Some Aspects of Different Types of Centers of Semirings

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY (SCIENCE) OF

JADAVPUR UNIVERSITY

2023

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Some aspects of different types of centers of semirings**" submitted by Sri **Ranjan Sarkar** who got his name registered on **5th September, 2018 (Index No.: 148/18/Maths./26)** for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Sukhendu Kar** and that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

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Dedicated to

my parents

Sri Ranjit Sarkar

and

Smt Manjuri Sarkar

Acknowledgements

Undertaking this Ph.D. thesis has been a truly life-changing experience for me and it was not possible to undertake such research work without the support and guidance of many people. I am grateful to those people who have and valuable opinions in undertaking this research work.

First and foremost, I would like to express my deepest appreciation to my respected supervisor, Prof. Sukhendu Kar for the continuous support of my Ph.D. study and research, for his patience, motivation, enthusiasm and immense knowledge. I thank him whole heartedly for providing me this opportunity. As my supervisor, he has constantly forced me to remain focused on achieving my goal. Thank him for his continuous guidance with all the useful discussions and brainstorming sessions, especially during the difficult conceptual development stage. Without his supervision, I would not have completed this Ph.D. thesis.

Apart from my Supervisor, I won't forget to express the gratitude to rest of my research committee : Prof. Tapan Kumar Dutta, Prof. Avishek Adhikary, for their encouragement, critical assessments and thought-provoking discussions. They generously gave their time to offer me valuable comments toward improving my work.

I am deeply indebted to Dr. Bijon Biswas. His analysis capability is so brilliant that he provides me lots suggestions. He always endows me help and support during my studying.

My heartiest congratulation to Dr. Sudipta Purkait for his valuable time and his precise advice which helped me my research work to move forward.

It has been a rewarding, penetrating, stimulating and prestigious experience which I have received from the renowned mathematician Prof. M.K. Sen. I am extremely lucky to receive his valuable advice and motivation during the course of research.

I am also thankful to the faculty members and staffs of the Department of Mathematics, as well as the officials and staffs of the Research Section, Jadavpur University, for all their co-operation.

My research work in the Department of Mathematics of Jadavpur University has been a wonderful experience for me. I must thank Dr. Phonindra Nath Das, Dr. Paltu Sarkar, Dr. Amit Mondal, Dr. Agni Roy, Soumen Pradhan, Debopriyo Das and Saikat Das for making this journey comfortable with all their love and comradeship.

Some special words of gratitude go to my friends who have always been a major source of support when things would get a bit discouraging : Shamit, Debangshu, Uttam, Salim, Goutam, Ramen, Parimal. I am thankful to have such dependable friends who are always there when I need them.

I pay my humble tribute to my grandmother-in-law Gayatri Mondal for her advice and rewarding encouragement during the course of research.

I would like to express my gratitude to my parents-in-law Amalendu Roy and Prabhabati Roy for their unfailing emotional support. I also thank for heart-warming kindness from the family of my brother-in-law: Abhrendu Roy.

I express my love and sincerest gratitude to my parents. Nothing would have been possible without their unconditional trust, unequivocal, timely encouragement and endless patience. My mother raised me with her caring and gentle love and she will be forever the compass of my life. I thank them for having faith in me and encouraging me in my academic pursuits.

As to my sweet little girl, Arasrika, my love and longing for her are beyond words. Birth of my beloved daughter inspired me to achieve this goal. She is the softest point of my heart. I am sorry for not being able to accompany or witness every step of her growing up in the first six months of her life.

Finally, but definitely not the least, I express my gratitude to my dear wife, Arpika Roy. I am so appreciative for her constant love, understanding and encouraging, for her taking up the whole responsibilities to our family and bearing the pressure both from working and living during my studying.

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(RANJAN SARKAR)

Abstract

Center plays an important role in the structure of group theory, algebraic geometry, structure of ring theory as well as the structure of semiring. There has been significant works on the center of semiring that had notable impact in derivation of semiring. In this thesis we survey and introduce some concepts of different types of centers of semirings. We aim to shed light on aspects of the structural properties of different class of centers of semirings. Here different characterizations for that centers of semirings have been done. Further some algebraic characterizations of certain classes of centers semirings are studied.

First, the concept of Birkhoff center of c-semiring is discussed. The concept of Birkhoff center of semigroup was intoduced by Swamy and Murti. In this thesis we have extended the Birkhoff center to a c-semiring S as the set of elements in S which are of the form $(1, 0)$ under some factorisation of S as direct product of c-semiring S_1 with identity element 1 and a c-semiring S_2 with zero element 0 and study various properties of the Birkhoff center of c-semiring. We have also showed that Birkhoff center of c-semiring froms a distributive lattice.

Let S be a semiring. An element $e \in S$ is called an almost idempotent if $e + e^2 = e^2$. The set of all almost idempotents of a semiring S will be denoted by $E_c(S)$. This was introdced by M.K. Sen and A.K. Bhuniya. The impetus behind the formation of the proposed class of center of semiring called almost idempotent center of semiring, which is the generalization of the set of almost idempotents and the center of semiring. We have analyzed the center of the semiring and established that almost idempotent center of semiring forms a distributive lattice in a certain condition.

Also we have studied some special center-like subsets which we call h-center of

semiring, k-center of semiring, generalized center of semiring and hypercenter of semiring. Besides we have characterized that centers of semirings. Also we have established relationship among the Birkhoff center of c-semiring, almost idempotent center of semiring, k-center of semiring, generalized center of semiring and hypercenter of semiring.

Let S be a semiring with center $Z(S)$. If S is commutative then $S = Z(S)$. This result have inspired us to develop a new type of semirings called almost idempotent central semiring, h-central semiring, k-central semiring, generalized central semiring and hypercentral semiring in which the whole semiring coincide with its corresponding center. Also we have studied some structural properties of those semirings. At long last, we have determined the connection between the almost idempotent central semiring, the h-central semiring and the k-central semiring.

List of Symbols

$\mathbb{Z}$	Set of all integers
$\mathbb{N}$	Set of all natural numbers
$\mathbb{N}_0$	Set of all non-negative integers
$\mathbb{Q}$	Set of all rationals
$\mathbb{R}$	Set of all real numbers
$\mathbb{Z}_n$	Congruent classes of integers modulo n
$\mathbb{F}_n$	Finite field having exactly n elements
$\|A\|$	Cardinality of a set A
$A - B$	All elements which are in set A but not in set B
$A \times B$	Cartesian product of A and B
$gcd\{a, b\}$	Greatest common divisor of integers a, b
$lcm\{a, b\}$	Least common multipliers of integers a, b
$Z(G)$	Center of a group G
$Z(R)$	Center of a ring R
$Z(S)$	Center of a semiring S
$\mathcal{P}(G)$	Set of all subsets of a group G
$B(S)$	Birkhoff center of a c-semiring S
$E_c(S)$	Almost idempotent center of a semiring S
$C_h(S)$	h-center of a semiring S
$C_k(S)$	k-center of a semiring S
$C_G(S)$	Generalized center of a semiring S
$T(S)$	Hypercenter of a semiring S

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Introduction and Preliminaries

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Introduction and Preliminaries

1.1 Introduction

Ring theory plays a key role in advanced algebra both in theory and application. Advanced algebra is characterized by its emphasis on the systematic investigation of abstract algebraic structures. A numerous research works have been done on different generalizations of ring theory. Among them, study of semirings has become a great interest of the recent researchers. Semirings, from an algebraic perspective, provide the most natural common generalization that unifies the theories of rings and distributive lattices. The concept of semirings was first introduced by Vandiver [67] in 1934. However, the developments of the theory in semirings have been taking place since 1950. Semirings abound in the Mathematical world around us. Semirings hold a fundamental place within the realm of Mathematics. Indeed the first mathematical structure we encounter the set of natural numbers is a semiring. Other semirings arise naturally in such diverse areas of mathematics as combinatories, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative and noncommutative ring theory, optimization theory, automata theory, formal language theory, the mathematical modeling of quantum physics and parallel computation system. The study of semiring structures has gained considerable attention from various researchers, including F. Pastijin, Y. Q. Guo, M. K. Sen, K. P. Shum and other contributors (See [9], [24], [50], [54], [58], [59], [60]).

The center of a group plays a fundamental role in the study of group properties and has various applications in different areas of mathematics. The center of a group G is the set of all those elements of G which commutes with all elements of G. The concept of a center was extended to rings, where the center $Z(R)$ of a ring R is defined as the set of elements that commute with all other elements in the ring. The center $Z(S)$ of a semiring S refers to a subset of the semiring that consists of elements that commute with every other element of the semiring. So, the concept of a center in semiring theory emerged as a generalization of the notion of a center in ring theory. Moreover, the center of a semiring plays a crucial role in the study of various algebraic structures and their properties. While the center of a semiring has not received as much attention as it has in other algebraic structures, there may be specific areas or research directions where the concept of the center in semirings has been explored in more depth. Several results concerning the center of rings have analogues in case of semirings. To get rid of these difficulties, we defined a more restricted class of center in a semiring, which we referred to as k-center. A k-center $C_k(S)$ of a semiring S is a subset of S such that whenever $a \in C_k(S)$, for any $x \in S\backslash\{0\}$, then $a + ax = ax$ and $ax = xa$. We note that $C_k(S)$ is a subsemiring of S, but it is not an ideal of S. It is interesting that $C_k(S)$ is an ideal of the center $Z(S)$ of S. In this thesis, we defined a still more restricted class of center in semirings, which we called h-center. A h-center $C_h(S)$ of a semiring S is a subset of S such that whenever $a \in C_h(S)$, for any $x \in S$, then $a + ax = a$ and $ax = xa$. It is clear that every element of $C_k(S)$ of a semiring S is left zero if and only if $C_k(S) = C_h(S)$.

Let S be a semigroup S. An element $a \in S$ is said to be a central element of S if there exist semigroups S_1 with 1_{S_1} and S_2 with 0_{S_2} and an isomorphism of S onto $S_1 \times S_2$ such that a is mapped onto $(1_{S_1}, 0_{S_2})$. The set of all central elements of S is called the Birkhoff center of S and is denoted by $B(S)$. Swamy and Murti [63] introduced the concept of the Birkhoff center of semigroup with 0 and 1 analogous to that of a bounded poset [11] and proved that it is a Boolean algebra in which the meet operation is the operation in S. Swamy and Murti [64] extended this concept for a semigroup with sufficiently many commuting idempotents and proved that it is a relatively complemented distributive lattice. Swamy and Pragathi [62] have also extended the above concept for an arbitrary semigroup S and proved that Birkhoff center of any semigroup S is a relatively complemented distributive lattice in which the meet operation in the semigroup S.

The notion mentioned above was investigated in semiring setting in this thesis. In [39], we introduce the notion of Birkhoff center of a c-semiring and prove that the Birkhoff center of a c-semiring forms a distributive lattice. Furthermore, we also in [52] study the structure of Birkhoff center of c-semiring. The notion of c-semiring was introduced to describe many constraint satisfaction schemes. There are some literature on c-semirings, for instance see [13], [14], [22] and [43].

In 1956, the notion of idempotent semirings was firstly introduced by S. C. Kleene on the theory of finite automata in [42]. An element $a \in S$ is called additive [resp. multiplicative] idempotent if it satisfies $a+a = a$ [resp. $aa = a$]. A semiring S is called additive [resp. multiplicative] idempotent semiring if its additive [resp. multiplicative] reduct $(S, +)$ [resp,$(S, \cdot)$] is an idempotent semigroup. If S is both additive idempotent and multiplicative idempotent then S is idempotent semiring. The max-plus semiring stands as the most widely recognized instance of an idempotent semiring. The concept of an idempotent semiring is a basic concept in idempotent analysis. This concept has many applications in different optimization problems (including dynamic programming), computer science, automata and formal language theory, numerical methods, parallel programming. Several works on idempotent semiring have been explored in literature, such as those presented in references [21], [25], [40], [44], [66], [69], [71]. In 2010, M.K. Sen and A.K. Bhuniya generalized [57] the concept of idempotent semiring to almost idempotent semiring. A semiring is said to be an almost idempotent semiring if its every elements is an almost idempotent. It plays a significant role in the field of real numbers in real analysis. In this thesis, we have developed the concept of almost idempotent semiring to almost idempotent central semiring. The almost idempotent central semirings may be considered as a generalization of the almost idempotent semirings. M.K. Sen and A.K. Bhuniya proved that the class of all almost idempotent semirings forms a variety. In this thesis, we also show that the class of all almost idempotent central semirings forms a variety.

A ring $(R, +, \cdot)$ is called additively cancellative if the semigroup $(R, +)$ is cancellative, i.e., if $a + x = a + y$ implies $x = y$ for all $a, x, y \in R$. The definition of additively cancellative semiring is similar to the definition of additively cancellative ring. This definition inspired us to construct a new class of center in a semiring, which we referred to as the generalized center of a semiring. It is denoted by $C_G(S)$. The generalized center $C_G(S)$ of a semiring S is the collection of those elements of 'a' in S such that $a + ab = a + ba$ for all $b \in S$. In the case of rings, the definition of generalized center $C_G(R)$ of a ring S is same as the generalized center $C_G(S)$ of a semiring S. In an additive cancellative semiring S, $C_G(S)$ concides with $Z(S)$, the center of semiring S. Similarly, in an additive cancellative ring S, no difference can be found between $C_G(R)$ and $Z(R)$, the center of ring R. Again, by the definition of the generalized center $C_G(S)$ of a semiring S, it is clear that $Z(S) \subseteq C_G(S)$ (resp. $Z(R) \subseteq C_G(R)$). Therefore, the generalized center is a proper generalization of the center in the semiring (resp. the ring). For this reason, we named this particular type of center as the "Generalized Center".

In 1975, I. N. Herstein [34] defined a certain subset of a ring R which is closely related to the center $Z(R)$ of the ring R. He named this certain subset as "hypercenter of a ring ". The hypercenter $T(R)$ of R is $T(R) = \{a \in R : ax^n = x^n a, n = n(x, a) \geq 1, \text{ all } x \in R\}$. In this thesis, we study the above notion in semiring setting. We introduce the notion of hypercenter of semiring and explore its structural properties.

Let S be a semiring with center $Z(S)$. If S is commutative then $S = Z(S)$. This result have inspired us to develop a new type of semirings called h-central semiring, k-central semiring, generalized central semiring and hypercentral semiring in which the whole semiring coincide with its corresponding center.

The present study "Some Aspects of Different Types of Centers of Semirings" was carried out to study the properties of different types of centers of semirings.

1.2 Overview of the Thesis

The entire thesis comprises seven chapters mentioned below. Unless otherwise stated, all results of Chapters 2, 3, 4, 5, 6 and 7 have actually been contributed by the author of the thesis himself under his Ph.D. Supervisor, some of which may be the upgradition of existing results for semirings.

Chapter 1 : Introduction and Preliminaries

In the first chapter, we discuss the background and motivation of this study and also present some preliminary definitions and important results which are relevant for this thesis. Additionally, it provides a brief introductory ideas about the thesis.

Chapter 2 : Birkhoff Center of c-Semirings

In this chapter, we introduce the notion of Birkhoff center $B(S)$ of a c-semiring and provide some illustrations of that center of a c-semiring. Next we initiate the notion of an important subset $E(S)$ consisting the set of all commuting idempotents of c-semiring S for the construction of the main characterization theorem for Birkhoff center of a c-semiring. The main purpose of this chapter to establish the main characterization theorem for Birkhoff center of a c-semiring : "Let S be a c-semiring. Then $a \in B(S)$ if and only if $a \in E(S)$ and there exists a homomorphism $f_a : S \longrightarrow S_a$ such that $x \longmapsto (ax, f_a(x))$ is an isomorphism of S onto $aS \times S_a$ and f_a is identity on S_a". We also prove that the Birkhoff center of a c-semiring forms a lattice. For this purpose, we are able to furnish that if S is a c-semiring, $B(S)$ is a c-subsemiring of S. Further we study different types of lattice structures of Birkhoff center of a c-semiring and also prove that if two c-semirings S_1 and S_2 are isomorphic, then their Birkhoff centers $B(S_1)$ and $B(S_2)$ are isomorphic but the converse is not necessarily true. Finally, we investigate several properties concerning the Birkhoff center of a c-semiring. We conclude this chapter by showing that if S is a c-semiring, then $B(S)$ is a subalgebra of $B(E(S))$.

Chapter 3 : Almost Idempotent Center of Semirings

Within this chapter, we present the concept of the almost idempotent center, denoted as $E_c(S)$, for a given semiring S. In some sense $E_c(S)$ is a center-like subset of a semiring S. Like center $Z(S)$ of a semiring S, $E_c(S)$ is not an ideal of S. In the section 3.2, we define the almost idempotent center $E_c(S)$ of a semiring S and provide a few instances of that center of a semiring S. In the section 3.3, we produce several fascinating properties of $E_c(S)$ of a semiring S. In the section 3.4, our main target is to establish the lattice structure for $E_c(S)$ of a semiring S under a certain condition. For achieving this goal, we introduce a partial order relation in $E_c(S)$ and finally, we prove the theorem as follows : "If S is a semiring with an additive absorbing element 1, then $E_c(S)$ forms a lattice." Furthermore, we analyze various lattice structures for $E_c(S)$. In the last section 3.5 of this chapter, we introduce the notion of almost idempotent central semiring in which $S = E_c(S)$ and generalize the concept of almost idempotent center of semiring which is introduced by M.K. Sen and A.K. Bhuniya in [57] and furnish some examples of that semiring. Furthermore, we investigate a range of fascinating properties associated with it. We concentrate on establishing some characterizations of that special class of semiring.

Chapter 4 : On h-Center of Semirings

In chapter 4, our focus is to describe investigate many results on center-like subset of a semiring which are analogous to the same direction in ring theory. For this purpose, we introduce the notion of a special center-like subset which we call h-center of semiring which is denoted by $C_h(S)$ in the section 4.2. Also in this section, we provide some examples of h-center of a semiring S and investigate the h-centers of power semiring and matrix semiring. In the section 4.3, we concentrate on the algebraic properties of $C_h(S)$ of a semiring S. In this section, we are able to show that if S be a division semiring with additive absorbing identity 1, then $C_h(S)$ also functions as a division semiring. Besides, in the section 4.4, we try to set up the lattice structure of $C_h(S)$ of a semiring S under certain condition on S and study different types of lattice structures of $C_h(S)$ of a semiring S. In the section 4.5, we look at some structural properties of $C_h(S)$ of a semiring S via structure preserving mapping and find out relation between h-centers of two semirings and their cartesian product semiring. Finally, we introduce the notion of h-central semiring and try to throw light on some characterizations of that semiring in the section 4.5.

Chapter 5 : k-Center of a Semiring

The chapter 5 is devoted for the study of k-center $C_k(S)$ of a semiring S. In the section 5.2 of this chapter begins with the definition of k-center $C_k(S)$ of a semiring S together with some basic results related to this definition. In the section 5.3, our focus shifts to exploring the algebraic structure of this k-center for different classes of semirings. In 1936, J. Von Neumann proved [49] the well known theorem "The center of regular ring is regular". We also generalize the above theorem for $C_k(S)$ of semiring. In the section 5.4, we discuss some properties of $C_k(S)$ of a semiring S. Additionally, we demonstrate that in the case of a semiring S with identity, $C_k(S)$ indeed constitutes a semilattice. In the section 5.5, we construct a newer type of semiring, referred to as the k-central semiring with the help of k-center of semiring S and provide some properties of that special class of semiring.

Chapter 6 : Generalized Center of a Semiring

Our aim of the chapter 6 is to generalize to the usual center of a semiring. To accomplish this objective, we introduce a novel concept called the "Generalized Center" of a semiring, unveiling a new type of central structure within its framework. In the section 6.2. we define the concept of generalized center $C_G(S)$ of a semiring S and exhibit some examples of that center of a semiring S. Furthermore, some attributes of $C_G(S)$ of a semiring S that arise naturally from its definition are shown in this section. In the section 6.3, we enlighten some structural properties of $C_G(S)$ of a semiring S. Moreover, we establish that when D is a division semiring with the unity element 1, $C_G(D)$ qualifies as a division subsemiring of D. Let R be a ring with center Z = Z(R). If R is commutative then $R = Z(R)$. Our objective is to identify the class of semirings where a semiring S coincides with its generalized center. To accomplish this, in the section 6.4 we introduce the concept of a generalized central semiring, where $S = C_G(S)$ holds true. Additionally, we offer a selection of examples to illustrate this notion. Mainly we characterize them by using generalized center of semiring.

Chapter 7 : On the Hypercenter of a Semiring

The chapter 7 concerns with hypercenter of a semiring. Section 7.1 may also be read as an introduction to hypercenter of semiring. In the section 7.2, we define the concept of the hypercenter of a semiring and provide several instances that illustrate the properties of that center of semiring. Some basic properties of hyperenter of a semiring are discussed in the 7.3. Additionally, we are dedicated to exploring the hypercenter for matrix semiring. In the last section 7.4, we introduce the notion of hypercentral semiring and explore some properties of that semiring. Lastly, we demonstrate that the class of all hypercentral semirings forms a variety

Chapter 8 : The Interrelation of Centers

Chapter 8 focuses on exploring the interconnectedness between various centers of semirings. The chapter concludes by establishing the interrelationship between the almost idempotent central semiring, the h-central semiring and the k-central semiring.

1.3 Algebraic Preliminaries

In the forthcoming section, we recall specific definitions and notions of semirings that will be utilized in this thesis.

Definition 1.3.1. *[32] A non-empty set S together with two binary operations "+" and "$\cdot$" is said to be a semiring if*

(i) $(S, +)$ is a commutative semigroup,

(ii) $(S, \cdot)$ is a semigroup,

(iii) Both operations are connected by the distributive laws : $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in S$.

A semiring S is called a semiring with zero element '0' if $a + 0 = 0 + a = a$ and $0.a = a.0 = 0$ for all $a \in S$. A semiring S is called a semiring with identity element 1 if $1 \cdot a = a \cdot 1 = a$ for all $a \in S$.

A semiring may or may not have a zero and an identity element.

We consider a semiring $(S, +, .)$ with zero element '0' and identity element '1' throughout this thesis.

We assume $1 \neq 0$. The zero [identity element] (if it exists) of a semiring S is called an absorbing zero [resp. absorbing identity element] if it satisfies $x \cdot 0 = 0 \cdot x = 0$

[resp. $x + 1 = 1 + x = 1$] for all $x \in S$. Unlike rings, zeros are not always absorbing here. A semiring S with an absorbing zero is called a hemiring.

Unless otherwise stated, a semiring $(S, +, .)$ will be denoted simply by S and multiplication "$\cdot$" will be denoted by juxtaposition.

By the product AB of two non-empty subsets A and B of a semiring S, we mean the set $\{\sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B\}$.

Let $(S, +, .)$ and $(T, +, .)$ be two semirings. Then a mapping $f : S \longrightarrow T$ is said to be a semiring homomorphism [32] of S into T if $f(x + y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$ for all $x, y \in S$. An injective homomorphism is called a monomorphism, a surjective homomorphism is called an epimorphism and a bijective homomorphism is called an isomorphism.

Definition 1.3.2. *[28] A semiring $(S, +, \cdot)$ is said to be commutative if $(S, \cdot)$ is commutative.*

Definition 1.3.3. *[32] A semiring $(S, +, \cdot)$ is called additively cancellative if the semigroup $(S, +)$ is cancellative, i.e., if $a + x = a + y$ implies $x = y$ for all $a, x, y \in S$. An additively cancellative hemiring is called a halfring.*

Definition 1.3.4. *[32] For each semiring $(S, +, \cdot)$ we introduce the notation S^* by defining $S^* = S\backslash\{0\}$ if $(S, +, \cdot)$ has a zero 0 and $S^* = S$ otherwise.*

Definition 1.3.5. *[32] A semiring $(S, +, \cdot)$ is called multiplicatively left cancellative if each element $a \in S^*$ is multiplicatively left cancellable in $(S, +, \cdot)$, i.e., left cancellable in $(S, \cdot)$.*

A semiring which is multiplicatively left and right cancellative is called multiplicatively cancellative.

Definition 1.3.6. *[28] Let I be a nonempty subset of a semiring S. Then I is said to be a left ideal (resp. right ideal) of S if $(I, +)$ is a subsemigroup of $(S, +)$ and $sa \in I$ (resp. $as \in I$) for all $s \in S$ and for all $a \in I$. On the other hand, I is said to be an ideal of S if it is both a left ideal and a right ideal of S.*

Definition 1.3.7. *[32] A left ideal (resp. right ideal, ideal) I of a semiring S is said to be a left k-ideal (resp. right k-ideal, k-ideal) of S if for any $x \in S$ and $y \in I$, $x + y \in I$ implies that $x \in I$.*

Lemma 1.3.8. *If I and J are two left k-ideals (resp. right k-ideals, k-ideals) of S then $I \cap J$ is also a left k-ideal (resp. right k-ideal, k-ideal) of S.*

Definition 1.3.9. *[32] Let A be a non-empty subset of a semiring S. Then the k-closure of A, denoted by $\overline{A}$, is defined as : $\overline{A} = \{a \in S : a + b = c$ for some $b, c \in A\}$.*

Lemma 1.3.10. *[32] Let S be a semiring. Then for any two non-empty subsets A, B of S, we have the following :*
(i) $A \subseteq \overline{A}$, (ii) $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$, (iii) $\overline{\overline{A}} = \overline{A}$ and (iv) $\overline{AB} = \overline{\overline{A}\,\overline{B}}$.

Lemma 1.3.11. *[32] A left ideal (resp. right ideal, ideal) I of a semiring S is a left k-ideal (resp. right k-ideal, k-ideal) of S if and only if $I = \overline{I}$.*

Definition 1.3.12. *[32] A non-zero element 'a' of a semiring S is said to be a zero divisor if there exists $0 \neq b \in S$ such that $ab = 0$.*

Definition 1.3.13. [18] *If S is a commutative semiring with identity element, then S is called a semidomain if $ab = 0$, $a, b \in S$ implies $a = 0$ or $b = 0$.*

Definition 1.3.14. [28] Let S be a semiring with identity. Then S is called a division semiring if every nonzero element of S is a unit.

Note that a commutative division semiring is a semifield.

Definition 1.3.15. *[28]* Let S be a semiring. Then the center of S, denoted by $Z(S)$ and is defined by $Z(S) = \{x \in S : xy = yx \text{ for all } y \in S\}$.

Now we present the definition of different class of semirings.

Definition 1.3.16. A semiring S is said to be a central semiring if $Z(S) = S$.

Definition 1.3.17. [72] *An element 'a' of a semiring S is said to be regular if there exists an element $x \in S$ such that $a = axa$.*

A semiring S is said to be regular if every element of S is regular.

Theorem 1.3.18. *[38] A semiring S is regular if and only if $R \cap L = RL$ for every right ideal R and every left ideal L of S.*

Definition 1.3.19. [15] *A semiring S is called a k-regular semiring if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.*

It is to be noted that Bourne preferred the term "regular semiring" in spite of "k-regular semiring". Subsequently, Adhikari, Sen and Weinert [1] renamed the Bourne regularity of a semiring as k-regularity to distinguish this form the notion of Von Neumann regularity.

Example 1.3.20. (i) *The semiring $(\mathbb{Z}^+, \oplus, \odot)$ is k-regular, where $\oplus$ and $\odot$ are defined by : $a \oplus b = max(a, b)$ and $a \odot b = min(a, b)$ for all $a, b \in \mathbb{Z}^+$.*

(ii) *The semiring $(\mathbb{Z}^+, \oplus, \odot)$ is k-regular where $\oplus$ and $\odot$ are defined by : $a \oplus b = max(a, b)$ and $a \odot b$ is the usual multiplication of integers, for all $a, b \in \mathbb{Z}^+$.*

Remark 1.3.21. *In general, while a regular semiring S can be considered a k-regular semiring, the reverse statement does not hold. Suppose a semiring S is regular and $a \in S$. Then there exists $x \in S$ such that $a = axa$. This shows that $a+aya = axa+aya$ for any $y \in S$. So, $a + aya = a(x + y)a$ i.e. $a + aya = aza$; where $z = x + y$. This shows that S is k-regular. But if we consider Example 1.3.20 (ii), we can check that $(\mathbb{Z}^+, \oplus, \odot)$ is not regular.*

Theorem 1.3.22. *[3] A semiring S is k-regular if and only if for every right k-ideal R and every left k-ideal L of S, $\overline{RL} = R \cap L$.*

Definition 1.3.23. *[3] An element 'x' of a semiring S is said to be intra-regular if $x = \sum_{i=1}^{m} a_i x^2 b_i$ for some $a_i, b_i \in S$ and $m \in \mathbb{N}$.*

A semiring S is said to be intra-regular if every element of S is intra-regular.

Theorem 1.3.24. *[3] A semiring S is intra-regular if and only if $L \cap R \subseteq LR$ for every left ideal L and right ideal R of S.*

Definition 1.3.25. *[3] An element 'x' of a semiring S is said to be k-intra-regular if $x + \sum_{i=1}^{m} a_i x^2 b_i = \sum_{j=1}^{n} c_j x^2 d_j$ for some $a_i, b_i, c_j, d_j \in S$.*

A semiring S is said to be k-intra-regular if every element of S is k-intra-regular.

Theorem 1.3.26. *[3] A semiring S is k-intra-regular if and only if $L \cap R \subseteq \overline{LR}$ for every left k-ideal L and right k-ideal R of S.*

Definition 1.3.27. [61] *An element $a \in S$ is called completely regular if there exists an element $x \in S$ satisfying the following conditions: (i) $a = a + x + a$, (ii) $a + x = x + a$, (iii) $a(a + x) = a + x$.*

A semiring S is called a completely regular semiring if every element a of S is completely regular.

Definition 1.3.28. [55] *Let S be a semiring and $a \in S$. Then 'a' is called completely k-regular if there exist $x, u \in S$ such that $a + axa = axa, ax + xua = xua$ and $xa + aux = aux$.*

If each element of S is completely k-regular then S is called a completely k-regular semiring.

