY_1,W_1) \\ &\quad -2g(IX_1,Y_1)g(IZ_1,W_1)]. \end{aligned} \tag{5.5.2}$$

Setting $X_1 = e_i, Y_1 = Ie_i, Z_1 = Z_1$, and $W_1 = IW_1$ in (5.5.2) and summing over i, $1 \leq i \leq 4n$, and also using the result of the theorem 5.3.3, we have

$$S(Z_1, W_1) = -\frac{3nr}{(n+4)(4n+2)} g(Z_1, W_1). \tag{5.5.3}$$

Hence the proof. □

From theorem 5.5.1, we have the following corollary:

Corollary 5.5.1. *A Bochner flat hyperKähler manifold is locally flat.*

Proof. Taking $Z_1 = W_1 = e_i$ in (5.5.3) and summing over i, $1 \leq i \leq 4n$, we acquire

$$r = 0, \quad \textit{provided} \quad 4n + 2 \neq 0. \tag{5.5.4}$$

Then, equation (5.5.3) becomes

$$S(Z_1, W_1) = 0.$$

So, the manifold is locally flat. □

5.6 Generalised W_2-flatness of hyperKähler manifold

Theorem 5.6.1. *A generalised W_2-flat hyperKähler manifold becomes Ricci flat, provided* $a \neq \left(b + \frac{c}{4n-7}\right)$.

Proof. Since the manifold is a $4n$ dimensional hyperKähler manifolds. Now for $4n > 8$, taking the scalar product of (1.2.13) with U_1, we acquire

$$\begin{aligned} g(\overline{W_2}(X_1, Y_1)Z_1, U_1) &= a\widetilde{R}(X_1, Y_1, Z_1, U_1) + \left(b + \frac{c}{4n-7}\right)[g(X_1, Z_1)S(Y_1, U_1) \\ &\quad - g(Y_1, Z_1)S(X_1, U_1)]. \end{aligned} \tag{5.6.1}$$

Now, as the manifold is $\overline{W}_2$-flat, we can rewrite (5.6.1) as

$$a\widetilde{R}(X_1, Y_1, Z_1, U_1) + \left(b + \frac{c}{4n-7}\right)[g(X_1, Z_1)S(Y_1, U_1) - g(Y_1, Z_1)S(X_1, U_1)] = 0. \tag{5.6.2}$$

Putting $X_1 = e_i, Y_1 = Ie_i, U_1 = IU_1$ in (5.6.2) and summing over i, $1 \leq i \leq 4n$ and operating the result of the theorem 5.3.2, we have

$$\left(b + \frac{c}{4n-7}\right) S(Z_1, U_1) = 0. \tag{5.6.3}$$

Then, we obtain $S(Z_1, U_1) = 0$, provided $\left(b + \frac{c}{4n-7}\right) \neq 0$, for any Z_1, $U_1 \in \chi(M)$. This completes the proof. □

The findings of theorem 5.6.1 lead to the subsequent corollary.

Corollary 5.6.1. *A quasi-W_2 flat hyperKähler manifold becomes Ricci flat, provided $c \neq 0$.*

The following examples are given in the paper [57]

Example 5.6.1. *A trivial example of hyperKähler manifold is $\mathbb{H}^n$. However, in contrast to the Kähler case, $\mathbb{H}P_n$ is not hyperKähler and neither do its generic quaternionic submanifolds.*

Example 5.6.2. *In the particular case $n = 1$, then $SP_1 = SU_2$ in SO_4, so a 4-dimensional Riemannian manifold is hyperKähler exactly when it is Kähler and Ricci flat. Specifically, this shows that any compact complex surface M of Kähler type with vanishing first Chern class is either a torus or simply connected and admits a unique complex-symplectic structure, i.e., is a so-called "K3-surface".*

Example 5.6.3. *A class of non-compact hyperKähler manifolds of real dimension 4 can be obtained by resolving the singularity of C^2/Γ for $\Gamma \subset SU_2$ a finite subgroup.*

Example 5.6.4. *Many examples of non-compact hyperKähler manifolds arise as moduli spaces of solutions to gauge-theoretic equations. The hyperKähler structure is obtained by a hyperKähler reduction from $\mathbb{H}^n$.*

These results can be verified in these examples.

6

Properties of some curvature tensors on paraKähler manifolds and paraKähler space-time

6.1 Introduction

A para-complex geometry is defined to be the geometry related to the algebra of complex numbers and para-complex structures [13] which is the study of structures on differentiable manifolds. A paraKähler structure and its variants have a compatible neutral pseudo-Riemannian metric.

In 1948 Rashevskij [54] introduced the properties of paraKähler manifolds. A scalar field was defined by him which he considered the metric of signature (m, m) defined from a potential function, on a stratified space which is an n-dimensional locally product manifold. Also, in 1949 Rozenfeld [55] explicitly defined the paraKähler manifold. A similarity between complex and para-complex manifolds was established by comparing Rashevskij's definition with Kähler definition in the complex case.

Moreover, the concept of space-time is linked to four-dimensional pseudo-Riemannian manifolds denoted as (M^4, g). These manifolds possess a Lorentz metric denoted as g, characterized by the signature $(-, +, +, +)$. The notion of the causal nature of vectors is crucial in the geometric study of Lorentz manifolds. This is because Lorentz manifolds become a covariant frame for the study of general relativity. V. R. Kaigorodov [40] studied the curvature structure of space-time in 1983. Many authors ([16, 17], and [47]) have extended these concepts of general relativity of space-time. U. C. De, and G. C. Ghosh

[17] derived some results by considering weakly Ricci symmetric space-time in 2004. The concepts of the weakly symmetric and the weakly Ricci symmetric manifolds were presented by the authors [5, 68] in their work. Additionally, the authors M. Prvanovic [52], U. C. De, and S. Bandyopdhayay [15] have examined various instances and provided substantial findings regarding Kähler manifolds that are weakly symmetric and weakly Ricci symmetric.

This kind of evolution prompted us to study the general relativity of space-time in paraKähler manifold admitting the space-time metric, also known as the paraKähler space-time.

This chapter contains nine sections of which the first two sections are the introduction and preliminaries. In the third section, we initiate the study of some curvature identities in paraKähler manifolds that are pseudo-quasi-conformally flat, pseudo-projectively flat, W_2-flat, and Bochner flat and prove that these manifolds are Einstein manifold or Ricci flat. Next in the fourth section, we establish the important results related to the sectional curvature in the paraKähler manifold. In the fifth section, we delve into the investigation of perfect fluid paraKähler space-time. We demonstrate that if such a space-time fulfils the Einstein equation alongside a cosmological constant, it is classified as an Einstein manifold. In the sixth section, our focus is on examining weakly symmetric paraKähler space-time. We demonstrate that in the case of a weakly symmetric perfect fluid paraKähler space-time that adheres to the Einstein equation alongside a cosmological constant, both ρ and $\overline{\rho}$ act as eigenvectors of the Ricci tensor, associated with the eigenvalue $\frac{r}{2}$. In the later section, we investigate weakly Ricci symmetric perfect fluid paraKähler space-time and establish that no such space-time exists that fulfils the Einstein equation alongside a cosmological constant and has a non-zero scalar curvature. Moreover, in the eighth section, we evolve previously mentioned curvature identities along with generalised W_2-flat curvature identity in paraKähler space-time, and prove that these manifolds are Einstein manifold or Ricci flat. In the last section, we contrive the important results related to the sectional curvature in paraKähler space-time.

6.2 Preliminaries

The first chapter already provides definitions and some properties of the paraKähler manifold, the Lorentzian manifold, space-time, perfect fluid in space-time and the paraKähler space-time manifold. Here, we would like to review some findings on these manifolds that were made by illustrious mathematicians and were helpful in reaching the conclusions in this chapter.

Furthermore, a paraKähler manifold satisfies the following properties:

$$R(FX_1, FY_1)Z_1 = -R(X_1, Y_1)Z_1, \tag{6.2.1}$$

$$R(FX_1, Y_1)Z_1 = -R(X_1, FY_1)Z_1, \tag{6.2.2}$$

$$S(FX_1, Y_1) = -S(FY_1, X_1), \tag{6.2.3}$$

$$S(FX_1, FY_1) = -S(X_1, Y_1). \tag{6.2.4}$$

If we consider Q to be the Ricci operator, then the Ricci tensor S in terms of Q is defined as

$$S(X_1, Y_1) = g(QX_1, Y_1),$$

where

$$rQY_1 = -\sum_i \epsilon_i R(e_i, Y_1)e_i,$$

where $\{e_1, e_2, ...e_n\}$ is an orthonormal basis and ϵ_i are the indicators of e_i. The Riemannian metric g is defined as $\epsilon_i = g(e_i, e_i) = 1$ in terms of e_i and ϵ_i.

6.3 Some results on curvature identities in paraKähler manifold

Theorem 6.3.1. *A pseudo-quasi-conformally flat paraKähler manifold becomes an Einstein manifold.*

Proof. In a paraKähler manifold, we interpret the Ricci tensor by

$$S(Z_1, W_1) = \frac{1}{2}\sum_{i=1}^{n} \epsilon_i \widetilde{R}(e_i, F(e_i), F(W_1)), \tag{6.3.1}$$

where $\{e_1, e_2, ...e_n\}$ is an orthonormal basis and n is the dimension of the manifold. Considering the scalar product of (1.2.11) with W_1, we obtain

$$\begin{aligned} g(\widetilde{V}(X_1,Y_1)Z_1,W_1) &= (p+d)\widetilde{R}(X_1,Y_1,Z_1,W_1) + \left(q - \frac{d}{n-1}\right)[S(Y_1,Z_1)g(X_1,W_1) \\ &\quad -S(X_1,Z_1)g(Y_1,W_1)] \\ &\quad +q[g(Y_1,Z_1)S(X_1,W_1) - g(X_1,Z_1)S(Y_1,W_1)] \\ &\quad -\frac{r}{n(n-1)}[p+2(n-1)q][g(Y_1,Z_1)g(X_1,W_1) \\ &\quad -g(X_1,Z_1)g(Y_1,W_1)]. \end{aligned} \tag{6.3.2}$$

Now, as the manifold is pseudo-quasi-conformally flat, the foregoing equation can be rewrite as

$$\begin{aligned} &(p+d)\widetilde{R}(X_1,Y_1,Z_1,W_1) + \left(q - \frac{d}{n-1}\right)[S(Y_1,Z_1)g(X_1,W_1) \\ &-S(X_1,Z_1)g(Y_1,W_1)] + q[g(Y_1,Z_1)S(X_1,W_1) - g(X_1,Z_1)S(Y_1,W_1)] \\ &-\frac{r}{n(n-1)}[p+2(n-1)q][g(Y_1,Z_1)g(X_1,W_1) \\ &-g(X_1,Z_1)g(Y_1,W_1)] = 0. \end{aligned} \tag{6.3.3}$$

Setting $X_1 = e_i, Y_1 = Fe_i, W_1 = FW_1$ in (6.3.3) and summing over $i = 1, 2, ..., n$, and applying (6.3.1), (6.2.3), we have

$$\begin{aligned} &2(p+d)S(Z_1,W_1) + \left(q - \frac{d}{n-1}\right)[-S(FZ_1,FW_1) + S(Z_1,W_1)] \\ &+q[-S(FZ_1,FW_1) + S(Z_1,W_1)] - \frac{r}{n(n-1)}[p+2(n-1)q] \\ &[-g(FY_1,FW_1) + g(Z_1,W_1)] = 0. \end{aligned} \tag{6.3.4}$$

Using (6.2.4) and (1.3.18) in (6.3.4), we acquire

$$\begin{aligned} &2(p+d)S(Z_1,W_1) + 2\left(q - \frac{d}{n-1}\right)S(Z_1,W_1) + 2qS(Z_1,W_1) \\ &-\frac{2r}{n(n-1)}[p+2(n-1)q]g(Z_1,W_1) = 0. \end{aligned} \tag{6.3.5}$$

which reduces to

$$\left[p + 2q + d\left(\frac{n-2}{n-1}\right)\right]S(Z_1,W_1) = \frac{r}{n(n-1)}[p+2(n-1)q]g(Z_1,W_1). \tag{6.3.6}$$

Thus the manifold becomes an Einstein manifold.

This completes the proof. □

Corollary 6.3.1. *A pseudo-quasi-conformally flat paraKähler manifold is Ricci flat, provided p+d $\neq$ 0.*

Proof. Utilizing $Z_1 = W_1 = e_i$ in (6.3.6) and taking sum over $i = 1, 2, ..., n$, we obtain

$$r(p+d)\left(\frac{n-2}{n-1}\right) = 0.$$

As a result

$$r = 0.$$

Then (6.3.6) becomes

$$S(Z_1, W_1) = 0.$$

This implies that the manifold becomes Ricci flat.
Hence the proof. □

Theorem 6.3.2. *A pseudo-projectively flat paraKähler manifold becomes an Einstein manifold.*

Proof. Applying the inner product of (1.2.5) with W_1, we have

$$\begin{aligned} g(\overline{P}(X_1, Y_1)Z_1, W_1) &= a\widetilde{R}(X_1, Y_1, Z_1, W_1) + b[S(Y_1, Z_1)g(X_1, W_1) - S(X_1, Z_1)g(Y_1, W_1)] \\ &- \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1)]. \quad (6.3.7) \end{aligned}$$

As the manifold is pseudo-projectively flat, so the (6.3.7) minimizes as

$$\begin{aligned} &a\widetilde{R}(X_1, Y_1, Z_1, W_1) + b[S(Y_1, Z_1)g(X_1, W_1) - S(X_1, Z_1)g(Y_1, W_1)] \\ &-\frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1)] = 0. \end{aligned} \quad (6.3.8)$$

Setting $X_1 = e_i, Y_1 = Fe_i, W_1 = FW_1$ in (6.3.8) and taking summation over $i = 1, 2, ..., n$, and utilizing (6.3.1), (6.2.3), we obtain

$$\begin{aligned} &2aS(Z_1, W_1) + b[-S(FZ_1, FW_1) + S(Z_1, W_1)] \\ &-\frac{r}{n}\left(\frac{a}{n-1} + b\right)[-g(FZ_1, FW_1) + g(Z_1, W_1)] = 0. \end{aligned} \quad (6.3.9)$$

Using (6.2.4) and (1.3.18) in (6.3.9), we get

$$(a+b)S(Z_1, W_1) = \frac{r}{n}\left(\frac{a}{n-1} + b\right)g(Z_1, W_1). \quad (6.3.10)$$

So the manifold is an Einstein manifold.
Consequently, the proof. □

Corollary 6.3.2. *A pseudo-projectively flat paraKähler manifold is Ricci flat, provided a $\neq 0$.*

Proof. Setting $Z_1 = W_1 = e_i$ in (6.3.10) and summing over $i = 1, 2, ..., n$, we have

$$ar\left(\frac{n-2}{n-1}\right) = 0.$$

It follows because

$$r = 0.$$

Then (6.3.10) becomes

$$S(Z_1, W_1) = 0.$$

According to this, the manifold is Ricci flat.
This completes the proof. □

Theorem 6.3.3. *In a W_2-flat paraKähler manifold, the scalar curvature tensor vanishes.*