Definition 1.3.29. [10] *An intra k- regular semiring S is a semiring whose additive reduct is a semilattice and for each $a \in S$ there exists $x \in S$ such that $a + xa^2x = xa^2x$.*

Definition 1.3.30. [31] *An element 'a' of a semiring S is called π-regular if there exist $x, y \in S$ and a positive number n such that $a^n + a^nxa^n = a^nya^n$.*

A semiring S is called π-regular if every element of S is π-regular.

Definition 1.3.31. [57] *A semiring S whose additive reduct is a semilattice is said to be an almost idempotent semiring if $a + a^2 = a^2$ for any $a \in S$.*

Definition 1.3.32. [57] *An almost idempotent semiring S is said to be rectangular if for any $a, b \in S$, there exists $x \in S$ such that $a + axbxa = axbxa$.*

Definition 1.3.33. [57] *An almost idempotent semiring S is called a left(right) zero almost idempotent semiring if for any $a, b \in S$, there exists $x \in S$ such that $a + axb = axb$ $(b + axb = axb)$.*

Definition 1.3.34. [53] *A semiring $(S, +, \cdot)$ is said to be a positive rational domain(PRD) if $(S, \cdot)$ is an abelian group.*

Definition 1.3.35. [35] *A semigroup S is called a rectangular band if $axa = a$ for all $a, x \in S$.*

Definition 1.3.36. [65] *A semigroup S is called E- inversive if for every $a \in S$ there exists $x \in S$ such that $(ax)^2 = ax$, that is, ax is an idempotent of S. These semigroups are also called E-dense; the latter name is sometimes used for E- inversive semigroups with commuting idempotents.*

Definition 1.3.37. [45] *A semiring S is called a b-lattice semiring if $(S, +)$ is a semilattice and $(S, \cdot)$ is a band.*

Definition 1.3.38. [28] *A semiring S with identity 1 is called a simple semiring if $1 + a = 1$ for all $a \in S$.*

Definition 1.3.39. *[32]* A semiring S is called a mono-semiring if $a + b = ab$ for all $a, b \in S$.

Definition 1.3.40. *[28]* A semiring $(S, +, \cdot)$ with multiplicatively zero 0 is said to be zero square semiring if $x^2 = 0$ for all $x \in S$.

Definition 1.3.41. *[68]* A semiring $(S, +, \cdot)$ with additive identity zero is said to be zerosumfree semiring if $x + x = 0$ for all $x \in S$. Zerosumfree semirings are also known as antirings [28].

Example 1.3.42. *The set $\mathbb{Z}^+$ of all nonnegative integers with the usual operations of addition and multiplication of integers is a zerosumfree semiring.*

Definition 1.3.43. *[35]* An element 'z'of a semigroup S is called a left zero if $zx = z$ for every x in S.

Definition 1.3.44. *[35]* Let $(S, +)$ be a semigroup. An element 'e' of a semigroup S is called a left identity if $e + x = x$ for every x in S.

Definition 1.3.45. *[29]* An element 'a' of a semiring S is called multiplicatively subidempotent if and only if $a + a^2 = a$ and a semiring S is multiplicatively subidempotent if and only if each of its element is multiplicatively subidempotent.

Definition 1.3.46. *[29]* A viterbi semiring S is a semiring in which S is an additively idempotent and multiplicatively subidempotent i.e. $a + a = a$ and $a + a^2 = a$ for all $a \in S$.

Definition 1.3.47. *[28]* Let S be a semiring with the zero element '0' and identity element 1 such that $0 \neq 1$. Let us define $P'(S) = \{0\} \cup \{r + 1 : r \in S\}$. Clearly, $P'(S)$ is a subsemiring of the semiring S.

A semiring S is said to be an antisimple semiring if $S = P'(S)$. Any ring is antisimple as a semiring.

Definition 1.3.48. *[28]* A semiring is called nil if every element of the semiring is nilpotent, that is, the semiring S is nil if for every element x of S, there is a positive integer n for which $x^n = 0$. More strongly, the semiring S is called nilpotent if there is a positive integer m such that $S^m = 0$.

Definition 1.3.49. A variety of algebras is a class of algebras of the same type that is closed under the formation of subalgebras, homomorphic images and direct products (see [16]). It is well known (Birkhoff's theorem) that a class of algebras of the same type is a variety if and only if it is an equational class. Thus, all semirings form a variety. A variety is formed by all idempotent semirings, as indicated by [70].

We now discuss some preliminary definitions and notions of c-semiring that are pertinent to this thesis. Additionally, we will explore the properties associated with c-semiring.

Definition 1.3.50. [14] A *c-semiring* is an algebraic system that consists of a non-empty set S together with two binary operations, called addition "+" and multiplication, denoted by juxtaposition such that

(i) S is an additively commutative monoid with identity 0_S,

(ii) S is a multiplicatively commutative monoid with identity 1_S,

(iii) $a0_S = 0_S a = 0_S$ for all $a \in S$.

(iv) $a + a = a$ and $a + 1_S = 1_S$ for all $a \in S$,

(v) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

Note 1.3.51. *Intuitively, the idempotence of "+" is needed to get a partial order over the elements of the semiring S; the commutativity of multiplication allows to consider sets of constraints (instead of ordered tuples); and the fact that 1_S is the absorbing element of "+" makes the element 1_S the maximum element of the partial order "$\leq_S$". This is useful for our treatment, since it gives us an upper bound to all the elements of S. Also 0_S is the minimum element of the partial order "$\leq_S$".*

Let S be a c-semiring. Now define a binary relation "$\leq_S$" on S by

$a \leq_S b$ if and only if $a + b = b$ for all $a, b \in S$. Then $(S, \leq_S)$ forms a partial ordered set.

Given a c-semiring S together with the partial order relation "$\leq_S$" defined above, the following results are proved in [14] :

Result I : $a \leq_S b \Longrightarrow a + c \leq_S b + c$ and $a \leq_S b \Longrightarrow ac \leq_S bc$ for all $a, b, c \in S$. (Addition and Multiplication are monotone over $\leq_S$).

Result II : $ab \leq_S a$ for all $a, b \in S$. (Multiplication is intensive).

Result III : $(S, \leq_S)$ is a complete lattice.

Result IV : $(S, \leq_S)$ is a distributive lattice, if S is idempotent (i.e. $a^2 = a$ for all $a \in S$).

Definition 1.3.52. *Let $(S, +, .)$ be a c-semiring and T be a non-empty subset of S. Then T is called a c-subsemiring of S if T itself forms a c-semiring w.r.t. the restricted operations of S i.e. T is a subsemiring of S such that $0_T, 1_T \in T$ and $(T, +)$ is a band having 1_T is the absorbing element of T w.r.t. "+".*

We now a present a few instances of c-semirings as illustrative examples.

Example 1.3.53. *Consider $S = \{0, 1, x, y\}$. Define the operations "+" and "." as follows :*

+	1	x	y	0
1	1	1	1	1
x	1	x	y	x
y	1	y	y	y
0	1	x	y	0

.	1	x	y	0
1	1	x	y	0
x	x	0	0	0
y	y	0	0	0
0	0	0	0	0

Then $(S, +, \cdot)$ forms a c-semiring.

Example 1.3.54. *Consider $S = \mathbb{R}_0^+ \cup \{+\infty\}$; where $\mathbb{R}_0^+$ is the set of non-negative real numbers.*

Then (i) $(S, \oplus, \odot)$ is a c-semiring with identity $1_S = 0$ and zero $0_S = +\infty$; where $a \oplus b = min(a, b)$ as addition on S and $a \odot b = max(a, b)$ as multiplication on S. Also $+\infty \oplus x = x$ for all $x \in S$, $+\infty \oplus +\infty = +\infty$, $+\infty \odot x = +\infty$ for all $x \in S$ and $+\infty \odot +\infty = +\infty$.

(ii) $(S, \oplus, \odot)$ is a c-semiring with identity $1_S = 0$ and zero $0_S = +\infty$; where $a \oplus b = min(a, b)$ as addition on S and $a \odot b$ for the usual addition on S. Also $+\infty \oplus x = x$ for all $x \in S$, $+\infty \oplus +\infty = +\infty$, $+\infty \odot x = +\infty$ for all $x \in S$ and $+\infty \odot +\infty = +\infty$.

Example 1.3.55. Let $A = (|A|,\ \oplus_A,\ \otimes_A,\ 0_A,\ 1_A)$ and $B = (|B|,\ \oplus_B,\ \otimes_B,\ 0_B,\ 1_B)$ be c-semiring. Define $\oplus_{A\times B},\ \otimes_{A\times B} : (|A| \times |B|) \times (|A| \times |B|) \to |A| \times |B|$ by $(a_1,\ b_1) \oplus_{A\times B} (a_2,\ b_2) = (a_1 \oplus_A a_2,\ b_{1\oplus_B} b_2)$ and $(a_1,\ b_1) \otimes_{A\times B} (a_2,\ b_2)$
$= (a_1 \otimes_A a_2,\ b_1 \otimes_B b_2)$.

Then $A \times B = (|A| \times |B|,\ \oplus_{A\times B},\ \otimes_{A\times B},\ (0_A,\ 0_B),\ (1_A,\ 1_B))$ is a c-semiring.

Definition 1.3.56. A c-semiring S is called a c-semifield if every nonzero element of S is unit.

1.4 Lattice-Theoretic Preliminaries

In this section, we present a brief overview of lattices and their connection to semirings, which will become apparent in the future. For instance, lattices serve as a valuable illustration for semirings that fulfill certain conditions. All the definitions and results in this section can be found in [11], [12], [17], [19], [20], [23], [30], [36], [41], [46].

Definition 1.4.1. *Let A and B be two nonempty sets. The Cartesian product of A and B, written $A \times B$, is defined to the set $A \times B = \{(a, b) : a \in A,\ b \in B\}$.*

Definition 1.4.2. *A subset R of $A \times B$ is called a relation from A to B. If $A = B$ we say R is a relation on A, denoted as $R \subseteq A \times B$. When $(a, b) \in R$, we can also express it as aRb.*

Definition 1.4.3. *Let R be a relation on a set A then R is called :*
(i) reflexive: if $(a, b) \in R$ for all $a \in A$.
(ii) symmetric: if $(a, b) \in R$, implies that $(b, a) \in R$.
(iii) antisymmetric: if $(a, b) \in R$ and $(b, a) \in R$ implies that $a = b$.
(vi) transitive: if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.
(v) equivalence: if R is reflexive, symmetric and transitive.
(iv) partial order: if R is reflexive, antisymmetric and transitive.

Definition 1.4.4. *A nonempty set P together with a partial order relation " $\leq$ " on P is called a partial order set or a poset, denoted by $(P, \leq)$.*

Example 1.4.5. *The set of integers $\mathbb{Z}$, under usual less than or equal " $\leq$ " relation is a poset.*

Example 1.4.6. *The set of natural numbers $\mathbb{N}$, under divisibility is a poset.*

Example 1.4.7. *For any nonempty set X, the power set of X, denoted by $P(X)$ is the set of all subsets of X under contained in "$\subseteq$" is a poset, such that for any $A, B \in P(X)$, $A \leq B$ means $A \subseteq B$.*

Definition 1.4.8. *Suppose S is a subset of a partially ordered set P and let $a \in P$ be an upper bound of S if $x \leq a$ for all $x \in S$. If a is an upper bound such that $a \leq b$ for all other upper bounds b, then a is referred to as the least upper bound (lub), denoted as sup $S = a$.*

Definition 1.4.9. *Suppose S is a subset of a partially ordered set P and let $a \in P$ be a lower bound of S, satisfying $a \leq x$ for all $x \in S$. If a is a lower bound such that any other lower bound b satisfies $b \leq a$, then a is referred to as the greatest lower bound (glb), denoted as inf $S = a$.*

Definition 1.4.10. *A poset $(L, \leq)$ is said to form a lattice, if for every $a, b \in L$, sup $\{a, b\}$ and inf $\{a, b\}$ and exist in L. We can represent the supremum as $a \vee b$, which is read as" a join b" and the infimum as $a \wedge b$, which is read as" a meet b."*

Example 1.4.11. *Let X be a nonempty set, then the power set of X, $\mathcal{P}(X)$, under contained in " $\subseteq$ " is a lattice, such that for any two sets A, B in $\mathcal{P}(X)$, we have $A \wedge B = A \cap B$, and $A \vee B = A \cup B$. Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and for any set C such that $C \subseteq A$, $C \subseteq B$, then $C \subseteq A \cap B$, then $A \wedge B = A \cap B$. Similarly $A \vee B = A \cup B$.*

Note 1.4.12. *By a least element of any subset X of a lattice L, we mean an element $a \in X$ such that $a \leq x$ for all $x \in X$ and a is said to be minimal element of X if there exists no $x \in X$ such that $x < a$. The concept of greatest element and maximal element can be defined dually. A least element must be minimal and a greatest element must be maximal; but converse is not true. For any $X \subseteq L$, there can be at most one least (greatest) element while there can be more than one minimal (maximal) element in X.*

Theorem 1.4.13. *[41] Let L be a poset, then L is a lattice if and only if every nonempty finite subset of L has sup and inf.*

Theorem 1.4.14. *[41] If L is any lattice, then for any a, $b, c, d \in L$, the following results holds.*

1. $a \wedge a = a = a \vee a$.
2. $a \wedge b = b \wedge a$ *and* $a \vee b = b \wedge a$.
3. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *and* $a \vee (b \vee c) = (a \vee b) \vee c$.
4. $a \wedge b \leq a$, $a \wedge b \leq b$, $a \leq a \vee b$ *and* $b \leq a \vee b$.
5. $a \leq b$ *if and only if* $a \wedge b = a$ *if and only if* $a \vee b = b$.
6. $a \wedge (a \vee b) = a$ *and* $a \vee (a \wedge b) = a$.
7. $a \leq b$ *and* $c \leq d$ *imply* $a \wedge c \leq b \wedge d$ *and* $a \vee c \leq b \vee d$.

Definition 1.4.15. *A nonempty subset S of a lattice L is called a sublattice of L if for any two elements a, b in S, both $a \wedge b$ and $a \vee b$ are also contained within S.*

Definition 1.4.16. *A poset $(P, \leq)$ is called a meet-semilattice [resp. join-semilattice] if for all $a, b \in P$, inf $\{a,b\}$ [resp. sup $\{a,b\}$] exists.*

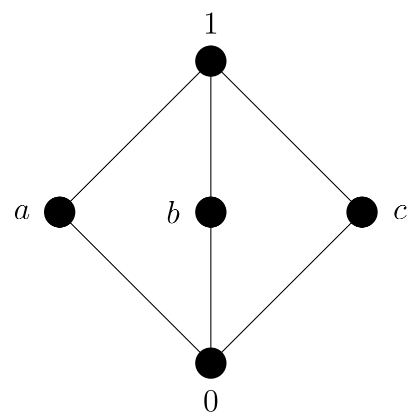
Clearly, a poset $(P, \leq)$ is a lattice iff it is a join and a meet semilattice.

Definition 1.4.17. *A lattice L is called a modular lattice simply modular, if for all $a, b, c \in L$ with $a \geq b$, implies $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $[= b \vee (a \wedge c)]$.*

Definition 1.4.18. *A lattice L is called a distributive lattice if for all $a, b, c \in L$, we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$*

Theorem 1.4.19. *[19] A distributive lattice is always modular lattice.*

However, it should be noted that the converse is not necessarily true, meaning that a modular lattice is not always a distributive lattice. This can be illustrated with the example of M_5, which is a modular lattice.



M_5

But M_5 is not a distributive lattice and $a \wedge (b \vee c) = a$, whereas $(a \wedge b) \vee (a \wedge c) = 0$.

Definition 1.4.20. *[19] A lattice $(L, \vee, \wedge)$ is called a complete lattice if every subset of L has l.u.b. and g.l.b.*

Note 1.4.21. • *In light of Theorem 1.4.13, every finite lattice is complete.*

• *The lattice formed by the set of all integers $\mathbb{Z}$ under the usual order relation " less than or equal to" $(\leq)$ is not complete due to the absence of an upper bound for the set $K = \{x \in \mathbb{Z} : x \geq 0\}$.*

Definition 1.4.22. *[19] Let L be a bounded lattice and $x \in L$. An element $y \in L$ is called a complement of x if $x \wedge y = 0$ and $x \vee y = 1$. L is called complemented lattice if all of its elements have complements.*

A bounded complemented distributive lattice is called a Boolean lattice.

Example 1.4.23. *Let S be a nonempty set. Then $(\mathcal{P}(S), \subseteq)$ forms a distributive lattice and each element has a complement. Thus $(\mathcal{P}(S), \subseteq)$ is a Boolean lattice.*

Definition 1.4.24. *[41] A subalgebra (or a Boolean subalgebra) is a non-empty subset of S of a Boolean algebra L such that $a,\ b \in S \Longrightarrow a \wedge b,\ a \vee b,\ a' \in S$.*

1.5 Literature Review and Important Results

In this section, we briefly give an outline of the earlier studies where center of rings have been associated with various algebraic structures. In particular, we list the results regarding center of rings which have greatly motivated us to define the new type of center of semirings which have been studied in the present thesis. Additionally, we include all the essential results on semirings that will be utilized extensively.

Note that for a semiring S, the center $Z(S)$ is not a semifield but in particular, we have the following result :

Theorem 1.5.1. *If D is a division semiring, then the center $Z(D)$ is a semifield.*

Proof. Let D be a division semiring and $Z(D)$ be the center of D. Then $Z(D)$ is a commutative semiring with identity 1, say. Let $a \neq 0$ be any element of $Z(D)$. Then Da is a nonzero left ideal of D. Since D is a division semiring, it follows that $Da = D$. Thus, there exists $d \in D$ such that $da = 1$. Since $a \in Z(D)$, we have $da = ad = 1$. We have to only show that $d \in Z(D)$. Now for any $x \in D$, we find that $a(xd) = (ax)d = (xa)d = x(ad) = x1 = 1x = (ad)x = a(dx)$. This implies that $xd = dx$ for all $x \in D$. So, $d \in Z(D)$. Thus, we find that $ad = da = 1$ for some $d \in Z(D)$. This shows that a is a unit in $Z(D)$. Therefore, $Z(D)$ is a semifield. □

We now present the following theorem that we generalize in this thesis.

Theorem 1.5.2. *[49] The center of a regular ring R is regular.*

Proof. Let a be a central element of R and let $x \in R$ be such that $a = axa = ax^2$. So, ax^2 is central. Let $z \in R$. Then $a^2xz = za^2x$ and hence $xa^2z = a^2zx$, i.e. a^2z commutes with x and so, it commutes with x^3. Therefore, $a^2x^3z = za^2x^3$, i.e. $y = a^2x^3$ is central. But, since $a^2x^2 = ax$, we have $y = ax^2$ and clearly, $aya = a^2x^2a = axa = a$. This completes the proof. □

Theorem 1.5.3. *[28] Let $S = (S, +, \cdot, 0, 1)$ be a semiring. Then S is a bounded distributive lattice if and only if S is commutative, simple and idempotent.*

Proof. The necessity of the conditions is clear. Conversely, assume the conditions to be true. Then $(S, +)$ and $(S, \cdot)$ are commutative semigroups. Let $a, b \in S$. Then $(a + b)a = aa + ba = a + ba = 1a + ba = (1 + b)a = (b + 1)a = 1a = a$ and $ab + a = ab + a1 = a(b + 1) = a1 = a$. Hence, the absorption laws hold and therefore, $(S, +, \cdot)$ is a lattice. Moreover, $0 + a = a$ and $a + 1 = 1$, i.e. $0 \leq a \leq 1$. Consequently, S is a bounded lattice. Since "$\cdot$" is distributive with respect to "$+$", S is a bounded distributive lattice. □

Chapter 2

Birkhoff Center of c-Semirings

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Birkhoff Center of c-Semirings

2.1 Introduction

The concept of center of a partially ordered set P is the set of all elements of P with an element 0 and an element 1 for which in some decomposition of P into a direct product with one of the components is 1 and the other is 0. The center of a partially ordered set with an element 0 and an element 1 forms a Boolean algebra. The concept of Birkhoff center of a semigroup with 0 and 1 was introduced by Swamy and Murti [63], analogous to that of a partially ordered set and proved that it is a Boolean algebra. Swamy and Pragathi [62] have also extended the above concept for an arbitrary semigroup S and proved that Birkhoff center of any semigroup S is a relatively complemented distributive lattice in which the meet operation is the operation in the semigroup S. In [64], Swamy and Murti also introduced the concept of Boolean center of a universal algebra which is another kind of notion of a center.

It is quite natural to ask whether we can generalize the notion of Birkhoff center in case of semiring to form some lattice structure. We are unable to construct the notion of Birkhoff center of an arbitray semiring to form a lattice structure. But we observed that if we consider a particular type of semiring, namely c-semiring inspite of an arbitrary semiring then we can construct the Birkhoff center of that semiring to form a lattice structure. In c-semiring, "c" stands for "constraint". The notion of c-semiring was introduced to describe many constraint satisfaction schemes.

More precisely, the notion of c-semiring was introduced by Bistarelli et al. [14] to tackle constraint solving over semirings. In fact, by considering c-semiring the elements of the chosen semiring can be interpreted in many ways : costs, levels of preference, uncertainties, probabilities, etc. This specific choice of c-semiring is used to different instances of framework which lead to some new constraint solving schemes.

On the other hand, some authors studied the notion of the Birkhoff center of dynamical system for topological aspects and some others also studied hyperbolic types of Birkhoff center of a diffeomorphism in a compact manifold. However, in this chapter, we confine ourselves wholly to study the algebraic aspects of Birkhoff center and we describe the Birkhoff center of a certain algebraic structure. Our main purpose in this chapter is to prove that the Birkhoff center of any c-semiring forms a distributive lattice.

2.2 Birkhoff Center of c-Semirings

In this section, we will present the concept of the Birkhoff center for a c-semiring and offer illustrative instances of this center. To begin with, let us establish a definition for the Birkhoff center of a c-semiring, which is as follows :

Definition 2.2.1. *Let S be a c-semiring. An element $a \in S$ is said to be a central element of S if there exist c-semirings S_1 with identity 1_{S_1} and S_2 with zero 0_{S_2} and an isomorphism of S onto $S_1 \times S_2$ such that a is mapped onto $(1_{S_1}, 0_{S_2})$.*

The set of all central elements of S is called Birkhoff center of S and is denoted by $B(S)$.

Example 2.2.2. *Consider the c-semirings S, S_1 and S_2 defined in Example 1.3.54(i). Now define $f : S \longrightarrow S_1 \times S_2$ by $f(x) = (x \oplus \alpha, x \oplus \beta)$ for all $x \in S$, where α and β are fixed positive real numbers.*

Now for all $x, y \in S$, $f(x \oplus y) = (x \oplus y \oplus \alpha, x \oplus y \oplus \beta) = (x \oplus y \oplus \alpha \oplus \alpha, x \oplus y \oplus \beta \oplus \beta) = (x \oplus \alpha \oplus y \oplus \alpha, x \oplus \beta \oplus y \oplus \beta) = (x \oplus \alpha, x \oplus \beta) + (y \oplus \alpha, y \oplus \beta) = f(x) + f(y)$ and $f(x \odot y) = ((x \odot y) \oplus \alpha, (x \odot y) \oplus \beta) = ((x \odot y) \oplus \alpha \oplus \alpha, (x \odot y) \oplus \beta \oplus \beta) =$

$((x\oplus\alpha)\odot(y\oplus\alpha),(x\oplus\beta)\odot(y\oplus\beta)) = (x\oplus\alpha,x\oplus\beta)(y\oplus\alpha,y\oplus\beta) = f(x)f(y)$. *Hence f is a homomorphism.*

To prove that f is one to one, let $x, y \in S$ be such that $f(x) = f(y)$. Then $(x\oplus\alpha, x\oplus\beta) = (y\oplus\alpha, y\oplus\beta)$. This implies that $x\oplus\alpha = y\oplus\alpha$ and $x\oplus\beta = y\oplus\beta$ i.e. $min(x,\alpha) = min(y,\alpha)$ and $min(x,\beta) = min(y,\beta)$. Thus $x = y$ for specific choice of α and β. This implies that f is one to one.

To prove that f is onto, let $(u,v) \in S_1 \times S_2$. Let γ be an arbitrary fixed element of $\mathbb{R}_0^+$ such that $u < \alpha < \gamma$ and $u < \beta < \gamma$. Then $f(u\oplus\gamma, v\oplus\gamma) = (u,v)$. This shows that f is onto. Hence, f is an isomorphism.

If $x \in B(S)$, then $f(x) = (1_{S_1}, 0_{S_2})$. This implies that $(x\oplus\alpha, x\oplus\beta) = (1_{S_1}, 0_{S_2})$. Therefore, $(x\oplus\alpha, x\oplus\beta) = (0,+\infty)$. Thus $x\oplus\alpha = 0$ and $x\oplus\beta = +\infty$.

Case (i): $x < \alpha$, $x < \beta$: Then $(min(x,\alpha),(min(x,\beta)) = (0,+\infty) \Longrightarrow (x,x) = (0,+\infty)$. Thus, $x = 0$ and $x = +\infty$.

Case (ii): $x < \alpha$, $x \geq \beta$: Then $(min(x,\alpha),(min(x,\beta)) = (0,+\infty) \Longrightarrow (x,\beta) = (0,+\infty)$. Thus $x = 0$ and $\beta = +\infty$.

Case (iii): $x \geq \alpha$, $x \geq \beta$: Then $(min(x,\alpha),(min(x,\beta)) = (0,+\infty) \Longrightarrow (\alpha,\beta) = (0,+\infty)$. Thus $\alpha = 0$ and $\beta = +\infty$.

Case (iv): $x > \alpha$, $x < \beta$: Then $(min(x,\alpha),(min(x,\beta)) = (0,+\infty) \Longrightarrow (\alpha,x) = (0,+\infty)$. Thus, $\alpha = 0$ and $x = +\infty$. From Case (iii) ($x \geq \alpha$, $x \geq \beta$), for any values of x, we get $f(x) = (1_{S_1}, 0_{S_1}) = (0,+\infty)$. Therefore, S has infinitely many central elements i.e. $B(S)$ contains infinitely many elements.

Example 2.2.3. *Consider $S = \mathbb{R}_0^- \cup \{-\infty\}$, where $\mathbb{R}_0^-$ is the set of non-positive real numbers. Then $(S,\oplus,\odot)$ is a c-semiring with identity $1_S = 0$ and zero $0_S = -\infty$, where $a\oplus b = max(a,b)$ as addition on S and $a\odot b = min(a,b)$ as multiplication on S. Also $-\infty\oplus x = x$ for all $x \in S$, $-\infty\oplus-\infty = -\infty$, $-\infty\odot x = -\infty$ for all $x \in S$ and $-\infty\odot-\infty = -\infty$. Then in a similar fashion, we can show that S has infinitely many central elements i.e. $B(S)$ contains infinitely many elements.*

Example 2.2.4. *Consider $S = \{0, 1, x, y, z\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	y	z	1
0	0	x	y	z	1
x	x	x	z	y	1
y	y	z	y	x	1
z	z	y	x	z	1
1	1	1	1	1	1

.	0	x	y	z	1
0	0	0	0	0	0
x	0	0	0	0	x
y	0	0	0	0	y
z	0	0	0	0	z
1	0	x	y	z	1

Then $(S, +, \cdot)$ *is a c-semiring and* $B(S) = \{0, 1\}$.

Lemma 2.2.5. *If* S *is a c-semiring then* $0_S, 1_S \in B(S)$.

Proof. We first define $f : S \longrightarrow \{0_S\} \times S$ by $f(x) = (0_S,\ x)$ for all $x \in S$. Let $x = y$ for some $x,\ y \in S$. Then $f(x) = (0_S,\ x)$ and $f(y) = (0,\ y)$. Since $x = y$, $(0_S,\ x) = (0_S,\ y) \Longrightarrow f(x) = f(y)$. Thus f is well defined. Now $f(x+y) = (0_S, x+y) = (0_S, x) + (0_S, y) = f(x) + f(y)$ and $f(xy) = (0_S, xy) = (0_S, x)(0_S, y) = f(x)f(y)$. Hence, f is a homomorphism. To establish the injectiveness of f, let's suppose that for any $x, y \in S,\ f(x) = f(y) \Longrightarrow (0_S, x) = (0_S, y) \Longrightarrow x = y$. Hence, f is one to one. Let us take an arbitrary element $(0_S, z) \in \{0_S\} \times S$, where $z \in S$. According to the definition of the function f, we obtain $f(x) = (0_S, x)$. Consequently, we can conclude that $f(z) = (0_S, z)$. This implies that f is onto. Hence f is an isomorphism from S onto $\{0_S\} \times S$, where $\{0_S\}$ is a c-semiring with 0_S as the identity element. Also $0_S \longmapsto (1_{\{0_S\}}, 0_S)$ and hence $0_S \in B(S)$.

Now we define $g : S \longrightarrow S \times \{1_S\}$ by $g(x) = (x, 1_S)$ for all $x \in S$. Let $x = y$ for some $x, y \in S$. Now $g(x) = (x, 1_S)$ and $g(y) = (y, 1_S)$. Since $x = y$, $(x, 1_S) = (y, 1_S) \Longrightarrow g(x) = g(y)$. Thus, g is well defined. Now $g(x+y) = (x+y, 1_S) = (x+y, 1_S+1_S)$ (since $(S, +)$ is a band) $= (x+1_S, y+1_S) = g(x)+g(y)$ and $g(xy) = (xy, 1_S) = (x, 1_S)\ (y, 1_S) = g(x)\ g(y)$. Hence, g is a homomorphism. To prove that g is one-one, suppose for any $x, y \in S$, g(x) = g(y)$\Longrightarrow (x, 1_S) = (y, 1_S) \Longrightarrow x = y$. Therefore, g is one to one. To prove that g is onto, let $(z, 1_S) \in S \times \{1_S\}$, where $z \in S$. So, by the definition of g, we get $g(x) = (x, 1_S)$. Therefore, $g(z) = (z, 1_S)$. Thus g is onto. Hence, g is an isomorphism from S onto $S \times \{1_S\}$, where $\{1_S\}$ is a c-semiring with 1_S as the zero element. Also $1_S \longmapsto (1_S, 0_{\{1_S\}})$ and hence $1_S \in B(S)$. □

Lemma 2.2.6. *If S is a c-semiring containing a single element 'a' i.e. if $S = \{a\}$ is a c-semiring, then $a \in B(S)$.*

Proof. If S is a c-semiring containing a single element 'a' i.e. if $S = \{a\}$ is a c-semiring, then 'a' is as an identity element as well as a zero element of S. Based on Lemma 2.2.5, we can conclude that $a \in B(S)$. □

2.3 The Characterization of $B(S)$

In this section we give a necessary and sufficient condition for $B(S)$ of a c-semiring S. To aid in the characterization of the central element of a c-semiring S, we introduce a important subset called $E(S)$.

The set of all commuting idempotents of a c-semiring is denoted by $E(S)$. Thus for a c-semiring S, the set $E(S)$ is defined by $E(S) = \{a \in S : a^2 = a \text{ and } ax = xa \text{ for all } x \in S\}$.

Theorem 2.3.1. *If S is a c-semiring, then $E(S)$ is a c-subsemiring of S.*

Proof. Since S is c-semiring, then $0 \in E(S)$. So $E(S)$ is non-empty. Let $a, b \in E(S)$. Now $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a + ab + b + ba = a(1+b) + b(1+a) = a1 + b1$ (since 1 is the absorbing element w.r.t. "+") $= a + b$. Again $(a+b)x = ax + bx = xa + xb = x(a+b)$ for all $x \in S$. Therefore, $a + b \in E(S)$. Also $(ab)^2 = abab = aabb = a^2b^2 = ab$ and $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$ for all $x \in S$. Therefore, $ab \in E(S)$. Thus, $E(S)$ is a subsemiring of S. Again $1_S, 0_S \in E(S)$. Since $E(S)$ is a subsemiring of S and $(S, +)$ is a band, $(E(S), +)$ is a band. Also 1_S is the absorbing element of $E(S)$ w.r.t. "+". Hence $E(S)$ is a c-subsemiring of S. □

Lemma 2.3.2. *Let S be a c-semiring. Then the following are equivalent :*

(*i*) $E(S) = \{a \in S : a^2 = a \text{ and } ax = xa \text{ for all } x \in S\}$

(*ii*) $E(S) = \{a \in S : a + a^2 = a^2 \text{ and } ax = xa \text{ for all } x \in S\}$.

Proof. $(i) \Longrightarrow (ii)$.

Let $a \in E(S)$. Then $a^2 = a \Longrightarrow a^2 = a1 \Longrightarrow a^2 = a(1+a)$ (since 1 is an absorbing element w.r.t. "+"). Therefore, $a^2 = a + a^2$. Since $a \in E(S)$, $ax = xa$ for all $x \in S$. Hence $E(S) = \{a \in S : a + a^2 = a^2 \text{ and } ax = xa \text{ for all } x \in S\}$.

$(ii) \Longrightarrow (i)$.

Let $a \in E(S)$. Then $a^2 = a + a^2 \Longrightarrow a^2 = a(1 + a) \Longrightarrow a^2 = a$ (since 1 is an absorbing element w.r.t. "+"). Hence, $E(S) = \{a \in S : a^2 = a \text{ and } ax = xa \text{ for all } x \in S\}$. □

Remark 2.3.3. *In a semiring S, an element 'a' of S is said to be almost idempotent if $a+a^2 = a^2$. Thus, we see that in a c-semiring, the notion of idempotent and almost idempotent coincides.*

Now we are going to prove that the Birkhoff center of a c-semiring S is contained in the Birkhoff center of $E(S)$.

For this, we need the following result :

Lemma 2.3.4. *Let S_1 and S_2 be two semirings. Then $E(S_1 \times S_2) = E(S_1) \times E(S_2)$.*

Proof. Let $e = (e_1, e_2) \in E(S_1 \times S_2)$. Then $e^2 = e$ and $ex = xe$ for all $x = (x_1, x_2) \in S_1 \times S_2$. Now $e^2 = e \Longrightarrow (e_1, e_2)^2 = (e_1, e_2) \Longrightarrow (e_1^2, e_2^2) = (e_1, e_2)$. Therefore, $e_1^2 = e_1$ and $e_2^2 = e_2$. Again $ex = xe$ for all $x = (x_1, x_2) \in S_1 \times S_2 \Longrightarrow (e_1, e_2)(x_1, x_2) = (x_1, x_2)(e_1, e_2) \Longrightarrow (e_1x_1, e_2x_2) = (x_1e_1, x_2e_2)$. This implies that $e_1x_1 = x_1e_1$ for all $x_1 \in S_1$ and $e_2x_2 = x_2e_2$ for all $x_2 \in S_2$. Hence, e_1, e_2 are elements in $E(S_1)$ and $E(S_2)$ respectively. Thus, $e = (e_1, e_2) \in E(S_1) \times E(S_2)$. This implies that $E(S_1 \times S_2) \subseteq E(S_1) \times E(S_2)$ (i).

Let $e = (e_1, e_2) \in E(S_1) \times E(S_2) \Longrightarrow e_1 \in E(S_1)$ and $e_2 \in E(S_2)$. Then $e_1^2 = e_1$ and $e_1x_1 = x_1e_1$ for all $x_1 \in S_1$ and $e_2^2 = e_2$ and $e_2x_2 = x_2e_2$ for all $x_2 \in S_2$. Now $e^2 = (e_1, e_2)^2 = (e_1, e_2)(e_1, e_2) = (e_1^2, e_2^2) = (e_1, e_2)$ (since $e_1 \in E(S_1)$ and $e_2 \in E(S_2)) = e$. Therefore, e is an idempotent element in $E(S_1 \times S_2)$. Again $ex = (e_1, e_2)(x_1, x_2) = (e_1x_1, e_2x_2) = (x_1e_1, x_2e_2)$ (since $e_1 \in E(S_1)$ and $e_2 \in E(S_2)$) $= (x_1, x_2)(e_1, e_2) = xe$ for all $x = (x_1, x_2) \in S_1 \times S_2$. Hence, e is a commuting idempotent element in $E(S_1 \times S_2)$. This implies that $e \in E(S_1 \times S_2)$. Consequently, we find that $E(S_1) \times E(S_2) \subseteq E(S_1 \times S_2)$ (ii).

From (i) and (ii), it follows that $E(S_1 \times S_2) = E(S_1) \times E(S_2)$. □

Theorem 2.3.5. *Let S be a c-semiring. Then $B(S) \subseteq B(E(S))$.*

Proof. Let $a \in B(S)$ and $x \in S$. Since $a \in B(S)$, there exist c-semirings S_1 with identity 1_{S_1} and S_2 with zero 0_{S_2} and there is an isomorphism $f : S \longrightarrow S_1 \times S_2$ such that $f(a) = (1_{S_1}, 0_{S_2})$. Let $f(x) = (x_1, x_2)$, where $x \in S$ and $x_1 \in S_1,\ x_2 \in S_2$. Then $f(a^2) = f(a)f(a) = (1_{S_1}, 0_{S_2})(1_{S_1}, 0_{S_2}) = (1_{S_1}, 0_{S_2}) = f(a)$. Since f is injective, it follows that $a^2 = a$. So, a is an idempotent element. Also $f(ax) = f(a)f(x) = (1_{S_1}, 0_{S_2})(x_1, x_2) = (x_1, 0_{S_2}) = (x_1, x_2)(1_{S_1}, 0_{S_2}) = f(x)f(a) = f(xa)$. Again since f is injective, we find that $ax = xa$. Thus, a is a commuting idempotent element in S and hence $a \in E(S)$. Consequently, it follows that $B(S) \subseteq E(S)$. Now the restriction of f to $E(S)$ becomes an isomorphism of $E(S)$ onto $E(S_1 \times S_2) = E(S_1) \times E(S_2)$ (by Lemma 2.3.4) and $f(a) = (1_{E(S_1)}, 0_{E(S_2)})$. Note that $1_{E(S_1)}$ and $0_{E(S_2)}$ are identity and zero in $E(S_1)$ and $E(S_2)$ respectively. Cosequently, $a \in B(E(S))$ and hence $B(S) \subseteq B(E(S))$. □

Let S be a c-semiring and $a \in S$. We define two sets

$aS = \{as : s \in S\}$ and $S_a = \{x \in S : ax = a \ \ and \ \ a + x = x\}$.

By utilizing the following two results, we can outline main characterization of the central elements.

Lemma 2.3.6. *Let S be a c-semiring and $a \in S$. Then S_a is a c-subsemiring of S with 'a' as the zero element.*

Proof. Let $x, y \in S_a$. Now $a(xy) = (ax)y = ay = a$ (since $ax = a$ and $ay = a$). Also $a + x = x \Longrightarrow ay + xy = xy \Longrightarrow a + xy = xy$ (since $ay = a$). Therefore, $xy \in S_a$. Again $a(x + y) = ax + ay = a + a = a$ (since $ax = a$, $ay = a$ and $(S, +)$ is a band). Also $a + (x + y) = (a + x) + y = x + y$ (since $a + x = x$). Therefore, $x + y \in S_a$. Also $a \in S_a$. Thus, S_a is a subsemiring of S with a as zero element. Again $1 \in S_a$. This implies that $a1 = 1a = a...(i)$ and $a + 1 = 1...(ii)$. From (i), we get that 1 is the identity element of S_a. Again from (ii), it follows that 1 is an absorbing element of S_a w.r.t. "+". Since S_a is a subsemiring of S and $(S, +)$ is a band, so $(S_a, +)$ is a band. Hence, S_a is a c-subsemiring of S with 'a' as the zero element. □

Lemma 2.3.7. *If $a \in E(S)$, then aS is a c-subsemiring of S with 'a' as the identity element and in this case $aS = \{x \in S : ax = x\}$.*

Proof. Let $x, y \in aS$. Then $x = ax_1$ for some $x_1 \in S$ and $y = ay_1$ for some $y_1 \in S$. Now $xy = (ax_1)(ay_1) = a(x_1ay_1) = as'$, where $s' = x_1ay_1 \in S$. Therefore, $xy \in aS$. Again $x + y = ax_1 + ay_1 = a(x_1 + y_1) \in aS$. Therefore, aS is a c-subsemiring of S. Let e be the identity of aS. So by definition we get $ze = z$ for some $z \in aS$. Since $z \in aS$, $z = as$ for some $s \in S$. Therefore, $ase = as$ (i).
We claim that 'a' is the identity of aS. Put $e = a$ in L.H.S. of (i). L.H.S. $=$ $ase = asa = a^2s = as$ (since $a \in E(S)$) $= as =$ R.H.S. Therefore, 'a' is the identity of aS. Again $0 \in aS$. It follows that $z0 = 0z = 0$ and $z + 0 = 0 + z = z$ for all $z \in aS$. So, 0 is the zero element of aS. Since aS is a subsemiring of S and $(S, +)$ is a band, $(aS, +)$ is a band. Also 'a' is the absorbing element of aS w.r.t. "+". Hence, aS is a c-subsemiring of S with 'a' as the identity element. □

We now present the main characterization theorem for Birkhoff center of a c-semiring, in fact, this is the main result of this chapter.

Theorem 2.3.8. *Let S be a c-semiring. Then $a \in B(S)$ if and only if $a \in E(S)$ and there exists a homomorphism $f_a : S \longrightarrow S_a$ such that $x \longmapsto (ax, f_a(x))$ is an isomorphism of S onto $aS \times S_a$ and f_a is identity on S_a.*

Proof. Suppose $a \in B(S)$. Then $a \in E(S)$ by Theorem 2.3.5. Thus there exists an isomorphism $\alpha : S \longrightarrow S_1 \times S_2$ such that $\alpha(a) = (1_{S_1}, 0_{S_2})$, where S_1 is a c-semiring with identity 1_{S_1} and S_2 is a c-semiring with zero 0_{S_2}. For any $x \in S$, let $\alpha(x) = (x_1, x_2)$, where $x_1 \in S_1$, $x_2 \in S_2$. Define $f_a : S \longrightarrow S_a$ by $f_a(x) = \alpha^{-1}(1_{S_1}, x_2)$. Let $x, y \in S$ be such that $x = y$. Then $\alpha(x) = \alpha(y) \Longrightarrow (x_1, x_2) = (y_1, y_2)$. This implies that $x_1 = y_1$ and $x_2 = y_2$. Now $f_a(x) = \alpha^{-1}(1_{S_1}, x_2) = \alpha^{-1}(1_{S_1}, y_2) = f_a(y)$. Thus, f_a is well defined. Now $af_a(x) = \alpha^{-1}(1_{S_1}, 0_{S_2})\, \alpha^{-1}(1_{S_1}, x_2) = \alpha^{-1}(1_{S_1}, 0_{S_2}) = a$

and $a+f_a(x) = \alpha^{-1}(1_{S_1}, 0_{S_2}) + \alpha^{-1}(1_{S_1}, x_2) = \alpha^{-1}(1_{S_1}+1_{S_1}, 0_{S_2}+x_2) = \alpha^{-1}(1_{S_1}, x_2) = f_a(x)$. So, $f_a(x) \in S_a$ for all $x \in S$. Also $f_a(x)f_a(y) = \alpha^{-1}(1_{S_1}, x_2)\ \alpha^{-1}(1_{S_1}, y_2) = \alpha^{-1}(1_{S_1}, x_2y_2) = f_a(xy)$ and $f_a(x) + f_a(y) = \alpha^{-1}(1_{S_1}, x_2) + \alpha^{-1}(1_{S_1}, y_2) = \alpha^{-1}(1_{S_1} + 1_{S_1}, x_2 + y_2) = \alpha^{-1}(1_{S_1}, x_2 + y_2) = f_a(x + y)$. Hence, f_a is a homomorphism. Let $\psi : S \longrightarrow aS \times S_a$ be defined by $\psi(x) = (ax, f_a(x))$ for all $x \in S$. We show that ψ is an isomorphism. Now $\psi(x)\psi(y) = (ax, f_a(x))(ay, f_a(y)) = (a^2xy, f_a(xy)) = (axy, f_a(xy)) = \psi(xy)$ and $\psi(x)+\psi(y) = (ax, f_a(x))+(ay, f_a(y)) = (a(x+y), f_a(x+y)) = \psi(x + y)$ for all $x, y \in S$. Hence, ψ is a homomorphism. To prove that ψ is one to one, suppose for any $x, y \in S$, $\psi(x) = \psi(y)$. Then $ax = ay$ and $f_a(x) = f_a(y)$. Now $ax = ay \Longrightarrow \alpha(ax) = \alpha(ay) \Longrightarrow \alpha(a)\alpha(x) = \alpha(a)\alpha(y) \Longrightarrow (1_{S_1}, 0_{S_2})\ (x_1, x_2) = (1_{S_1}, 0_{S_2})\ (y_1, y_2) \Longrightarrow (x_1, 0_{S_2}) = (y_1, 0_{S_2}) \Longrightarrow x_1 = y_1$. Also $f_a(x) = f_a(y) \Longrightarrow \alpha^{-1}(1_{S_1}, x_2) = \alpha^{-1}(1_{S_1}, y_2) \Longrightarrow x_2 = y_2$. Therefore, $\alpha(x) = (x_1, x_2) = (y_1, y_2) = \alpha(y) \Longrightarrow x = y$. Hence, ψ is one to one.

To prove that ψ is onto, let $(ax, y) \in aS \times S_a$. As earlier, let $\alpha(x) = (x_1, x_2)$ and $\alpha(y) = (y_1, y_2)$ with $x_i, y_i \in S_i,\ i = 1, 2$. Then $ay = a$ and hence $(1_{S_1}, 0_{S_2}) = \alpha(a) = \alpha(ay) = \alpha(a)\alpha(y) = (1_{S_1}, 0_{S_2})(y_1, y_2) = (y_1, 0_{S_2})$. Therefore, $y_1 = 1_{S_1}$. Choose $z \in S$ such that $\alpha(z) = (x_1, y_2)$. Then $\alpha(az) = \alpha(a)\alpha(z) = (1_{S_1}, 0_{S_2})(x_1, y_2) = (x_1, 0_{S_2}) = (1_{S_1}, 0_{S_2})(x_1, x_2) = \alpha(a)\alpha(x) = \alpha(ax)$ and hence $az = ax$. Also $f_a(z) = \alpha^{-1}(1_{S_1}, y_2) = \alpha^{-1}(y_1, y_2)$ (since $y_1 = 1_{S_1}$) $= \alpha^{-1}\alpha(y) = y$. Therefore, ψ is onto. Thus, ψ is an isomorphism of S onto $aS \times S_a$. Also for any $x \in S_a$, $ax = a$ and hence $\alpha(a)\alpha(x) = \alpha(a)$ i.e. $(1_{S_1}, 0_{S_2})(x_1, x_2) = (1_{S_1}, 0_{S_2})$ i.e. $x_1 = 1_{S_1}$. Now $\alpha(f_a(x)) = (1_{S_1}, x_2) = (x_1, x_2) = (x_1, x_2) = \alpha(x)$ and therefore $f_a(x) = x$. Thus, f_a restricted to S_a is identity.

Conversely, suppose the given condition is satisfied. We have to prove that $a \in B(S)$. Since $a \in E(S)$, aS is a subsemiring with a as identity element and S_a is a semiring with a as the zero element. Since $a \in E(S)$, $a^2 = a$ and $f_a(a) = a$, it follows that $x \longmapsto (ax, f_a(x))$ is an isomorphism of S onto $aS\times S_a$ such that a is mapped onto (a, a). Take $S_1 = aS$ and $S_2 = S_a$. Define $\alpha : S \longrightarrow S_1 \times S_2$ by $\alpha(x) = (ax, f_a(x))$. Then α is an isomorphism and $\alpha(a) = (1_{S_1}, 0_{S_2})$ in $S_1 \times S_2$. Thus, $a \in B(S)$. □

Now we prove some lemmas.

Let $a, b \in B(S)$. Then by the above theorem, there exist maps $f_a : S \to S_a$ and $f_b : S \to S_b$ such that $x \longmapsto (ax, f_a(x))$ is an isomorphism of S onto $aS \times S_a$ and $x \longmapsto (bx, f_b(x))$ is an isomorphism of S onto $bS \times S_b$, f_a is identity on S_a and f_b is identity on S_b. Put $c = f_a(b)$. Then we have the following.

Lemma 2.3.9. *Let $f_b : S \to S_b$ be a map such that $x \mapsto (bx, f_b(x))$ is an isomorphism of S onto $bS \times S_b$ and f_b is identity on S_b. Then $f_b(S_a) \subseteq S_a$.*

Proof. $x \in S_a \Longrightarrow ax = a$ and $a + x = x \Longrightarrow f_b(ax) = f_b(a)$ and $f_b(a + x) = f_b(x) \Longrightarrow f_b(a)\ f_b(x) = f_b(a)$ and $f_b(a) + f_b(x) = f_b(x) \Longrightarrow f_b(a)f_b^2(x) = f_b(a)$ and $f_b(a) + f_b^2(x) = f_b^2(x)$ (since $f_b(x) \in S_b$ and f_b is identity on S_b and $f_b^2(x) = f_b(f_b(x)) = f_b(x)) \Longrightarrow f_b(af_b(x)) = f_b(a)$ and $f_b(a+f_b(x)) = f_b(f_b(x)) \Longrightarrow af_b(x) = a$ and $a + f_b(x) = f_b(x)$. This implies that $f_b(x) \in S_a$. Consequently, $f_b(S_a) \subseteq S_a$. □

Lemma 2.3.10. $S_c = S_a \cap S_b = (S_a)_b = (S_b)_a := \{x \in S_b : ax = a\}$.

Proof. Since $c = f_a(b) \in S_a$, we have that $ac = a$ and $a + c = c$. Also, since $b \in S_a$ and $f_a(S_b) \subseteq S_b$, we have that $c \in S_b$ and hence $bc = b$ and $b + c = c$. Now, $x \in S_c \Longrightarrow cx = c \Longrightarrow acx = ac$ and $bcx = bc \Longrightarrow ax = a$ and $bx = b$ *(i)*.
Again $x \in S_c \Longrightarrow c+x = x \Longrightarrow a+(c+x) = a+x$ and $b+(c+x) = b+x \Longrightarrow x = a+x$, since $a \in S_a$ and $c \in S_a$, $a + c \in S_a$ and $x = b + x$ *(ii)*,
since $b \in S_a$ and $c \in S_a$, $b + c \in S_a$. From (i) and (ii), we get $ax = a$, $a + x = x$ and $bx = b$, $b+x = x$. Therefore, $x \in S_a$ and $x \in S_b \Longrightarrow x \in S_a \cap S_b$. Hence, $S_c \subseteq S_a \cap S_b$.

Conversely, let $x \in S_a \cap S_b \Longrightarrow ax = a$ and $bx = b \Longrightarrow f_a(bx) = f_a(b) \Longrightarrow f_a(b)\ f_a(x) = f_a(b) \Longrightarrow cx = c$ *(iii)*,
since $x \in S_a$, $f_a(x) = x$. Again $x \in S_a \cap S_b \Longrightarrow x \in S_a$ and $x \in S_a \Longrightarrow a + x = x$ and $b + x = x \Longrightarrow f_a(b + x) = f_a(x) \Longrightarrow f_a(b + x) = x$ (since $c = f_a(b) \in S_a$, $c + x = f_a(b + x) \in S_a$) $\Longrightarrow c + x = x$ *(iv)*,
since $c = f_a(b) \in S_a$, $c + x = f_a(b + x) \in S_a$. From (iii) and (iv), we find that $cx = x$ and $c + x = x$. Therefore, $x \in S_c$. Accordingly, $S_a \cap S_b \subseteq S_c$ and hence $S_c = S_a \cap S_b$. □

Lemma 2.3.11. *Let $a, b \in B(S)$. Then $f_a(b) = f_b(a)$.*

Proof. Put $c = f_a(b)$ and $d = f_b(a)$. By Lemma 2.3.10, $S_c = S_a \cap S_b = S_b \cap S_a = S_d$. Since $c \in S_c$ and $d \in S_d$, we have $c = cd = dc = d$ and $c = c + d = d + c = d$. Thus, $f_a(b) = f_b(a)$. □

2.4 Lattice Structure of $B(S)$

Our next goal is to find out the lattice structure of $B(S)$ of a c-semiring S. To achieve this, we first prove the following technical result.

Theorem 2.4.1. *If S is a c-semiring, then $B(S)$ is a c-subsemiring of S.*

Proof. Let $a, b \in B(S)$. Then there exist c-semirings S_1 with identity 1_{S_1} and S_2 with zero 0_{S_2} and an isomorphism $f : S \longrightarrow S_1 \times S_2$ such that $f(a) = (1_{S_1}, 0_{S_2})$. Similarly, there exist c-semirings T_1 with 1_{T_1} and T_2 with 0_{T_2} and an isomorphism $g : S \longrightarrow T_1 \times T_2$ such that $g(b) = (1_{T_1}, 0_{T_2})$. Let $g(a) = (t_1, t_2)$; where $t_1 \in T_1$ and $t_2 \in T_2$ and $f(b) = (w_1, w_2)$; where $w_1 \in S_1$ and $w_2 \in S_2$. Since $B(S) \subseteq E(S)$, 'a' is a commuting idempotent in S. So, $a^2 = a$ for all $x \in S$. Then $g(a^2) = g(a) \Longrightarrow g(a)\ g(a) = g(a) \Longrightarrow (t_1, t_2)\ (t_1, t_2) = (t_1, t_2) \Longrightarrow (t_1^2, t_2^2) = (t_1, t_2)$. Thus, $t_1^2 = t_1$ and $t_2^2 = t_2$. Again for any $a \in S$, $ax = xa$ for all $x \in S$. Therefore, $g(ax) = g(xa) \Longrightarrow g(a)\ g(x) = g(x)\ g(a) \Longrightarrow (t_1, t_2)\ (x_1, x_2) = (x_1, x_2)\ (t_1, t_2) \Longrightarrow (t_1x_1, t_2x_2) = (x_1t_1, x_2t_2)$; where $g(x) = (x_1, x_2)$, $x_i \in T_i$ for $i = 1, 2$. So, $t_1x_1 = x_1t_1$ and $t_2x_2 = x_2t_2$. Hence, for any $t_1 \in T_1$, $t_1^2 = t_1$ and $t_1x_1 = x_1t_1$ for all $x_1 \in T_1$ and for any $t_2 \in T_2$, $t_2^2 = t_2$ and $t_2x_2 = x_2t_2$ for all $x_2 \in T_2$. Thus, t_1 and t_2 are commuting idempotent in T_1 and T_2, respectively. Now we put $R_1 = t_1T_1$ and $R_2 = t_2T_2 \times S_2$. Then R_1 is a c-semiring (c-subsemiring of T_1) with t_1 as identity and R_2 is a c-semiring with $(0_{T_2}, 0_{S_2})$ as zero (since T_2 and S_2 are so). For any $x \in S$, let $f(x) = (s_1, s_2)$ and $g(x) = (x_1, x_2)$; where $s_i \in S_i$ and $x_i \in T_i$ for $i = 1, 2$. Define $h : S \longrightarrow R_1 \times R_2$ by $h(x) = (t_1x_1, (t_2x_2, s_2))$. We now prove that h is an isomorphism and $h(ab) = (1_{R_1}, 0_{R_2})$. Let $y \in S$ and $g(y) = (y_1, y_2)$ and $f(y) = (r_1, r_2)$; where $r_i \in S_i$, $y_i \in T_i$ for $i = 1, 2$. Then $g(xy) = g(x)\ g(y) = (x_1, x_2)\ (y_1, y_2) = (x_1y_1, x_2y_2)$ and $f(xy) = f(x)\ f(y) = (s_1, s_2)\ (r_1, r_2) = (s_1r_1, s_2r_2)$. Now $h(x + y) = (t_1(x_1 + y_1), (t_2(x_2 + y_2), s_2 + r_2)) =$