Proof. Applying the inner product of (1.2.9) with U_1, we acquire

$$g(W_2(X_1, Y_1)Z_1, U_1) = \widetilde{R}(X_1, Y_1, Z_1, U_1) + \frac{1}{n-1}[g(X_1, Z_1)S(Y_1, U_1) - g(Y_1, Z_1)S(X_1, U_1)]. \tag{6.3.11}$$

As the manifold is W_2-flat, so (6.3.11) contracts as

$$\widetilde{R}(X_1, Y_1, Z_1, U_1) + \frac{1}{n-1}[g(X_1, Z_1)S(Y_1, U_1) - g(Y_1, Z_1)S(X_1, U_1)] = 0. \tag{6.3.12}$$

Putting $X_1 = e_i, Y_1 = Fe_i, U_1 = FU_1$ in (6.3.12) and taking summation over $i = 1, 2, ..., n$, and by virtue of (6.3.1), (6.2.3), we obtain

$$S(Z_1, U_1) + \frac{1}{n-1}[-S(Z_1, U_1) + S(FZ_1, FU_1)] = 0. \tag{6.3.13}$$

Again utilizing (6.2.4) in (6.3.13), we have

$$(n-3)S(Z_1, U_1) = 0. \tag{6.3.14}$$

Hence, since n is even, we get $S(Z_1, U_1) = 0$, for every Z_1, $U_1 \in \chi(M)$. As a result, $r = 0$ can be deduced from the equation above.
The evidence is now complete. □

Theorem 6.3.4. *A Bochner flat paraKähler manifold is an Einstein manifold.*

Proof. Considering the scalar product of (1.2.10) with W_1, we acquire

$$\begin{aligned} g(B(X_1,Y_1)Z_1,W_1) &= \widetilde{R}(X_1,Y_1,Z_1,W_1) - \frac{1}{n+4}[g(Y_1,Z_1)S(X_1,W_1) \\ &\quad -g(X_1,Z_1)S(Y_1,W_1) + S(Y_1,Z_1)g(X_1,W_1) \\ &\quad -S(X_1,Z_1)g(Y_1,W_1) + g(FY_1,Z_1)S(FX_1,W_1) \\ &\quad -g(FX_1,Z_1)S(FY_1,W_1) + S(FY_1,Z_1)g(FX_1,W_1) \\ &\quad -S(FX_1,Z_1)g(FY_1,W_1) - 2S(FX_1,Y_1)g(FZ_1,W_1) \\ &\quad -2g(FX_1,Y_1)S(FZ_1,W_1)] \\ &\quad +\frac{r}{(n+2)(n+4)}[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1) \\ &\quad +g(FY_1,Z_1)g(FX_1,W_1) - g(FX_1,Z_1)g(FY_1,W_1) \\ &\quad -2g(FX_1,Y_1)g(FZ_1,W_1)]. \end{aligned} \tag{6.3.15}$$

As the manifold is Bochner flat, the foregoing equation can be rewritten as

$$\begin{aligned} \widetilde{R}(X_1,Y_1,Z_1,W_1) &= \frac{1}{n+4}[g(Y_1,Z_1)S(X_1,W_1) - g(X_1,Z_1)S(Y_1,W_1) \\ &\quad +S(Y_1,Z_1)g(X_1,W_1) - S(X_1,Z_1)g(Y_1,W_1) \\ &\quad +g(FY_1,Z_1)S(FX_1,W_1) - g(FX_1,Z_1)S(FY_1,W_1) \\ &\quad +S(FY_1,Z_1)g(FX_1,W_1) - S(FX_1,Z_1)g(FY_1,W_1) \\ &\quad -2S(FX_1,Y_1)g(FZ_1,W_1) - 2g(FX_1,Y_1)S(FZ_1,W_1)] \\ &\quad -\frac{r}{(n+2)(n+4)}[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1) \\ &\quad +g(FY_1,Z_1)g(FX_1,W_1) - g(FX_1,Z_1)g(FY_1,W_1) \\ &\quad -2g(FX_1,Y_1)g(FZ_1,W_1)]. \end{aligned} \tag{6.3.16}$$

Setting $X_1 = e_i, Y_1 = Fe_i, W_1 = FW_1$ in (6.3.16) and taking sum over $i = 1, 2, ..., n$, and applying (6.3.1), (6.2.3) and (6.2.4), we have

$$\left(1 + \frac{n}{n+4}\right) S(Z_1,W_1) = -\frac{r}{n+4}\left(1 - \frac{n}{n+2}\right) g(Z_1,W_1).$$

This indicates

$$S(Z_1,W_1) = -\frac{r}{(n+2)^2} g(Z_1,W_1). \tag{6.3.17}$$

This is a representation of an Einstein manifold.

The proof is now done. □

Corollary 6.3.3. *A Bochner flat paraKähler manifold is Ricci flat.*

Proof. Setting $Z_1 = W_1 = e_i$ in (6.3.17) and summing over $i = 1, 2, ..., n$, we obtain

$$\left[1 + \frac{n}{(n+2)^2}\right] r = 0.$$

This implies

$$r = 0.$$

Then (6.3.17) becomes

$$S(Z_1, W_1) = 0.$$

This suggest that the manifold is Ricci flat.
Therefore, the proof. □

6.4 Sectional curvature in paraKähler manifold

The sectional curvature of Riemannian manifolds with dimensions greater than one is one approach to characterize the curvature of these manifolds in Riemannian geometry. For a point c' of the manifold, a two-dimensional linear subspace $\sigma_{c'}$ of the tangent space determines the sectional curvature $K(\sigma_{c'})$. It may be described geometrically as the Gaussian curvature of the surface, generated from geodesics starting at c' and extending in the directions of $\sigma_{c'}$, that has the plane $\sigma_{c'}$ as a tangent plane at c'. The curvature tensor is entirely determined by the sectional curvature.
When two linearly independent tangent vectors are presented at the same points X_1 and Y_1 on M, the sectional curvature is determined by the formula

$$K(X_1, Y_1) = \frac{R(X_1, Y_1, X_1, Y_1)}{g(X_1, X_1)g(Y_1, Y_1) - g(X_1, Y_1)^2},$$

where X_1 and Y_1 represents the tangent vectors, R denotes the Riemann curvature tensor, and g represents the metric tensor.

Theorem 6.4.1. *In pseudo-quasi-conformal flat paraKähler manifold the sectional curvature determined by X_1, Y_1 is*

$$K\left(X_1, Y_1\right) = \frac{r}{n}\left[\frac{p + 2(n-1)q}{(n-1)(p+2q) + (n-2)d}\right].$$

Proof. From the equation (6.3.6), setting the value of $S(Z_1, W_1)$ in the equation (6.3.3), we obtain

$$(p+d)\widetilde{R}(X_1,Y_1,Z_1,W_1) + \left[\frac{r\{p+2(n-1)q\}}{n(n-1)\{p+2q+d(\frac{n-2}{n-1})\}}\right]\left(2q - \frac{d}{n-1}\right)$$
$$[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)] - \frac{r}{n(n-1)}[p+2(n-1)q]$$
$$[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)] = 0. \qquad (6.4.18)$$

which reduces to

$$\widetilde{R}(X_1,Y_1,Z_1,W_1) - \frac{r}{n}\left[\frac{p+2(n-1)q}{(n-1)(p+2q)+(n-2)d}\right]$$
$$[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)] = 0. \qquad (6.4.19)$$

This indicates

$$K(X_1,Y_1) = \frac{r}{n}\left[\frac{p+2(n-1)q}{(n-1)(p+2q)+(n-2)d}\right].$$

The proof is now finalized. □

Theorem 6.4.2. *In pseudo-projectively flat paraKähler manifold the sectional curvature determined by X_1, Y_1 is*

$$K(X_1,Y_1) = \frac{r}{n(n-1)}\left[\frac{a+(n-1)b}{a+b}\right].$$

Proof. From the equation (6.3.10), putting the value of $S(Z_1, W_1)$ in (6.3.8), we have

$$a\widetilde{R}(X_1,Y_1,Z_1,W_1) + \frac{br}{n(a+b)}\left(\frac{a}{n-1}+b\right)[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)$$
$$-\frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)] = 0. \qquad (6.4.20)$$

This implies that

$$\widetilde{R}(X_1,Y_1,Z_1,W_1) - \frac{r}{n(n-1)}\left[\frac{a+(n-1)b}{a+b}\right][g(Y_1,Z_1)g(X_1,W_1) - g(X_1,Z_1)g(Y_1,W_1)] = 0. \qquad (6.4.21)$$

which means

$$K(X_1,Y_1) = \frac{r}{n(n-1)}\left[\frac{a+(n-1)b}{a+b}\right].$$

Henceforth, the proof. □

Theorem 6.4.3. *In W_2-flat paraKähler manifold the sectional curvature is Zero.*

Proof. From theorem 6.3.3, using the value of $S(Z_1, W_1)$ in the equation (1.2.9), we get

$$\widetilde{R}(X_1, Y_1, Z_1, W_1) = 0.$$

which minimizes

$$K(X_1, Y_1) = 0.$$

The conclusive proof is now complete. ☐

Theorem 6.4.4. *In a Bochner flat paraKähler manifold the sectional curvature determined by X_1, Y_1 is*

$$K(X_1, Y_1) = -\frac{r}{(n+2)^2}.$$

Proof. From the equation (6.3.17), setting the value of $S(Z_1, W_1)$ in the equation (6.3.16), we obtain

$$\begin{aligned} \widetilde{R}(X_1, Y_1, Z_1, W_1) = & -\frac{2r}{(n+2)^2(n+4)}[g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1) \\ & +g(FY_1, Z_1)g(FX_1, W_1) - g(FX_1, Z_1)g(FY_1, W_1) \\ & -2g(FX_1, Y_1)g(FZ_1, W_1)] \\ & -\frac{r}{(n+2)(n+4)}[g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1) \\ & +g(FY_1, Z_1)g(FX_1, W_1) - g(FX_1, Z_1)g(FY_1, W_1) \\ & -2g(FX_1, Y_1)g(FZ_1, W_1)]. \end{aligned} \tag{6.4.22}$$

which reduces to

$$\begin{aligned} \widetilde{R}(X_1, Y_1, Z_1, W_1) = & -\frac{r}{(n+2)^2}[g(Y_1, Z_1)g(X_1, W_1) - g(X_1, Z_1)g(Y_1, W_1) \\ & +g(FY_1, Z_1)g(FX_1, W_1) - g(FX_1, Z_1)g(FY_1, W_1) \\ & -2g(FX_1, Y_1)g(FZ_1, W_1)]. \end{aligned} \tag{6.4.23}$$

$$\therefore K(X_1, Y_1) = -\frac{r}{(n+2)^2}.$$

Thus the evidence. ☐

6.5 Perfect fluid paraKähler space-time

In this section, we study the nature of the perfect fluid paraKähler space-time admitting the Einstein equation with a cosmological constant.

Theorem 6.5.1. *A paraKähler space-time with a perfect fluid, which fulfills the Einstein equation while incorporating a cosmological constant, can be classified as an Einstein manifold.*

Proof. Substituting X_1, Y_1 with $\overline{X_1}$, $\overline{Y_1}$ respectively in (1.3.19) and applying (1.3.21) and (6.2.4), we acquire

$$S(X_1, Y_1) - \frac{r}{2}g(X_1, Y_1) + \lambda g(X_1, Y_1) = c[\tilde{p}g(X_1, Y_1) - (\sigma + \tilde{p})\omega(X_1)\omega(Y_1)]. \tag{6.5.1}$$

The result of subtracting (1.3.19) from (6.5.1) is

$$c(\sigma + \tilde{p})[\omega(\overline{X}_1)\omega(\overline{Y}_1) - \omega(X_1)\omega(Y_1)] = 0. \tag{6.5.2}$$

By inserting $Y_1 = \rho$ in (6.5.2), we obtain

$$c(\sigma + \tilde{p})\omega(X_1) = 0. \tag{6.5.3}$$

The fluid acts as a cosmological constant since $c \neq 0$ and $\omega(X_1) \neq 0$, as shown by the equation

$$\sigma + \tilde{p} = 0. \tag{6.5.4}$$

Additionally, from (6.5.4) we have $\sigma = -\tilde{p}$, which in cosmology stands for inflation—a fast expansion of space-time. Now applying (6.5.4), the formula (1.3.19) exhibits

$$S(X_1, Y_1) = (\frac{r}{2} - \lambda + c.\tilde{p})g(X_1, Y_1). \tag{6.5.5}$$

Using $X_1 = Y_1 = e_i$, $1 \leq i \leq 4$ in (6.5.5) and adding over i, we may quickly arrive

$$\lambda - c.\tilde{p} = \frac{r}{4}. \tag{6.5.6}$$

From (6.5.5) and (6.5.6), we achieve

$$S(X_1, Y_1) = \frac{r}{4}g(X_1, Y_1). \tag{6.5.7}$$

Hence, the proof. □

6.6 Weakly symmetric perfect fluid paraKähler space-time

This section focuses on the fundamental aspects of a weakly symmetric perfect fluid paraKähler space-time that allows for the inclusion of the Einstein equation alongside a cosmological constant.

Theorem 6.6.1. *If a weakly symmetric perfect fluid paraKähler space-time fulfills the Einstein equation alongside a cosmological constant, then the eigenvectors of the Ricci tensor are ρ and $\overline{\rho}$, corresponding to the eigenvalue $\frac{r}{2}$.*

Proof. In a weakly symmetric paraKähler space-time M, we conclude that

$$R(\overline{Y_1}, \overline{Z_1}, U_1, V_1) = R(Y_1, Z_1, U_1, V_1). \tag{6.6.8}$$

By using the covariant derivative of (6.6.8), we get easily obtain

$$(\nabla_{X_1} R)(\overline{Y_1}, \overline{Z_1}, U_1, V_1) = (\nabla_{X_1} R)(Y_1, Z_1, U_1, V_1). \tag{6.6.9}$$

Utilizing (1.2.3) in equation (6.6.9), we acquire

$$\begin{aligned} &\omega(Y_1)R(X_1, Z_1, U_1, V_1) + \omega(Z_1)R(Y_1, X_1, U_1, V_1) \\ &+\omega(\overline{Y_1})R(X_1, \overline{Z_1}, U_1, V_1) + \omega(\overline{Z_1})R(\overline{Y_1}, X_1, U_1, V_1) = 0. \end{aligned} \tag{6.6.10}$$

Setting $X_1 = e_i, Z_1 = Fe_i, V_1 = FV_1$ in the aforementioned equation and adding the terms over i, $1 \leq i \leq 4$, and applying (6.2.4), (6.5.5), we get at the conclusion that

$$2\omega(Y_1)S(U_1, V_1) + \omega(\overline{Y_1})S(U_1, \overline{V_1}) + R(Y_1, \overline{\rho}, U_1, \overline{V_1}) + R(\overline{Y_1}, \rho, U_1, \overline{V_1}) = 0. \tag{6.6.11}$$

Putting $U_1 = V_1 = e_i$, $1 \leq i \leq 4$ in (6.6.11), and summing over i, we have

$$S(Y_1, \rho) = \frac{r}{2}\omega(Y_1). \tag{6.6.12}$$

or

$$S(Y_1, \rho) = \frac{r}{2}g(Y_1, \rho). \tag{6.6.13}$$

Substituting ρ by $\overline{\rho}$ in (6.6.13), we can express

$$S(Y_1, \overline{\rho}) = \frac{r}{2}g(Y_1, \overline{\rho}).$$

Hence the evidence. □

6.7 Weakly Ricci symmetric perfect fluid paraKähler space-time

The objective of this section is to explore the characteristics of a weakly Ricci symmetric perfect fluid paraKähler space-time by introducing the Einstein equation alongside a cosmological constant.

Theorem 6.7.1. *No perfect fluid paraKähler space-time that is weakly Ricci symmetric, satisfies the Einstein equation with a cosmological constant, and has a non-zero scalar curvature exists.*

Proof. For a perfect fluid paraKähler space-time, we know $(\nabla_{X_1} S)(Y_1, Z_1) = 0$, and applying the relation (1.2.4), we obtain

$$A(X_1)S(Y_1, Z_1) + \omega(Y_1)S(X_1, Z_1) + \omega(Z_1)S(Y_1, X_1) = 0, \tag{6.7.14}$$

for weakly Ricci symmetric perfect fluid paraKähler space-time. Applying (6.5.7) in (6.7.14), we can represent

$$\frac{r}{4}[A(X_1)g(Y_1, Z_1) + \omega(Y_1)g(X_1, Z_1) + \omega(Z_1)g(Y_1, X_1)] = 0. \tag{6.7.15}$$

This implies $r = 0$ or

$$A(X_1)g(Y_1, Z_1) + \omega(Y_1)g(X_1, Z_1) + \omega(Z_1)g(Y_1, X_1) = 0. \tag{6.7.16}$$

Now, by substituting Y_1, Z_1 with $\overline{Y_1}$, $\overline{Z_1}$ in (6.7.15) and applying (1.3.21), we have

$$A(X_1)g(Y_1, Z_1) - \omega(\overline{Y_1})g(\overline{Z_1}, X_1) - \omega(\overline{Z_1})g(X_1, \overline{Y_1}) = 0. \tag{6.7.17}$$

By minus (6.7.16) from (6.7.17), we get at

$$\omega(Y_1)g(Z_1, X_1) + \omega(\overline{Y_1})g(\overline{Z_1}, X_1) + \omega(Z_1)g(X_1, Y_1) + \omega(\overline{Z_1})g(X_1, \overline{Y_1}) = 0. \tag{6.7.18}$$

Setting $X_1 = Z_1 = e_i$, in the above equation and adding the sums across i, $1 \leq i \leq 4$, and using (6.2.4) and (6.5.5), we obtain

$$\omega(Y_1) = 0, \tag{6.7.19}$$

which is contradictory because $g(\rho, \rho) = -1$.
Therefore, the proof. □

6.8 Some results on curvature tensors in paraKähler space-time

In this section, we will discuss some important results in paraKähler space-time using some curvature tensors. For different curvature tensors, we shall show that the paraKähler space-time characterizes as an Einstein manifold or Ricci flat.

Theorem 6.8.1. *In a pseudo-quasi-conformally flat paraKähler space-time that fulfills Einstein's field equation alongside a cosmological constant, the cosmological constant can be expressed as* $c\tilde{p} + \frac{r}{4}\left(\frac{5p+6q+4d}{3p+6q+2d}\right)$.

Proof. Considering scalar product of (1.2.11) with T_1, we acquire

$$\begin{aligned} g(\widetilde{V}(X_1, Y_1)Z_1, T_1) &= (p+d)\widetilde{R}(X_1, Y_1, Z_1, T_1) + \left(q - \frac{d}{3}\right)[S(Y_1, Z_1)g(X_1, T_1) \\ &\quad -S(X_1, Z_1)g(Y_1, T_1) + q[g(Y_1, Z_1)S(X_1, T_1) - g(X_1, Z_1)S(Y_1, T_1)] \\ &\quad -\frac{r}{12}(p+6q)[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1)]. \end{aligned} \tag{6.8.1}$$

As the manifold is pseudo-quasi-conformally flat, we can rewrite the equation as

$$\begin{aligned} &(p+d)\widetilde{R}(X_1, Y_1, Z_1, T_1) + \left(q - \frac{d}{3}\right)[S(Y_1, Z_1)g(X_1, T_1) \\ &-S(X_1, Z_1)g(Y_1, T_1)] + q[g(Y_1, Z_1)S(X_1, T_1) - g(X_1, Z_1)S(Y_1, T_1)] \\ &-\frac{r}{12}(p+6q)[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1)] = 0. \end{aligned} \tag{6.8.2}$$

Setting $X_1 = e_i, Y_1 = Fe_i$, and $T_1 = FT_1$ in (6.8.2) and summing over i, $1 \leq i \leq 4$, and using (1.2.11), (6.2.3), we have

$$\begin{aligned} &2(p+d)S(Z_1, T_1) + \left(q - \frac{d}{3}\right)[-S(FZ_1, FT_1) + S(Z_1, T_1)] \\ &+q[-S(FZ_1, FT_1) + S(Z_1, T_1)] - \frac{r}{12}(p+6q) \\ &[-g(FY_1, FT_1) + g(Z_1, T_1)] = 0. \end{aligned} \tag{6.8.3}$$

Again, by virtue of (6.2.4) and (1.3.18) in the above equation, we obtain

$$(p+d)S(Z_1, T_1) + \left(q - \frac{d}{3}\right)S(Z_1, T_1) + qS(Z_1, T_1) - \frac{r}{12}(p+6q)g(Z_1, T_1) = 0. \tag{6.8.4}$$

which reduces to

$$\left(p + 2q + \frac{2d}{3}\right)S(Z_1, T_1) = \frac{r}{12}(p+6q)g(Z_1, T_1). \tag{6.8.5}$$

Now, using (6.5.1) in (6.8.5), we achieve

$$\lambda = c\tilde{p} + \frac{r}{4}\left(\frac{5p+6q+4d}{3p+6q+2d}\right).$$

This completes the proof. □

Corollary 6.8.1. *A pseudo-quasi-conformally flat paraKähler space-time is Ricci flat, provided* $p+d \neq 0$.

Proof. Now, putting $Z_1 = T_1 = e_i$ in (6.8.5) and the summing over i, $1 \leq i \leq 4$, we obtain

$$(p+d)r = 0.$$

This implies

$$r = 0.$$

Then (6.8.5) becomes

$$S(Z_1, T_1) = 0.$$

This implies that the manifold is Ricci flat.
Hence proof is presented. □

Theorem 6.8.2. *In a pseudo-projectively flat paraKähler space-time that adheres Einstein's field equation alongside a cosmological constant, the formula for the cosmological constant can be expressed as* $c\tilde{p} + \frac{r}{12}\left(\frac{5a+3b}{a+b}\right)$.

Proof. Taking the inner product in (1.2.5) by T_1, we acquire

$$\begin{aligned} g(\overline{P}(X_1,Y_1)Z_1,T_1) &= a\widetilde{R}(X_1,Y_1,Z_1,T_1) + b[S(Y_1,Z_1)g(X_1,T_1) - S(X_1,Z_1)g(Y_1,T_1)] \\ &\quad -\frac{r}{4}\left(\frac{a}{3}+b\right)[g(Y_1,Z_1)g(X_1,T_1) - g(X_1,Z_1)g(Y_1,T_1)]. \end{aligned} \tag{6.8.6}$$

Now, as the manifold is pseudo-projectively flat, so from (6.8.6) we obtain

$$\begin{aligned} &a\widetilde{R}(X_1,Y_1,Z_1,T_1) + b[S(Y_1,Z_1)g(X_1,T_1) - S(X_1,Z_1)g(Y_1,T_1)] \\ &-\frac{r}{4}\left(\frac{a}{3}+b\right)[g(Y_1,Z_1)g(X_1,T_1) - g(X_1,Z_1)g(Y_1,T_1)] = 0. \end{aligned} \tag{6.8.7}$$

Writing $X_1 = e_i, Y_1 = Fe_i$, and $T_1 = FT_1$ in (6.8.7) and adding over i, $1 \leq i \leq 4$, and applying (6.3.1), (6.2.3), we get

$$\begin{aligned} &2aS(Z_1,T_1) + b[-S(FZ_1,FT_1) + S(Z_1,T_1)] \\ &-\frac{r}{4}\left(\frac{a}{3}+b\right)[-g(FZ_1,FT_1) + g(Z_1,T_1)] = 0. \end{aligned} \tag{6.8.8}$$

Again, by applying (6.2.4) and (1.3.18) in (6.8.8), we have

$$(a+b)S(Z_1,T_1) = \frac{r}{4}\left(\frac{a}{3}+b\right)g(Z_1,T_1). \tag{6.8.9}$$

Now, using (6.5.1) in the equation (6.8.9), we achieve

$$\lambda = c\tilde{p} + \frac{r}{12}\left(\frac{5a+3b}{a+b}\right).$$

This completes the proof. □

Corollary 6.8.2. *A pseudo-projectively flat paraKähler space-time is Ricci flat, provided* $a \neq 0$.

Proof. Setting $Z_1 = T_1 = e_i$ in the (6.8.9) and summing over i, $1 \leq i \leq 4$, and we acquire

$$ar = 0.$$

This implies

$$r = 0.$$

Then (6.8.9) becomes

$$S(Z_1, T_1) = 0.$$

This implies that the manifolds are Ricci flat.
Hence the proof at once. □

Theorem 6.8.3. *A generalised* W_2*-flat paraKähler space-time is Ricci flat, provided* $a \neq \left(b + \frac{c}{3}\right)$.

Proof. The scalar product of (1.2.13) with T_1 leads to

$$g(\overline{W_2}(X_1, Y_1)Z_1, T_1) = a\widetilde{R}(X_1, Y_1, Z_1, T_1) + \left(b + \frac{c}{3}\right)[g(X_1, Z_1)S(Y_1, T_1) - g(Y_1, Z_1)S(X_1, T_1)]. \tag{6.8.10}$$

As the manifold is $\overline{W}_2$-flat, we can rewrite (6.8.10) as

$$a\widetilde{R}(X_1, Y_1, Z_1, T_1) + \left(b + \frac{c}{3}\right)[g(X_1, Z_1)S(Y_1, T_1) - g(Y_1, Z_1)S(X_1, T_1)] = 0. \tag{6.8.11}$$

Putting $X_1 = e_i, Y_1 = Fe_i$, and $T_1 = FT_1$ in (6.8.11) and summing over i, $1 \leq i \leq 4$, and utilizing (1.2.13), (6.2.3), and (6.2.4), we get

$$\left(a - b - \frac{c}{3}\right) S(Z_1, T_1) = 0. \tag{6.8.12}$$

Then, we have $S(Z_1, T_1) = 0$, provided $a \neq (b + \frac{c}{3})$, for any Z_1, $T_1 \in \chi(M)$. The proof is now concluded. □

Based on theorem 6.8.3, we derive the following corollary.

Corollary 6.8.3. *A quasi-W_2 flat paraKähler space-time is Ricci flat, provided $a \neq \frac{c}{3}$.*

Theorem 6.8.4. *In a Bochner flat paraKähler space-time that obeys to Einstein's field equation alongside a cosmological constant, the formula for the cosmological constant is given by $c\tilde{p} + \frac{19r}{36}$.*

Proof. Considering the inner product if (1.2.10) with T_1, we acquire

$$\begin{aligned} g(B(X_1, Y_1)Z_1, T_1) &= \widetilde{R}(X_1, Y_1, Z_1, T_1) - \frac{1}{8}[g(Y_1, Z_1)S(X_1, T_1) - g(X_1, Z_1)S(Y_1, T_1) \\ &\quad + S(Y_1, Z_1)g(X_1, T_1) - S(X_1, Z_1)g(Y_1, T_1) + g(FY_1, Z_1)S(FX_1, T_1) \\ &\quad - g(FX_1, Z_1)S(FY_1, T_1) + S(FY_1, Z_1)g(FX_1, T_1) \\ &\quad - S(FX_1, Z_1)g(FY_1, T_1) - 2S(FX_1, Y_1)g(FZ_1, T_1) \\ &\quad - 2g(FX_1, Y_1)S(FZ_1, T_1)] \\ &\quad + \frac{r}{48}[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1) \\ &\quad + g(FY_1, Z_1)g(FX_1, T_1) - g(FX_1, Z_1)g(FY_1, T_1) \\ &\quad - 2g(FX_1, Y_1)g(FZ_1, T_1)]. \end{aligned} \tag{6.8.13}$$

Now, as the manifold is Bochner flat, then (6.8.13) reduces to

$$\begin{aligned} \widetilde{R}(X_1, Y_1, Z_1, T_1) &= \frac{1}{8}[g(Y_1, Z_1)S(X_1, T_1) - g(X_1, Z_1)S(Y_1, T_1) \\ &\quad + S(Y_1, Z_1)g(X_1, T_1) - S(X_1, Z_1)g(Y_1, T_1) \\ &\quad + g(FY_1, Z_1)S(FX_1, T_1) - g(FX_1, Z_1)S(FY_1, T_1) \\ &\quad + S(FY_1, Z_1)g(FX_1, T_1) - S(FX_1, Z_1)g(FY_1, T_1) \\ &\quad - 2S(FX_1, Y_1)g(FZ_1, T_1) - 2g(FX_1, Y_1)S(FZ_1, T_1)] \\ &\quad - \frac{r}{48}[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1) \\ &\quad + g(FY_1, Z_1)g(FX_1, T_1) - g(FX_1, Z_1)g(FY_1, T_1) \\ &\quad - 2g(FX_1, Y_1)g(FZ_1, T_1)]. \end{aligned} \tag{6.8.14}$$

Putting $X_1 = e_i, Y_1 = Fe_i$, and $T_1 = FT_1$ in (6.8.14), summing over i, $1 \leq i \leq 4$, applying (6.2.3), and (6.2.4), we acquire

$$S(Z_1, T_1) = -\frac{r}{36}g(Z_1, T_1). \tag{6.8.15}$$

Now, using (6.5.1) in the equation (6.8.15), we achieve

$$\lambda = c\tilde{p} + \frac{19r}{36}.$$

The proof is now ended. □

Corollary 6.8.4. *A Bochner flat paraKähler space-time is Ricci flat.*

Proof. Now, taking $Z_1 = T_1 = e_i$ in (6.8.15) and summing over i, $1 \leq i \leq 4$, we get

$$r = 0.$$

Then (6.8.15) becomes

$$S(Z_1, T_1) = 0.$$

This implies the manifold is Ricci flat.

The proof is now over. □

6.9 Sectional curvature in paraKähler space-time

Theorem 6.9.1. *In a pseudo-quasi-conformal flat paraKähler space-time, the sectional curvature is* $\frac{r}{4}\left(\frac{p+6d}{3p+6q+2d}\right)$.

Proof. From (6.8.5), putting the value of $S(Z_1, T_1)$ in (6.8.2), we obtain

$$\begin{aligned} &(p+d)\widetilde{R}(X_1, Y_1, Z_1, T_1) + \left[\frac{r(p+6q)}{12(p+2q+\frac{2d}{3})}\right]\left(2q - \frac{d}{3}\right)[g(Y_1, Z_1)g(X_1, T_1) \\ &-g(X_1, Z_1)g(Y_1, T_1)] - \frac{r}{12}(p+6q)[g(Y_1, Z_1)g(X_1, T_1) \\ &-g(X_1, Z_1)g(Y_1, T_1)] = 0. \end{aligned} \tag{6.9.16}$$

which leads to

$$\widetilde{R}(X_1, Y_1, Z_1, T_1) - \frac{r}{4}\left(\frac{p+6d}{3p+6q+2d}\right)[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1)] = 0. \tag{6.9.17}$$

This implies

$$K(X_1, Y_1) = \frac{r}{4}\left(\frac{p+6d}{3p+6q+2d}\right).$$

The proof is now done. □

Theorem 6.9.2. *In a pseudo-projectively flat paraKähler space-time, the sectional curvature is* $\frac{r}{12}\left(\frac{a+3b}{a+b}\right)$.