$(t_1x_1,(t_2x_2,s_2))+(t_1y_1,(t_2y_2,r_2)) = h(x)+h(y)$ and $h(xy) = (t_1x_1y_1,(t_2x_2y_2,s_2r_2)) = (t_1x_1t_1y_1,(t_2x_2y_2,s_2r_2)) = (t_1x_1,(t_2x_2,s_2))\ (t_1y_1,(t_2y_2,r_2)) = h(x)h(y)$. So, h is a homomorphism. To prove that h is one to one, let $h(u) = h(v)$ for some $u, v \in S$. Then $(t_1u_1,(t_2u_2,m_2)) = (t_1v_1,(t_2v_2,n_2)) \Longrightarrow t_1u_1 = t_1v_1,\ t_2u_2 = t_2v_2$ and $m_2 = n_2 \Longrightarrow u_1 = v_1,\ u_2 = v_2$ and $m_2 = n_2$ (since $t_1u_1 = u_1,\ t_1v_1 = v_1,\ t_2u_2 = u_2$ and $t_2v_2 = v_2$). Now $(u_1,u_2) = (v_1,v_2) \Longrightarrow g(u) = g(v) \Longrightarrow u = v$ (since g is one to one). Therefore, h is one to one. To prove that h is onto, suppose $(t_1p_1,(t_2p_2,q_2)) \in R_1\times R_2$; where $p_1 \in T_1,\ p_2 \in T_2,\ q_2 \in S_2$. Since $(p_1,p_2) \in T_1 \times T_2$ and $g : S \longrightarrow T_1 \times T_2$ is an isomorphism, there exists $y \in S$ such that $g(y) = (p_1,p_2)$. Now $g(ay) = g(a)g(y) = (t_1,t_2)(p_1,p_2) = (t_1p_1,t_2p_2) = (p_1,p_2)$ (since $t_1p_1 = p_1, t_2p_2 = p_2$) $= g(y)$. This imlpies that $ay = y$. So $f(ay) = f(y) \Longrightarrow f(a)f(y) = f(y) \Longrightarrow (1_{S_1},0_{S_2})(y_1,y_2) = (y_1,y_2)$ (since $f(y) = (y_1,y_2)$). Therefore, $y_2 = 0_{S_2}$. Since f is an isomorphism, there exits $x \in S$ such that $f(x) = (y_1,q_2)$. Then $f(ax) = f(a)f(x) = (1_{S_1},0_{S_2})(y_1,q_2) = (y_1,0_{S_2}) = (y_1,y_2)$ (since $y_2 = 0_{S_2}$) $= f(y)$. Hence, $ax = y$. Let $g(x) = (z_1,z_2)$. Then $(t_1p_1,t_2p_2) = g(y) = g(ax) = g(a)g(x) = (t_1,t_2)\ (z_1,z_2) = (t_1z_1,t_2z_2)$. Therefore, $h(x) = (t_1z_1,(t_2z_2,q_2)) = (t_1p_1,(t_2p_2,q_2))$. Hence, h is onto and h is an isomorphism. Also $g(ab) = g(a)g(b) = (t_1,0_{T_2})$ and $f(ab) = f(a)f(b) = (w_1,0_{S_2})$. This implies that $h(ab) = (t_1,(0_{T_2},0_{S_2})) = (1_{R_1},0_{R_2})$. Thus, $ab \in B(S)$. We now prove that $a+b \in B(S)$. Put $R_1' = t_1T_1 \times t_2T_2$ and $R_2' = t_2 + T_2$. Then R_1' is a c-semiring with (t_1,t_2) as identity and R_2' is a c-semiring with $(t_2 + 0_{T_2})$ as zero with respect to the following operations : $(t_2+p)+(t_2+q) = t_2+(p+q)$ and $(t_2+p)(t_2+q) = t_2+pq$ for all $p, q \in T_2$. Define $h' : S \longrightarrow R_1' \times R_2'$ by $h'(x') = ((t_1x_1',t_2x_2'),t_2+x_2')$. For any $x' \in S$, let $g(x') = (x_1',x_2')$ and $f(x') = (s_1',s_2')$; where $x_1' \in T_1,\ x_2' \in T_2,\ s_1' \in S_1,\ s_2' \in S_2$. For any $y' \in S$, let $g(y') = (y_1',y_2')$ and $f(y') = (r_1',r_2')$, where $y_1' \in T_1,\ y_2' \in T_2,\ r_1' \in S_1,\ r_2' \in S_2$. We have $g(x'+y') = g(x')+g(y') = (x_1',x_2')+(y_1',y_2') = (x_1'+y_1',x_2'+y_2')$ and $f(x'+y') = f(x')+f(y') = (s_1',s_2')+(r_1',r_2') = (s_1'+r_1',s_2'+r_2')$. Now $h'(x')h'(y') = ((t_1x_1',t_2x_2'),t_2+x_2')((t_1y_1',t_2y_2'),t_2+y_2') = ((t_1x_1't_1y_1',t_2x_2't_2y_2'),t_2+x_2'y_2') = ((t_1x_1'y_1',t_2x_2'y_2'),t_2+x_2'y_2') = h'(x'y')$ (since $t_1y_1' = y_1'$ and $t_1y_1' = y_1'$) and $h'(x')+h'(y') = ((t_1x_1',t_2x_2'),t_2+x_2')+((t_1y_1',t_2y_2'),\ t_2+y_2') = ((t_1(x_1'+y_1'),t_2(x_2'+y_2')),t_2+(x_2'+y_2')) = h'(x'+y')$. Hence, h' is a homomorphism. To prove that

h' is one to one, let $h'(u') = h'(v')$ for some $u', v' \in S \Longrightarrow ((t_1u'_1, t_2u'_2), t_2 + u'_2) = ((t_1v'_1, t_2v'_2), t_2 + v'_2) \Longrightarrow t_1u'_1 = t_1v'_1, t_2u'_2 = t_2v'_2$ and $t_2 + u'_2 = t_2 + v'_2 \Longrightarrow t_1u'_1 = t_1v'_1, t_2u'_2 = t_2v'_2$ and $t_2+u'_2 = t_2+v'_2 \Longrightarrow u'_1 = v'_1, u'_2 = v'_2$ and $t_2+u'_2 = t_2+v'_2$ (since $t_1u'_1 = u'_1t_2u'_2 = u'_2t_1v'_1 = v'_1, t_2v'_2 = v'_2$). Now $(u'_1, u'_2) = (v'_1, v'_2) \Longrightarrow g(u') = g(v')$. Therefore, $u' = v'$ (since g is one to one). Thus h' is one to one. To prove that h' is onto, suppose $((t_1p'_1, t_2p'_2), t_2 + q'_2) \in R'_1 \times R'_2$, where $p'_1 \in T_1, p'_2 \in T_2, q'_2 \in T_2$. Since $(p'_1, p'_2) \in T_1 \times T_2$ and $g : S \longrightarrow T_1 \times T_2$ is an isomorphism, there exists $p' \in S$ such that $g(p') = (p'_1, p'_2)$. Similarly, since $(p'_1, q'_2) \in T_1 \times T_2$, there exists $q' \in S$ such that $g(q') = (p'_1,\ q'_2)$. Now $g(aq') = g(a)g(q') \Longrightarrow g(aq') = (t_1, t_2)\ (p'_1, q'_2) = (t_1p'_1, t_2q'_2) = (p'_1, q'_2)$ (since $t_1p'_1 = p'_1$ and $t_1q'_2 = q'_2$) $= g(q')$. Since g is one to one, so we find that $aq' = q'$. This implies that $h'(aq') = h'(q') \Longrightarrow h'(a)h'(q') = h'(q') \Longrightarrow ((t_1, t_2), t_2)((t_1p'_1, t_2q'_2), t_2 + q'_2) = ((t_1p'_1, t_2q'_2), t_2 + q'_2) \Longrightarrow ((t_1p'_1, t_2q'_2), t_2 + 0_{T_2}) = ((t_1p'_1, t_2q'_2), t_2+q'_2) \Longrightarrow t_2+q'_2 = t_2+0_{T_2}$. Again $h'(a+b) = (1_{R'_1}, 0_{R'_2}) = ((t_1, t_2), t_2 + 0_{T_2})$. Now $h'((a + b)p') = h'(a + b)h'(p') = ((t_1, t_2), t_2 + 0_{T_2})((t_1p'_1, t_2p'_2), t_2 + p'_2) = ((t_1p'_1, t_2p'_2), t_2+0_{T_2}) = ((t_1p'_1, t_2p'_2), t_2+q'_2)$ (since $t_2+0_{T_2} = t_2+q'_2$). Hence, h' is onto and h' is an isomorphism. Also $f(a + b) = f(a) + f(b) = (w_1 + 1_{S_1}, w_2) = (1_{S_1}, w_2)$ (since 1_{S_1} is absorbing with respect to "+") and $g(a + b) = g(a) + g(b) = (t_1 + 1_{T_1}, t_2) = (1_{T_1}, t_2)$ (since 1_{T_1} is absorbing with respect to "+"); from which we get $h'(a+b) = ((t_1(t_1 + 1_{T_1}), t_2t_2), t_2 + t_2) = ((t_1, t_2), t_2) = ((t_1, t_2), t_2 + 0_{T_2}) = (1_{R'_1}, 0_{R'_2})$. Thus, $a + b \in B(S)$. Hence, $B(S)$ is a subsemiring of S. Again $0_S, 1_S \in B(S)$ (by Lemma 2.2.5). Since $B(S)$ is a subsemiring of S and $(S, +)$ is a band, $(B(S), +)$ is a band. Also 1_S is the absorbing element of $B(S)$ w.r.t. "+". Hence, $B(S)$ is a c-subsemiring of S. □

Now we are going to set up the lattice structure for $B(S)$. For that we start with the following note.

Note 2.4.2. *A subset may fail to be a sublattice even when it forms a lattice under the operations of the parent lattice restricted to the sublattice.*

Example 2.4.3. *Let G be a group. Then $\mathcal{P}(G)$ = Set of all subsets of G forms a lattice under subset relation $\subseteq$ and where $H \wedge K = H \cap K$ and $H \vee K$ = Intersection* *of all subsets of G containing both H and K ($= H \cup K$). Let τ be set of all subgroups of G. Then $\tau \subseteq \mathcal{P}(G)$ and τ is not a sublattice of $\mathcal{P}(G)$ as for $H, K \in \tau$, $H \vee K =$ Intersection of all subsets containing both H and K will be $H \cup K$ which may not be a subgroup and thus $H \vee K \notin \tau$.*

But in case of c-semiring, a c-subsemiring of a c-semiring S forms a lattice structure w.r.t. the same partial order relation defined on S.

It is already in the literature that a c-semiring S forms a lattice structure w.r.t. a parial order relation "$\leq_S$" on S defined by $a \leq_S b$ if and only if $a + b = b$ for all $a, b \in S$.

In this lattice $a \vee b$ and $a \wedge b$ are nothing but $a + b$ and ab respectively. So, any c-subsemiring of a c-semiring S forms a sublattice. Since $B(S)$ is a c-subsemiring of a c-semiring S, it is a sublattice of S.

Theorem 2.4.4. *If S is a c-semiring, then $B(S)$ forms a distributive lattice.*

Proof. Let $a, b, c \in B(S)$. Then $a \wedge (b \vee c) = a \wedge (b + c)$ (since $b \vee c = b + c$) $= a(b + c)$ (since $a \wedge b = ab$) $= ab + ac = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in B(S)$. Hence $B(S)$ is a distributive lattice. □

A distributive lattice is always modular lattice. Also from Theorem 2.4.4, it follows that $B(S)$ is a distributive lattice for a c-semiring S. So we have the following result:

Lemma 2.4.5. *If S is a c-semiring, then $B(S)$ forms a modular lattice.*

Theorem 2.4.6. *If S is a c-semiring, then $B(S)$ forms a complete lattice.*

Proof. Let S be a c-semiring. Define a partial order relation "$\leq_S$" on S by $a \leq_S b$ if and only if $a + b = b$ for all $a, b \in S$. Then from [14], it follows that $(S, \leq_S)$ is a complete lattice. From Theorem 2.4.1, we know that $B(S)$ is a c-subsemiring of S. So, we conclude that $(B(S), \leq_S)$ is a complete lattice. □

Theorem 2.4.7. *If S is a c-semiring, then $B(S)$ forms a Boolean lattice.*

Proof. Let S be a c-semiring with identity 1_S and zero element 0_S. Suppose that $\alpha \in B(S)$, $\alpha \neq 0_S, 1_S$. Then there exist c-semirings S_1 with identity 1_{S_1}, S_2 with zero element 0_{S_2} and an isomorphism $f : S \longrightarrow S_1 \times S_2$ such that $f(\alpha) = (1_{S_1}, 0_{S_2})$. Since f is surjective and $(0_{S_1}, 1_{S_2}) \in S_1 \times S_2$, there exists $\beta \in S$ such that $f(\beta) = (0_{S_1}, 1_{S_2})$. Now $f(\alpha+\beta) = f(\alpha)+f(\beta) = (1_{S_1}, 0_{S_2})+(0_{S_1}, 1_{S_2}) = (1_{S_1}, 1_{S_2}) = f(1_S)$. This implies that $\alpha + \beta = 1_S$. Again $f(\alpha\beta) = f(\alpha)f(\beta) = (1_{S_1}, 0_{S_2})(0_{S_1}, 1_{S_2}) = (0_{S_1}, 0_{S_2}) = f(0_S)$. Thus, $\alpha\beta = 0_S$. So, β is a complement of α. We show that $\beta \in B(S)$. Let us construct a function $g : S \longrightarrow S_2 \times S_1$ defined by $g(x) = (c, d)$, where $f(x) = (d, c)$. To prove that g is homomorphism, suppose for $x, y \in S$, $g(x + y) = (c, d)$. Then $f(x + y) = (d, c) \Longrightarrow f(x) + f(y) = (d, c)$ (since f is homomorphism) $\Longrightarrow (a_1, b_1) + (a_2, b_2) = (d, c)$, where $f(x) = (a_1, b_1)$ and $f(y) = (a_2, b_2) \Longrightarrow (a_1+a_2, b_1+b_2) = (d, c)$. This implies that $a_1+a_2 = d$ and $b_1+b_2 = c$. Again $g(x) = (b_1, a_1)$ and $g(y) = (b_2, a_2)$. Now $g(x) + g(y) = (b_1, a_1) + (b_2, a_2) = (b_1 + b_2, a_1 + a_2) = (c, d) = g(x + y)$. Again $g(xy) = (c_1, d_1)$. Then by the definition of g, $f(xy) = (d_1, c_1) \Longrightarrow f(x)f(y) = (d_1, c_1)$ (since f is homomorphism) $\Longrightarrow (a_1, b_1)(a_2, b_2) = (d_1, c_1)$, where $f(x) = (a_1, b_1)$ and $f(y) = (a_2, b_2) \Longrightarrow (a_1a_2, b_1b_2) = (d_1, c_1)$. Thus, $a_1a_2 = d_1$ and $b_1b_2 = c_1$. Again $g(x) = (b_1, a_1)$ and $g(y) = (b_2, a_2)$. Now $g(x)g(y) = (b_1, a_1)(b_2, a_2) = (b_1b_2, a_1a_2) = (c_1, d_1) = g(xy)$. Hence, g is a homomorphism. To prove that g is one to one, suppose for any $x, y \in S$, $g(x) = g(y)$. Then $(c_1, d_1) = (c_2, d_2) \Longrightarrow c_1 = c_2$ and $d_1 = d_2$. Now $(d_1, c_1) = (d_2, c_2) \Longrightarrow f(x) = f(y) \Longrightarrow x = y$ (since f is one to one). Hence, g is one to one. To prove that $g : S \longrightarrow S_2 \times S_1$ is onto, consider $(a, b) \in S_2 \times S_1$. Then $(b, a) \in S_1 \times S_2$. Since $f : S \longrightarrow S_1 \times S_2$ is surjective, there exists $z \in S$ such that $f(z) = (b, a)$. This implies that $g(z) = (a, b)$. Therefore, g is onto. Thus, g is an isomorphism. Therefore, $\beta \in B(S)$. So, the complement of any element of $B(S)$ is also the member of $B(S)$. Hence, $B(S)$ is a complemented lattice. From Theorem 2.4.4, it follows that $B(S)$ is a distributive lattice. Again $0_S, 1_S \in B(S)$. Thus, $B(S)$ is a bounded lattice and consequently, $B(S)$ forms a Boolean lattice. □

Corollary 2.4.8. *If S is a finite c-semiring, then the number of elements of $B(S)$ is 2^n for some $n \in \mathbb{N}$.*

Proof. From the above, we find that $B(S)$ forms a Boolean lattice. From [19], it follows that every finite Boolean lattice is isomorphic to a set with 2^n elements for some $n = 0, 1, 2, ...$ So, the number of elements of $B(S)$ is 2^n for some $n \in \mathbb{N}$. □

2.5 Properties of Birkhoff Center of c-Semiring

The following section is dedicated to investigating various properties associated with the Birkhoff center $B(S)$ of a c-semiring S.

Lemma 2.5.1. *Let S be a c-semiring. If x is a nonzero and nonidentity element of S such that $x \in B(S)$, then x is a divisor of zero in S.*

Proof. Let S be a c-semiring and $x \in B(S)$. Then there exist two c-semirings S_1 with identity 1_{S_1} and S_2 with zero 0_{S_2} and an isomorphism $f : S \longrightarrow S_1 \times S_2$ such that $f(x) = (1_{S_1}, 0_{S_2})$. Again $f(0_S) = (0_{S_1}, 0_{S_2})$ and $f(1_S) = (1_{S_1}, 1_{S_2})$. Suppose $f(y) = (0_{S_1}, 1_{S_2})$, where $y(\neq 0) \in S$. Now $f(xy) = (0_{S_1}, 0_{S_2}) = f(0_S)$. This implies that $xy = 0_S$ (since f is a isomorphism). Therefore, x is a zero divisor in S. □

The following example shows that the converse of the above result is not true i.e. a divisor of zero in a c-semiring S is not necessarily a member of $B(S)$.

Example 2.5.2. *Consider the set $S_1 = \{0_{S_1}, 1_{S_1}\}$. Define the operations "$+$" and "$.$" on S by means of the followings two tables :*

$+$	0_{S_1}	1_{S_1}
0_{S_1}	0_{S_1}	1_{S_1}
1_{S_1}	1_{S_1}	1_{S_1}

$.$	0_{S_1}	1_{S_1}
0_{S_1}	0_{S_1}	0_{S_1}
1_{S_1}	0_{S_1}	1_{S_1}

Then $(S_1, +, \cdot)$ is a c-semiring and $B(S_1) = \{0_{S_1}, 1_{S_1}\}$.

Consider $S_2 = \{0_{S_2}, 1_{S_2}, x\}$. Define the operations "$+$" and "$.$" on T as follows :

$+$	0_{S_2}	x	1_{S_2}
0_{S_2}	0_{S_2}	x	1_{S_2}
x	x	x	1_{S_2}
1_{S_2}	1_{S_2}	1_{S_2}	1_{S_2}

$.$	0_{S_2}	x	1_{S_2}
0_{S_2}	0_{S_2}	0_{S_2}	0_{S_2}
x	0_{S_2}	0_{S_2}	x
1_{S_2}	0_{S_2}	x	1_{S_2}

Then $(S_2, +, \cdot)$ *is a c-semiring and* $B(S_2) = \{0_{S_2}, 1_{S_2}\}$.

Let $S = S_1 \times S_2 = \{(0_{S_1}, 0_{S_2}), (0_{S_1}, 1_{S_2}), (0_{S_1}, x), (1_{S_1}, 0_{S_2}), (1_{S_1}, 1_{S_2}), (1_{S_1}, x)\}$ *with two binary operations "+" and "." on* S*, defined by componentwise. Then* $(S, +, \cdot)$ *is a c-semiring and* $B(S) = \{(0_{S_1}, 0_{S_2}), (1_{S_1}, 1_{S_2}), (1_{S_1}, 0_{S_2}), (0_{S_1}, 1_{S_2})\}$.

Now $|S| = |S_1 \times S_2| = 6$ *and* $|B(S)| = |B(S_1 \times S_2)| = 4$*. Again* $(0_{S_1}, x), (1_{S_1}, x) \in S$ *and* $(0_{S_1}, x) \cdot (1_{S_1}, x) = (0_{S_1}, x \cdot x) = (0_{S_1}, 0_{S_2})$*. This implies that* $(0_{S_1}, x)$ *and* $(1_{S_1}, x)$ *are divisors of zero. But* $(0_{S_1}, x)$*,* $(1_{S_1}, x) \notin B(S)$*.*

Theorem 2.5.3. *If* $S = \{0_S, 1_S, x_1, x_2, ..., x_{n-2}\}$ *is a finite set of* n *elements together with two binary operations addition and multiplication (denoted by juxtaposition) satifying the following conditions :* $x_i + x_i = x_i$ *for all* $x_i \in S$*,* $x_i + x_j = 0$*,* $x_i x_j = 0$ *for all* $i \neq j$*,* $x_i, x_j \in S$*,* $1_S x_i = x_i$ *for all* $x_i \in S$*,* $0_S x_i = 0_S$ *for all* $x_i \in S$*,* $1_S + x_i = 1_S$ *for all* $x_i \in S$*, then* S *forms a c-semiring and* $B(S) = \{0_S, 1_S\}$*.*

Proof. It can be easily shown that S forms a c-semiring with multiplicative identity 1_S and additive identity 0_S. Again 1_S is an absorbing element of S with respect to addition. If $x \in B(S)$, then there exists an element $y \in B(S)$ such that x, y are complement to each other i.e. $x + y = 1_S$ and $xy = 0_S$, since $B(S)$ forms a Boolean lattice. From hypothesis, it is only possible when either $x = 1_S$, $y = 0_S$ or $x = 0_S$, $y = 1_S$. Hence, $B(S) = \{0_S, 1_S\}$. □

Theorem 2.5.4. *If* S *is a finite c-semiring with prime cardinality, then* $B(S) = \{0_S, 1_S\}$*.*

Proof. Suppose that S is a finite c-semiring with prime cardinality i.e. $|S| = p$, where p is a prime number. Let $x \in B(S)$. Then there exist two c-semirings S_1 with identity 1_{S_1} and S_2 with zero 0_{S_2} and an isomorphism $f : S \longrightarrow S_1 \times S_2$ such that $f(x) = (1_{S_1}, 0_{S_2})$. Since $0_S, 1_S \in B(S)$, $f(0_S) = (0_{S_1}, 0_{S_2})$ and $f(1_S) = (1_{S_1}, 1_{S_2})$. Again since $S \simeq S_1 \times S_2$, so $|S| = |S_1 \times S_2| = |S_1| \times |S_2|$. Now $|S| = p$ implies that $|S| = |S_1| \times |S_2| = p$. This implies either $|S_1| = 1, |S_2| = p$ or $|S_1| = p, |S_2| = 1$. Let $S_1 = \{0_{S_1}\}$ and $S_2 = \{0_{S_2}, 1_{S_2}, x_1, \ldots, x_{p-2}\}$. Since $f(x) = (1_{S_1}, 0_{S_2}) = (0_{S_1}, 0_{S_2}) = f(0_S)$. This implies that $x = 0_S$. Taking $|S_1| = p$ and $|S_2| = 1$, we can prove that $x = 1_S$. Hence, $B(S) = \{0_S, 1_S\}$. □

Theorem 2.5.5. *If S_1 and S_2 are two c-semirings, then $|B(S_1 \times S_2)| \geq 4$.*

Proof. Let S_1 be a c-semiring with identity 1_{S_1} and zero 0_{S_1}, S_2 be another c-semiring with identity 1_{S_2} and zero 0_{S_2}. Let $f : S_1 \times S_2 \longrightarrow S_1 \times S_2$ be the identity isomorphism. We know that $(1_{S_1}, 1_{S_2}), (0_{S_1}, 0_{S_2}) \in B(S_1 \times S_2)$. Also $f(1_{S_1}, 0_{S_2}) = (1_{S_1}, 0_{S_2})$ implies that $(1_{S_1}, 0_{S_2}) \in B(S_1 \times S_2)$. Now $(0_{S_1}, 1_{S_2})$ is a complement of $(1_{S_1}, 0_{S_2})$. Hence, $(1_{S_1}, 0_{S_2}) \in B(S_1 \times S_2)$, $(0_{S_1}, 1_{S_2}) \in B(S_1 \times S_2)$. Therefore, $|B(S_1 \times S_2)| \geq 4$. □

Theorem 2.5.6. *If S is a c-semiring with cardinality $|S| \leq 7$, then $|B(S)| \leq 4$.*

Proof. If S is a singleton c-semiring, then $B(S)$ is a singleton. Again from Theorem 2.5.4, it follows that if S is a c-semiring with prime cardinality then $B(S) = \{0_S, 1_S\}$, where 0_S is the zero element of S and 1_S is the identity element of S. If S is a c-semiring with cardinality 4 then the cardinality of $B(S)$ is either 2 or 4. Again if S is a c-semiring with cardinality 6, then the cardinality of $B(S)$ is either 2 or 4. Therefore, $|B(S)| \leq 4$. □

Remark 2.5.7. *If $S = S_1 \times S_2$, where S_1 and S_2 are two c-semirings and $|S| \leq 7$ then by Theorem* 2.5.5 *and Theorem* 2.5.6*, we conclude that $|B(S)| = 4$.*

Theorem 2.5.8. *If S_1 and S_2 are two c-semirings, then $B(S_1) \times B(S_2) \subseteq B(S_1 \times S_2)$.*

Proof. Let $(a, b) \in B(S_1) \times B(S_2)$. Then as $a \in B(S_1)$, there exist two c-semirings T_1 with identity 1_{T_1} and T_2 with zero 0_{T_2} and an isomorphism $f : S_1 \longrightarrow T_1 \times T_2$ such that $f(a) = (1_{T_1}, 0_{T_2})$. Also as $b \in B(S_2)$, there exist two c-semirings T_1' with identity $1_{T_1'}$ and T_2' with zero $0_{T_2'}$ and an isomorphism $g : S_2 \longrightarrow T_1' \times T_2'$ such that $g(b) = (1_{T_1'}, 0_{T_2'})$. Now we consider the c-semiring $T_1 \times T_1'$ with identity $(1_{T_1}, 1_{T_1'})$ and the c-semiring $T_2 \times T_2'$ with zero $(0_{T_2}, 0_{T_2'})$ and define a mapping $F : S_1 \times S_2 \longrightarrow (T_1 \times T_1') \times (T_2 \times T_2')$ by $F(x, y) = ((s_1, s_1'), (t_2, t_2'))$, where $f(x) = (s_1, t_2)$, $g(y) = (s_1', t_2')$ and $s_1 \in T_1$, $s_1' \in T_1'$, $t_2 \in T_2$, $t_2' \in T_2'$. At first we show that F is well defined. Let $(x, y), (p, q) \in S_1 \times S_2$ such that $(x, y) = (p, q)$. Then $x = p$ and $y = q$. Since f, g are well defined, $x = p \implies f(x) = f(p)$ and $y = q \implies g(y) = g(q)$. Let $f(x) = f(p) = (a, b)$ and $g(y) = g(q) = (c, d)$, where $a \in T_1$, $b \in T_2$, $c \in T_1'$ and $d \in T_2'$. So, $F(x, y) = ((a, c), (b, d)) =$

$F(p,q)$. Thus, F is well defined. To show that F is a homomorphism, consider $(x_1,y_1),(x_2,y_2) \in S_1 \times S_2$. Now we assume that $f(x_1) = (s_1,s_2)$, $f(x_2) = (t_1,t_2)$, $g(y_1) = (s'_1,s'_2)$ and $g(y_2) = (t'_1,t'_2)$, where $s_1,t_1 \in T_1$, $s_2,t_2 \in T_2$, $s'_1,t'_1 \in T'_1$ and $s'_2,t'_2 \in T'_2$. Since f is a homomorphism, it follows that $f(x_1+x_2) = f(x_1)+f(x_2) = (s_1,s_2)+(t_1,t_2) = (s_1+t_1,s_2+t_2)$ and $f(x_1x_2) = f(x_1)f(x_2) = (s_1,s_2)(t_1,t_2) = (s_1t_1,s_2t_2)$. Again, since g is a homomorphism, we have $g(y_1+y_2) = g(y_1)+g(y_2) = (s'_1,s'_2)+(t'_1,t'_2) = (s'_1+t'_1,s'_2+t'_2)$ and $g(y_1y_2) = g(y_1)g(y_2) = (s'_1,s'_2)(t'_1,t'_2) = (s'_1t'_1,s'_2t'_2)$. So, we find that $F((x_1,y_1)+(x_2,y_2)) = F(x_1+x_2,y_1+y_2) = ((s_1+t_1,s_2+t_2),(s'_1+t'_1,s'_2+t'_2)) = ((s_1,t_1),(s_1,t'_1))+((s_2,t_2),(s'_2,t'_2)) = F(x_1,y_1)+F(x_2,y_2)$. Again $F((x_1,y_1)(x_2,y_2)) = F(x_1x_2,y_1y_2) = ((s_1s_2,t_1t_2),(s'_1s'_2,t'_1t'_2)) = ((s_1,t_1),(s'_1,t'_1))((s_2,t_2),(s'_2,t'_2)) = F(x_1,y_1)F(x_2,y_2)$. This shows that F is a homomorphism. To prove that F is one to one, suppose that for any $(x,y),(p,q) \in S_1 \times S_2, F(x,y) = F(p,q)$. Then we have $((s_1,s'_1),(t_2,t'_2)) = ((p_1,p'_1),(q_2,q'_2))$, where $f(x) = (s_1,t_2)$, $g(y) = (s'_1,t'_2)$, $f(p) = (p_1,q_2)$ and $g(q) = (p'_1,q'_2)$. This implies that $s_1 = p_1$, $t_2 = q_2$, $s'_1 = p'_1$ and $t'_2 = q'_2$. Since $s_1 = p_1, s'_1 = p'_1 \Longrightarrow f(x) = f(p) \Longrightarrow x = p$ (since f is one to one) (i).

Again $t_2 = q_2, t'_2 = q'_2 \Longrightarrow f(y) = f(q) \Longrightarrow y = q$ (since f is one to one) (ii).

From (i) and (ii), we find that $(x,y) = (p,q)$ for all $(x,y),(p,q) \in S_1 \times S_2 \Longrightarrow F$ is one to one. To prove that F is onto, let $((a_1,a'_1),(a_2,a'_2)) \in (T_1 \times T'_1) \times (T_2 \times T'_2)$, where $a_1 \in T_1$, $a'_1 \in T'_1$, $a_2 \in T_2$ and $a'_2 \in T'_2$. Since $(a_1,a_2) \in T_1 \times T_2$ and $f : S_1 \longrightarrow T_1 \times T_2$ is an isomorphism, there exists $x \in S_1$ such that $f(x) = (a_1,a_2)$. Again since $(a'_1,a'_2) \in T'_1 \times T'_2$ and $g : S_2 \longrightarrow T'_1 \times T'_2$ is an isomorphism, there exists $y \in S_2$ such that $g(y) = (a'_1,a'_2)$. Therefore, $F(x,y) = ((a_1,a'_1),(a_2,a'_2))$. This implies that F is onto and cosequently, F is an isomorphism. Since $f(a) = (1_{T_1},0_{T_2})$ and $g(b) = (1_{T'_1},0_{T'_2})$, $F(a,b) = ((1_{T_1},1_{T'_1}),(0_{T_2},0_{T'_2}))$, where $(1_{T_1},1_{T'_1})$ is the identity of $T_1 \times T'_1$ and $(0_{T_2},0_{T'_2})$ is the zero element of $T_2 \times T'_2$. So, $(a,b) \in B(S_1 \times S_2)$. Hence, $B(S_1) \times B(S_2) \subseteq B(S_1 \times S_2)$. □

Now we show that the following theorem can be used to strengthen theorem 2.5.8.

Theorem 2.5.9. *If S_1 and S_2 are two c-semirings without zero divisor, then $B(S_1 \times S_2) = B(S_1) \times B(S_2)$.*

Proof. Let S_1 be c-semiring with identity 1_{S_1}, zero 0_{S_1} and S_2 be c-semiring identity 1_{S_2}, zero 0_{S_2}. Since S_1 and S_2 be two c-semirings without zero divisor, then the set of zero divisor of $S_1 \times S_2$ is $(S_1^* \times \{0_{S_2}\}) \cup (\{0_{S_1}\} \times S_2^*)$, where S_i^* denotes $S_i \backslash 0_{S_i}$ for $i = 1, 2$. From Lemma 2.5.1, we get if $(a, b) \in B(S_1 \times S_2) \setminus \{(1_{S_1}, 1_{S_2}), (0_{S_1}, 0_{S_2})\}$, then (a, b) must be zero divisor. So, $B(S_1 \times S_2) \setminus \{(1_{S_1}, 1_{S_2}), (0_{S_1}, 0_{S_2})\} \subseteq (S_1^* \times \{0_{S_2}\}) \cup (\{0_{S_1}\} \times S_2^*)$. This implies that a central element of $S_1 \times S_2$ except $(1_{S_1}, 1_{S_2})$ and $(0_{S_1}, 0_{S_2})$ must be of the form either $(x, 0_{S_2})$ or $(0_{S_1}, y)$. Since $(x, 0_{S_2}) \in B(S_1 \times S_2)$, then there exists an element $(0_{S_1}, y) \in B(S_1 \times S_2)$ (Since $B(S_1 \times S_2)$ forms a Boolean lattice) such that $(x, 0_{S_2}) + (0_{S_1}, y) = (1_{S_1}, 1_{S_2})$. So, $(x, y) = (1_{S_1}, 1_{S_2})$. This implies that $x = 1_{S_1}$ and $y = 1_{S_2}$. Therefore, $(x, 0_{S_2}) = (1_{S_1}, 0_{S_2})$. Similarly, we show that $(0_{S_1}, y) = (0_{S_1}, 1_{S_2})$. Therefore, $B(S_1 \times S_2) = \{(0_{S_1}, 0_{S_2}), (1_{S_1}, 1_{S_2}), (1_{S_1}, 0_{S_2}), (0_{S_1}, 1_{S_2})\}$. Again $B(S_1) = \{0_{S_1}, 0_{S_2}\}$ and $B(S_2) = \{0_{S_2}, 1_{S_2}\}$. Hence, $B(S_1 \times S_2) = B(S_1) \times B(S_2)$. □

Theorem 2.5.10. *If two c-semirings are isomorphic, then their Birkhoff centers are isomorphic.*

Proof. Consider S and T are two c-semirings which are isomorphic. Let $f : S \longrightarrow T$ be an isomorphism. Consider $y \in B(T)$. By the definition of Birkhoff center, there exists an isomorphism $g : T \longrightarrow T_1 \times T_2$ such that $g(y) = (1_{T_1}, 0_{T_2})$, where T_1 is a c-semiring with identity 1_{T_1} and T_2 with zero 0_{T_2}. Since $f : S \longrightarrow T$ and $g : T \longrightarrow T_1 \times T_2$ are isomorphism, $gof : S \longrightarrow T_1 \times T_2$ is an isomorphism. Now $(gof)(x) = g(f(x)) = g(y) = (1_{T_1}, 0_{T_2})$. Thus, $x \in B(S)$. For any $y \in B(T)$, there exists a unique $x \in B(S)$ such that $f(x) = y$. This implies that $|B(S)| = |B(T)|$. The restriction of f to $B(T)$ i.e. $f_{|_{B(S)}} : B(S) \longrightarrow B(T)$ is an isomorphism. Therefore, $B(S) \simeq B(T)$. □

The following example shows that the converse of the above theorem is not true i.e. if $B(S)$ and $B(T)$ are isomorphic then the c-semirings S and T may not be isomorphic.

Example 2.5.11. *Consider the set $S = \{0, 1, a\}$. Define the operations "+" and "." on S by means of the followings two tables :*

+	0	a	1
1	1	1	1
a	a	a	1
0	0	a	1

.	0	a	1
1	0	a	1
a	0	0	a
0	0	0	0

Then $(S, +, \cdot)$ is a c-semiring and $B(S) = \{0, 1\}$.

Consider $T = \{0, 1, x, y\}$. Define the operations "+" and "." on T as follows :

+	0	x	y	1
1	1	1	1	1
x	x	x	y	1
y	y	y	y	1
0	0	x	y	1

.	0	x	y	1
1	0	x	y	1
x	0	0	0	x
y	0	0	0	y
0	0	0	0	0

Then $(T, +, \cdot)$ is a c-semiring and $B(T) = \{0, 1\}$. Thus, $B(S) \simeq B(T)$ although S is not isomorphic to T because $|S| = 3$ and $|T| = 4$.

Now we are trying to find out the class of c-semirings where the converse of the above Theorem 2.5.10 is true.

Remark 2.5.12. *We provide a class of c-semirings where the converse of the above Theorem 2.5.10. is true. Let $S = S_2 \times S_2 \times \cdots \times S_2$ (k-times) and $T = S_2 \times S_2 \times \cdots \times S_2$ (m-times) be two c-semirings such that $B(S) \simeq B(T)$, where S_2 is a c-semiring of order 2. From Theorem 2.5.13, we have $B(S) = S$ and $B(T) = T$. Further from Corollary 2.4.8, it follows that $|B(S)| = 2^k$ and $|B(T)| = 2^m$. Since $|B(S)| = B(T)|$, $k = m$. So, $S = S_2 \times S_2 \times \cdots \times S_2$ (k-times) and $T = S_2 \times S_2 \times \cdots \times S_2$ (k-times). Therefore, $S \simeq T$.*

Theorem 2.5.13. *Let S be a c-semiring. If S is isomorphic to $S_2 \times S_2 \times ... \times S_2$ (n times, $n \in \mathbb{N}$), then $S = B(S)$, where S_2 is a c-semiring of order 2.*

Proof. If two c-semirings are isomorphic then their Birkhoff centers are isomorphic. Since $S \simeq S_2 \times S_2 \times .... \times S_2$, $B(S) \simeq B(S_2 \times S_2 \times .... \times S_2)$. Again since $B(S)$

forms a Boolean lattice, the order of $B(S)$ is 2^n. Since $B(S)$ is a c-subsemiring of S, $B(S_2 \times S_2 \times \ldots\ldots\ldots \times S_2) \subseteq S_2 \times S_2 \times \ldots\ldots\ldots \times S_2$. Then $|B(S_2 \times S_2 \times \ldots\ldots\ldots \times S_2)| \leq |S_2 \times S_2 \times \ldots\ldots\ldots \times S_2| = 2^n$ (since S_2 is a c-semiring of order 2) $\quad (i)$.
$S = S_2 \times S_2 \times \ldots\ldots\ldots \times S_2$ (n times) $= (S_2 \times S_2 \times \ldots\ldots\ldots \times S_2)(n-1$ times$) \times S_2 = S_1 \times S_2$, where $S_1 = (S_2 \times S_2 \times \ldots\ldots\ldots \times S_2)$ $(n-1$ times$)$. Since S_1 and S_2 are two c-semirings, $B(S_1) \times B(S_2) \subseteq B(S_1 \times S_2)$. It follows that $|B(S_1)| \times |B(S_2)| \leq |B(S_1 \times S_2)|$. Therefore, $2^n \leq |B(S_1 \times S_2)| = |B(S_2 \times S_2 \times \ldots\ldots\ldots \times S_2)| \quad (ii)$.

From (i) and (ii), we get $2^n \leq |B(S_2 \times S_2 \times \ldots\ldots\ldots \times S_2)| \leq 2^n$. Consequently, $|B(S_2 \times S_2 \times \ldots\ldots\ldots \times S_2)| = 2^n$. Hence, $S = B(S)$. □

Theorem 2.5.14. *If S is a c-semiring, then $B(S)$ is a subalgebra of $B(E(S))$.*

Proof. Suppose $a \in B(S)$. Then $a \in E(S)$ and there exists a map $f_a : S \longrightarrow S_a$ such that $x \longmapsto (ax, f_a(x))$ is an isomorphism of S onto $aS \times S_a$ and f_a is identity on S_a. Since $a \in B(S)$, there exist two c-semirings S_1 with identity 1_{S_1} and S_2 with zero 0_{S_2} and there is an isomorphism $f : S \longrightarrow S_1 \times S_2$ such that $f(a) = (1_{S_1}, 0_{S_2})$. For any $x \in S$, consider $f(x) = (x_1, x_2)$, where $x_1 \in S_1$ and $x_2 \in S_2$. Define $f_a : S \longrightarrow S_a$ by $f_a(x) = f^{-1}(1_{S_1}, x_2)$. For any $e \in S$, define $\alpha : E(S) \longrightarrow aE(S) \times E(S)_a$ by $\alpha(e) = (ae, f_e(x))$. To prove that α is one to one, suppose for any $x, y \in S$, $\alpha(x) = \alpha(y)$. Then $ax = ay$ and $f_a(x) = f_a(y)$. Now $ax = ay \Longrightarrow f(ax) = f(ay)$(since f is an isomorphism)$\Longrightarrow f(a)f(x) = f(a)f(y) \Longrightarrow (1_{S_1}, 0_{S_2})(x_1, x_2) = (1_{S_1}, 0_{S_2})(y_1, y_2) \Longrightarrow (x_1, 0_{S_2}) = (y_1, 0_{S_2}) \Longrightarrow x_1 = y_1$. Also, $f_a(x) = f_a(y) \Longrightarrow f^{-1}(1_{S_1}, x_2) = f^{-1}(1_{S_1}, y_2) \Longrightarrow x_2 = y_2$. Therefore, $f(x) = (x_1, x_2) = (y_1, y_2) = f(y) \Longrightarrow x_2 = y_2$(since f is an isomorphism). Consequently, α is one to one. To prove that α is onto, suppose $(ac, d) \in aE(S) \times E(S)_a$. Choose $x \in S$ such that $ax = ac$ and $f_a(x) = d = f_a(d)$. Now $ax^2 = (ax)^2 = (ac)^2 = ac = ax$ and $f_a(x^2) = f_a(d^2) = f_a(d) = f_a(x)$ and hence $x^2 = x$ (since f is an one to one) i.e. x is an idempotent. Let $y \in S$. Now, $f_a(xy) = f_a(x)f_a(y) = f_a(d)f_a(y) = f_a(dy) = f_a(yd) = f_a(y)f_a(d) = f_a(yx)$. Therefore, $xy = yx$ for all $y \in S$. Therefore, $x \in E(S) \Longrightarrow \alpha$ is onto. So α is an isomorphism. Thus, $a \in B(E(S))$. Hence, $B(S)$ is a subalgebra of $B(E(S))$. □

Conclusion : In this chapter, we have proved that if two c-semirings are isomorphic, then their Birkhoff centers are isomorphic but the converse of this result is not necessarily true. It will be more interesting to study different types of c-semirings where the converse of this result is also true.

Chapter 3

Almost Idempotent Center of Semirings

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Almost Idempotent Center of Semirings

3.1 Introduction

In 2001, M.K. Sen et al. initially coined [56] the term "k-idempotent" to describe a unique class of elements in a k-regular semiring. However, during subsequent research involving this concept, they recognized that the term "almost idempotent" was a more fitting designation. Consequently, they officially adopted the term "almost idempotent" to refer to these elements. An element $e \in S$ is called an almost idempotent if $e + e^2 = e^2$. Almost idempotent has a significant impact on k-Clifford semirings, as well as left k-Clifford semirings and other notable subclasses within the realm of k-regular semirings. This notion is a proper generalization of multiplicative idempotent elements in the semirings whose additive reduct is a semilattice. We were inspired by the concepts of almost idempotent and the center of semiring, which led us to propose a fresh variation known as the "almost idempotent center" of a semiring. The almost idempotent center of a semiring S consists of almost idempotents of S that are also the elements of usual center $Z(S)$ of S. So, almost idempotent center is the intersection of the set of all idempotent elements and the center of semirings. The objective of this chapter is to explore numerous findings that demonstrate a strong resemblance between the properties of the almost idempotent center in a semiring and those in ring theory. Additionally, within this chapter, we present the concept of an almost idempotent central semiring, expanding on the notion of an almost idempotent semiring. Moreover, we employ the assistance of the almost idempotent center of the semiring to effectively characterize the properties of the almost idempotent central semiring.

3.2 $E_c(S)$ of a Semiring S

This section begins by introducing the concept of the almost idempotent center of a semiring. We present several examples and fundamental findings that are not only beneficial for subsequent results but also for the upcoming sections. To commence, we define the almost idempotent center as follows :

Definition 3.2.1. *Let S be a semiring. A subset $E_c(S)$ of a semiring S is called an almost idempotent center of S which is defined by $E_c(S) = \{a \in S : a + a^2 = a^2 \text{ and } ab = ba \text{ for all } b \in S\}$.*

Example 3.2.2. *Consider $(\mathbb{N}, \oplus, \odot)$ is a semiring with $a \oplus b = min\{a, b\} = a$ as addition on $\mathbb{N}$; where $b > a$ and $a \odot b$ for the usual multiplication. For any $b \in \mathbb{N}$, $a \oplus a^2 = min\{a, aa\} = a$ and $ab = ba$. Then $E_c(\mathbb{N}) = \mathbb{N}$.*

Example 3.2.3. *Suppose that $(\mathbb{N}, \oplus, \odot)$ is a semiring; where $a \oplus b = max\{a, b\}$ and $a \odot b = min\{a, b\}$. In this case, it can be observed that $E_c(\mathbb{N}) = \mathbb{N}$.*

Example 3.2.4. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Therefore, we can conclude that $(S, +, \cdot)$ is a semiring and $E_c(S) = \{0, x, 1\} = S$.

Example 3.2.5. *Consider $S = \{0, 1, 2, 3\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	1	2	3
0	0	1	2	3
1	1	2	3	3
2	2	3	3	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

Then $(S, +, \cdot)$ is a commutative semiring. It is worth noting that $Z(S) = S = \{0, 1, 2, 3\}$. But $E_c(S) = \{0, 2, 3\}$. Hence, we can conclude that $E_c(S) \subsetneq Z(S)$.

Example 3.2.6. *Consider $S = \{0, a, b, c\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	c	c

Consequently, $(S, +, \cdot)$ forms a non-commutative semiring. Moreover, $Z(S) = E_c(S) = \{0, a\}$.

Example 3.2.7. *$\{$A class of finite semiring$\}$: Let n, i be integers such that $2 \leq n$, $0 \leq i < n$, and $B(n, i) = \{0, 1, 2, ..., n-1\}$. We define addition and multiplication in $B(n, i)$ by the following equations (let $m = n - i$) :*

$$x + y = \begin{cases} x + y, & \text{if } x + y \leq n - 1 \\ l, & \text{if } x + y \geq n \text{ ; where } l \equiv (x + y) mod m \text{ and } i \leq l \leq n - 1. \end{cases}$$

$$x \cdot y = \begin{cases} xy, & \text{if } xy \leq n - 1 \\ l, & \text{if } xy \geq n \text{ ; where } l \equiv (xy) mod m \text{ and } i \leq l \leq n - 1. \end{cases}$$

Then the set $B(n, i)$ is a commutative semiring with zero (0) and identity (1) under addition ("+") and multiplication ("·"). In particular, let $n = 4$ and $i = 1$, then we *have* $B(4,1) = \{0,1,2,3\}$. *The operations "+" and "·" on* $B(4,1)$ *by means of the following tables :*

+	0	1	2	3
0	0	1	2	3
1	1	2	3	1
2	2	3	1	2
3	3	1	2	3

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	1	3
3	0	3	3	3

As a result, the set $B(4,1)$ *forms a commutative semiring, leading to the conclusion that* $E_c(B(4,1)) = \{0,3\}$.

Example 3.2.8. *Let* G *be a group and* S *be the set of all subsets of* G. *Let us define "+" and "·" in* S *as :* $A + B = A \cup B$ *and* $A \cdot B = \{ab \mid a \in A, b \in B\}$ *for all* $A, B \in S$. *Then* $(S, +, \cdot)$ *is an additively idempotent semiring where* ϕ *is the zero element and* $\{e\}$ [e *denotes the identity element of the group* G] *is the unity of the semiring* S. *Consider* $G = k_4 = \{e, a, b, c\}$ *(Klein's 4- group).*

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

In this case $S = Z(S) = \{\phi, \{e\}, \{e,a\}, \{e,b\}, \{e,c\}, \{e,a,b\}, \{e,b,c\}, \{e,a,c\}, \{a,b,c\}, \{e,a\,b,c\}\}$. *It is important to note that* $E_c(S) = \{\phi, \{e\}, \{e,a\}, \{e,b\}, \{e,c\}, \{e,a,b\}, \{e,b,c\}, \{e,a,c\}, \{a,b,c\}, \{e,a\,b,c\}\}$. *Therefore,* $E_c(S) \neq Z(S)$.

We now construct an interseting example of almost idempotent center of power set of a semiring. The construction is outlined as follows :

Example 3.2.9. *Let* S *be a commutative semigroup with identity and* $\mathcal{P}(S)$ *be the set of all subsets of* S. *Define addition and multiplication on* $\mathcal{P}(S)$ *by :* $U + V = U \cup V$ *and* $U \cdot V = \{ab : a \in U, b \in V\}$ *for all* $U, V \in \mathcal{P}(S)$. *Then* $(\mathcal{P}(S), +, \cdot)$ *forms a semiring.*

Let A be a subsemigroup of S and $A \in E_c(\mathcal{P}(S))$. Then $A + A^2 = A^2$ and $AB = BA$ for all $B \in \mathcal{P}(S) \Longrightarrow A \cup A^2 = A^2$ and $AB = BA$ for all $B \in P(S) \Longrightarrow A \subseteq A^2$ (i).

Since A is a subsemigroup of S, so $A^2 = A \cdot A \subseteq A$ (ii).

From (i) and (ii), we obtain $A = A^2$. So, $E_c(\mathcal{P}(S)) \subseteq \{A \in \mathcal{P}(S) : A^2 = A\}$. If we take $B \in \{A \in P(S) : A^2 = A\} \Longrightarrow B^2 = B$. Now $B + B^2 = B + B = B \cup B = B = B^2 \Longrightarrow B + B^2 = B^2$. Since S is commutative, so $AB = BA$ for all $A \in \mathcal{P}(S)$. Consequently, $\{A \in \mathcal{P}(S) : A^2 = A\} \subseteq E_c(P(S))$. Therefore, we conclude that $E_c(\mathcal{P}(S)) = \{A \in \mathcal{P}(S) : A^2 = A\}$.

Theorem 3.2.10. *Let S be a semiring with identity. Then $a \in E_c(S)$ if and only if $aI_n \in E_c(M_n^d(S))$; where $M_n^d(S)$ is the set of all $n \times n$ diagonal matrices of the form :*

$$M_n^d(S) = \left\{ \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} : a_{ii} \in S \right\}$$

Proof. Let $a \in E_c(S)$. Then $a + a^2 = a^2$ and $ab = ba$ for all $b \in S$.

Now $aI_n + (aI_n)^2$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} a^2 & 0 & \cdots & 0 \\ 0 & a^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^2 \end{pmatrix}$$

$$= \begin{pmatrix} a + a^2 & 0 & \cdots & 0 \\ 0 & a + a^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a + a^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & 0 & \cdots & 0 \\ 0 & a^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^2 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$$

$= (aI_n)^2$, since $a \in E_c(S)$.

Again $(aI_n)B = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$

$$= \begin{pmatrix} ab_1 & 0 & \cdots & 0 \\ 0 & ab_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ab_n \end{pmatrix}$$

$$= \begin{pmatrix} b_1a & 0 & \cdots & 0 \\ 0 & b_2a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_na \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$$

$= B(aI_n)$, since $a \in E_c(S)$.

Therefore, it follows that $aI_n \in E_c(M_n^d(S))$.

Conversely, let's assume that $aI_n \in E_c(M_n^d(S))$. Our task now is to demonstrate that $a \in E_c(S)$. Let $x \in S$. Now consider B $= \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}$. Due to the fact that $aI_n \in E_c(M_n(S))$, we can determine that $aI_n + (aI_n)^2 = (aI_n)^2$ and $(aI_n)B = B(aI_n)$. Comparing both sides, it becomes evident that $a + a^2 = a^2$ and $ax = xa$. Consequently, it can be deduced that $a \in E_c(S)$. □

We adapt some of basic results to almost idempotent center of semirings. We will now introduce a lemma that will be applicable in the subsequent theorem.

Lemma 3.2.11. *If S is a semiring with zero element 0, then $0 \in E_c(S)$.*

Proof. Considering an element $a \in E_c(S)$, we observe that $a + a^2 = a^2$ and $ax = xa$ hold for all $x \in S$. Furthermore, for any $a \in S$, we have $0+0^2 = 0^2$ and $a \cdot 0 = 0 = a \cdot 0$. As a result, we can conclude that $0 \in E_c(S)$. □

Theorem 3.2.12. *The almost idempotent center of a semiring S forms a subsemiring of S.*

Proof. By applying Lemma 3.2.11, it follows that if S is a semiring with a zero element 0, then we can deduce that $0 \in E_c(S)$. So, the almost idempotent center $E_c(S)$ of a semiring S is non-empty. Let S be a semiring and $a, b \in E_c(S)$. Our objective is to prove that $a + b \in E_c(S)$ and $ab \in E_c(S)$. Since $a \in E_c(S)$, $a + a^2 = a^2$ and $ab = ba$ for all $b \in S$ and $b \in E_c(S)$, $b + b^2 = b^2$ and $ba = ab$ for all $a \in S$. Now $(a+b)+(a+b)^2 = a+b+(a+b)(a+b) = a+b+a^2+ab+ba+b^2 = a+a^2+b+b^2+ab+ba = a^2+b^2+ab+ba = a^2+ab+b^2+ba = a(a+b)+b(b+a) = (a+b)^2$. Hence, for every $x \in S$, $(a+b)x = ax+bx = xa+xb = x(a+b)$. Therefore, $a+b \in E_c(S)$.

Again $(ab)^2 = (ab)(ab) = a(ba)b = aabb = a^2b^2 = (a+a^2)(b+b^2) = ab + ab^2 + a^2b + a^2b^2 = ab + a^2b^2 + a^2b$ (since $a + a^2 = a^2 \Rightarrow ab^2 + a^2b^2 = a^2b^2$)$= ab + a^2b^2$ (since $b + b^2 = b^2 \Rightarrow a^2b + a^2b^2 = a^2b^2$)$= ab + (ab)^2$. For any $x \in S$, $(ab)x = abx = axb = x(ab)$. Hence, it follows that $ab \in E_c(S)$. Consequently, we can conclude that $E_c(S)$ is a subsemiring of S. □

Remark 3.2.13. *The Example 3.2.7 shows that the almost idempotent center of a semiring S is not an ideal of S, as evidenced by the fact that $2 \cdot 3 = 2 \notin E_c(B(4,1)) = \{0,3\}$.*

Proposition 3.2.14. *If S is an additively cancellation semiring, then $E_c(S) = \{0\}$.*

Proof. Let $a \in E_c(S)$. Then $a + a^2 = a^2$ and $ab = ba$ for all $b \in S$. If S is an additively cancellation semiring, then $a = \{0\}$. Since a is an arbitrary element of $E_c(S)$, it follows that $E_c(S) = \{0\}$. □

3.3 The Structural Properties of $E_c(S)$

In this section, we delve into the fundamental structural properties of $E_c(S)$ for a semiring S that are typically assumed across conventional semirings.

Now, we observe the following outcome :

Theorem 3.3.1. *If φ is an epimorphism from S to S', then $\varphi(E_c(S)) \subseteq E_c(S')$.*

Proof. Suppose S is a semiring and $\varphi : S \longrightarrow S'$ is an epimorphism. Let $\varphi(E_c(S)) = \{\varphi(s) : s \in E_c(S)\}$. Our goal is to prove that $\varphi(E_c(S)) \subseteq E_c(S')$. Let $\varphi(e) \in \varphi(E_c(S))$ and consider $x \in S'$. Given that φ is a surjective function, we can conclude that there exists an element y in set S for which there exists a preimage x in set S' such that $\varphi(y) = x$. Furthermore, since $e \in E_c(S)$, we can observe that $e+e^2 = e^2$ and $ey = ye$. Now for any $x \in S'$, $\varphi(e)+\varphi(e)^2 = \varphi(e)+\varphi(e^2) = \varphi(e+e^2) = \varphi(e^2) = \varphi(e)^2$ and $\varphi(e)x = \varphi(e)\varphi(y) = \varphi(ey) = \varphi(ye) = \varphi(y)\varphi(e) = x\varphi(e)$, utilizing the fact that φ is an epimorphism. As a consequence, we can conclude that $\varphi(e) \in E_c(S')$. Therefore, we can deduce that $\varphi(E_c(S)) \subseteq E_c(S')$. □

Theorem 3.3.2. *Let S and S' be two semiring. If $f : S \to S'$ is a monomorphism, then $f(E_c(S)) = E_c(f(S))$.*

Proof. Let's assume $x \in f(E_c(S))$. This means that there exists some y in $E_c(S)$ such that $x = f(y)$. Our goal is to prove two statements: $f(y) + f(y)^2 = f(y)^2$ and $f(y)s = sf(y)$ for all s in $f(S)$.

For any s in $f(S)$, we have $f(y) + f(y)^2 = f(y) + f(y)f(y) = f(y) + f(yy) = f(y + y^2) = f(y^2)$. Similarly, $f(y)s = f(y)f(r) = f(yr) = f(ry) = f(r)f(y) = sf(y)$. Thus, we have shown that $x = f(y) \in E_c(f(S))$. Therefore, $f(E_c(S)) \subseteq E_c(f(S))$.

Now, let's consider x' in $E_c(f(S))$. This implies that $x' = f(r')$ for some r' in S. We need to prove that $r' \in E_c(S)$. Since $x' \in E_c(f(S))$, we can deduce that for any $f(s) \in f(S)$, $x' + x'^2 = x'^2 \implies f(r') + f(r')^2 = f(r')^2 \implies f(r' + r'^2) = f(r'^2) \implies r' + r'^2 = r'^2$, since f is a monomorphism and $x'f(s) = f(s)x' \implies f(r')f(s) = f(s)f(r') \implies f(r's) = f(sr')$. Applying the monomorphism property of f, we get $r's = sr'$. Consequently, we can conclude that $r' \in E_c(S)$. Thus, we have proven that $E_c(f(S)) \subseteq f(E_c(S))$.

In summary, we have shown that $E_c(f(S)) = f(E_c(S))$. □

Theorem 3.3.3. *If two semirings S_1 and S_2 are isomorphic, then their almost idempotent centers $E_c(S_1)$ and $E_c(S_2)$ are isomorphic.*

Proof. Consider two semirings S_1 and S_2 which are isomorphic. Then there is an isomorphism $f : S_1 \longrightarrow S_2$. Let $x \in E_c(S_1)$. Then for any $s_1 \in S_1$, $x + x^2 = x^2$ and $xs_1 = s_1x$. Let $f(x) = y$; where $y \in S_2$. Since f is an isomorphism, for any $s_2 \in S_2$, there exists $s_1 \in S_1$ such that $f(s_1) = s_2$. Thus, $y + y^2 = f(x) + f(x)^2 = f(x) + f(x^2) = f(x + x^2) = f(x^2) = f(x)^2 = y^2$ and $ys_2 = f(x)f(s_1) = f(xs_1) = f(s_1x) = f(s_1)f(x) = s_2y$, since $x \in E_c(S_1)$. Hence, we can conclude that $y \in E_c(S_2)$. This implies that $f(E_c(S_1)) \subseteq E_c(S_2)$. Again, let $b \in E_c(S_2)$. Then $b = f(a)$; where $a \in S_1$. Since f is an isomorphism, for any $y \in S_2$, there exists $x \in S_1$ such that $y = f(x)$. Since $b \in E_c(S_2)$, we have $b + b^2 = b^2$ and $by = yb$. Consequently, we can deduce that $b + b^2 = b^2 \implies f(a) + f(a)^2 = f(a)^2 \implies f(a) + f(a)f(a) = f(a)f(a) \implies f(a) + f(a^2) = f(a^2) \implies f(a + a^2) = f(a^2) \implies a + a^2 = a^2$ and $by = yb \implies f(a)f(x) = f(x)f(a) \implies f(ax) = f(xa) \implies ax = xa$, since f is an isomorphism. Therefore, we conclude that $a \in E_c(S_1)$. Consequently, $b = f(a) \in f(E_c(S_1))$. Thus, $E_c(S_2)) \subseteq f(E_c(S_1))$ and hence $E_c(S_2) = f(E_c(S_1))$. As a result, $g = f_{|_{E_c(S_1)}} : E_c(S_1) \longrightarrow E_c(S_2)$ is well defined and it is an isomorphism from $E_c(S_1)$ onto $E_c(S_2)$. □

In general, it is important to note that the converse of the above Theorem 3.3.3 does not hold true. This can be illustrated by the following example :

Example 3.3.4. *$E_c(\mathbb{Z}_0^+) = \{0\}$ and $E_c(\mathbb{R}_0^+) = \{0\}$. However, it should be noted that $\mathbb{Z}_0^+$ and $\mathbb{R}_0^+$ are not isomorphic.*

Theorem 3.3.5. *If S_1 and S_2 are two semirings, then $E_c(S_1 \times S_2) = E_c(S_1) \times E_c(S_2)$.*

Proof. Let S_1 and S_2 be two semirings. Assume $e = (e_1, e_2) \in E_c(S_1 \times S_2)$. Thus, we have $e + e^2 = e^2$ and $ex = xe$ for all $x = (x_1, x_2) \in S_1 \times S_2$. By expanding the equations, we obtain $(e_1, e_2) + (e_1, e_2)^2 = (e_1, e_2)^2 \Longrightarrow (e_1, e_2) + (e_1^2, e_2^2) = (e_1^2, e_2^2) \Longrightarrow (e_1+e_1^2, e_2+e_2^2) = (e_1^2, e_2^2)$. Consequently, we conclude that $e_1+e_1^2 = e_1^2$ and $e_2+e_2^2 = e_2^2$. Furthermore, we observe that $ex = xe$ for all $x = (x_1, x_2) \in S_1 \times S_2$. This implies that $(e_1, e_2)(x_1, x_2) = (x_1, x_2)(e_1, e_2) \Longrightarrow (e_1x_1, e_2x_2) = (x_1e_1, x_2e_2)$, which leads to $e_1x_1 = x_1e_1$ for all $x_1 \in S_1$ and $e_2x_2 = x_2e_2$ for all $x_2 \in S$. Hence, we can conclude that e_1, e_2 are elements in $E_c(S_1)$ and $E_c(S_2)$ respectively. Consequently, $e = (e_1, e_2) \in E_c(S_1) \times E_c(S_2)$. This implies that $E_c(S_1 \times S_2) \subseteq E_c(S_1) \times E_c(S_2)$ (i).

For reverse part, let $e = (e_1, e_2) \in E_c(S_1) \times E_c(S_2)$. Therefore, we have $e_1 \in E_c(S_1)$ and $e_2 \in E_c(S_2)$. Then $e_1 + e_1^2 = e_1^2$ and $e_1x_1 = x_1e_1$ for all $x_1 \in S_1$ and $e_2 + e_2^2 = e_2^2$ and $e_2x_2 = x_2e_2$ for all $x_2 \in S_2$. Now, $e + e^2 = (e_1, e_2) + (e_1, e_2)^2 = (e_1, e_2) + (e_1^2, e_2^2) = (e_1 + e_1^2, e_2 + e_2^2) = (e_1^2, e_2^2) = (e_1, e_2)^2 = e^2$, since $e_1 \in E_c(S_1)$ and $e_2 \in E_c(S_2)$. Again $ex = (e_1, e_2)(x_1, x_2) = (e_1x_1, e_2x_2) = (x_1e_1, x_2e_2) = (x_1, x_2)(e_1, e_2) = xe$ for all $x = (x_1, x_2) \in S_1 \times S_2$. This implies that $e \in E_c(S_1 \times S_2)$ and consequently, we find that $E_c(S_1) \times E_c(S_2) \subseteq E_c(S_1 \times S_2)$ (ii).

From (i) and (ii), it follows that $E(S_1 \times S_2) = E_c(S_1) \times E_c(S_2)$. □

With the application of Theorem 3.3.5, we construct an example of an almost idempotent center of semiring.