Proof. Again, from (6.8.9), putting the value of $S(Z_1, T_1)$ in (6.8.7), we acquire

$$a\widetilde{R}(X_1, Y_1, Z_1, T_1) + \frac{br}{4(a+b)}\left(\frac{a}{3}+b\right)[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1)$$
$$-\frac{r}{4}\left(\frac{a}{3}+b\right)[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1)] = 0. \quad (6.9.18)$$

This implies that

$$\widetilde{R}(X_1, Y_1, Z_1, T_1) - \frac{r}{12}\left(\frac{a+3b}{a+b}\right)[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1)] = 0. \quad (6.9.19)$$

which implies

$$K(X_1, Y_1) = \frac{r}{12}\left(\frac{a+3b}{a+b}\right).$$

The proof is now prepared. □

Theorem 6.9.3. *In a generalised* W_2 *and a quasi-*W_2 *flat paraKähler space-time, the sectional curvature is* 0.

Proof. Also, from theorem 6.8.3, setting the value of $S(Z_1, T_1)$ in (1.2.13), we obtain

$$R(X_1, Y_1, Z_1, T_1) = 0.$$

which reduces

$$K(X_1, Y_1) = 0.$$

The proof is now complete. □

Theorem 6.9.4. *In a Bochner flat paraKähler space-time, the sectional curvature is* $-\frac{r}{36}$.

Proof. Also, from (6.8.15), putting the value of $S(Z_1, T_1)$ in (6.8.14), we get

$$\begin{aligned}\widetilde{R}(X_1, Y_1, Z_1, T_1) \;=\; & -\frac{r}{144}[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1) \\ & +g(FY_1, Z_1)g(FX_1, T_1) - g(FX_1, Z_1)g(FY_1, T_1) \\ & -2g(FX_1, Y_1)g(FZ_1, T_1)] \\ & -\frac{r}{48}[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1) \\ & +g(FY_1, Z_1)g(FX_1, T_1) - g(FX_1, Z_1)g(FY_1, T_1) \\ & -2g(FX_1, Y_1)g(FZ_1, T_1)]. \quad (6.9.20)\end{aligned}$$

which reduces to

$$\begin{aligned}\widetilde{R}(X_1, Y_1, Z_1, T_1) \quad = \quad & -\frac{r}{36}[g(Y_1, Z_1)g(X_1, T_1) - g(X_1, Z_1)g(Y_1, T_1) \\ & +g(FY_1, Z_1)g(FX_1, T_1) - g(FX_1, Z_1)g(FY_1, T_1) \\ & -2g(FX_1, Y_1)g(FZ_1, T_1)]. \qquad (6.9.21)\end{aligned}$$

$$\therefore K(X_1, Y_1) = -\frac{r}{36}.$$

The proof is now established. □

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Reprints

Some Weakly Symmetric Kählher Manifolds

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Abstract

In this paper we have studied weakly symmetric Kähler manifold which is pseudo-projectively flat and quasi-conformally flat.

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1 Introduction

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + C(Z)R(Y,X,U,V) \\ &\quad + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X), \qquad (1.1)\end{aligned}$$

where A,B,C,D,E are simultaneously non-vanishing 1-forms and X,Y,Z,U,V are vector fields and ∇ be the operator of covariant differentiation with respect to the Riemannian metric g. The 1- forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$.

In 1995 M. Prvanovic [1] proved that if the manifold M is a weakly symmetric manifold satisfying (1.1) then $B = C = D = E$. In this paper we consider $B = C = D = E = \omega$ and then (1.1) becomes

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + \omega(Y)R(X,Z,U,V) + \omega(Z)R(Y,X,U,V) \\ &\quad + \omega(U)R(Y,Z,X,V) + \omega(V)R(Y,Z,U,X), \qquad (1.2)\end{aligned}$$

where $g(X, \rho) = \omega(X)$ and $g(X, \alpha) = A(X)$. where ρ and α are vector fields. In 2002 Prasad [2] defined and studied a tensor field $\overline{P}$ on a Riemannian manifold of dimension $n(n > 2)$ which includes the projective curvature tensor P. This tensor field $\overline{P}$ is known as pseudo-projective curvature tensor and is given by

$$\begin{aligned}\overline{P}(X,Y,Z) &= aR(X,Y,Z) + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y],\end{aligned}$$

(1.3)

where a and b are constants such that $a, b \neq 0, R$ is the curvature tensor, S is the Ricci tensor and r is the scalar curvature. A non-pseudo projectively flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be weakly pseudo projectively symmetric manifold if the pseudo-projective curvature tensor $\overline{P}$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X \overline{P})(Y, Z, U, V) &= A(X)\overline{P}(Y, Z, U, V) + B(Y)\overline{P}(X, Z, U, V) + C(Z)\overline{P}(Y, X, U, V) \\ &\quad + D(U)\overline{P}(Y, Z, X, V) + E(V)\overline{P}(Y, Z, U, X), \qquad (1.4)\end{aligned}$$

for all vectors fields X, Y, Z, U, V and A, B, C, D, E are non-vanishing 1-forms. Such an n-dimensional manifold is denoted by $(WPPS)_n$. Also, we give the definition of quasi-conformal curvature tensor given by Yano and Sawaki [8] as follows

$$\begin{aligned}\overline{C}(X, Y, Z) &= aR(X, Y, Z) + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y, Z)X - g(X, Z)Y], \qquad (1.5)\end{aligned}$$

where a and b are non zero constants. If $a = 1$ and $b = -\frac{1}{n-2}$, then quasi-conformal curvature tensor is reduced to the conformal curvature tensor. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be weakly quasi-conformally symmetric manifold if the quasi-conformally curvature tensor $\overline{C}$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X \overline{C})(Y, Z, U, V) &= A(X)\overline{C}(Y, Z, U, V) + B(Y)\overline{C}(X, Z, U, V) + C(Z)\overline{C}(Y, X, U, V) \\ &\quad + D(U)\overline{C}(Y, Z, X, V) + E(V)\overline{C}(Y, Z, U, X), \qquad (1.6)\end{aligned}$$

for all vectors fields X, Y, Z, U, V and A, B, C, D, E are non-vanishing 1-forms such an n-dimensional manifold is denoted by $(WQCS)_n$.

In this paper we have considered two types of Kähler manifold namely weakly pseudo projectively symmetric Kähler manifold and weakly quasi-conformally symmetric Kähler manifold.

2 Preliminaries

First of all, we define Kähler manifold in this section. A Kähler manifold is a Riemannian manifold M of even dimension n with complex structure F on the tangent space of M at each point satisfies the following relation

$$F^2(X) = -X, g(\overline{X}, \overline{Y}) = g(X, Y), (\nabla_X F)(Y) = 0$$

where F is a tensor field of type $(1, 1)$ such that $F(X) = \overline{X}$, g is a Riemannian metric and ∇ is the Levi-Civita Connection. Also in this section we derive some formulae which will be required to study of $(WPPS)_n$ and $(WQCS)_n$. Let $e_i, i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then from (1.3), we have the following:-

(a) $\sum_{i=1}^{n} \overline{P}(e_i, Y, Z, e_i) = [a + (n-1)b][S(Y, Z) - \frac{r}{n} g(Y, Z)]$

(b) $\sum_{i=1}^{n} \overline{P}(X, Y, e_i, e_i) = 0$

(c) $\sum_{i=1}^{n} \overline{C}(e_i, Y, Z, e_i) = [a + (n-2)b][S(Y, Z) - \frac{r}{n} g(Y, Z)]$

(d) $\sum_{i=1}^{n} \overline{C}(X, Y, e_i, e_i) = 0$

Now we have proved the following proposition:

Proposition (2.1.) In a Riemannian manifold $(M^n, g)(n > 2)$ the pseudo-projective curvature tensor and quasi-conformally curvature tensor satisfies the following relation:

$$(I) \quad \overline{P}(X, Y, Z, U) + \overline{P}(Y, Z, X, U) + \overline{P}(Z, X, Y, U) = 0$$

$$(II) \quad \overline{P}(X, Y, U, Z) + \overline{P}(Y, Z, U, X) + \overline{P}(Z, X, U, Y) = 0$$

$$(III) \quad \overline{C}(X, Y, Z, U) + \overline{C}(Y, Z, X, U) + \overline{C}(Z, X, Y, U) = 0$$

$$(IV) \quad \overline{C}(X, Y, U, Z) + \overline{C}(Y, Z, U, X) + \overline{C}(Z, X, U, Y) = 0$$

3 Weakly Pseudo Projectively Symmetric K$\ddot{a}$hler Manifold

If the manifold M is a weakly pseudo projectively symmetric Kähler manifold, then we have proved

$$\overline{P}(\overline{Y}, \overline{Z}, U, V) = \overline{P}(Y, Z, U, V). \tag{3.1}$$

Taking the covariant derivative, we get

$$(\nabla_X \overline{P})(\overline{Y}, \overline{Z}, U, V) = (\nabla_X \overline{P})(Y, Z, U, V). \tag{3.2}$$

Using(1.2) and (1.4) in (3.2), we obtain

$$\omega(Y)\overline{P}(X, Z, U, V) + \omega(Z)\overline{P}(Y, X, U, V) = \omega(\overline{Y})\overline{P}(X, \overline{Z}, U, V) + \omega(\overline{Z})\overline{P}(\overline{Y}, X, U, V).$$

(3.3)

By, putting $Z = U = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\begin{aligned}
&(a-b)\omega(Y)S(X, V) - \frac{(a-1)br}{n}\omega(Y)g(X, V) - aR(Y, X, V, \rho) + bg(Y, V)S(X, \rho) \\
&-2bg(X, V)S(Y, \rho) + \frac{2r}{n}\left[\frac{a}{n-1} + b\right] g(X, V)g(Y, \rho) - \frac{r}{n}\left[\frac{a}{n-1} + b\right] g(Y, V)g(X, \rho) \\
&= (a+b)\omega(\overline{Y})S(X, \overline{V}) - \frac{r}{n}\left[\frac{a}{n-1} + b\right] g(X, \overline{V})\omega(\overline{Y}) + aR(\overline{Y}, X, V, \overline{\rho}) \\
&+\frac{r}{n}\left[\frac{a}{n-1} + \frac{(r-n)b}{r}\right] g(\overline{Y}, V)S(X, \overline{\rho}).
\end{aligned} \tag{3.4}$$

Again, putting $X = V = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$rg(Y,\rho)\left[a(1-b)+(2-\frac{1}{n})(\frac{a}{n-1}+b)\right] = S(Y,\rho)\left[2a+2b(n-1)+\frac{ar}{n(n-1)}+\frac{br}{n}\right].$$

(3.5)

We get,

$$S(Y,\rho) = fg(Y,\rho).$$

(3.6)

This is an Einstein manifold for every vector field ρ.

Thus we state the following theorem:

Theorem 3.1. *A weakly pseudo projectively symmetric Kähler manifold is an Einstein manifold with respect to vector field ρ defined by $g(X,\rho) = \omega(X)$.*

From theorem (3.1.), we have the following corollary

Corollary 3.1. *For a weakly pseudo projectively symmetric Kähler manifold if the vector field ρ is a unit vector field and Y=ρ, then the expression for scalar curvature is, $r = \frac{2nh[a+(n-1)b]}{2na+(n-h-1)(a+b)}$ provided $2na + (n-h-1)(a+b) \neq 0$ where $h = S(\rho,\rho)$. In addition if $a + (n-1)b = 0$, then the scalar curvature vanishes.*

4 Pseudo-Projectively Flat Weakly Symmetric K$\ddot{a}$hler manifold

For pseudo-projectively flat curvature tensor, $\overline{P}(Y,Z,U,V) = 0$, then

$$\begin{aligned} aR(Y,Z,U,V)+bS(Z,U)g(Y,V) &- bS(Y,U)g(Z,V)-\frac{r}{n}\left[\frac{a}{n-1}+b\right]g(Z,U)g(Y,V) \\ &+ \frac{r}{n}\left[\frac{a}{n-1}+b\right]g(Y,U)g(Z,V) = 0. \end{aligned} \tag{4.1}$$

Then

$$\begin{aligned} R(Y,Z,U,V) &= -\frac{b}{a}[S(Z,U)g(Y,V)-S(Y,U)g(Z,V)] \\ &+\frac{r}{an}\left[\frac{a}{n-1}+b\right][g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]. \end{aligned} \tag{4.2}$$

Taking covariant differentiation w.r.t. X,we get

$$(\nabla_X R)(Y,Z,U,V) = -\frac{b}{a}[g(Y,V)(\nabla_X S)(Z,U)-g(Z,V)(\nabla_X S)(Y,U)], \tag{4.3}$$

then (4.3) reduces to

$$A(X)R(Y,Z,U,V)+\omega(Y)R(X,Z,U,V)+\omega(Z)R(Y,X,U,V)+\omega(U)R(Y,Z,X,V)$$
$$+\omega(V)R(Y,Z,U,X)=-\frac{b}{a}[g(Y,V)A(X)S(Z,U)+\omega(Z)S(X,U)+\omega(U)S(Z,X)$$
$$-g(Z,V)A(X)S(Y,U)+\omega(Y)S(X,U)+\omega(U)S(Y,X)]. \tag{4.4}$$

By, putting $Y=V=e_i$, $1\le i\le n$ and summing over i, we get

$$\left[1+\frac{b}{a}(n-1)\right][A(X)S(Z,U)+\omega(Z)S(X,U)+\omega(U)S(Z,X)]=0. \tag{4.5}$$

Again,taking $X=U=e_i$, $1\le i\le n$ and summing over i, we get

$$\left[1+\frac{b}{a}(n-1)\right][S(Z,\alpha)+r\omega(Z)+S(Z,\rho)]=0, \tag{4.6}$$

for any vector field ρ defined by $g(X,\rho)=\omega(X)$ and $g(X,\alpha)=A(X)$,then we have

$$S(Z,\alpha)+S(Z,\rho)=-r\omega(Z). \tag{4.7}$$

Then we get the theorem,

Theorem 4.1. *In a pseudo projectively flat weakly symmetric Kähler manifold, the Ricci tensor satisfies the relation* $S(Z,\alpha)+S(Z,\rho)=-r\omega(Z)$.

5 Weakly Quasi-Conformally Symmetric K$\ddot{a}$hler manifold

If the manifold M is a weakly quasi-conformally symmetric Kähler manifold, then we have proved

$$\overline{C}(\overline{Y},\overline{Z},U,V)=\overline{C}(Y,Z,U,V). \tag{5.1}$$

Taking the covariant derivative, we get

$$(\nabla_X\overline{C})(\overline{Y},\overline{Z},U,V)=(\nabla_X\overline{C})(Y,Z,U,V). \tag{5.2}$$

Using(1.5) in (5.2), we obtain

$$\omega(Y)\overline{C}(X,Z,U,V)+\omega(Z)\overline{C}(Y,X,U,V)=\omega(\overline{Y})\overline{C}(X,\overline{Z},U,V)+\omega(\overline{Z})\overline{C}(\overline{Y},X,U,V).$$

(5.3)

By, putting $Z = U = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\begin{aligned}
&[a+(n-4)b]\omega(Y)S(X,V) - \frac{r}{n}[a+(n-2)b]\omega(Y)g(X,V) - aR(Y,X,V,\rho) \\
&+bg(Y,V)S(X,\rho) - 2bg(X,V)S(Y,\rho) + bg(X,\rho)S(Y,V) \\
&= (a+2b)\omega(\overline{Y})S(X,\overline{V}) - \frac{r}{n}\left[\frac{a}{n-1}+2b\right]\omega(\overline{Y})g(X,\overline{V}) + aR(\overline{Y},X,V,\overline{\rho}) \\
&-bg(Y,\overline{V})S(X,\overline{\rho}) + \frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(\overline{Y},V)g(X,\overline{\rho}) - bg(X,\overline{\rho})S(\overline{Y},V). \qquad (5.4)
\end{aligned}$$

Again, putting $X = V = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$-rg(Y,\rho)\left[2b + \frac{a}{n(n-1)} + \frac{2b}{n}\right] = S(Y,\rho)[2a-3b+2bn]. \tag{5.5}$$

We get

$$S(Y,\rho) = fg(Y,\rho). \tag{5.6}$$

This is again an Einstein manifold for every vector field ρ.

Theorem 5.1. *A weakly quasi-conformally symmetric Kähler manifold is an Einstein manifold with respect to vector field ρ defined by $g(X,\rho) = \omega(X)$.*

From theorem (5.1.), we have the following corollary

Corollary 5.1. *For a weakly quasi-conformally symmetric Kähler manifold if the vector field ρ is a unit vector field and Y=ρ, then the expression for scalar curvature is, $r = -\frac{n(n-1)h[2a+(2n-3)b]}{a+(n^2-1)2b}$ provided $a+(n^2-1)2b \neq 0$ where $h = S(\rho,\rho)$. In addition if $2a+(2n-3)b = 0$, then the scalar curvature vanishes.*

6 Quasi-Conformally Flat Weakly Symmetric K$\ddot{a}$hler manifold

For quasi-conformally flat curvature tensor, $\overline{C}(Y,Z,U,V) = 0$, then

$$\begin{aligned}
aR(Y,Z,U,V) + bS(Z,U)g(Y,V) &- bS(Y,U)g(Z,V) + g(Z,U)g(QY,V) - bg(Y,U)g(QZ,V) \\
&- \frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(Z,U)g(Y,V) \\
(6.1) \qquad &+ \frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(Y,U)g(Z,V) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
R(Y,Z,U,V) = &- \frac{b}{a}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) + g(Z,U)g(QY,V) - g(Y,U)g(QZ,V)] \\
(6.2) \qquad &+ \frac{r}{an}\left[\frac{a}{n-1}+2b\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)].
\end{aligned}$$

Taking covariant differentiation w.r.t. X,we get

$$(6.3)\quad (\nabla_X R)(Y,Z,U,V) = -\frac{b}{a}[g(Y,V)(\nabla_X S)(Z,U) - g(Z,V)(\nabla_X S)(Y,U)],$$

then

$$\begin{aligned} &A(X)R(Y,Z,U,V) + \omega(Y)R(X,Z,U,V) + \omega(Z)R(Y,X,U,V) + \omega(U)R(Y,Z,X,V) \\ &+\omega(V)R(Y,Z,U,X) = -\frac{b}{a}[g(Y,V)A(X)S(Z,U) + \omega(Z)S(X,U) + \omega(U)S(Z,X) \\ (6.4)\quad &-g(Z,V)A(X)S(Y,U) + \omega(Y)S(X,U) + \omega(U)S(Y,X)]. \end{aligned}$$

By, putting $Y = V = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$(6.5)\quad \left[1 + \frac{b}{a}(n-1)\right][A(X)S(Z,U) + \omega(Z)S(X,U) + \omega(U)S(Z,X)] = 0.$$

Again,taking $X = U = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$(6.6)\quad \left[1 + \frac{b}{a}(n-1)\right][S(Z,\alpha) + r\omega(Z) + S(Z,\rho)] = 0,$$

for any vector field ρ defined by $g(X,\rho) = \omega(X)$ and $g(X,\alpha) = A(X)$,then we have

$$(6.7)\quad S(Z,\alpha) + S(Z,\rho) = -r\omega(Z).$$

Thus we state the following theorem:

Theorem 6.1. *In a quasi-conformally flat weakly symmetric Kähler manifold, the Ricci tensor satisfies the relation* $S(Z,\alpha) + S(Z,\rho) = -r\omega(Z)$.

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Some Curvature Identities on Nearly Kӓhler Manifolds

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In this paper we have studied and obtained expressions of some curvature identities on nearly Kӓhler manifold which is con-circularly flat and projectively flat. Also we got interesting results on 6-dimensional nearly Kӓhler manifold with an example.

Key words : Kӓhler manifold, nearly Kӓhler manifold, con-circularly flat spaces, projectively flat spaces, Einstein manifold.
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1. Introduction

An almost Hermite manifold (M,g,F) is said to be nearly Kӓhler manifold if $(\nabla_X F)(X) = 0$ is satisfies for all vector fields X on M, where ∇ denotes the Livi-Civita connection associated with the metric g. A nearly Kӓhler manifold is called strict if $\nabla_X(F) \neq 0$ for any non-vanishing vector field $X \in TM$, where TM denotes the tangent bundle of M. On the other hand, Nagy proved in [11, 12] that, in the compact case, his study amounts to that of quaternion-Kӓhler manifolds with positive scalar curvature [13] and nearly Kӓhler manifolds of dimension 6. Thus our focus on the study of such manifolds of dimension 6 can be justified by his results.

Definition : Let M be an almost Hermite manifold with almost complex structure F and Riemannian metric g. Then

$$F^2 = -I, \quad g(F(X), F(Y)) = g(X, Y),$$

for all vector fields X and Y on M. We denote by ∇ the operator of covariant differentiation with respect to g. If the almost complex structure F on M satisfies

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0, \tag{1}$$

for any vector fields X and Y on M, then the manifold M is called a nearly Kӓhler manifold or an almost Tachibana manifold.

Putting X for Y in (1.1) we get

$$(\nabla_X F)(X) = 0.$$

If in an almost Tachibana manifold, Nijenhuis tensor vanishes, then it is called a Tachibana manifold.

2. Preliminaries

In this section, we explain our notation and write down some important curvature identities. Let (M,g,F) be a connected almost Hermitian manifold. Then we have $g(FX, FY) = g(X, Y)$ for all X and Y in TM. Throughout this paper we shall assume that (M,g,F) is nearly Kähler, that is $(\nabla_X F)(X) = 0$ for all $X \in TM$.
Let R denote the curvature tensor defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for any vector fields X and Y in TM. Let $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ denote the value of the curvature tensor for every X, Y, Z and W in TM. Then we have the following identities [1,2,3]:

$$(\nabla_X F)(Y) + (\nabla_{FX} F)(FY) = 0; \tag{2}$$

$$(\nabla_X F)(FY) + F((\nabla_X F)(Y)) = 0; \tag{3}$$

$$R(W, X, Y, Z) - R(W, X, FY, FZ) = g((\nabla_W F)(X), (\nabla_Y F)(Z)), \tag{4}$$

$$and \;\; R(W, X, Y, Z) = R(FW, FX, FY, FZ). \tag{5}$$

We now define linear transformations R_1 and R_1^* by

$$Ric(X, Y) = g(R_1(X), Y) = \sum_{i=1}^{2n} R(X, e_i, Y, e_i) \;\; and$$

$$Ric^*(X, Y) = g(R_1^*(X), Y) = \frac{1}{2}\sum_{i=1}^{2n} R(X, FY, e_i, Fe_i)$$

respectively, where $\{e_1, ..., e_{2n}\}$ denotes a local orthonormal frame field on M. We shall call *Ric* the *Ricci* tensor of the metric and *Ric** the *Ricci** tensor respectively. Now note that $Ric - Ric^*$ is given by the formula

$$(Ric - Ric^*)(X, Y) = \sum_{i=1}^{2n} g((\nabla_X F)e_i, (\nabla_Y F)e_i)$$

for all vector fields X, Y on M [5]. Furthermore, Gray [3] proved that

$$\sum_{i,j=1}^{2n}(Ric - Ric^{*})(e_i, e_j)(R(X, e_i, Y, e_j) - 5R(X, e_i, FY, Fe_j)) = 0.$$

So using the above results we have proved

Theorem 2.1. A necessary and sufficient condition for an almost Hermite manifold to be an almost nearly Kähler manifold is

$$\nabla_X F(Y) + \nabla_Y F(X) = F(\nabla_X Y) + F(\nabla_Y X).$$

Proof : First we suppose that an almost Hermite manifold is an almost nearly Kähler manifold. Then

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0$$

$$or,\ \nabla_X F(Y) - F(\nabla_X Y) + \nabla_Y F(X) - F(\nabla_Y X) = 0,$$

$$or,\ \nabla_X F(Y) + \nabla_Y F(X) = F(\nabla_X Y) + F(\nabla_Y X).$$

Conversely, we suppose that

$$\nabla_X F(Y) + \nabla_Y F(X) = F(\nabla_X Y) + F(\nabla_Y X)$$

$$or,\ \nabla_X F(Y) - F(\nabla_X Y) + \nabla_Y F(X) - F(\nabla_Y X) = 0,$$

$$or,\ (\nabla_X F)(Y) + (\nabla_Y F)(X) = 0.$$

Hence the manifold is an almost nearly Kähler manifold.

Proposition 2.1.[5] (i) For a nearly Kähler manifold

$N(X, Y) = 2M(X, Y) = -4F((\nabla_X F)(Y)) = 4F((\nabla_Y F)(X)) = 4F((\nabla_{F(X)} F)F(Y))$,

where $M(X, Y) = \nabla_{F(X)} F(Y) - \nabla_X Y - F(\nabla_{F(X)} Y) - F(\nabla_X F(Y))$.

(ii) If M is nearly Kähler manifold then $N(X, Y) = F(\nabla_X F)Y$,

where $4N(X, Y) = [X, Y] - [FX, FY] + F[FX, Y] + F[X, FY]$.

Theorem 2.2. If the Nijenhuis tensor N of a nearly Kähler manifold M vanishes, then M is Kähler manifold.

Proof : From the Proposition (2.1) we obtain

$$N(X,Y) = -4F((\nabla_X F)(Y)).$$

If $N(X,Y) = 0$, then $F((\nabla_X F)(Y)) = 0$. That is, $F^2(\nabla_X F)Y = 0$.
Hence $(\nabla_X F)(Y) = 0$.
Therefore the manifold is a Kähler manifold.

Theorem 2.3. On a nearly Kähler manifold $divF = 0$.

Proof : On a nearly Kähler manifold we have

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0.$$

Now contracting X and Y we have

$$(\nabla_X F)(X) = 0.$$

That is, $divF = 0$.

3. Curvature identities on nearly Kähler manifold

In this section we prove some curvature identities for a nearly Kähler manifold.

Theorem 3.1. For a con-circularly flat nearly Kähler manifold the following relation holds

$$\begin{aligned}&2g(F(R(X,Y)Z,W)) + g[(\nabla_X F)(\nabla_Y Z), W] - g[(\nabla_Y F)(\nabla_X Z), W]\\ &= \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].\end{aligned}$$

Proof : In an n-dimensional Riemannian manifold the con-circular curvature tensor is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{6}$$

so (3.1) can be written as

$$\begin{aligned}\widetilde{C}(X,Y,Z,W) = {} & \widetilde{R}(X,Y,Z,W)\\ & -\frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{7}$$

where

$$\widetilde{C}(X,Y,Z,W) = g(C(X,Y)Z,W), \widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$$

and r is the scalar curvature. Now for con-circularly flat manifold, we have $\widetilde{C}(X,Y,Z,W) = 0$. Hence from (3.2) we get

$$\widetilde{R}(X,Y,Z,W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{8}$$

Now putting $Z = F(Z)$ in (3.3) we get

$$g(\nabla_X \nabla_Y F(Z), W) - g(\nabla_Y \nabla_X F(Z), W) - g(\nabla_{[X,Y]} F(Z), W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) \\ -g(X,Z)g(Y,W)]. \qquad (9)$$

By using

$$\nabla_X F(Y) = (\nabla_X F)Y + F(\nabla_X Y)$$

and nearly Kähler condition

$$(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0$$

we have

$$\begin{aligned}
&-g[\nabla_X(\nabla_Z F)Y, W] + g[(\nabla_X F)(\nabla_Y Z), W] + g(F(\nabla_X \nabla_Y Z), W) \\
&+g[\nabla_Y(\nabla_Z F)X, W] - g[(\nabla_Y F)(\nabla_X Z), W] - g(F(\nabla_Y \nabla_X Z), W) \\
&-g[(\nabla_{[X,Y]} F)Z, W] - g(F(\nabla_{[X,Y]} Z), W) \\
&= \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],
\end{aligned}$$

this implies

$$\begin{aligned}
&2g(F(R(X,Y)Z, W)) + g[(\nabla_X F)(\nabla_Y Z), W] - g[(\nabla_Y F)(\nabla_X Z), W] \\
&= \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].
\end{aligned}$$

Theorem 3.2. For a con-circularly flat nearly Kähler manifold the following expression holds

$$\sum_{i=1}^{2n} g((\nabla_{e_i} F)(e_i), (\nabla_{e_i} F)(e_i)) = 0.$$

Proof : we know in a nearly Kähler manifold the curvature tensor $\widetilde{R}$ satisfies,

$$\widetilde{R}(X,Y,X,Y) = \widetilde{R}(X,Y,F(X),F(Y)) + g((\nabla_X F)(Y), (\nabla_X F)(Y)),$$

where $\widetilde{R}(X,Y,X,Y) = g(R(X,Y)X,Y)$.