Example 3.3.6. *Consider two semiring $(\mathbb{N}, \oplus, \odot)$; where $a \oplus b = min\{a, b\}$ and $\odot$ is usual multiplication and $(\mathbb{R}_0^+, +, \cdot)$; where "$+$" is usual addition and "$\cdot$" is usual multiplication. Now we take a semiring $(\mathbb{N} \times \mathbb{R}_0^+, +, \cdot)$ with component-wise addition and multiplication. Then $E_c(\mathbb{N} \times \mathbb{R}_0^+) = \mathbb{N} \times \{0\}$.*

3.4 Lattice Structure of $E_c(S)$

The objective of this section is to establish the lattice structure for $E_c(S)$. To achieve this, we start by considering the following example :

Example 3.4.1. *Consider the ring $\mathbb{Z}_{12}$ and let $\Omega_{\mathbb{Z}_{12}}$ be the set of all ideals of $\mathbb{Z}_{12}$. Specifically, $\Omega_{\mathbb{Z}_{12}} = \{<0>, <1>, <2>, <3>, <4>, <6>\}$. Define $\oplus$ and $\odot$ by $I_1 \oplus I_2 = \{a_1 + b_1 : a_1 \in I_1,\ b_1 \in I_2\}$ and $I_1 \odot I_2 = \left\{\sum_{i=1}^{n} a_i b_i : a_1 \in I_1,\ b_1 \in I_2\right\}$ for all $I_1, I_2 \in \Omega_{\mathbb{Z}_{12}}$; where $n \in \mathbb{N}$. We can observe that $(\Omega_{\mathbb{Z}_{12}}, \oplus, \odot)$ forms a semiring. Here, $E_c(\Omega_{\mathbb{Z}_{12}}) = \{<0>, <1>, <3>, <4>\} \neq \{0\}$. This particular example serves as a significant non-trivial illustration of an almost idempotent center of a semiring that contains an additive absorbing element denoted by 1.*

Theorem 3.4.2. *If S is a semiring with an additive absorbing element 1, then $E_c(S)$ is a multiplicatively band.*

Proof. Given that 1 is an additive absorbing element of S, so $1 + a = 1$ for all $a \in S$. Let $a \in E_c(S)$. Then $a + a^2 = a^2$, which can be rewritten as $a(1 + a) = a^2$. Simplifying further, we have $a \cdot 1 = a^2$, leading to the conclusion that $a = a^2$. Thus, for all $a \in E_c(S)$, we find that $a^2 = a$. Consequently, $E_c(S)$ can be regarded as a multiplicatively band. □

We would like to introduce a lemma that will assist us in constructing the lattice structure for $E_c(S)$.

Lemma 3.4.3. *Let S be a semiring with an additive absorbing element 1. Define a binary relation "$\leq_S$" on $E_c(S)$ by "$a \leq_S b$" if and only if $ab = a$ for all a, b in $E_c(S)$. Then $(E_c(S), \leq_S)$ forms a partial ordered set.*

Proof. For any elements $a, b \in E_c(S)$, we have $a \leq_S b \Longleftrightarrow ab = a$. To demonstrate this, let's take an element a from $E_c(S)$. Since S is a semiring with an additive absorbing identity element 1, according to Theorem 3.4.2, we can conclude that $a^2 = a$ holds true for all $a \in E_c(S)$. Consequently, we find that $a \cdot a = a^2 = a$. This observation implies that $a \leq_S a$. Therefore, the relation "$\leq_S$" is reflexive. Consider elements a and b from $E_c(S)$ such that $a \leq_S b$ and $b \leq_S a$. This implies that $ab = a$ and $ba = b$ for $a, b \in E_c(S)$. Since $a, b \in E_c(S)$, we can deduce that $ab = ba$. From this, we can conclude that $a = b$. Hence, the relation "$\leq_S$" is antisymmetric.

Now, let's assume $a \leq_S b$ and $b \leq_S c$ hold for $a, b, c \in E_c(S)$. We know that $a \leq_S b$ implies $ab = a$ and $b \leq_S c$ implies $bc = b$. By substituting $bc = b$ in $ab = a$, we get $a(bc) = a$. Simplifying this further, we obtain $(ab)c = a$. Consequently, we have $ac = a$, since $ab = a$. Therefore, we can conclude that $a \leq_S c$ for all $a, c \in E_c(S)$. Hence, the relation "$\leq_S$" is transitive. Consequently, "$\leq_S$" forms a partial order relation on $E_c(S)$ and therefore, $(E_c(S), \leq_S)$ constitutes a partially ordered set. □

Now, we unveil the core outcome of this section.

Theorem 3.4.4. *If S is a semiring with an additive absorbing element 1, then $E_c(S)$ forms a lattice.*

Proof. Suppose S is a semiring with an additive absorbing element 1. Define a binary relation denoted as "$\leq_S$" on $E_c(S)$ as follows : $a \leq_S b$ if and only if $ab = a$ for all $a, b \in E_c(S)$. By Lemma 3.4.3, it is ensured that $(E_c(S), \leq_S)$ forms a partially ordered set. Let's consider the expression $abb = ab^2 = ab$, since $b \in E_c(S)$ and $E_c(S)$ is multiplicatively band. This implies that $ab \leq_S b$ for all $a, b \in E_c(S)$. Additionally, we have $aba = (aa)b$ (since $a \in E_c(S)$) $= a^2b = ab$, given that $a \in E_c(S)$ and $E_c(S)$ is multiplicatively band. Hence, we can conclude that $ab \leq_S a$ for all $a, b \in E_c(S)$. Therefore, ab serves as a lower bound for both a and b.

We will now demonstrate that ab is the greatest lower bound of a and b. Suppose g is another lower bound of a and b. This means that $g \leq_S a$ holds true if and only if $ga = g$ for all $g, a \in E_c(S)$ and similarly, $g \leq_S b$ if and only if $gb = g$ for all $g, b \in E_c(S)$. Now, we observe that $gab = (ga)\ b = gb$ (since $ga = g$) $= g$, as $gb = g$. Therefore, we can conclude that $g \leq_S ab$ for all $a, b \in E_c(S)$. Consequently, ab is the greatest lower bound of a and b.

We now determine the least upper bound of both a and b in $E_c(S)$. By utilizing the fact that $E_c(S)$ is a multiplicatively band with the property $a\ (a+b) = a^2+ab = a+ab$ (since $a \in E_c(S)$), we can simplify this further to $a\ (1 + b) = a1 = a$, as 1 acts as an additive absorbing element in S. Based on this, we can conclude that $a \leq_S a+b$ for any $a, b \in E_c(S)$. Therefore, $a+b$ serves as an upper bound for both a and b.

Next, we aim to demonstrate that $a+b$ is the least upper bound of a and b. Let's assume that $l \in E_c(S)$ is another upper bound of a and b. Consequently, $a \leq_S l$ implies $al = a$ for all $a, l \in E_c(S)$ and $b \leq_S l$ implies $bl = b$ for all $b, l \in E_c(S)$. By applying these conditions, we find that $(a+b)\, l = al + bl = a + b$ (due to $al = a$ and $bl = b$). Therefore, $a + b \leq_S l$ holds true for all $a, b \in E_c(S)$. Consequently, we can confirm that $a+b$ is indeed the least upper bound of a and b. As a result, we have demonstrated that $E_c(S)$ forms a lattice. □

Our focus now shifts to exploring the different lattice structures that are associated with $E_c(S)$.

Theorem 3.4.5. *If S is a semiring that has an additive absorbing element 1, then $E_c(S)$ constitutes a distributive lattice.*

Proof. Given that S is a semiring with an additive absorbing element 1, it is known that $E_c(S)$ forms a lattice. For any two elements $a, b \in E_c(S)$, we have $a \vee b = a + b$ and $a \wedge b = ab$. Let $a, b, c \in E_c(S)$. Thus, we can deduce that $a \wedge (b \vee c) = a \wedge (b + c)$, which follows from the fact that $b \vee c = b + c$. This can be further simplified as $a(b + c)$, utilizing the property that $a \wedge (b + c) = a(b + c)$. Consequently, we obtain $ab + ac = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in E_c(S)$. As a result, it can be concluded that $E_c(S)$ constitutes a distributive lattice. □

Remark 3.4.6. *A distributive lattice is always modular lattice. Consequently, if S is an additive absorbing element 1, then $E_c(S)$ forms a modular lattice.*

Remark 3.4.7. *If S is a finite semiring with an additive absorbing element 1, then $E_c(S)$ forms a complete lattice.*

We know that every finite set has supremum. As a result, if S is a finite set, then every subset of $E_c(S)$ possesses its supremum within $E_c(S)$. Correspondingly, any subset of $E_c(S)$ also has its infimum within $E_c(S)$. Thus, we can assert that $E_c(S)$ forms a complete lattice.

Theorem 3.4.8. *If S is a finite semiring with a zero element 0 and an additive absorbing element 1 which is also an identity element, then $E_c(S)$ forms a bounded lattice.*

Proof. In the case of a finite semiring S with an additive absorbing element 1, it is known that the set $E_c(S)$ forms a lattice. Considering an element $a \in E_c(S)$, we observe that $a \cdot 1 = a$ due to the identity property of 1 and $a \cdot 0 = 0$, since 0 is the zero element. Consequently, we can establish that $a \leq 1$ and $0 \leq a$ for all $a \in E_c(S)$. As a result, 1 acts as the greatest element of $E_c(S)$ while 0 acts as the least element. Hence, we conclude that $E_c(S)$ constitutes a bounded lattice. □

3.5 The Structure of Almost Idempotent Central Semirings

Within this section, we introduce almost idempotent central semiring as a generalization of almost idempotent semiring was previously introduced by M.K. Sen et al. [57]. We begin by outlining a definition and subsequently explore various properties associated with this semiring.

Definition 3.5.1. *A semiring S is said to be almost idempotent central semiring if $E_c(S) = S$.*

We now provide several instances of almost idempotent central semiring.

Example 3.5.2. *Consider the set of integers $\mathbb{Z}^+$ with the operations $a+b = lcm\{a, b\}$ and $a \cdot b = ab$. Then $(\mathbb{Z}^+, +, \cdot)$ is a semiring with zero element 1. Then $E_c(\mathbb{Z}^+) = \mathbb{Z}^+$. Therefore, $(\mathbb{Z}^+, +, \cdot)$ is an almost idempotent central semiring.*

Example 3.5.3. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	1
1	0	1	1

Then $(S, +, \cdot)$ *is a semiring and* $E_c(S) = \{0, x, 1\} = S$. *Therefore,* $(S, +, \cdot)$ *is an almost idempotent central semiring.*

Example 3.5.4. *Consider* $S = \{0, x, y, 1\}$. *Define the operations "+" and "." on* S *by means of the following tables :*

+	0	x	y	1
0	0	x	y	1
x	x	x	y	x
y	y	y	y	y
1	1	x	y	1

.	0	x	y	1
0	0	0	0	0
x	0	x	x	x
y	0	x	y	y
1	0	x	y	1

Then $(S, +, \cdot)$ *is a semiring and* $E_c(S) = \{0, x, y, 1\} = S$. *Therefore,* $(S, +, \cdot)$ *is an almost idempotent central semiring.*

Remark 3.5.5. *If* S *is an almost idempotent central semiring, then* $E_c(S) = Z(S)$.

Our focus now shifts to identifying certain conditions under which a semiring can be classified as an almost idempotent central semiring.

Theorem 3.5.6. *Every commutative idempotent semiring is an almost idempotent central semiring.*

Proof. Let S be an idempotent semiring. Then $a + a = a$ and $a^2 = a$ for all $a \in S$. Clearly, $E_c(S) \subseteq S$. Take an arbitrary non-zero element a in S. For any $a \in S$, we have $a + a^2 = a^2 + a^2 = a^2$, as S is an idempotent semiring. Furthermore, since S is a commutative semiring, satisfying $ab = ba$ for all $a, b \in S$, it follows that $a(\neq 0) \in E_c(S)$. Additionally, 0 is also an element of $E_c(S)$. Thus, we can conclude that $S \subseteq E_c(S)$ and therefore, $S = E_c(S)$. Consequently, S is an an almost idempotent central semiring. □

Theorem 3.5.7. *Every additively idempotent mono-semiring is an almost idempotent central semiring.*

Proof. Assume S to be an additively idempotent semiring. For any $a \in S$, if $a + a = a$, then we can deduce that $a^2 = a$, utilizing the fact that S is a mono-semiring. Consequently, S is a multiplicatively band. It is evident that $E_c(S) \subseteq S$. Let's choose an arbitrary non-zero element a from S. We observe that for any a in S, if $a + a = a$, then we find that $a + a^2 = a^2$, utilizing the fact that S is a multiplicatively band. Additionally, considering any elements a and b from S, we find that $ab = a+b = b+a = ba$, since S is a mono-semiring. Hence, $a(\neq 0) \in E_c(S)$. Additionally, we note that 0 is an element of $E_c(S)$. Consequently, we can conclude that $S \subseteq E_c(S)$ and as a result, $S = E_c(S)$. This establishes S as an almost idempotent central semiring. □

Theorem 3.5.8. *If S is a mono-semiring with identity element 1 and $(S, \cdot)$ is a rectangular band, then S is an almost idempotent central semiring.*

Proof. Given that $E_c(S)$ is a subsemiring of S, it can be deduced that $E_c(S) \subseteq S$. Assume $a(\neq 0)$ to be any arbitrary element within S. As $(S, \cdot)$ is a rectangular band, for any $a, b \in S$, we have $aba = a$. In particular, $a1a = a$, which implies $a^2 = a$. Since S is a mono-semiring, we can conclude that $a+a = a$. Therefore, we have $a+a^2 = a^2$ for all a in S. Moreover, the commutative property $ab = a+b = b+a = ba$ is satisfied by all elements a and b in S, as S is a mono-semiring. Consequently, $a(\neq 0) \in E_c(S)$. Furthermore, $0 \in E_c(S)$. Therefore, we can establish that $S \subseteq E_c(S)$, leading to the conclusion that $S = E_c(S)$. As a result, S is an almost idempotent central semiring. □

In the following, we outline a characterization theorem pertaining to almost idempotent central semiring.

Theorem 3.5.9. *Let S be a multiplicatively subidempotent semiring. If S forms a semilattice with respect to multiplication if and only if S is an almost idempotent central semiring.*

Proof. Let's establish the necessity. It is evident that $E_c(S) \subseteq S$. Take an arbitrary element $a(\neq 0)$ from S. Since S forms a semilattice with respect to multiplication, we have $a = a^2$ and $ab = ba$ for all $a, b \in S$. Furthermore, due to S being a multiplicatively subidempotent semiring, we can deduce that for any $a \in S$, $a + a^2 = a \implies a + a^2 = a^2$, as S is also multiplicatively band. Additionally, the property $ab = ba$ holds for all $a, b \in S$, as S is a semilattice with respect to multiplication. Consequently, $a(\neq 0) \in E_c(S)$. Furthermore, we have $0 \in E_c(S)$. Thus, we can conclude that $S \subseteq E_c(S)$, which implies $S = E_c(S)$. As a result, S represents an almost idempotent central semiring.

Now, let's prove the converse. Suppose S is an almost idempotent central semiring, meaning $E_c(S) = S$. Consider an element $a \in E_c(S)$. Then we have $a + a^2 = a^2$ (i).

Moreover, since S is a multiplicatively subidempotent semiring, we can also deduce $a + a^2 = a$ (ii).

Combining equations (i) and (ii), we conclude that $a = a^2$. Additionally, since $a \in E_c(S)$, we have $ab = ba$ for all $a, b \in S$. As a result, S is commutative and multiplicative band. Hence, S forms a semilattice with respect to multiplication. □

Theorem 3.5.10. *If S is a mono-semiring, then any almost idempotent central semiring can be considered as a k-regular semiring.*

Proof. Suppose S is an almost idempotent central semiring. We can deduce that $E_c(S) = S$. Additionally, due to S being a mono-semiring, we have $a + b = ab$ for all $a, b \in S$. By multiplying both sides of this equation by 'a', we derive $a^2 + ba = aba$. Further simplification yields $a + a^2 + ba = a + aba$, which can be further reduced to $a^2 + ba = a + aba$, as S is an almost idempotent semiring. Using the fact that S is a mono-semiring, we can conclude that $(a + b)a = a + aba$, resulting in $aba = a + aba$. As a consequence, S is k-regular semiring. Consequently, $E_c(S)$ is a k-regular semiring. □

The subsequent pair of examples illustrates that the converse of the Theorem 3.5.10 is not true i.e. every k-regular semiring is not almost idempotent central semiring.

Example 3.5.11. *The semiring $(\mathbb{Z}_0^+, +, \cdot)$ is k-regular; where "+" and "$\cdot$" are usual addition and multiplication respectively. In this case $E_c(\mathbb{Z}_0^+) = \{0\} \neq S$. As a result, it is evident that $(\mathbb{Z}_0^+, +, \cdot)$ is not almost idempotent central semiring.*

Example 3.5.12. *Consider $S = \{0, x, 1\}$. Define the operations "+" and "." on S_1 by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	1	1

Therefore, $(S, +, \cdot)$ is a k-regular semiring. However, $E_c(S) = \{0\} \neq S$. So, $(S, +, \cdot)$ is not an almost idempotent central semiring.

Theorem 3.5.13. *Let S be an almost idempotent central semiring. If S is an additively regular semiring, then S is an additively idempotent semiring.*

Proof. Given that S is an almost idempotent central semiring, we can deduce that $S = E_c(S)$. So, $a + a^2 = a^2$ and $ab = ba$ for all $a, b \in S$. Multiplying both sides by 'a', we obtain $a + a^2 + a = a + a^2$ (i).

Additionally, S is an additively regular semiring, we can assert that $a + a^2 + a = a$. Therefore, the equation (i) simplifies to $a = a + a^2$. By adding 'a' on both sides of this equation, we get $a + a = a + a^2 + a$ for all $a \in S$. From this, we can deduce that $a + a = a$ for all $a \in S$. Consequently, we conclude that S is an additively idempotent semiring. □

The subsequent results yield the different properties of almost idempotent central semirings.

Proposition 3.5.14. *Suppose S is an almost idempotent central semiring. Then the following statements are true.*

(a) *If $(S, \cdot)$ is a band, then $(S, +)$ is a band.*

(b) *If $(S, +)$ is a band and possesses cancellative properties, then $(S, \cdot)$ is a band.*

Proof. (a) Assume that S is an almost idempotent central semiring i.e. $S = E_c(S)$. Let $a \in E_c(S) = S$. For any $a \in S$, $a + a^2 = a^2$ (i).

As $(S, \cdot)$ is a band, we can conclude that $a^2 = a$ for all $a \in S$. Based on equation (i), we can deduce that $a + a = a$ for all $a \in S$. Hence, $(S, +)$ is a band.

(b) Consider $(S, +)$ to be a band. Given that S is an almost idempotent central semiring, we can deduce that $S = E_c(S)$. Let $a \in E_c(S)$. For any $a \in S$, we have $a + a^2 = a^2$ (i).

Since $(S, +)$ is a band, we can conclude that $a^2 + a^2 = a^2$ for all $a^2 \in S$ (ii).

Combining equations (i) and (ii), we can see that $a + a^2 = a^2 + a^2$. Applying the cancellative law, we obtain $a = a^2$ for all $a \in S$. Thus, we can conclude that $(S, \cdot)$ is a band. □

Theorem 3.5.15. *Let S be an almost idempotent central semiring. If S is a zero square semiring, then $S = \{0\}$.*

Proof. Suppose S is an almost idempotent central semiring i.e. $E_c(S) = S$. Let $a \in E_c(S)$. We have $a + a^2 = a^2$, which implies $a + 0 = 0$, since S is a zero square semiring. As 'a' is chosen arbitrarily from S, we can conclude that $S = \{0\}$. □

Theorem 3.5.16. *If S is an almost idempotent central semiring with identity element 1 and $(S, \cdot)$ is cancellative, then the equation $b + ab = ab$ holds true for all elements a and b in S.*

Proof. Assuming S is an almost idempotent central semiring, represented by $E_c(S) = S$. Let $a \in E_c(S)$. Consequently, for any $a \in S$, we have $a + a^2 = a^2$, which implies $a(1 + a) = a \cdot a$. By the cancellative law, we obtain $1 + a = a$ (i).
Subsequently, multiplying both sides of equation (i) by 'b', we obtain $(1 + a)b = ab$, which can be rearranged to yield $b + ab = ab$ for all $a, b \in S$. □

Theorem 3.5.17. *Every simple and almost idempotent central semiring is also a multiplicatively idempotent semiring.*

Proof. Suppose S is an almost idempotent central semiring, denoted by $E_c(S) = S$. Consider an element a that belongs to $E_c(S) = S$. Then for any $a \in S$, we have

$a + a^2 = a^2$. By simplifying the equation using the fact that S is a simple semiring, we can deduce $a(1+a) = a^2$, which can be further reduced to $a \cdot 1 = a^2$. Consequently, we arrive at the conclusion that $a^2 = a$, thereby establishing S as a multiplicatively idempotent semiring. $\square$

Our next target is to show that the class of all almost idempotent central semirings is a variety. In order to accomplish this, we commence by establishing the subsequent Lemma.

Lemma 3.5.18. *Let S be an almost idempotent central semiring and S' be an subsemiring of S. Then S' is an almost idempotent central semiring.*

Proof. Let S be an almost idempotent central semiring, denoted as $E_c(S) = S$. Our objective is to show that S' is an almost idempotent central semiring, denoted as $E_c(S') = S'$. Suppose $a \in S' \subseteq S = E_c(S)$. Consequently, $a + a^2 = a^2$ and $ab = ba$ for all $b \in S$. As S' is a subsemiring of S, we can infer that $a + a^2 = a^2$ and $ab = ba$ for all $b \in S'$. Hence, $a \in E_c(S')$, indicating that $S' \subseteq E_c(S')$. Furthermore, since $E_c(S')$ serves as a subsemiring of S', it follows that $E_c(S') \subseteq S'$. Thus, we conclude that $E_c(S') = S'$, affirming that S' is an almost idempotent central semiring. $\square$

Lemma 3.5.19. *Every homomorphic image of an almost idempotent central semiring is also an almost idempotent central semiring*

Proof. Let S be an almost idempotent central semiring with identity 1_S and S' be an almost idempotent central semiring with identity $1_{S'}$. Let $f : S \to S'$ be onto homomorphism. Then S' is the homomorphic image of the the almost idempotent semiring S. We have to show that S' is an almost idempotent semiring. Since f is an onto homomorphism, so $S' = \{f(a) : a \in S\}$. Furthermore, $f(1_S) = 1_{S'}$ acts as the identity element of S'. Let $s' \in S'$. Then there exists $a \in S$ such that $f(a) = s'$. As S is an almost idempotent semiring, we know that $a + a^2 = a^2$ and $ab = ba$ for all $a, b \in S$. So, for any $f(b) \in S'$, we have $s' + s'^2 = f(a) + f(a)^2 = f(a) + f(a)f(a) = f(a) + f(aa) = f(a) + f(a^2) = f(a + a^2) = f(a^2) = f(a)^2 = s'^2$ and $s'f(b) = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = f(b)s'$. Therefore, we conclude that $s' = f(a) \in E_c(S') = S'$. Since s' is an arbitrary element of S', it follows that S' is an almost idempotent central semiring. □

Lemma 3.5.20. *Let $\{S_i : i = 1, 2, ..., n\}$ be a finite family of semirings. Then the direct product of semirings $S = \prod_{i=1}^{n} S_i$ is almost idempotent central semiring if and only if each semiring S_i is almost idempotent central semiring.*

Proof. Let's consider a family of semirings $\{S_i : i = 1, 2, ..., n\}$; where each semiring S_i is almost idempotent central semiring. Now, suppose we have an element $(x_1, x_2, ..., x_n) \in S$, where each $x_i \in S_i$. Since each S_i is almost idempotent central semiring, for any $y_i \in S_i$, we have $x_i + x_i^2 = x_i^2$ and $x_i y_i = y_i x_i$ for all $i = 1, 2, ..., n$. Consequently, it follows that $(x_1, x_2, ..., x_n) + (x_1, x_2, ..., x_n)^2$
$= (x_1, x_2, ..., x_n) + (x_1, x_2, ..., x_n)(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n) + (x_1^2, x_2^2, ..., x_n^2)$
$= (x_1 + x_1^2, x_2 + x_2^2, ..., x_n + x_n^2) = (x_1^2, x_2^2, ..., x_n^2) = (x_1, x_2, ..., x_n)(x_1, x_2, ..., x_n)$
$= (x_1, x_2, ..., x_n)^2$ and $(x_1, x_2, ..., x_n)(y_1, y_2, ..., y_n) = (x_1 y_1, x_2 y_2, ..., x_n y_n)$
$= (y_1 x_1, y_2 x_2, ..., y_n x_n) = (y_1, y_2, ..., y_n)(x_1, x_2, ..., x_n)$. As a result, we conclude that S is an almost central idempotent semiring

Conversely, suppose that $S = \prod_{i=1}^{n} S_i$ is almost idempotent central semiring. Our objective is to demonstrate that each semiring S_i is almost idempotent central semiring. To accomplish this, we will examine the mapping $\pi : S \to S_i$ defined by $\pi((x_1, x_2, ..., x_n)) = x_i$ for all $(x_1, x_2, ..., x_n) \in S$. It can be observed that π is an onto homomorphism from $S = \prod_{i=1}^{n} S_i$ to S_i. Consequently, according to Lemma 3.5.19, S_i is an almost idempotent central semiring for all $i = 1, 2, ..., n$. This completes the proof. □

Theorem 3.5.21. *The class of all almost idempotent central semirings is a variety.*

Proof. By utilizing the Lemmas 3.5.18, 3.5.19 and 3.5.20, we have proved that the class of almost idempotent central semirings is closed under taking subsemirings, homomorphic images and direct products. Therefore, it can be concluded that the class of all almost idempotent central semirings is a variety. □

Chapter 4

On h-Center of Semirings

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On h-Center of Semirings

4.1 Introduction

Center has left indelible mark on the structure theory of rings and useful for many purposes. A center-like subset refers to a subset within certain classes of rings that is defined based on a commutativity condition and coincides with the center of the ring. There are many results in the literature of ring theory about that direction. The hypercenter theorem, introduced by Herstein [34], is the first notable result in this particular context. Several authors have investigated the center-like subsets of rings or special classes of rings (See [6], [7], [8], [26], [48]). Some recent works on center-like subsets of rings can be found in [37], [47], [73]. The motivation for constructing a special type of center, namely "h-center" of a semiring came from the concept of center-like subsets of a ring.

The aim of this chapter is to explore various findings concerning a center-like subset of a semiring that bear resemblance to similar concepts in ring theory, using the assistance of the h-center of the semiring. In some sense, the h-center, represented as $C_h(S)$, is a center-like subset of a semiring S. Like center $Z(S)$ of a semiring S, $C_h(S)$ is not an ideal of S. But we can establish that $C_h(S)$ is indeed an ideal of $Z(S)$. If S is an additively cancellative semiring, it is noteworthy that $C_h(S) = Ann(S) \cap Z(S)$; where $Ann(S)$ denote the annihilator of semiring S. Consequently, the h-center serves as a generalization of both the semiring's annihilator and its center. In the final section of this chapter we introduce an exclusive kind of semiring known as the "h-central semiring" utilizing the concept of h-center. Additionally, we delve into several structural aspects pertaining to this particular semiring.

4.2 $C_h(S)$ of a Semiring S

Within this section, we will provide a formal definition for the concept of h-center of a semiring S, accompanied by a few illustrative examples. Additionally, an investigation into the h-centers of the power semiring and matrix semiring will be conducted.

Definition 4.2.1. *Let S be a semiring. A subset $C_h(S)$ of a semiring S is called h-center of S if $C_h(S) = \{a \in S : a + ab = a \text{ and } ab = ba \text{ for all } b \in S\}$.*

Note 4.2.2. *If S be a semiring with zero element 0, then from the definition of h-center, we see that $0 \in C_h(S)$.*

Example 4.2.3. *Note that $\mathbb{Z}_0^+ = \{x \in \mathbb{Z} : x \geq 0\}$, $\mathbb{Q}_0^+ = \{x \in \mathbb{Q} : x \geq 0\}$ and $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$ are commutative semirings with zero. In this case, $C_h(\mathbb{Z}_0^+) = \{0\}$, $C_h(\mathbb{Q}_0^+) = \{0\}$ and $C_h(\mathbb{R}_0^+) = \{0\}$.*

Example 4.2.4. *Let's consider the semiring $(\mathbb{N}, \oplus, \odot)$; where for any $b > a$, $a \oplus b = min\{a, b\} = a$ as addition on $\mathbb{N}$ and $a \odot b$ for the usual multiplication on $\mathbb{N}$. Then for any $b \in \mathbb{N}$, we have $a \oplus ab = min\{a, ab\} = a$ and $ab = ba$. Consequently, we can conclude that $C_h(\mathbb{N}) = \mathbb{N}$.*

Example 4.2.5. *Consider $S = \{0, a, b, c\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	c	c

Therefore, $(S,+,\cdot)$ is a non-commutative semiring. In this case, $Z(S) = \{0,a\}$. However, $C_h(S) = \{0,a\}$. Hence, we can observe that $Z(S) = C_h(S)$.

Let us now investigate the characteristics of $C_h(S)$ within the framework of a power semiring.