Also for con-circularly flat manifold, we have $\widetilde{C}(X,Y,Z,W)=0$. So

$$\widetilde{R}(X,Y,Z,W)=\frac{r}{n(n-1)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]. \tag{10}$$

Now
from (3.5) and putting $X=Y=e_i$, $1\leq i\leq 2n$ and summing over i
we obtain

$$\sum_{i=1}^{2n} g((\nabla_{e_i}F)(e_i),(\nabla_{e_i}F)(e_i))=0.$$

Note: For a conformally flat, projectivly flat, con-harmonic flat and Bochner flat nearly Kăhler manifold the following relations holds

$$\sum_{i=1}^{2n} g((\nabla_{e_i}F)(e_i),(\nabla_{e_i}F)(e_i))=0.$$

Theorem 3.3. If a nearly Kăhler manifold M is of constant holomorphic sectional curvature c at every point P in M and con-circularly flat, then

$$\sum_{i=1}^{2n} g((\nabla_X F)(Y),(\nabla_{e_i}F)(e_i))=0.$$

Proof : We know that in a nearly Kähler manifold M of constant holomorphic sectional curvature c at every point P in M, the Riemannian curvature tensor of M is of the form

$$\begin{aligned}\widetilde{R}(X,Y,Z,W)= &\ \frac{c}{4}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)]\\ , &\ +g(X,F(W))g(Y,F(Z))\\ &-g(X,F(Z))g(Y,F(W))\\ &-2g(X,F(Y))g(Z,F(W))]\\ &+\frac{1}{4}[g((\nabla_X F)W,(\nabla_Y F)Z)-g((\nabla_X F)Z,(\nabla_Y F)W)\\ &-2g((\nabla_X F)Y,(\nabla_Z F)W)].\end{aligned}$$

Also for con-circularly flat manifold, we have $\widetilde{C}(X,Y,Z,W)=0$. So

$$\widetilde{R}(X,Y,Z,W)=\frac{r}{n(n-1)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]. \tag{11}$$

Now
from (3.6) and putting $Z = W = e_i$, $1 \leq i \leq 2n$ and summing over i
we have

$$\sum_{i=1}^{2n} g((\nabla_X F)(Y), (\nabla_{e_i} F)(e_i)) = 0.$$

Note: For a conformally flat, projectivly flat, con-harmonic flat and Bochner flat nearly Kӓhler manifold M is of constant holomorphic sectional curvature c at every point P in M, then the following expression holds

$$\sum_{i=1}^{2n} g((\nabla_X F)(Y), (\nabla_{e_i} F)(e_i)) = 0.$$

4. Curvature identities in 6 - dimensional nearly Kӓhler manifolds

In a lower dimensions, the nearly Kӓhler manifolds are widely determined. If M is nearly Kӓhler manifold with dim$M \leq 4$, then M is Kӓhler manifold. If dim$M = 6$, then we have the following [2,3,6,14].

Proposition 4.1. [10] Let (M,g,F) be a 6-dimensional, strict, nearly Kӓhler manifold. Then we have

(i) ∇F has constant type; that is,

$$g((\nabla_X F)(Y), (\nabla_X F)(Y)) = \frac{r}{30}(g(X,X)g(Y,Y) - g(X,Y)^2 - g(FX,Y)^2)$$

for all vector fields X and Y,
(ii) the first Chern class of (M, F) vanishes, and
(iii)M is Einstein manifold;

$$Ric = \frac{r}{6}g, Ric^* = \frac{r}{30}g.$$

Furthermore, from this proposition we have the following lemma [2,3,14].

Lemma 4.1. For vector fields W, X, Y and Z, we have

$$\begin{aligned} g((\nabla_W F)(X), (\nabla_Y F)(Z)) = \frac{r}{30}&[g(W,Y)g(X,Z) - g(W,Z)g(X,Y) \\ &-g(W,FY)g(X,FZ) + g(W,FZ)g(X,FY)] \end{aligned}$$

and

$$g((\nabla_W \nabla_Z X), Y) = \frac{r}{30}(g(W,Z)g(FX,Y) - g(W,X)g(FZ,Y) + g(W,Y)g(FZ,X)).$$

We can easily verify that for a 6-dimensional nearly Kӓhler manifold the con-circular curvature tensor takes the form

$$\widetilde{C}(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) - \frac{r}{30}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

The dimension of the manifold can be verified by using Lemma (4.1) and concircularly flatness conditions.

We also deduce the following result

Result 4.1. *For a projectively flat 6-dimensional nearly Kähler manifold Ricci curvature tensor is S(X,Y) =* $\frac{r}{6}g(X,Y)$. *So the manifold is an Einstein manifold.*

5. Example of nearly Kähler manifold

A 6-dimensional unit sphere S^6 has an almost complex structure F defined by the vector cross product in the space of purely imaginary Cayley numbers. This almost complex structure is not integrable and satisfies $(\nabla_X F)(X) = 0$, for any vector field X on S^6. Hence S^6 is a nearly Kähler manifold which is not Kähler.

A structure on an n-dimensional manifold M given by a non-null tensor field f satisfies $f^3 + f = 0$, is called an f-structure. Then the rank of f is a constant, say r. If $r = n$, then the f-structure gives an almost complex structure of the manifold M. In this case n is even.

The results can be verified in the above example.

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Some Curvature Identities on Kähler-Norden Manifolds

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Abstract. In this paper we have studied some curvature identities on Kähler-Norden manifolds that are quasi-conformally flat, pseudo-projectively flat, Weyl-conformally flat and Bochner flat. Additionally, we show that a Kähler-Norden manifold is symmetric if and only if it is locally symmetric. We have also conducted a study on semi-symmetric Kähler-Norden manifolds.

Key words : Kähler manifold, Kähler-Norden manifold, quasi-conformally flat Kähler spaces, pseudo-projectively flat spaces, Weyl-conformally flat spaces, Boch
ner curvature tensor, symmetric Kähler-Norden manifold, semi-symmetric Kähler-Norden manifold

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1. Introductrion. An even dimensional differentiable manifold M^n, where n =2m, $m \geq 2$ is said to be an Kähler-Norden manifold (anti-Kähler manifold) [3] if there is an almost complex structure F and an anti-Hermitian metric g such that $\nabla F = 0$ where ∇ is the Levi-Civita connection of g. The metric g is called anti-Hermitian if it satisfied $g(FX, FY) = -g(X, Y)$ for all vector fields X and Y on M^n. Then the metric g has necessarily a neutral signature (m, m) and M^n is a complex manifold and there exist a holomorphic metric on M^n [9]. This fact gives us some topological obstructions to an anti-Kähler manifold, for instance, all its odd Chern numbers vanish because its holomorphic metric gives us a complex isomorphism between the complex tangent bundle and its dual and a compact simply connected Kähler manifold cannot be anti-Kähler because it does not admit a holomorphic metric. In the present paper we have studied some curvature identities on Kähler-Norden manifolds.

2. Preliminaries. Let M be a connected differentiable manifold of dimension $n = 2m, m \geq 2$, F be a (1,1)-tensor field and g be a pseudo-Riemannian metric on M. Then (M,F,g) is said to be a Kähler-Norden manifold if the following conditions hold:

$$F^2 = -I, \quad g(FX, FY) = -g(X, Y), \quad \nabla F = 0$$

for any $X, Y \in TM$, being the Lie algebra of vector fields on M, ∇ is the Levi-Civita connection of g and I is the identity operator. In a Kähler-Norden manifold (M,F,g), the Riemannian curvature operator R, the Riemannian curvature tensor $\widetilde{R}$, the Ricci tensor S, the scalar curvature r and the r^* curvature are defined by:

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

$$\widetilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W),$$

$$S(X, Y) = trace \ \ of \ \ Z \to R(Z, X)Y,$$

$$r = trace \ \ S, \tag{2.1}$$

$$r^* = S(Fe_i, e_i). \tag{2.2}$$

Also, the following properties are satisfied in a Kähler-Norden manifold:

$$R(FX, FY)Z = -R(X, Y)Z, \tag{2.3}$$

$$R(FX, Y)Z = R(X, FY)Z, \tag{2.4}$$

$$S(FX, Y) = S(FY, X), \tag{2.5}$$

$$S(FX, FY) = -S(X, Y). \tag{2.6}$$

If we take Q as the Ricci operator then the Ricci tensor of type $(0, 2)$ in terms of Q is defined as

$$S(X, Y) = g(QX, Y), \tag{2.7}$$

where

$$rQY = -\sum_i \epsilon_i R(e_i, Y)e_i,$$

and $\{e_1, e_2, ...e_n\}$ is an orthonormal basis and ϵ_i are the indicators of e_i. The Riemannian metric g in terms of e_i and ϵ_i are given by

$$\epsilon_i = g(e_i, e_i) = \pm 1, \tag{2.8}$$

$$g(Fe_i, e_i) = 0. \tag{2.9}$$

3. Some results on Curvature Identities on Kähler-Norden manifold

Definition : *The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki* [19] *and is defined by*:

$$\begin{aligned} C(X,Y)Z &= \alpha R(X,Y)Z + \beta[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{n}\left[\frac{\alpha}{n-1} + 2\beta\right][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

where α, β are constants, Q is the Ricci operator, defined by $g(QX,Y) = S(X,Y)$ and n is the dimension of the manifold. Moreover, if $\alpha = 1$ and $\beta = -\frac{1}{n-2}$, the above equation reduces to conformal curvature tensor [8]. A manifold (M^n, g) where $n > 3$, is said to be quasi-conformally flat if $C = 0$. Using the above definition we prove the following:

Theorem 3.1 *In a quasi-conformally flat Kähler-Norden manifold, the Ricci tensor satisfies the relation* $S(Y,W) = \frac{\beta r^*}{\alpha - 2\beta} g(FY, W)$, *provided* $\alpha \neq 2\beta$.

Proof: In an n-dimensional Kähler-Norden manifold, we can define the Ricci tensor S by

$$S(X,Y) = \sum_{i=1}^{n} \epsilon_i \widetilde{R}(F(e_i), F(Y), e_i, W), \tag{3.2}$$

where $\{e_1, e_2, ...e_n\}$ is an orthonormal basis and ϵ_i is the indicator of e_i, $\epsilon_i = g(e_i, e_i) = \pm 1$. Taking inner product in (3.1) by W, we get

$$\begin{aligned} g(C(X,Y)Z,W) &= \alpha \widetilde{R}(X,Y,Z,W) + \beta[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &\quad + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ &\quad - \frac{r}{n}\left[\frac{\alpha}{n-1} + 2\beta\right][g(Y,Z)g(X,W) \\ &\quad - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.3}$$

Now, as the manifold is quasi-confomally flat, then the above equation reduces to

$$\begin{aligned}&\alpha\widetilde{R}(X,Y,Z,W)+\beta[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\\&+g(Y,Z)S(X,W)-g(X,Z)S(Y,W)]\\&-\frac{r}{n}\left[\frac{\alpha}{n-1}+2\beta\right][g(Y,Z)g(X,W)\\&-g(X,Z)g(Y,W)]=0.\end{aligned}\tag{3.4}$$

Putting $X=Fe_i, Y=FY, Z=e_i$ in the above equation and summing over $i=1,2,...,n$, and using (3.2), (2.2), (2.5), (2.6) and (2.9), we have

$$(\alpha-2\beta)S(Y,W)-\beta r^*g(FY,W)+\frac{r}{n}\left(\frac{\alpha}{n-1}+2\beta\right)g(Y,W)=0.\tag{3.5}$$

Taking $Y=W=e_i$ in the above equation and summing over $i=1,2,...,n$, and applying (2.1), we obtain

$$\alpha nr=0.$$

This implies

$$r=0,\quad provided\quad \alpha\neq 0.$$

Then (3.13) becomes

$$(\alpha-2\beta)S(Y,W)-\beta r^*g(FY,W)=0.$$

This implies

$$S(Y,W)=\frac{\beta r^*}{\alpha-2\beta}g(FY,W),\quad provided\quad \alpha\neq 2\beta.$$

This completes the proof.

Definition : *The pseudo-projective curvature tensor* $\overline{P}$ [15] *is given by*:

$$\begin{aligned}\overline{P}(X,Y)Z&=aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y]\\&-\frac{r}{n}\left[\frac{a}{n-1}+b\right][g(Y,Z)X-g(X,Z)Y],\end{aligned}\tag{3.6}$$

where a, and $b\neq 0$ are constants. Also, a manifold (M^n,g) is said to be pseudo-projectively flat if $\overline{P}=0$.

Theorem 3.2 *In a pseudo-projectively flat Kähler-Norden manifold, the Ricci tensor satisfies the relation* $S(Y,W) = \frac{br^*}{a-b} g(FY,W)$*, provided* $a \neq b$*.*

Proof: Taking inner product in (3.6) by W, we get

$$\begin{aligned} g(\overline{P}(X,Y)Z,W) = {} & a\widetilde{R}(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ & -\frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)g(X,W) \\ & -g(X,Z)g(Y,W)]. \end{aligned} \tag{3.7}$$

Now, as the manifold is pseudo-projectively flat, then the above equation reduces to

$$\begin{aligned} & a\widetilde{R}(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ & -\frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)g(X,W) \\ & -g(X,Z)g(Y,W)] = 0. \end{aligned} \tag{3.8}$$

Setting $X = Fe_i, Y = FY, Z = e_i$ in the above equation and summing over $i = 1, 2, ..., n$, and using (3.2), (2.2), (2.5), (2.6) and (2.9), we have

$$(a-b)S(Y,W) - br^* g(FY,W) + \frac{r}{n}\left(\frac{a}{n-1} + b\right) g(Y,W) = 0. \tag{3.9}$$

Taking $Y = W = e_i$ in the above equation and summing over $i = 1, 2, ..., n$, and applying (2.1), we obtain

$$anr = 0.$$

This implies

$$r = 0, \quad \text{provided} \quad a \neq 0.$$

Then (3.9) becomes

$$(a-b)S(Y,W) - br^* g(FY,W) = 0.$$

This implies

$$S(Y,W) = \frac{br^*}{a-b} g(FY,W), \quad \text{provided} \quad a \neq b.$$

Hence the proof.

Definition : *The Weyl-conformal curvature tensor* ($n > 3$) [14] *is given by:*

$$\begin{aligned} W(X,Y)Z = {} & R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY \\ & + S(Y,Z)X - S(X,Z)Y] \\ & + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.10}$$

So, a manifold is Weyl-conformal flat if $\widetilde{W}(X,Y,Z,U) = g(W(X,Y)Z,U) = 0$.

Theorem 3.3 *In a Weyl-conformally flat Kähler-Norden manifold, the Ricci tensor satisfies the relation* $S(Y,U) = -\frac{r^*}{n}g(FY,U)$, *provided* $n \neq 0$.

Proof: Taking inner product in (3.10) by U, we get

$$\begin{aligned} g(W(X,Y)Z,U) = {} & \widetilde{R}(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ & + S(Y,Z)g(X,U) - S(X,Z)g(Y,U)] \\ & + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{3.11}$$

Now, as the manifold is Weyl-confomally flat, then the above equation reduces to

$$\begin{aligned} & \widetilde{R}(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ & + S(Y,Z)g(X,U) - S(X,Z)g(Y,U)] \\ & + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] = 0. \end{aligned} \tag{3.12}$$

Putting $X = Fe_i, Y = FY, Z = e_i$ in the above equation and summing over $i = 1, 2, ..., n$, and using (3.2), (2.2), (2.5), (2.6) and (2.9), we get

$$\frac{n}{n-2}S(Y,U) + \frac{r^*}{n-2}g(FY,U) - \frac{r}{(n-1)(n-2)}g(Y,U) = 0. \tag{3.13}$$

Taking $Y = U = e_i$ in the above equation and summing over $i = 1, 2, ..., n$, and applying (2.1) we have,

$$nr = 0.$$

This implies

$$r = 0, \quad \text{provided} \quad n \neq 0.$$

Then (3.13) becomes

$$\frac{n}{n-2}S(Y,U)+\frac{r^*}{n-2}g(FY,U)=0.$$

This implies

$$S(Y,U)=-\frac{r^*}{n}g(FY,U),\quad \text{provided}\quad n\neq 0.$$

This completes the proof.

Definition : *The notion of Bochner curvature tensor* [4] *is defined by*:

$$\begin{aligned}B(X,Y)Z = R(X,Y)Z &- \frac{1}{n+4}[g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X\\ &-S(X,Z)Y+g(FY,Z)QFX-g(FX,Z)QFY\\ &+S(FY,Z)FX-S(FX,Z)FY-2S(FX,Y)FZ\\ &-2g(FX,Y)QFZ]+\frac{r}{(n+2)(n+4)}[g(Y,Z)X-g(X,Z)Y\\ &+g(FY,Z)FX-g(FX,Z)FY-2g(FX,Y)FZ],\end{aligned} \tag{3.14}$$

where Q is the Ricci operator, defined by $g(QX,Y) = S(X,Y)$ and n is the dimension of the manifold. Moreover, a manifold is Bochner flat if $\widetilde{B}(X,Y,Z,U) = g(B(X,Y)Z,U)=0$.

Theorem 3.4 *In a Bochner flat Kähler-Norden manifold, the Ricci tensor satisfies the relation* $S(Y,W)=-\frac{r^*}{2(n+4)}g(FY,W)$, *provided* $n+4\neq 0$.

Proof: Considering the inner product in (3.14) by W, we get

$$\begin{aligned}g(B(X,Y)Z,W) = \widetilde{R}(X,Y,Z,W) &- \frac{1}{n+4}[g(Y,Z)S(X,W)-g(X,Z)S(Y,W)\\ &+S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(FY,Z)S(FX,W)\\ &-g(FX,Z)S(FY,W)+S(FY,Z)g(FX,W)\\ &-S(FX,Z)g(FY,W)-2S(FX,Y)g(FZ,W)\\ &-2g(FX,Y)S(FZ,W)]+\frac{r}{(n+2)(n+4)}[g(Y,Z)g(X,W)\\ &-g(X,Z)g(Y,W)+g(FY,Z)g(FX,W)\\ &-g(FX,Z)g(FY,W)-2g(FX,Y)g(FZ,W)].\end{aligned} \tag{3.15}$$

Now, as the manifold is Bochner flat, then the above equation reduces to

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & \frac{1}{n+4}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + S(Y,Z)g(X,W) \\ & -S(X,Z)g(Y,W) + g(FY,Z)S(FX,W) \\ & -g(FX,Z)S(FY,W) + S(FY,Z)g(FX,W) \\ & -S(FX,Z)g(FY,W) - 2S(FX,Y)g(FZ,W) \\ & -2g(FX,Y)S(FZ,W)] - \frac{r}{(n+2)(n+4)}[g(Y,Z)g(X,W) \\ & -g(X,Z)g(Y,W) + g(FY,Z)g(FX,W) \\ & -g(FX,Z)g(FY,W) - 2g(FX,Y)g(FZ,W)]. \qquad (3.16)\end{aligned}$$

Putting $X = Fe_i, Y = FY, Z = e_i$ in the above equation and summing over $i = 1, 2, ..., n$, and using (3.2), (2.1), (2.2), (2.5), (2.6) and (2.9), we have

$$S(Y,W) = -\frac{r^*}{2(n+4)}g(FY,W), \quad \text{provided} \quad n+4 \neq 0.$$

Hence the proof.

Corollary 3.5 *In a Bochner Kähler-Norden manifold, the scalar curvature vanishes.*

Proof: Setting $Y = W = e_i$ in the above equation, and taking the summation over $i = 1, 2, ..., n$, we obtain $r = 0$.

Therefore, the proof is complete.

4. Symmetric Kähler-Norden manifold. Let (M,g) be a Riemannian manifold and ∇ be the Levi-Civita connection of (M,g) then a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g).

A pseudo-projectively curvature tensor is said to be parallel if the covariant derivative of pseudo-projective curvature tensor vanishes i.e. $\nabla\overline{P} = 0$, and this type of manifold is called pseudo-projectively symmetric manifold.

Theorem 4.1 *A Kähler-Norden manifold is pseudo-projectively symmetric if and only if it is locally symmetric.*

Proof: Taking the covariant derivative of equation (3.6) and putting $X = Fe_i, Y = FY, Z = e_i, W = W$ and also using $\nabla\overline{P} = 0$, we get

$$\begin{aligned}&(a-b)(\nabla_X S)(Y,W) - bdr^*(X)g(FY,W)\\&+\frac{dr(X)}{n}\left(\frac{a}{n-1}+b\right)g(Y,W) = 0.\end{aligned} \tag{4.1}$$

Now, putting $Y = W = e_i$ in above equation

$$an(dr(X)) = 0. \tag{4.2}$$

Since $a \neq 0$, which implies

$$dr(X) = 0. \tag{4.3}$$

Using (4.3) in (4.1), we obtain

$$(\nabla_X S)(Y,W) = \frac{b}{a-b}dr^*(X)g(FY,W). \tag{4.4}$$

Setting $Y = FY$ in the above equation, we get

$$(\nabla_X S)(FY,W) = -\frac{b}{a-b}dr^*(X)g(Y,W). \tag{4.5}$$

Again replacing Y and W in (4.5) by e_i, we have

$$\left(1+\frac{bn}{a-b}\right)dr^*(X) = 0, \tag{4.6}$$

this implies

$$dr^*(X) = 0. \tag{4.7}$$

Applying (4.7) in (4.4), we get

$$(\nabla_X S)(Y,W) = 0. \tag{4.8}$$

Now, taking the covariant derivative of (3.6) and using (4.3) and (4.8), we obtain

$$(\nabla_X\overline{P})(Y,Z,U,V) = a(\nabla_X R)(Y,Z,U,V), \text{ where } a \neq 0.$$

This proves the theorem.

From theorem 4.1, we have the following corollary:

Corollary 4.2 *Kähler-Norden manifolds are quasi-conformally symmetric, Weyl-conformally symmetric and Bochner symmetric if and only if these are all locally symmetric.*

5. Semi-symmetric Kähler-Norden manifold. Let (M,g) be a Riemannian manifold and a Riemannian or pseudo-Riemannian manifold is said to be semi-symmetric [16] if $R(X,Y).R = 0$, Ricci semi-symmetric [11] if $R(X,Y).S = 0$, where $R(X,Y)$ denote the derivation in the tensor algebra at each point of the manifold.

Theorem 5.1 *A Kähler-Norden manifold is pseudo-projectively semi-symmetric if and only if it is semi-symmetric.*

Proof: From equation (3.6) and putting $X = Fe_i, Y = FY, Z = e_i, W = W$, we obtain

$$\sum_{i=1}^{n} \epsilon_i \overline{P}(Fe_i, FY)e_i = (a-b)QY - br^* FY + \frac{r}{n}\left(\frac{a}{n-1} + b\right)Y, \tag{5.1}$$

where r^* is the $*$-scalar curvature which is defined by trace of FQ. If pseudo-projectively curvature tensor in Kähler-Norden manifold satisfies $R.\overline{P} = 0$ then from equation (5.1) $R.Q = 0$ and hence $R.S = 0$. Since we know that the Ricci tensors are defined by $S(X,Y) = g(QX,Y)$ and $S(FX,Y) = g(QFX,Y)$ then from equation (3.6) if $R.\overline{P} = 0$ and $R.S = 0$ then we obtain $R.R = 0$. Conversely if

$$R.R = 0 \Rightarrow R.S = 0 \Rightarrow R.Q = 0, \tag{5.2}$$

then from (5.1), we have $R.\overline{P} = 0$. Hence the proof.

From theorem 5.1, we have the following corollary:

Corollary 5.2 *Kähler-Norden manifolds are quasi-conformally semi-symmetric, Weyl-conformally semi-symmetric and Bochner semi-symmetric if and only if these are all semi-symmetric.*

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SOME CURVATURE IDENTITIES ON HYPERKÄHLER MANIFOLDS

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Abstract. The object of the present paper is to study some curvature identities on hyperKähler manifold which is locally symmetric. Also, we study conformal flatness, Bochner flatness and generalised W_2-flatness of a hyperKähler manifold. Finally, we gave some examples of a hyperKähler manifold.

1. Introduction

A hyperKähler manifold [15] is a Riemannian $4n$-manifold with a family of almost complex structures which act under composition like the multiplication, pure-imaginary, unit quaternions and which are covariantly constant with respect to the Levi-Civita connection. If we only requisite that these almost complex structures exist locally and that the Levi-Civita connection preserves this family generally, then we obtain a quaternionic Kähler structure, at least if $n \geq 2$. Thus hyperKähler manifolds are a special case of quaternionic Kähler manifolds. Although, note that quaternionic Kähler manifold need not be Kähler.

Remember that a Riemannian manifold which has just one such automorphism is called a Kähler manifold. The name "hyperKähler", which established with E. Calabi [8], is a proper description-the metric is Kählerian for several complex structures-even though it does recall Grassmann's "hypercomplex numbers" rather than Hamilton's quaternions. There is, however, an essential difference between Kähler and hyperKähler manifolds. A Kähler metric on a given complex manifold can be modified to another one simply by adding a hermitian form $\partial\bar{\partial}f$ for an arbitrary sufficiently small C^∞ function f. Thus the space of Kähler metrics is infinite dimensional. It is also easy to find examples of Kähler manifolds. Any complex submanifold of CP_n inherits a Kähler metric and so simple writing down algebraic equations for a projective variety gives a vast number of examples.

By contrast, hyperKähler metrics are much more rigid. On a compact manifold, if one such metric exists, then up to isometry there is only a finite dimensional

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space of them. Nor is it easy to find examples. Obviously, we will never find them as quaternionic submanifolds of the quaternionic projective space $\mathbb{H}P_n$ [6].

The idea of a hyperKähler manifold arose first in 1955 with M.Berger's classification of the holonomy groups of Riemannian manifolds. On a hyperKähler manifold, parallel translation preserves I, J and K (since they are covariant constant) and so the holonomy group is contained in both orthogonal group O_{4n} and the group $GL(n, \mathbb{H})$ of quaternionic invertible matrices (i.e., those linear transformations which commute with right multiplication by i, j and k). The maximal such intersection in SP_n, the group of $n \times n$ quaternionic unitary matrices. This group performed in Berger's list.

The group SP_n is also an intersection of U_{2n} and $SP(2n, C)$, the linear transformations of C^{2n} which preserve a non-degenerate skew form. Thus a hyperKähler manifold is naturally a complex manifold with a holomorphic symplectic form. One can see explicitly by taking the three Kähler two-forms, $\omega_1(X,Y) = g(IX,Y)$, $\omega_2(X,Y) = g(JX,Y)$, $\omega_3(X,Y) = g(KX,Y)$ for X, Y $\in TM$, defined for the complex structures I, J and K. As to complex structures I, J and K, the complex form $\omega_+ = \omega_2 + i\omega_3$ is non-degenerate and covariant constant, hence it is closed and holomorphic.

Now, also we define the generalised W_2-curvature tensor $(4n > 8)$ as follows:

$$\overline{W_2}(X,Y)Z = aR(X,Y)Z + \left(b + \frac{c}{4n-7}\right)[g(X,Z)QY - g(Y,Z)QX], \quad (1.1)$$

where $a, b, c \neq 0$. In particular, if $a = 1, b = 0, c = 1$; then it reduces to W_2-curvature tensor. Again if $b = 0$, we call the $\overline{W}_2$ tensor as quasi-W_2 tensor and is denoted by $\widetilde{W_2}$. So a manifold is generalised W_2-flat if $g(\overline{W_2}(X,Y)Z, W) = 0$.

The interest in some curvature identities on hyperKähler manifold is motivated by our study [13] and [15] of hyperKähler manifold and hypercomplex structures in $4n$-dimensional Riemannian manifolds, which is locally symmetric, conformally flat, Bochner flat and generalised W_2-flat.

2. PRELIMINARIES

Let (M, g) be a Riemmanian manifold with I, J, K compatible almost complex structures parallel for the Levi-Civita connection and with $IJ = K = -JI$. Consequently, (a) I, J, K are Integrable, (b) $\omega_1 = g(I.,.)$ etc. are symplectic forms. Let $\mathbb{H} = \mathbb{R}^4$ with basis $\{1, i, j, k\}$, $i^2 = -1 = j^2 = k^2$, quaternion division algebra. In $\mathbb{H}^n$, $Iq = -qi$ holds, with standard inner product. We also know $SP_1 = SU_2 = \{ai + bj + ck : a^2 + b^2 + c^2 = 1\}$ acts on the right. $SP_n = \{A \in M_n(\mathbb{H}) \mid \overline{A}^T A = 1_n\}$ is centraliser in SO_{4n} of SP_1.
A hyperKähler manifold is a Rieminnian $4n$-manifold with holonomy in SP_n.

Now, we have the following propositions:

Proposition 2.1. [14] *A hyperKähler manifold M is a complex manifold with a holomorphic symplectic form. Conversely , any compact Kähler manifold* [10] *with a holomorphic symplectic form is hyperKähler.*

Proposition 2.2. [14] *A hyperKähler manifold is a C^∞ Riemannian manifold together with three covariantly constant orthogonal endomorphisms I, J and K of the tangent bundle which satisfy the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$.*

Note that, I, J and K give each tangent space the structure of a quaternionic vector space, so the dimension of a hyperKähler manifold is divisible by 4. Since I, J and K are covariantly constant, a parallel transport commutes with the quaternionic multiplication and so the holonomy group is contained in $O_{4n} \bigcap GL_n(\mathbb{H}) \cong SP_n$, the group of quaternionic unitary $n \times n$ matrices. In particular, since $SP_n \subseteq SU_{2n}$ every hyperKähler manifold is Calabi-Yau [9]. Assume that M is an almost hypercomplex manifold. Define the Nijenhuis tensor N of I, J and K by

$$\begin{aligned} N_I(X,Y) &= [IX, IY] - I[IX, Y] - I[X, IY] - [X, Y], \\ N_J(X,Y) &= [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \ \textit{and} \\ N_K(X,Y) &= [KX, KY] - K[KX, Y] - K[X, KY] - [X, Y], \end{aligned}$$

for all vector fields X, Y. M is said to be hypercomplex if $N_I = N_J = N_K = 0$. Suppose that g is a pseudo Riemannian metric on M satisfying the condition

$$\begin{aligned} g(IX, Y) + g(X, IY) &= 0, \\ g(JX, Y) + g(X, JY) &= 0, \\ g(KX, Y) + g(X, KY) &= 0, \end{aligned} \tag{2.1}$$

for $X, Y \in \chi(M)$. Define the 2-forms $\widetilde{I}(X,Y) = g(X, IY)$, $\widetilde{J}(X,Y) = g(X, JY)$, $\widetilde{K}(X,Y) = g(X, KY)$ for all $X, Y \in \chi(M)$. M is said to be hyperKähler manifold if it is a hypercomplex and the respective 2-forms are closed and $\nabla I = \nabla J = \nabla K = 0$, where ∇ is the Levi-Civita connection on M are equivalent.

3. Main results of Some Curvature Identities on hyperKähler Manifolds

We are investigated some properties of curvature tensors and Ricci tensors of hyperKähler manifold. Let M be a hyperKähler manifold and R denotes the curvature tensor of M.