Definition 4.2.6. *Let S be a semigroup and $\mathcal{P}(S)$ be the set of all subsets of S. Define addition and multiplication on $\mathcal{P}(S)$ by : $U+V = U \cup V$ and $U \cdot V = \{ab : a \in U,\ b \in V\}$ for all $U, V \in \mathcal{P}(S)$. Then $(\mathcal{P}(S),+,\cdot)$ is a semiring whose additive reduct is a semilattice.*

Theorem 4.2.7. *Let S be a commutative semigroup and $(\mathcal{P}(S),+,\cdot)$ is a semiring. Then $C_h(\mathcal{P}(S)) = \{I \in \mathcal{P}(S) : I$ is an ideal of $S\}$.*

Proof. Let I be an ideal of S and consider $A \in \mathcal{P}(S)$. We can observe that $I + IA = I \cup IA = I$, since $IA \subset I$ due to I being an ideal of S. Additionally, since S is commutative, we have $IA = \{ia : i \in I, a \in A\} = AI$. Consequently, $I \in C_h(\mathcal{P}(S))$. Moreover, if $X \in C_h(\mathcal{P}(S))$, then $X + XA = X$ and $XA = AX$ for any $A \in \mathcal{P}(S)$. We claim that X is an ideal of S. If possible, let X be not an ideal of S. Then there exists some $a \in S$ such that $x_i a \notin X$ for some $x_i \in X$. Now $X + X\{a\} = X \cup X\{a\} \neq X$ as $x_i a \notin X$ which contradicts that $X \in C_h(\mathcal{P}(S))$. Thus, we conclude that X is indeed an ideal of S. Consequently, we can state that $C_h(\mathcal{P}(S))$ is the collection of all ideals of S, which can be expressed as $C_h(\mathcal{P}(S)) = \{I \in \mathcal{P}(S) : I$ is an ideal of $S\}$. □

The formation of the h-center in a matrix semiring is shown by the following theorem.

Theorem 4.2.8. *Let S be a semiring with identity. Then $a \in C_h(S)$ if and only if $aI_n \in C_h(M_n^d(S))$; where $M_n^d(S)$ is the set of all $n \times n$ diagonal matrices of the form:*

$$M_n^d(S) = \left\{ \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} : \ a_{ii} \in S \right\}$$

Proof. Let $a \in C_h(S)$. Then $a + ab = a$ and $ab = ba$ for all $b \in S$.

Now $aI_n + (aI_n)B$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} ab_1 & 0 & \cdots & 0 \\ 0 & ab_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ab_n \end{pmatrix}$$

$$= \begin{pmatrix} a+ab_1 & 0 & \cdots & 0 \\ 0 & a+ab_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a+ab_n \end{pmatrix} = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} = aI_n,$$

since $a \in C_h(S)$.

$$\text{Again } (aI_n)B = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$$

$$= \begin{pmatrix} ab_1 & 0 & \cdots & 0 \\ 0 & ab_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ab_n \end{pmatrix} = \begin{pmatrix} b_1a & 0 & \cdots & 0 \\ 0 & b_2a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_na \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} = B(aI_n), \text{ since } a \in C_h(S).$$

Consequently, it can be deduced that $aI_n \in C_h(M_n^d(S))$.

Conversely, suppose that $aI_n \in C_h(M_n^d(S))$. Our aim now is to demonstrate that $a \in C_h(S)$. Let $x \in S$. Now consider B $= \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}$. Since $aI_n \in C_h(M_n^d(S))$, it follows that $aI_n + (aI_n)B = aI_n$ and $(aI_n)B = B(aI_n)$. Comparing both sides, we can deduce that $a + ax = a$ and $ax = xa$. Consequently, we can conclude that $a \in C_h(S)$. □

By virtue of the Theorem 4.2.8, we arrive at the following Corollary :

Corollary 4.2.9. *Let S be a semiring with identity. Then the h-center of S i.e. $C_h(M_n(D)) = \{A \in M_n(D) : A + AB = A$ and $AB = BA$ for all $B \in M_n(D)\}$; where $M_n(D)$ is the set of all $n \times n$ diagonal matrices.*

4.3 Algebraic Properties of $C_h(S)$

In this section, we establish that $C_h(S)$ constitutes a subsemiring within S and elaborates on further algebraic characteristics of $C_h(S)$. We wrap up this section by examining $C_h(S)$ in the context of a different class of semiring.

Theorem 4.3.1. *The h-center of a semiring S with zero 0 is a subsemiring of S.*

Proof. Since S is a semiring with zero element 0, then $0 \in C_h(S)$. So, the h-center of a semiring S is non-empty. Let $x, y \in C_h(S)$. We now show that $x + y \in C_h(S)$ and $xy \in C_h(S)$. For any k in S, we have $(x+y)+(x+y)k = x+xk+y+yk = x+y$ (since $x \in C_h(S)$ and $y \in C_h(S)$) and $(x+y)k = xk+yk = kx+ky$ (since $x, y \in C_h(S)$) $=$ $k(x+y)$. Therefore, we conclude that $x+y \in C_h(S)$. Furthermore, we observe that $xy+xyk = x(y+yk) = xy$ (since $y \in C_h(S)$) and $xyk = xky$ (since $y \in C_h(S)$)$= kxy$ (since $x \in C_h(S)$). Hence, we can deduce that $xy \in C_h(S)$. Consequently, we can establish that $C_h(S)$ forms a subsemiring of S. □

According to above Theorem 4.3.1, we arrive the following corollary :

Corollary 4.3.2. *Let S be a semiring with identity element 1. Then $C_h(S)$ is an additively idempotent subsemiring of S.*

Proof. We can conclude that $C_h(S)$ is a subsemiring of S based on Theorem 4.3.1. Let's assume a belongs to $C_h(S)$. Consequently, $a + ax = a$ and $ax = xa$ for all $x \in S$. If we choose $x = 1$, we obtain $a + a = a$, which indicates that 'a' is an additive idempotent element. Since 'a' is an arbitrary element of $C_h(S)$, it follows that $C_h(S)$ is an additively idempotent subsemiring of S. □

Remark 4.3.3. *Consider a semiring S. In this context, $C_h(S)$ is a multiplicatively subidempotent semiring. Let $a \in C_h(S)$. Then $a + ax = a$ and $ax = xa$ for all $x \in S$. If $x = a$, then $a + a^2 = a$. This implies that 'a' is a multiplicatively subidempotent element of S. Since 'a' is arbitrary, we can conclude that $C_h(S)$ is a multiplicatively subidempotent semiring.*

Proposition 4.3.4. *If S is a semiring with identity element 1, then $C_h(S)$ is a viterbi semiring.*

Proof. If we let $a \in C_h(S)$, we observe that $a + ab = a$ and $ab = ba$ for every $b \in S$. By choosing $b = 1$ as the multiplicative identity in S, it becomes evident that $a + a = a$ and by setting $b = a$, we find that $a + a^2 = a$. Since a is chosen arbitrarily, we can conclude that $C_h(S)$ forms a viterbi semiring. □

Now we present the desired characterization theorem for h-center of a semiring.

Theorem 4.3.5. *Let S be a semiring with identity element 1. Then $1 \in C_h(S)$ if and only if 1 is an absorbing element of S with respect to addition.*

Proof. First suppose that $1 \in C_h(S)$. By the definition of h-center of a semiring, we have $1 + 1 \cdot a = 1$ for all $a \in S$. As a result, we can deduce that $1 + a = 1$ for every element a in S, indicating that 1 functions as an additively absorbing element of S.

Conversely, let's assume that 1 is an absorbing element of S with respect to addition. Thus we have $1 + a = 1$ for all $a \in S$. This implies that $1 + 1.a = 1$ for all $a \in S$. Hence, $a \in C_h(S)$. □

By using Theorem 4.3.5, we can easily deduce the subsequent Corollary.

Corollary 4.3.6. *Let S be a semiring with identity element 1. Then $1 \in C_h(S)$ if and only if S is a semiring with additively absorbing 1.*

We will now introduce another characterization theorem that applies to the h-center of a c-semiring.

Theorem 4.3.7. *Let S be a c-semiring. Then $1 \in C_h(S)$ if and only if $|S| = 2$.*

Proof. Given that S is a c-semiring, we can deduce that $0, 1 \in S$. Let $1 \in C_h(S)$. Then $1 + b = 1$ for all $b \in S$. Since S is a c-semiring, we have $1 + x = 1$ for all $x \in S$. This implies that $b = 1$. Thus, we have $S = \{0, 1\}$.

Conversely, suppose that $|S| = 2$. Since S is a c-semiring, so $0, 1 \in S$. Thus, it follows that $S = \{0, 1\}$. Additionally, since S is a c-semiring, we can deduce that $1 + x = 1$ for all $x \in S$. Hence, $1 \in C_h(S)$. □

The following theorems explore the characteristics and properties of $C_h(S)$ in different classes of semirings.

Theorem 4.3.8. *If S is a division semiring with unity 1, then $C_h(S)$ is a simple semiring.*

Proof. Let $z \in C_h(S)$ and $z \neq 0$. As $C_h(S) \subseteq S$, so $z \in S$. Moreover, S being a division semiring implies that there exists an inverse w for z in S, satisfying $zw = 1 = wz$. Since $z \in C_h(S)$, we have $z + zr = z$ for all $r \in S$ (i). By substituting $r = z$ into equation (i), we get $z + z^2 = z$. This can be further simplified to $wz + wz^2 = wz$, which leads to $1 + wz \cdot z = wz$. Simplifying further, we get $1 + z = 1$, given that $wz = 1$. As z is arbitrary, we can conclude that $C_h(S)$ constitutes a simple semiring. □

It can be observed in the following theorem that a division semiring's h-center, within a semiring with an additive absorbing identity, is itself a division semiring.

Theorem 4.3.9. *Let S be a division semiring with additive absorbing identity 1. Then $C_h(S)$ is a division semiring.*

Proof. Let $a(\neq 0) \in C_h(S) \subseteq S$. Since S is a division semiring, there exists $d \in S\backslash\{0\}$ such that $ad = da = 1$. By the existence of an additive absorbing identity 1 in S, we can apply Theorem 4.3.5 to conclude that $1 \in C_h(S)$. We have to only show that $d \in C_h(S)$. Since $a \in C_h(S)$, $a + ax = a$ and $ax = xa$ for all $x \in S$. Now multiplying d from right side, we have $(a + ax)d = ad$ and $axd = xad$. This implies that $ad+axd = ad$ and $axd = xad$. So, we have $1+axd = 1$ and $axd = x$, since $ad = 1$. Next multiplying d from left side, we have $d+daxd = d$ and $daxd = dx \Longrightarrow d+xd = d$ and $xd = dx$ for all $x \in S$, since $da = 1$. Therefore, we conclude that $d \in C_h(S)$.

In summary, we have shown that $ad = da = 1$ for some $d \in C_h(S)$, demonstrating that a is a unit in $C_h(S)$. Hence, $C_h(S)$ is a division semiring. □

Theorem 4.3.10. *Let S be an additively cancellative semiring. Then S is a mono-semiring if and only if $C_h(S)$ is additively rectangular band.*

Proof. Let S be a mono-semiring. By applying Theorem 4.3.1, we can conclude that $C_h(S)$ forms a subsemiring of S. Let $a \in C_h(S)$. Then $a + ab = a$ for all $b \in S$. This implies that $a + a + b = a$, since S is mono-semiring. Thus, $a + b + a = a$ for all $a \in C_h(S)$. Therefore, $C_h(S)$ is additively rectangular band.

Conversely, suppose that $C_h(S)$ is additively rectangular band. Then $a+b+a = a$ for all $a, b \in C_h(S)$. Since $C_h(S)$ is a subsemiring of S, we can infer that both a and b are elements of S. Therefore, we have $a + b + a = a$ for all $a,\ b \in S$ (i).
Additionally, if $a \in C_h(S)$, we can further state that $a + ab = a$ for all $b \in S$ (ii).

By combining equations (i) and (ii), we deduce that for any a and b in S, the equality $a + ab = a + b + a$ holds. This can be rearranged as $a + ab = a + a + b$, and by utilizing the additively cancellative property of S, we arrive at the conclusion that $ab = a+b$. Thus, we establish that $ab = a+b$ holds for all a and b in S. Consequently, we can affirm that S is a mono-semiring. □

Similar to ring theory, it can be shown that the center of a regular semiring is a regular semiring. With this in mind, we can now generalize this concept of center of semiring to h-center of semiring.

Theorem 4.3.11. *The h-center of a regular semiring is a regular semiring.*

Proof. Suppose S is a regular semiring. We can then conclude that the center $Z(S)$ is also a regular semiring. Let $x \in C_h(S)(\subseteq Z(S))$. Since $Z(S)$ is a regular semiring, we can find an element y in $Z(S)$ such that $x = xyx$. By expanding this expression, we obtain $x = xyx = (xyx)yx = x(yxy)x$. We aim to showcase that $yxy(= z) \in C_h(S)$. We know that $C_h(S)$ is an ideal of $Z(S)$. Since $x \in C_h(S)$, $y \in Z(S)$ and $C_h(S)$ is an ideal of $Z(S)$, so $z = yxy \in C_h(S)$. Thus for any $x \in C_h(S)$, there exists $z \in C_h(S)$ such that $x = xzx$. Therefore, $C_h(S)$ is a regular subsemiring of S. □

In general, the converse of the aforementioned theorem does not hold. This can be demonstrated by the following counterexample.

Example 4.3.12. *We consider* $(\mathbb{Z}_0^+, +, \cdot)$ *is a semiring; where "+" and "·" are usual addition and multiplication respectively. In this context,* $C_h(\mathbb{Z}_0^+) = \{0\}$ *which is regular. However, it is important to note that* $(\mathbb{Z}_0^+, +, \cdot)$ *is not a regular semiring.*

4.4 The Lattice Structure in the Context of $C_h(S)$

Within this section, our focus lies on proving that under specific conditions on S, $C_h(S)$ of a semiring S exhibits a lattice structure.

Theorem 4.4.1. *Let S be a semiring with identity* 1*. If* $(S, \cdot)$ *is a rectangular band, then* $C_h(S)$ *is a b-lattice semiring.*

Proof. We know that if S is a semiring with identity element 1, then $C_h(S)$ is an additively idempotent semiring. It is evident that $C_h(S)$ is also additively commutative. Consequently, $(C_h(S), +)$ is a semilattice. Let $a \in C_h(S)$. For any $b \in S$, the equation $a + ab = a$ implies $(a + ab)\ a = a \cdot a$, which further leads to $a^2 + aba = a^2$. Since $(S, \cdot)$ is a rectangular band, we can simplify this expression as $a^2 + a = a^2\ (i)$. By substituting $b = a$ into $a+ab = a$, we find that $a+a\cdot a = a$ results in $a+a^2 = a(ii)$.

From (i) and (ii), we can deduce that $a^2 = a$. Therefore, $(C_h(S), .)$ forms a band. As a result, $C_h(S)$ is a b-lattice semiring. □

We are now prepared to demonstrate that the set $C_h(S)$ indeed constitutes a partial order set when subjected to a particular binary relation.

Lemma 4.4.2. *Let S be a semiring with identity element 1. Define a binary relation "$\leq_S$" on $C_h(S)$ by "$a \leq_S b$" if and only if $a + b = a$ for all a, b in $C_h(S)$. Then $(C_h(S), \leq_S)$ forms a partial ordered set.*

Proof. The relation $\leq_S$ on the set $C_h(S)$ satisfies the following properties :

Reflexivity : For any $a \in C_h(S)$, $a + ax = a$ holds for all $x \in S$. In particular, for $x = 1$, we have $a + a1 = a$, which simplifies to $a + a = a$. Therefore, $a \leq_S a$ for all $a \in C_h(S)$. Thus "$\leq_S$" is reflexive.

Antisymmetry : If $a \leq_S b$ and $b \leq_S a$ for $a, b \in C_h(S)$, then $a + b = a$ and $b + a = b$. Combining these equations, we find that $a = b$, showing that the relation "$\leq_S$" is antisymmetric.

Transitivity : Suppose $a \leq_S b$ and $b \leq_S c$ for $a, b, c \in C_h(S)$. This means $a + b = a$ and $b + c = b$. Now $a + c = a + b + c = a + b = a$. Thus, $a \leq_S c$ holds for all $a, c \in C_h(S)$. Hence "$\leq_S$" is transitive.

As a result, we conclude that the relation "$\leq_S$" is a partial order on the set $C_h(S)$. Therefore, $(C_h(S), \leq_S)$ forms a partially ordered set. □

Theorem 4.4.3. *If S is a semiring with identity element 1, then $C_h(S)$ forms a semilattice.*

Proof. Assuming that S is a semiring with identity element 1, we can deduce from Corollary 4.3.2 that $C_h(S)$ constitutes an additively idempotent semiring. In order to establish a binary relation denoted as "$\leq_S$" on $C_h(S)$, we define $a \leq_S b$ if and only if $a + b = a$ holds for all $a, b \in C_h(S)$. By virtue of Theorem 4.4.2, we ascertain that $(C_h(S), \leq_S)$ forms a partially ordered set.

Further, for any $a, b \in C_h(S)$, we observe that $a + b + a = a + a + b = a + b$, since $C_h(S)$ is an additively idempotent semiring. This implies that $a + b \leq_S a$. Similarly, $a + b + b = a + b + b = a + b$, due to the additively idempotent property of $C_h(S)$. Consequently, $a + b \leq_S b$. Therefore, $a + b$ serves as a lower bound for both a and b.

Next, we aim to demonstrate that $a + b$ constitutes the greatest lower bound of a and b. Let g be another lower bound of a and b. This implies that $g \leq_S a$ if and only if $g + a = g$ holds for all $g, a \in C_h(S)$, and similarly, $g \leq_S b$ if and only if

$g + b = g$ holds for all $g, b \in C_h(S)$. Proceeding from $g + a = g$, we can deduce that $g + a + b = g + b$, and by virtue of $g + b = g$, we obtain $g + a + b = g$. Therefore, $g \leq_S a + b$ holds for all $a, b \in C_h(S)$. Consequently, $a + b$ represents the greatest lower bound of a and b. Hence, we can conclude that $C_h(S)$ forms a semilattice. □

Through the utilization of Lemma 4.4.2 alongside Theorem 4.4.3, we establish the subsequent theorem, which represents our principal objective concerning the lattice structure of the h-center of a semiring.

Theorem 4.4.4. *If S is a multiplicatively idempotent semiring with identity element 1, then $C_h(S)$ forms a lattice.*

Proof. In the multiplicatively idempotent semiring S with identity element 1, we define a binary relation denoted by "$\leq_S$" on $C_h(S)$; where "$a \leq_S b$" if and only if $a + b = a$ for all $a, b \in C_h(S)$. The Lemma 4.4.2 implies that $(C_h(S), \leq_S)$ forms a partially ordered set. Furthermore, from Theorem 4.4.3, we deduce that $a + b$ serves as the greatest lower bound for a and b. Let $a \in C_h(S)$. Then, for every b in S, we have $a + ab = a$, which implies that $a \leq_S ab$. Similarly, if $b \in C_h(S)$, we can observe that $b + ba = b$ for all $b \in S$, leading to $b \leq_S ab$. Consequently, ab acts as an upper bound for a and b. To establish that ab is the least upper bound for a and b, let $l \in C_h(S)$ be another lower bound of a and b in $C_h(S)$. Thus, $a \leq_S l$ if and only if $a + l = l$ for all $a, l \in C_h(S)$, and $b \leq_S l$ if and only if $b + l = b$ for all $b, l \in C_h(S)$. By utilizing the fact that S is a multiplicatively idempotent semiring, we can deduce that $(a+l)(b+l) = ab \Longrightarrow ab + al + bl + l^2 = ab \Longrightarrow ab + al + bl + l = ab$. Simplifying this expression, we find $ab + al + l = ab \Longrightarrow ab + l = ab$, as $l \in C_h(S)$. Consequently, we can conclude that $ab \leq_S l$ holds for all $a, b \in C_h(S)$. Therefore, ab represents the least lower bound of a and b, establishing that $C_h(S)$ forms a lattice. □

Our immediate goal is to determine the classification of the lattice of $C_h(S)$.

Theorem 4.4.5. *If S is a multiplicatively idempotent semiring with identity element 1, then $C_h(S)$ forms a distributive lattice.*

Proof. We know that if S is a multiplicatively idempotent semiring with identity element 1, then $C_h(S)$ forms a lattice. For any two elements a, b in $C_h(S)$, we have $a \vee b = ab$ and $a \wedge b = a + b$. Let $a, b, c \in C_h(S)$. Thus, we have $a \wedge (b \vee c) = a \wedge (bc)$ (since $b \vee c = bc$), which further simplifies to $a + bc$ (as $a \wedge b = a + b$). Similarly, $(a \wedge b) \vee (a \wedge c) = (a + b) \vee (a + c) = a^2 + ac + ba + bc = a + ac + ab + bc = a + bc$ for all $a, b, c \in C_h(S)$. Consequently, we can deduce that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all a, b, and c in $C_h(S)$. Hence, we can conclude that $C_h(S)$ is a distributive lattice. □

Remark 4.4.6. *A distributive lattice is always modular lattice. If S represents a multiplicatively idempotent semiring with the identity element 1, then $C_h(S)$ forms a modular lattice.*

Remark 4.4.7. *If S is a finite multiplicatively idempotent semiring, then $C_h(S)$ forms a complete lattice.*

Every finite set possesses a supremum, therefore, if S is finite, any subset of $C_h(S)$ also has its supremum within $C_h(S)$. Likewise, every subset of $C_h(S)$ has its infimum in $C_h(S)$. Hence, $C_h(S)$ constitutes a complete lattice.

Theorem 4.4.8. *Let S be a finite multiplicatively idempotent semiring with zero element 0 and identity 1. If 1 is additively absorbing, then $C_h(S)$ forms a bounded lattice.*

Proof. Suppose S is a finite multiplicatively idempotent semiring. If S possesses an identity element 1, then $C_h(S)$ forms a lattice. Let a be an element of $C_h(S)$. It follows that $1 + a = 1$ due to 1 being an absorbing element under addition, and $a + 0 = a$ as 0 acts as the additive identity. Consequently, we can deduce that $1 \leq a$ and $a \leq 0$ for every a in $C_h(S)$. Therefore, 1 serves as the least element, while 0 acts as the greatest element in $C_h(S)$. Thus, we can conclude that $C_h(S)$ is a bounded lattice. □

4.5 Structural Characteristics of $C_h(S)$

Within this section, we arrange several structural properties of $C_h(S)$ of a semiring S by means of structure preserving mapping, and find out relation between the h-centers of two semirings and their semiring formed by taking their Cartesian product.

Theorem 4.5.1. *Let S and S' be two semirings and $\varphi : S \longrightarrow S'$ be an epimorphism. Then $\varphi(C_h(S)) \subseteq C_h(S')$.*

Proof. Suppose we have an epimorphism $\varphi : S \longrightarrow S'$ of semirings. Let $\varphi(C_h(S)) = \{\varphi(s) : s \in C_h(S)\}$. Our objective is to demonstrate that $\varphi(C_h(S)) \subseteq C_h(S')$. Consider an element $\varphi(s_1) \in \varphi(C_h(S))$; where $s_1 \in C_h(S)$, and let x be an arbitrary element of S'. As φ is onto, there exists $y \in S$ such that y has a preimage x of S' such that $\varphi(y) = x$. Since $s_1 \in C_h(S)$, we have $s_1 + s_1y = s_1$ and $s_1y = ys_1$. Now for any $x \in S'$, we observe the following : $\varphi(s_1) + \varphi(s_1)x = \varphi(s_1) + \varphi(s_1)\varphi(y) = \varphi(s_1) + \varphi(s_1y) = \varphi(s_1 + s_1y) = \varphi(s_1)$ and $\varphi(s_1)x = \varphi(s_1)\varphi(y) = \varphi(s_1y) = \varphi(ys_1) = \varphi(y)\varphi(s_1) = x\varphi(s_1)$, since φ is an epimorphism. Consequently, we conclude that $\varphi(s_1) \in C_h(S')$. Hence, we have established that $\varphi(C_h(S)) \subseteq C_h(S')$. □

Theorem 4.5.2. *Let S and S' be two semirings. If $f : S \to S'$ is a monomorphism, then $f(C_h(S)) = C_h(f(S))$.*

Proof. Suppose $x \in f(C_h(S))$. Then $x = f(y)$ for some $y \in C_h(S)$. We have to show that $f(y) + f(y)s = f(y)$ and $f(y)s = sf(y)$ for all $s \in f(S)$. For any $s \in f(S)$, we have $f(y) + f(y)s = f(y) + f(y)f(r) = f(y + yr) = f(y)$ and $f(y)s = f(y)f(r) = f(yr) = f(ry) = f(r)f(y) = sf(y)$. Therefore, $x = f(y) \in C_h(f(S))$. Consequently, $f(C_h(S)) \subseteq C_h(f(S))$. Now, let $x' \in C_h(f(S))$. Then $x' = f(r')$ for some $r' \in S$. Our objective is to show that $r' \in C_h(S)$. Since $x' \in C_h(f(S))$, for any $f(s) \in f(S)$, it follows that $x' + x'f(s) = x' \implies f(r') + f(r')f(s) = f(r') \implies f(r' + r's) = f(r') \implies r' + r's = r'$, since f is a monomorphism and $x'f(s) = f(s)x' \implies f(r')f(s) = f(s)f(r') \implies f(r's) = f(sr') \implies r's = sr'$, as f is a monomorphism. Therefore, $r' \in C_h(S)$. Thus, $C_h(f(S)) \subseteq f(C_h(S))$ and hence $C_h(f(S)) = f(C_h(S))$. □

Theorem 4.5.3. *If two semirings S_1 and S_2 are isomorphic, then their h-centers $C_h(S_1)$ and $C_h(S_2)$ are isomorphic.*

Proof. Consider two semirings S_1 and S_2 which are isomorphic. Then there is an isomorphism $f : S_1 \longrightarrow S_2$. Let $x \in C_h(S_1)$. Then for any $s_1 \in S_1$, $x + xs_1 = x$ and $xs_1 = s_1x$. Let $f(x) = y$; where $y \in S_2$. Since f is an isomorphism, for any $s_2 \in S_2$, there exists $s_1 \in S_1$ such that $f(s_1) = s_2$. Consequently, we have $y + ys_2 = f(x) + f(x)f(s_1) = f(x + xs_1) = f(x) = y$ and $ys_2 = f(x)f(s_1) = f(xs_1) = f(s_1x) = f(s_1)f(x) = s_2y$, since $x \in C_h(S_1)$. Hence, we can deduce that $y \in C_h(S_2)$, demonstrating that $f(C_h(S_1)) \subseteq C_h(S_2)$.

Now, let's consider an element b in $C_h(S_2)$. Then $b = f(a)$; where $a \in S_1$. As f is an isomorphism, for any $y \in S_2$, there exists $x \in S_1$ such that $y = f(x)$. Since $b \in C_h(S_2)$, we conclude that $b + by = b$ and $by = yb$. Consequently, we have $f(a) + f(a)f(x) = f(a)$, which implies $f(a + ax) = f(a)$. This further leads to $a + ax = a$ and $by = yb$. Additionally, we can infer that $f(a)f(x) = f(x)f(a)$, resulting in $f(ax) = f(xa)$ and $ax = xa$, given that f is an isomorphism. Therefore, we can deduce that $a \in C_h(S_1)$. As a result, we obtain $b = f(a) \in f(C_h(S_1))$. This implies that $C_h(S_2) \subseteq f(C_h(S_1))$, and consequently, $C_h(S_2) = f(C_h(S_1))$. Hence, we can conclude that the restricted function $g = f_{|C_h(S_1)} : C_h(S_1) \longrightarrow C_h(S_2)$ is well-defined and serves as an isomorphism from $C_h(S_1)$ onto $C_h(S_2)$. □

The converse of the Theorem 4.5.3 does not hold, meaning that the isomorphism between $C_h(S_1)$ and $C_h(S_2)$ does not guarantee isomorphism between S_1 and S_2, as illustrated by the following example.

Example 4.5.4. $C_h(\mathbb{Z}_0^+) = \{0\}$ *and* $C_h(\mathbb{R}_0^+) = \{0\}$. *But* $\mathbb{Z}_0^+$ *and* $\mathbb{R}_0^+$ *do not exhibit isomorphism.*

Theorem 4.5.5. *If S_1 and S_2 are two semirings, then $C_h(S_1 \times S_2) = C_h(S_1) \times C_h(S_2)$.*

Proof. Let's consider S_1 and S_2 as two semirings. Assuming $z \in C_h(S_1 \times S_2)$, we have $z = (x, y) \in S_1 \times S_2$ and for any $(a, b) \in S_1 \times S_2$, $(x, y) + (x, y)(a, b) = (x, y)$ and $(x, y)(a, b) = (a, b)(x, y)$, which implies $(x + xa, y + yb) = (x, y)$ and $(xa, yb) = (ax, by)$.

By comparing both sides, we obtain $x+xa = x$, $y+yb = y$, $xa = ax$, and $yb = by$. This implies that $x \in C_h(S_1)$ and $y \in C_h(S_2)$. Consequently, $z = (x, y) \in C_h(S_1) \times C_h(S_2)$, thus establishing $C_h(S_1 \times S_2) \subseteq C_h(S_1) \times C_h(S_2)$ (i).

For the reverse part, let $(a, b) \in C_h(S_1) \times C_h(S_2)$. This implies that $a \in C_h(S_1)$ and $b \in C_h(S_2)$. Thus, for any $x \in S_1$, we have $a+ax = a$ and $ax = xa$, and for any $y \in S_2$, we have $b+by = b$ and $by = yb$. Now, $(a, b)+(a, b)(x, y) = (a+ax, b+by) = (a, b)$ and $(a, b)(x, y) = (ax, by) = (xa, yb) = (x, y)(a, b)$, since $a \in C_h(S_1)$ and $b \in C_h(S_2)$. This implies that $(a, b) \in C_h(S_1 \times S_2)$, establishing $C_h(S_1) \times C_h(S_2) \subseteq C_h(S_1 \times S_2)$ (ii).

From (i) and (ii), we conclude that $C_h(S_1 \times S_2) = C_h(S_1) \times C_h(S_2)$. □

The above theorem helps us to produce several examples of h-center of a semiring. We provide one such example in the following.

Example 4.5.6. *Consider two semirings* $(\mathbb{N}, \oplus, \odot)$*; where* $a \oplus b = min\{a, b\}$ *and* $\odot$ *is usual multiplication and* $(\mathbb{Z}_0^+, +, \cdot)$*; where "+" is usual addition and "·" is usual multiplication. Now we take a semiring* $(\mathbb{N} \times \mathbb{Z}_0^+, +, \cdot)$ *with component-wise addition and multiplication. Then* $C_h(\mathbb{N} \times \mathbb{Z}_0^+) = C_h(\mathbb{N}) \times C_h(\mathbb{Z}_0^+) = \mathbb{N} \times \{0\}$.