Theorem 3.1. *The curvature tensor R satisfies*

$$\begin{aligned}
&(i) R(X,Y)IZ = IR(X,Y)Z,\\
&(ii) R(IX,IY)Z = R(X,Y)Z,\\
&(iii) R(IX,Y)Z + R(X,IY)Z = 0,\\
&(iv) \widetilde{R}(IX,IY,IZ,IW) = \widetilde{R}(X,Y,Z,W),\\
&(v) \widetilde{R}(IX,Y,IZ,W) = \widetilde{R}(X,IY,Z,IW),\\
&(vi) \widetilde{R}(X,Y,IZ,JW) = -\widetilde{R}(IX,IY,Z,IJW),\\
&(vii) \widetilde{R}(IX,IY,JZ,JW) = \widetilde{R}(X,Y,IJZ,IJW),
\end{aligned}$$

where $\widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Proof. (i) Since I is parallel, i.e., $(\nabla_X I)(Y) = 0$, we get

$$\nabla_X I(Y) = I(\nabla_X Y).$$

Now,

$$\begin{aligned}
R(X,Y)I(Z) &= \nabla_X\nabla_Y I(Z) - \nabla_Y\nabla_X I(Z) - \nabla_{[X,Y]} I(Z)\\
&= \nabla_X I(\nabla_Y Z) - \nabla_Y I(\nabla_X Z) - I(\nabla_{[X,Y](Z)})\\
&= I(\nabla_X\nabla_Y Z) - I(\nabla_Y\nabla_X Z) - I(\nabla_{[X,Y]} Z)\\
&= I(R(X,Y)Z).
\end{aligned}$$

(ii) Since $g(R(X,Y)V,U) = g(R(U,V)Y,X)$, we have

$$\begin{aligned}
g(R(IX,IY)V,U) &= g(R(U,V)IY,IX)\\
&= g(I(R)(U,V)Y,IX)\\
&= -g(R(U,V)Y,I^2(X)),\ [\mathit{since}\ g(IX,Y) = -g(X,IY)]\\
&= g(R(U,V)Y,X),\ [\mathit{since}\ I^2 = J^2 = K^2 = -1\\
&\qquad\qquad \mathit{and}\ IJ = -K = JI]\\
&= g(R(X,Y)V,U).
\end{aligned}$$

Hence, $R(IX,IY)V = R(X,Y)V$.

(iii) Putting $X = IX$ in (ii) we obtain (iii).

(iv) Now,

$$\begin{aligned}
g(R(IX,IY)IZ,IW) &= -g(I(R)(IX,IY)IZ,W),\\
&\qquad [\mathit{since}\ g(IX,Y) = -g(X,IY)]\\
&= -g(R(IX,IY)Z,W)\\
&= g(R(X,Y)Z,W),\\
&\qquad [\mathit{using}\ g(R(IX,IY)V,U) = g(R(X,Y)V,U)]
\end{aligned}$$

Therefore, $\widetilde{R}(IX,IY,IZ,IW) = \widetilde{R}(X,Y,Z,W)$.

(v) Setting $Y = IY$, $W = IW$ in (iv) we get (v).

(vi) Putting $X = IX$, $Y = IY$, $W = KW$, where $IJ = K = -JI$ in equation (iv) then we obtain

$$\widetilde{R}(X, Y, IZ, JW) = -\widetilde{R}(IX, IY, Z, IJW).$$

(vii) Again putting $Z = KZ$, $W = KW$, where $IJ = K = -JI$ in equation (iv) then we have $\widetilde{R}(IX, IY, JZ, JW) = \widetilde{R}(X, Y, IJZ, IJW)$. □

Remark 3.2. Accordingly, Theorem 3.1 holds for operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above curvature identities also hold for the operators IJ and JI.

Let S be the Ricci tensor of M, i.e.,

$$S(Y, Z) = trace\{X \to R(X, Y, Z)\} = \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, Y, Z, e_i),$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis for M and $\epsilon_i = g(e_i, e_i) = 1$.

Theorem 3.3. *The Ricci tensor of a hyperKähler manifold satisfies*

$$(i) S(IX, IY) = S(X, Y),$$
$$(ii) S(IX, Y) + S(X, IY) = 0.$$

Proof.

$$\begin{aligned} S(IX, IY) &= trace\{Z \to R(Z, IX)IY\} \\ &= trace\{IZ \to R(IZ, IX)IY\} \\ &= trace\{IZ \to R(Z, X)IY\}, \ [by\ (ii)\ of\ Theorem\ 3.1] \\ &= trace\{IZ \to IR(Z, X)Y\}, \ [since IR = RI] \\ &= trace\{Z \to R(Z, X)Y\} \\ &= S(X, Y), \end{aligned}$$

which proves (i).

Now setting $X = IY$ in (i) we obtain (ii). □

Remark 3.4. In parallel, Theorem 3.3 holds for the operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above Ricci tensor of a hyperKähler manifold also satisfy for the operators IJ and JI.

Theorem 3.5. *For a hyperKähler manifold of dimension* $4n$ *the following relation holds, i.e.,* $\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), X, I(Y)) = 0$.

Proof. We have

$$\begin{aligned}
S(X,Y) &= \sum_{i=1}^{4n} \epsilon_i g(R(e_i, X)Y, e_i) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(I(e_i), I(X))Y, e_i) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(e_i, Y)I(X), I(e_i)) \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(Y, e_i)I(e_i), I(X)) \\
&= \sum_{i=1}^{4n} \epsilon_i g(R(Y, e_i)I(X), I(e_i))
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{4n} \epsilon_i g(R(I(X), e_i)Y, I(e_i)) + \sum_{i=1}^{4n} \epsilon_i g(R(Y, I(X))e_i, I(e_i)), \\
&\qquad\qquad\qquad\qquad [using\ Bianchi's\ identities] \\
&= -\sum_{i=1}^{4n} \epsilon_i g(R(e_i, I(X))Y, I(e_i)) - \sum_{i=1}^{4n} \epsilon_i g(R(I(X), Y)e_i, I(e_i)) \\
&= \sum_{i=1}^{4n} \epsilon_i g(I(R)(e_i, I(X))Y, e_i) + \sum_{i=1}^{4n} \epsilon_i g(I(R)(I(X), Y)e_i, e_i) \\
&= \sum_{i=1}^{4n} \epsilon_i g(R(e_i, I(X))I(Y), e_i) + \sum_{i=1}^{4n} \epsilon_i g(R(I(X), Y)I(e_i), e_i) \\
&= S(IX, IY) - \sum_{i=1}^{4n} \epsilon_i g(R(Y, I(X))I(e_i), e_i) \\
&= S(X,Y) + \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), X, I(Y)).
\end{aligned}$$

So this implies, $\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), X, I(Y)) = 0.$ □

Remark 3.6. Comparably, Theorem 3.5 holds for the operators J, K. Since $I^2 = J^2 = K^2 = -1$ and $IJ = -K = JI$, so the above global form of curvature tensors of a hyperKähler manifold also satisfy for the operators IJ and JI, i.e.

$$(i) \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, J(e_i), X, J(Y)) = 0.$$

$$(ii) \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, IJ(e_i), X, IJ(Y)) = 0.$$

$$(iii) \sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, JI(e_i), X, JI(Y)) = 0.$$

3.1. **Conformal flatness of hyperKähler manifold.** Since any pseudo-Riemannian as well as Riemannian manifold of dimension 3 is conformally flat, we are focused in dimension $4n \geq 4$. Utilizing the identities from the previous section, we prove the following theorem.

Theorem 3.7. *Let M be a conformally flat hyperKähler manifold. Then*
(i) M is locally flat if dim $M \geq 4$,
(ii) M is locally symmetric and its scalar curvature vanishes identically if dim $M = 4$.

Proof. By the vanishing of conformal curvature tensor, we have

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) =& \frac{1}{2n-2}[-g(X,Z)S(Y,W) - g(Y,W)S(X,Z) + g(X,W)S(Y,Z) \\ &+ g(Y,Z)S(X,W)] - \frac{r}{(2n-1)(2n-2)}[g(X,Z)g(Y,W) \\ &- g(X,W)g(Y,Z)], \qquad (3.1)\end{aligned}$$

r being the scalar curvature of M. From the above equation with the help of Theorem 3.3 and Theorem 3.1, we get

$$\sum_{i=1}^{4n} \epsilon_i \widetilde{R}(e_i, I(e_i), Z, I(W)) = -\frac{2}{n-1} S(Z,W) - \frac{r}{(2n-1)(n-1)} g(Z,W).$$

Then using the result of Theorem 3.5, we obtain

$$S(Z,W) = -\frac{r}{2(2n-1)} g(Z,W). \qquad (3.2)$$

Now setting $Z = W = e_i, 1 \leq i \leq n$ and summing over i then from Equation (3.2), we have $(4n-1)r = 0$, where $r = \sum_{i=1}^{4n} \epsilon_i S(e_i, e_i)$. Then this implies $r = 0$, when $4n \geq 4$. Now putting the value of $r = 0$ in the Equation (3.2), we get $S = 0$.

So from Equation (3.1) it follows that the manifold is locally flat. Now, also if $4n = 4$ then $r = 0$, then its scalar curvature vanishes identically. Next we proof the manifold is locally symmetric. Now we assume that conformally flatness implies,

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{6}[(X_r)g(Y,Z) - (Y_r)g(X,Z)]. \qquad (3.3)$$

Putting $r = 0$ in the Equation (3.3), we obtain

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{3.4}$$

On the other hand, from Theorem 3.3 it follows that

$$(\nabla_X S)(Y, I(Z)) + (\nabla_X S)(Z, I(Y)) = 0. \tag{3.5}$$

Using the equality and the Equation (3.4), we get

$$(\nabla_X S)(Y, I(Z)) + (\nabla_Y S)(Z, I(X)) = 0. \tag{3.6}$$

Applying the result of the Theorem 3.3 on the above equation, we obtain $\nabla S = 0$. Now from the Equation (3.1) and $\nabla S = 0$, implies that $\nabla R = 0$, i.e., the manifold is locally symmetric. ☐

From Theorem 3.7, we have the following Corollary:

Corollary 3.8. A conformally flat hyperKähler manifold of dimension $4n$ is an Einstein manifold.

3.2. Bochner flatness and generalised W_2-flatness of hyperKähler manifold.

The notion of Bochner curvature tensor [2] is defined by:

$$\begin{aligned} B(X, Y)Z =& R(X, Y)Z - \frac{1}{2n+4}[g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X \\ & - S(X, Z)Y + g(IY, Z)QIX - g(IX, Z)QIY + S(IY, Z)IX \\ & - S(IX, Z)IY - 2S(IX, Y)IZ - 2g(IX, Y)QIZ] \\ & + \frac{r}{(2n+2)(2n+4)}[g(Y, Z)X - g(X, Z)Y + g(IY, Z)IX \\ & - g(IX, Z)IY - 2g(IX, Y)IZ], \end{aligned} \tag{3.7}$$

where Q is the Ricci operator, defined by $g(QX, Y) = S(X, Y)$ and n is the dimension of the manifold. Moreover a manifold is Bochner flat if $\widetilde{B}(X, Y, Z, U) = g(B(X, Y)Z, U) = 0$.

Theorem 3.9. *A Bochner flat hyperKähler manifold is an Einstein manifold.*

Proof. Taking inner product in the Equation (3.7) by W, we get

$$\begin{aligned} g(B(X, Y)Z, W) = & \widetilde{R}(X, Y, Z, W) - \frac{1}{2n+4}[g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\ & + S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(FY, Z)S(IX, W) \\ & - g(IX, Z)S(IY, W) + S(IY, Z)g(IX, W) - S(IX, Z)g(IY, W) \\ & - 2S(IX, Y)g(IZ, W) - 2g(IX, Y)S(IZ, W)] \\ & + \frac{r}{(2n+2)(2n+4)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ & + g(IY, Z)g(IX, W) - g(IX, Z)g(IY, W) - 2g(IX, Y)g(IZ, W)]. \end{aligned} \tag{3.8}$$

Now as the manifold is Bochner flat then the above equation reduces to

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) =& \frac{1}{2n+4}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + S(Y,Z)g(X,W) \\ & - S(X,Z)g(Y,W) + g(IY,Z)S(IX,W) - g(IX,Z)S(IY,W) \\ & + S(IY,Z)g(IX,W) - S(IX,Z)g(IY,W) - 2S(IX,Y) \\ & g(IZ,W) - 2g(IX,Y)S(IZ,W)] - \frac{r}{(2n+2)(2n+4)}[g(Y,Z) \\ & g(X,W) - g(X,Z)g(Y,W) + g(IY,Z)g(IX,W) - g(IX,Z) \\ & g(IY,W) - 2g(IX,Y)g(IZ,W)]. \end{aligned} \tag{3.9}$$

Setting $X = e_i, Y = Ie_i, Z = Z$ and $W = IW$ in the above equation and taking summation over i, $1 \leq i \leq n$ and also using the result of the Theorem 3.5, we obtain

$$S(Z,W) = -\frac{r}{2n+6}g(Z,W). \tag{3.10}$$

Hence the proof. □

From Theorem 3.9 we have the following Corollary:

Corollary 3.10. A Bochner flat hyperKähler manifold is locally flat.

Proof. Taking $Z = W = e_i$ in the above equation and summing over i, $1 \leq i \leq n$ we obtain

$$r = 0, \ \textit{provided}\ 3n + 6 \neq 0. \tag{3.11}$$

Then the Equation (3.10) becomes

$$S(Z,W) = 0. \tag{3.12}$$

So the manifold is locally flat. □

Theorem 3.11. A generalised W_2-flat hyperKähler manifold is Ricci flat, provided $a \neq \left(b + \frac{c}{4n-7}\right)$.

Proof. Taking inner product in the Equation (1.1) by W, we get

$$\begin{aligned} g(\overline{W_2}(X,Y)Z,W) =& a\widetilde{R}(X,Y,Z,W) + \left(b + \frac{c}{4n-7}\right)[g(X,Z)S(Y,W) \\ & - g(Y,Z)S(X,W)]. \end{aligned} \tag{3.13}$$

Now as the manifold is $\overline{W}_2$-flat then the above equation reduces to

$$a\widetilde{R}(X,Y,Z,W) + \left(b + \frac{c}{4n-7}\right)[g(X,Z)S(Y,W) - g(Y,Z)S(X,W)] = 0. \tag{3.14}$$

Putting $X = e_i, Y = Ie_i, W = IW$ in the above equation and summing over i, $1 \leq i \leq 4n$ and operating the result of the Theorem 3.3 we have

$$\left(b + \frac{c}{4n-7}\right) S(Z,W) = 0. \tag{3.15}$$

Then we have $S(Z,W) = 0$, for any $Z, W \in \chi(M)$, being the Lie algebra of vector fields on M. This completes the proof. □

From Theorem 3.11 we have the following Corollary:

Corollary 3.12. A quasi-W_2 flat hyperKähler manifold is Ricci flat, provided $c \neq 0$.

The following examples are given in the paper [14]

Example 3.13. A trivial example is $\mathbb{H}^n$. However, in contrast to the Kähler case, $\mathbb{H}P_n$ is not hyperKähler and neither do its generic quaternionic submanifolds.

Example 3.14. In the particular case $n = 1$, then $SP_1 = SU_2$ in SO_4, so a 4-dimensional Riemannian manifold is hyperKähler exactly when it is Kähler and Ricci flat. Specifically, this shows that any compact complex surface M of Kähler type with vanishing first Chern class is either a torus or simply connected and admits a unique complex-symplectic structure, i.e., is a so-called "K3-surface".

Example 3.15. A class of non-compact hyperKähler manifolds of real dimension 4 can be obtained by resolving the singularity of C^2/Γ for $\Gamma \subset SU_2$ a finite subgroup.

Example 3.16. Many examples of non-compact hyperKähler manifolds arise as moduli spaces of solutions to gauge-theoretic equations. The hyperKähler structure is obtained by a hyperKähler reduction from $\mathbb{H}^n$.

These results can be verified in these examples.

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