4.6 On h-Central Semiring

This section presents the definition of an h-central semiring S based on its h-center form, and subsequently explores several characterizations of h-central semiring S in relation to its h-center.

Definition 4.6.1. *A semiring S is said to be an h-central semiring if* $C_h(S) = S$.

Note 4.6.2. *If S is an h-central semiring, then* $C_h(S) = S = Z(S)$ *i.e. S is a central semiring. However, in the case of a central semiring S, it does not necessarily imply that S is an h-central semiring. For instance, we consider a semiring* $(\mathbb{R}^+, +, .)$*. Then* $Z(\mathbb{R}^+) = \mathbb{R}^+$ *where as* $C_h(\mathbb{R}^+) = \{0\}$*. Hence,* $(\mathbb{R}^+, +, .)$ *is a central semiring but not an h-central semiring.*

Example 4.6.3. *Consider the set of integers $\mathbb{Z}^+$ with the operations $a+b = gcd\{a,b\}$ and $a \cdot b = ab$. Then $(\mathbb{Z}^+, +, \cdot)$ is a semiring with zero element 1. Then $C_h(\mathbb{Z}^+) = \mathbb{Z}^+$. Therefore, $(\mathbb{Z}^+, +, \cdot)$ is an h-central semiring.*

Example 4.6.4. *Consider $S = \{0, x, y, 1\}$. Define the operations "+" and "." on S by the following tables :*

+	0	x	y	1
0	0	x	y	1
x	x	x	y	1
y	y	y	y	1
1	1	1	1	1

.	0	x	y	1
0	0	0	0	0
x	0	x	x	x
y	0	x	y	y
1	0	x	y	1

Then $(S, +, \cdot)$ is a semiring and $C_h(S) = \{0, x, y, 1\} = S$. Consequently, S is an h-central semiring.

Example 4.6.5. *Let R be a ring. Let Ω_R be the set of all ideals of R. Define $\oplus$ and $\odot$ by $I_1 \oplus I_2 = \{a_1 + b_1 : a_1 \in I_1,\ b_1 \in I_2\}$ and $I_1 \odot I_2 = \left\{ \sum_{i=1}^{n} a_i b_i : a_1 \in I_1,\ b_1 \in I_2 \right\}$ for all $I_1, I_2 \in \Omega_R$, where $n \in \mathbb{N}$. Then $(\Omega_R, \oplus, \odot)$ forms a semiring. If R is commutative, then $(\Omega_R, \oplus, \odot)$ is an h-central semiring.*

Evidently, $C_h(\Omega_R) \subseteq \Omega_R$. Our aim is to establish that $\Omega_R \subseteq C_h(\Omega_R)$. Let $I \in \Omega_R$. To show that $I \in C_h(\Omega_R)$ i.e. to show that $I \oplus (I \odot I_1) = I$ and $I \odot I_1 = I_1 \odot I$ for all $I_1 \in \Omega_R$. Note that $I \odot I_1$ is an ideal of R. Suppose that $I \odot I_1 = I_2$. Clearly, $I \subseteq I \oplus I_2$ (i).

We need to demonstrate that $I \oplus I_2 \subseteq I$. Let $x = p + q \in I \oplus I_2$; where $p \in I$, $q \in I_2 = I \odot I_1$. We can express $q = \sum_{i=1}^{n} p_i q_i$; where $p_i \in I$, $q_i \in I_1$ for all $i = 1, ..., n$. Therefore, we have $x = p + q = p + \sum_{i=1}^{n} p_i q_i = p + p_1 q_1 + p_2 q_2 + ... + p_n q_n$. Now, we know that $p \in I$, $p_1 \in I$, $q_1 \in I_1 \subseteq R$. Since I is an ideal of R, we can conclude that $p_1 q_1, p_2 q_2, p_3 q_3, ..., p_n q_n \in I$. Consequently, we have $x = p + p_1 q_1 + p_2 q_2 + ... + p_n q_n = p + q \in I$. Since $x \in I \oplus I_2$ was arbitrarily chosen, we can affirm that $I \oplus I_2 \subseteq I$(ii).

Combining (i) and (ii), we obtain $I \oplus I_2 = I \implies I \oplus (I \odot I_1) = I$ (as $I_2 = I \odot I_1$). Given that R is commutative, we can deduce that $I \odot I_1 = I_1 \odot I$. Therefore,

$I \in C_h(\Omega_R)$. As I represents any arbitrary element of Ω_R, we can conclude that $\Omega_R \subseteq C_h(\Omega_R)$, leading to $C_h(\Omega_R) = \Omega_R$. Thus, we can assert that Ω_R is an h-central semiring.

We are currently in the process of determining the relationship between h-central semiring and k-regular semiring.

Theorem 4.6.6. *If S be a multiplicatively idempotent semiring, then every h-central semiring is a k-regular semiring.*

Proof. Let S be a h-central semiring. Then $C_h(S) = S$. Consider $a \in C_h(S)$. Then $a^2 \in C_h(S)$, since $C_h(S)$ is a subsemiring of S. So, for any $x \in S$, we get $a^2+a^2 \cdot x = a^2$. This can be rewritten as $a+a \cdot a \cdot x = a \cdot a$ (using the property that S is multiplicatively band). Further simplification yields $a + a \cdot x \cdot a = a \cdot a \cdot 1$ (since S is a semiring with identity element 1). Finally, we have $a + a \cdot x \cdot a = a \cdot 1 \cdot a$. Consequently, we can conclude that $C_h(S)$ is a k-regular semiring. □

The subsequent pair of instances demonstrate that the converse of Theorem 4.6.6 mentioned above does not hold, indicating that every k-regular semiring is not h-central semiring.

Example 4.6.7. *The semiring $(\mathbb{Z}_0^+, +, \cdot)$ is k-regular; where "+" and "." are usual addition and multiplication respectively. However, $C_h(\mathbb{Z}_0^+) = \{0\} \neq S$. Hence, we can conclude that $(\mathbb{Z}_0^+, +, \cdot)$ is not h-central semiring.*

Example 4.6.8. *Consider $S = \{0, x, 1\}$. Define the operations "+" and "." on S_1 by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	1	1

Therefore, $(S, +, \cdot)$ is a k-regular semiring. However, $C_h(S) = \{0\} \neq S$, thereby indicating that $(S, +, \cdot)$ is not a h-central semiring.

We now introduce a characterization theorem that establishes the relationship between an h-central semiring and a simple semiring.

Theorem 4.6.9. *If S is a commutative semiring with identity 1, it is an h-central semiring if and only if it is a simple semiring.*

Proof. Let S be an h-central semiring. Then we have $C_h(S) = S$. Let $a \in C_h(S)$. Then $a + ab = a$ and $ab = ba$ for all $b \in S$. In particular, we have $1 + b = 1$ for all $b \in S$. Therefore, we can conclude that S is a simple semiring.

Conversely, suppose that S is a simple semiring with identity element 1. It is clear that $C_h(S) \subseteq S$. Let $a(\neq 0)$ be an arbitrary element in S. Now for $b \in S$, $a = a1 = a(1 + b) = a + ab$, considering the fact that S is a simple semiring. Additionally, since S is commutative, we can deduce that $ab = ba$. Thus, we have shown that $a(\neq 0) \in C_h(S)$. Furthermore, $0 \in C_h(S)$ as well. Consequently, we can conclude that $S \subseteq C_h(S)$, leading to the equality $S = C_h(S)$. As a result, S is an h-central semiring. □

We now present another version of the above theorem.

Theorem 4.6.10. *Let S be a commutative semirng with identity 1. Then S is an h-central semiring if and only if S is a semiring with additive absorbing 1.*

Proof. Suppose that S is an h-central semiring. Then $C_h(S) = S$. Thus $1 + 1b = 1$ and $1b = b1$ for all $b \in S$. This implies that $1 + b = 1$ for all $b \in S$. This shows that 1 is an additively absorbing element.

Conversely, suppose that S is a semiring additive absorbing 1. Clearly, $C_h(S) \subseteq S$. Let $a(\neq 0)$ be an arbitrary element in S. Now for $b \in S$, $a+ab = a1+ab = a(1+b) = a1 = a$ (as 1 is an additive absorbing element). So, for any $b \in S$, $a + ab = a$ and $ab = ba$, as S is commutative. Thus, $a(\neq 0) \in C_h(S)$. Also $0 \in C_h(S)$. Therefore, $S \subseteq C_h(S)$ and hence $S = C_h(S)$. Consequently, S is an h-central semiring. □

We proceed to set up another characterization theorem for an h-central semiring. In order to do so, we present the following theorem.

Theorem 4.6.11. *Suppose S is a mono-semiring. Then S is an h-central semiring if and only if $(S, +)$ is a rectangular band.*

Proof. Let S be a mono-semiring. Then $a+b = ab$ holds true for all $a, b \in S$. Suppose that S is an h-central semiring, denoted by $C_h(S) = S$, then for any $a \in S = C_h(S)$, we have $a + ab = a$ and $ab = ba$ for all $b \in S$. To establish that $(S, +)$ is a rectangular band, we need to demonstrate that $a+b+a = a$ holds for all $a, b \in S$. By utilizing the fact that $a + b + a = ab + a$ (due to the mono-semiring property of S), we can deduce that $a + b + a = a$, as S satisfies the conditions of an h-central semiring. Hence, we conclude that $(S, +)$ is indeed a rectangular band

Conversely, let us assume that $(S, +)$ is a rectangular band. It is evident that $C_h(S)$ is a subset of S. Consider an arbitrary element $a(\neq 0)$ in S. For any $b \in S$, we have $a + ab = a + a + b$ (due to the mono-semiring property of S) $= a + b + a = a$ (since $(S, +)$ is a rectangular band). Therefore, for any $b \in S$, we have $a + ab = a$, and consequently, $ab = a + b = b + a = ba$, since S is a mono-semiring. Thus, we can conclude that $a(\neq 0) \in C_h(S)$. Additionally, $0 \in C_h(S)$. Hence, it follows that $S \subseteq C_h(S)$, leading to the conclusion that $S = C_h(S)$. As a result, S is as an h-central semiring. □

Proposition 4.6.12. *Let S be an h-central semiring. The following statements hold true :*

(i) If $(S, \cdot)$ is a band, then $(S, +)$ is a band.

(ii) If $(S, +)$ is a band and is cancellative, then $(S, \cdot)$ is a band.

Proof. (i) Assume $(S, \cdot)$ is a band, and let S be an h-central semiring, implying $S = C_h(S)$. Consider an element a belonging to $C_h(S)$. For any b in S, $a + ab = a$. By choosing $b = a$, we find that $a + aa = a$, which can be further simplified to $a + a^2 = a$, since $(S, \cdot)$ is a band. Consequently, $a + a = a$. Thus, $(S, +)$ is a band.

(ii) Suppose $(S, +)$ is a band, and let's assume that S is an h-central semiring, which implies $S = C_h(S)$. Consider an element $a \in C_h(S)$. For any $b \in S$, we have $a + ab = a$. In particular, $a + aa = a$, which can be rewritten as $a + a^2 = a$. Since $(S, +)$ is a band, we can simplify this further to $a + a^2 = a + a$. Consequently, we deduce that $a^2 = a$, as cancellation holds in $(S, +)$. Therefore, for any $a \in S$, it holds that $a^2 = a$. Thus, $(S, \cdot)$ also forms a band. $\square$

Theorem 4.6.13. *Let S be an h-central semiring. If $(S, \cdot)$ is a rectangular band, then the following are true :*

(a) $(S, \cdot)$ is a band.

(b) $(S, +)$ is a band.

Proof. (a) Suppose S is an h-central semiring. In this case, we have $C_h(S) = S$. Let's consider an element a belonging to $C_h(S)$. For any b in $S, a + ab = a$ holds. We can further simplify this expression as follows: $(a + ab)a = aa$, which can be written as $a^2 + aba = a^2$, which further simplifies to $a^2 + a = a^2$ $\quad (i)$, since $(S, \cdot)$ is a rectangular band. Moreover, if we substitute $b = a$ in the equation $a + ab = a$, we obtain $a + aa = a$, which can be rewritten as $a + a^2 = a$ $\quad (ii)$.

Based on the implications of equations (i) and (ii), we can conclude that $a^2 = a$. Consequently, we can say that $(S, \cdot)$ is a band.

(b) Consider $a \in C_h(S)$. Consequently, we have $a + ab = a$ for all $a, b \in S$. By setting $b = a$, we can deduce that $a + a^2 = a$, which simplifies to $a + a = a$, since $(S, \cdot)$ is a band for all $a \in S$. Therefore, $(S, +)$ is a band. $\square$

Theorem 4.6.14. *If S be a h-central semiring, then the following are true :*

(i) If $(S, \cdot)$ has a left zero, then $(S, +)$ is a band.

(ii) If $(S, \cdot)$ has a left identity, then $(S, +)$ has a left zero.

Proof. (i) Assuming S is an h-central semiring, meaning $C_h(S) = S$, and let $a \in C_h(S)$. For any $b \in S$, $a + ab = a \Longrightarrow a + a = a$, since $(S, \cdot)$ has a left zero. Thus, it follows that $(S, +)$ forms a band.

(ii) Consider $a \in C_h(S)$. Then $a + ab = a$ for all $b \in S$. Since $(S, \cdot)$ has a left identity, $ab = b$ for all a, b in S. This implies that $a + b = a$. As a result, we can conclude that $(S, +)$ is left zero. $\square$

Proposition 4.6.15. *Let S be a zerosumfree semiring. If S is an h-central semiring, then S is a zero square semiring.*

Proof. Suppose that S is an h-central semiring. Then $C_h(S) = S$. Let $a \in C_h(S) = S$. For any $a, b \in S$, $a+ab = a$ and $ab = ba$. Taking $b = a$, we get $a+aa = a \Longrightarrow a+a^2 = a \Longrightarrow a+a+a^2 = a+a \Longrightarrow 0+a^2 = 0$(since S is a zerosumfree semiring)$\Longrightarrow a^2 = 0$. Thus $a^2 = 0$ for all all $a \in S$. Accordingly, S is a zero square semiring. □

Theorem 4.6.16. *If S is a zero square and h-central semirng with additive identity 0, then $aba = 0$ for all $a, b \in S$.*

Proof. Assume S is an h-central semiring with $C_h(S) = S$. Take $a \in S = C_h(S)$. For any $b \in S$, if $a + ab = a$, then we have $a^2 + aba = a^2$, which simplifies to $0 + aba = 0$ due to S being a zero square semiring. Consequently, we can deduce that $aba = 0$. □

Theorem 4.6.17. *Let S be a zerosumfree semirng with additive identity 0. Then S is an h-central semiring if and only if $ab = 0$ for all $a, b \in S$.*

Proof. Suppose that S is an h-central semiring i.e. $C_h(S) = S$. Let $a \in S = C_h(S)$. Then for any $b \in S$, $a+ab = a \Longrightarrow a+ab+a = a+a \Longrightarrow a+a+ab = a+a \Longrightarrow ab = 0$, since S is a zerosumfree semirng.

Conversely, suppose that $ab = 0$ for all $a, b \in S$. This implies that $ab + a = 0 + a \Longrightarrow ab + a = a$ and $ab = 0 = ba$ for all $a, b \in S$. Thus, S is an h-central semiring. □

Proposition 4.6.18. *Let S be h- central semiring. If $(S, \cdot)$ is a band, then $(S, +)$ is E-inversive semigroup.*

Proof. Since S is an h-central semiring and $(S, \cdot)$ is a band, we need to demonstrate that $(S, +)$ forms an E-inversive semigroup. Let's consider an element a belonging to the set $C_h(S)$. For any $a, b \in S$, if $a + ab = a$, then $(a + ab)\ b = ab$, which further simplifies to $ab + ab^2 = ab$. Utilizing the fact that $(S, \cdot)$ is a band, we can conclude that $ab + ab = ab$. This equality holds for all ab in $E[+]$; where $E[+]$ represents the set of all additive idempotents in $(S, +)$. Hence, there exists an element b in S such that $ab + ab = ab$. This implies that a is an E-inversive element. Consequently, we can establish that $(S, +)$ forms an E-inversive semigroup. $\square$

We conclude this chapter by investigating the behaviour of a h-central semiring in the context of PRD.

Proposition 4.6.19. *Let S be a PRD. If S be a h-central semiring, then $1 + a = a$ for all a in S.*

Proof. Consider an h-central semiring S and let $a \in C_h(S)$. In this case, for any element $b \in S$, the equation $a + ab = a$ holds true. Since the operation $(S, \cdot)$ forms an abelian group, it guarantees the existence of an inverse a^{-1} for every element $a \in S$. Let's choose b to be a^{-1}, resulting in $a + aa^{-1} = a$. Consequently, we can conclude that $a + 1 = a$ holds for all elements a in S. $\square$

By virtue of Proposition 4.6.19, we arrive at the following Corollary :

Corollary 4.6.20. *Let S be a PRD. If S be a h-central semiring, then $a + b = a$ for all $a, b \in S$.*

Proof. By applying proposition 4.6.19, we establish that if S is a h-central semiring, then for all $a \in S$, the equation $1 + a = a$ holds. Multiplying both sides of the equation $a + 1 = a$ by 'b', we can deduce that $ab + b = ab$ implies $a + ab + b = a + ab$, which further implies $a + b = a$. This deduction relies on the fact that S is a h-central semiring and $a \in S = C_h(S)$. $\square$

Chapter 5

k-Center of a Semiring

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5.1 Introduction

Ideal of semiring plays a prominant role in structure theory and contributes significantly to the development of several branches within semiring theory. However, it is important to note that the notion of a semiring theoretical ideal does not generally align with the concept of a ring theoretical ideal. In fact, many results pertaining to ideals in rings have no analogues in semirings. Henriksen highlighted this discrepancy by observing that not every ideal I of a semiring S can be regarded as the kernel of a homomorphism. To address these challenges, in 1958, he [33] defined a more restricted class of ideals in a semiring, which he called this special kind of ideals a k-ideal or subtractive. An ideal I of a semiring S is referred to as a k-ideal or subtractive ideal if, for any two elements a in I and x in S such that $a + x$ is in I, x must also be in I. The significant findings derived from the study of k-ideals motivated us to define new type of center of semiring namely "k-center" of semiring S. The k-center, denoted by $C_k(S)$, is defined as the set of elements a in S that satisfy the conditions $a + ab = ab$ and $ab = ba$ for all nonzero elements b in S. In 1936, J. Von Neumann established [49] the well known theorem "The center of regular ring is regular". Additionally, we have also extended the above theorem for k-center of regular semiring. Furthermore, we analyze the concept of k-center of semiring S and and derive several noteworthy outcomes concerning that center of semiring. In the concluding section, we construct an interesting class of semiring namely, k-central semiring and delve into the structural properties of that semiring with the assistance of k-center of semiring S.

5.2 $C_k(S)$ of a Semiring S

In this segment, the concept of k-center in a semiring S is introduced, accompanied by relevant examples and fundamental findings that will prove beneficial in the following sections. To begin, we establish the definition of the k-center of a semiring.

Definition 5.2.1. Let S be a semiring. Then $C_k(S) = \{a \in S : a + ab = ab$ and $ab = ba$ for all $b \in S\backslash\{0\}\}$ is called k-center of S.

Example 5.2.2. *Consider the semiring $(\mathbb{N}, \oplus, \odot)$ with $a \oplus b = max\{a, b\} = b$ as addition on $\mathbb{N}$; where $b > a$ and $a \odot b$ for the usual multiplication of natural numbers. For any $b \in \mathbb{N}$, $a \oplus ab = max\{a, ab\} = ab$ and $ab = ba$. Then $C_k(\mathbb{N}) = \mathbb{N}$.*

Example 5.2.3. *Let's examine the semiring $(\mathbb{N}, \oplus, \odot)$; where $a \oplus b$ is defined as the maximum of a and b, and $a \odot b$ is defined as the minimum of a and b for any $a, b \in \mathbb{N}$. In this context, the set $C_k(\mathbb{N})$ is found to be 1.*

Example 5.2.4. *Consider the set $S = \{0, 1, x\}$. Define two operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S, +, \cdot)$ forms a semiring and $C_k(S) = \{0, x, 1\} = S$.

Example 5.2.5. *$\{$A class of finite semiring$\}$: Let n, i be integers such that $2 \leq n$, $0 \leq i < n$, and $B(n, i) = \{0, 1, 2, ..., n-1\}$. We define addition and multiplication in $B(n, i)$ by the following equations (let $m = n - i$) :*

$$x + y = \begin{cases} x + y, & \text{if } x + y \leq n - 1 \\ l, & \text{if } x + y \geq n \text{ ; where } l \equiv (x + y) mod m \text{ and } i \leq l \leq n - 1. \end{cases}$$

$$x \cdot y = \begin{cases} xy, & \text{if } xy \leq n-1 \\ l, & \text{if } xy \geq n \text{ ; where } l \equiv (xy) mod m \text{ and } i \leq l \leq n-1. \end{cases}$$

Then the set $B(n,i)$ is a commutative semiring with zero (0) and identity (1) under addition and multiplication.

In particular, let $n = 3$ and $i = 1$, then we have $B(3,1) = \{0,1,2\}$. The set $B(3,1)$ is a commutative semiring under addition ("+") and multiplication (".") . The operations "+" and "." on S_1 by means of the following tables :

+	0	1	2
0	0	1	2
1	1	2	1
2	2	1	2

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

Then $C_k(B(3,1)) = \{0,2\}$.

Example 5.2.6. *Let $S = \{0,a,b,c\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	c	c

Thus, $(S,+,\cdot)$ forms a non-commutative semiring. Additionally, in this case, $C_k(S) = Z(S) = \{0,a\}$.

Proposition 5.2.7. *If S is a semiring with zero element 0, then $0 \in C_k(S)$.*

Proof. Let $a \in C_k(S)$. Since $a \in C_k(S)$, $a + ax = ax$ and $ax = xa$ for all $x \in S \backslash \{0\}$. For any $a \in S \backslash \{0\}$, $0 + 0 \cdot a = 0 = 0 \cdot a$ and $a \cdot 0 = 0 = a \cdot 0$. Hence, $0 \in C_k(S)$. □

Theorem 5.2.8. *The k-center of a semiring S is a subsemiring of S.*

Proof. Given a semiring S with a zero element 0, we can conclude that $0 \in C_k(S)$. As a result, $C_k(S)$ is guaranteed to be non-empty. Consider a semiring S, and let $x, y \in C_k(S)$. Our task is to demonstrate that $x+y$ and xy also belong to $C_k(S)$.

For any s in S excluding 0, we can observe that $(x+y)+(x+y)s = x+xs+y+ys = xs+ys = (x+y)s$. Moreover, $(x+y)s = xs+ys = sx+sy = s(x+y)$, utilizing the fact that $x, y \in C_k(S)$. Thus, we conclude that $x+y \in C_k(S)$.

Furthermore, we have $xy+xys = x(y+ys) = x(ys) = xys$ and $xys = xsy = sxy$, given that $x, y \in C_k(S)$. Consequently, we can deduce that $xy \in C_k(S)$. Thus, we have shown that $C_k(S)$ is a subsemiring of S. □

In light of Theorem 5.2.8, we can deduce the following outcome :

Corollary 5.2.9. *If S is a semiring with identity element 1, then $C_k(S)$ is an additively idempotent subsemiring of S.*

Proof. By utilizing Theorem 5.2.8, it becomes evident that $C_k(S)$ forms a subsemiring within the semiring S. Suppose $a \in C_k(S)$, then, we have the conditions $a+ax = ax$ and $ax = xa$ for all $x \in S\backslash\{0\}$. Notably, this implies that $a + a \cdot 1 = a \cdot 1 = a$. Consequently, we can conclude that $C_k(S)$ constitutes an additively idempotent subsemiring of S. □

Proposition 5.2.10. *If a semiring S holds additively cancellation property, then $C_k(S) = \{0\}$.*

Proof. Let $a \in C_k(S)$. Since $a \in C_k(S)$, $a+ab = ab$ and $ab = ba$ for all $b \in S\backslash\{0\}$. If S holds additively cancellation property, then $a = \{0\}$. Therefore, $C_k(S) = \{0\}$. □

Remark 5.2.11. *Let S be a semiring and $a \in C_k(S)$. Since $a \in C_k(S)$, $a+ax = ax$ and $ax = xa$ for all $x \in S\backslash\{0\}$. If $x = a$, then $a+a^2 = a^2$. This implies that 'a' is a almost idempotent of S. Thus we see that in a semiring, every element of k-center is almost idempotent.*

The converse is not true. For example, we know that $\mathcal{P}_f(F)$ is an almost idempotent semiring if and only if F is a band. Take $b \in F$ as a non-absorbing element. So,

$bx \neq b$ for all $x \in F$. Again $\{b\} \cup \{b\}\{x\} = \{x\}\{b\} \Longrightarrow \{b, bx\} \neq \{xb\}$. Consequently, this suggests that $\{b\} \notin C_k(\mathcal{P}_f(F))$.

If S is a semiring with identity element 1, then $1 \in Z(S)$ but $1 \notin C_k(S)$.

However, we have the following result for a c-semiring S :

Proposition 5.2.12. *Let S be a c-semiring. Then $1 \in C_k(S)$ if and only if $|S| = 2$.*

Proof. Given that S is a c-semiring, we can deduce that $0, 1 \in S$. Considering $1 \in C_k(S)$, we observe that $1 + 1b = 1b$ holds true for all $b \in S \backslash \{0\}$. As S is a c-semiring, we can conclude that $x + 1 = 1$ for all $x \in S$, which implies $b = 1$. Consequently, we have $S = \{0, 1\}$.

On the other hand, let us assume that the cardinality of S is 2. Since S is a c-semiring, we can establish that $0, 1 \in S$. Consequently, we find that $S = \{0, 1\}$. As S is a c-semiring, we have $x + 1 = 1$ for all $x \in S$. As a result, we can infer that $1 \in C_k(S)$. □

The next corollary is an easy consequence of Proposition 5.2.12.

Corollary 5.2.13. *If S is a c-semifield, then $C_k(S) = \{0, 1\}$.*

5.3 $C_k(S)$ of Different Class of Semirings

In this section, we analyze the algebraic structure of the k-center in various types of semirings.

The notion that the center of a regular ring constitutes a regular ring is widely recognized in the field of ring theory. The following theorem serves to illustrate that this result extends to k-center in semiring theory as well.

Theorem 5.3.1. *The k-center of a regular semiring is a regular.*

Proof. Let S be a regular semiring, and a be an arbitrary element in $C_k(S)$. Our goal is to prove that there exists $y \in C_k(S)$ such that $a = aya$. We begin by noting that since $a \in C_k(S)$, we have $a \in S$. As S is regular, we can write $a = axa = aax = a^2x$, for some $x \in S$, since $a \in C_k(S)$. Thus, $a^2x\ (= a) \in C_k(S)$. Let $z \in S$ be arbitrary. Since $a^2x \in C_k(S)$, we have $a^2xz = za^2x$ which implies $xa^2z = a^2zx$ since $a \in C_k(S)$. Therefore, a^2z commutes with x. Next, we aim to show that a^2z commutes with x^3. We have $a^2zx^3 = a^2zxx^2 = xa^2zx^2 = xa^2zxx = xxa^2zx = xxxa^2z = x^3a^2z$. Thus, $a^2zx^3 = x^3a^2z$, and since $a^2 \in C_k(S)$, we obtain $za^2x^3 = a^2x^3z$. Hence, a^2x^3 commutes with z. Let $y = a^2x^3$. Then y commutes with z. Since $a^2x = a$, we have $y = a^2x^3 = a^2xx^2 = ax^2$. Thus, ax^2 commutes with z. For any $z \in S \setminus \{0\}$, we have $ax^2 + ax^2z = ax^2 + zax^2 = (a + za)x^2 = (a + az)x^2 = azx^2 = zax^2 = ax^2z$, since $a \in C_k(S)$, which implies $za = az$. Hence, $ax^2 + ax^2z = ax^2z$ and $zax^2 = ax^2z$. Thus, $ax^2 \in C_k(S)$, i.e., $y \in C_k(S)$. Now, we can calculate aya as follows : $aya = a(ax^2)a = a^2x^2a = a^2xxa = axa = a$, since $a^2x = a = axa$. Therefore, we have shown that $aya = a$ for some $y \in C_k(S)$. Since a was chosen arbitrarily, we conclude that every element of $C_k(S)$ is regular. Hence, $C_k(S)$ is a regular semiring. □

However, it should be noted that the converse of Theorem 5.3.1 does not hold in general. For instance, let's examine Example 5.2.3; where $C_k(\mathbb{N}) = \{1\}$. In this case, $C_k(\mathbb{N})$ is considered regular. However, upon further inspection, it can be verified that $(\mathbb{N}, \oplus, \odot)$ is not regular.

Proposition 5.3.2. *If S is a semiring, then $C_k(S)$ is a k-regular semiring.*

Proof. Let $a \in C_k(S)$. Then $a^2 \in C_k(S)$, since $C_k(S)$ is a subsemiring of S. Additionally, we have $a + a \cdot a^2 = a^2 \cdot a$, indicating that $a + a \cdot a \cdot a = a \cdot a \cdot a$. Therefore, we can conclude that $C_k(S)$ is a k-regular semiring. □

It is important to observe that a k-regular semiring is a π-regular. Consequently, the following corollary can be derived :

Corollary 5.3.3. *If S is a semiring, then $C_k(S)$ is a π-regular.*

Proposition 5.3.4. *If S is a semiring, then $C_k(S)$ is completely k-regular.*

Proof. Assume S is a semiring. In this case, $C_k(S)$ is known to be k-regular. Consider an element $a \in C_k(S)$. It follows that there exists an element $x \in C_k(S)$ such that

$a + axa = axa$ (i).

By multiplying both sides of equation (i) by x, we obtain $ax + axax = axax \implies ax + xaxa = xaxa$. Letting $ax = u$, we have $ax + xua = xua$. Once again, multiplying both sides of equation (i) by x leads us to $xa + xaxa = xaxa \implies xa + axax = axax$. Setting $xa = u$, we get $xa + aux = aux$. As a result, we can conclude that S is completely k-regular. □

Proposition 5.3.5. *If S is a semiring, then $C_k(S)$ is intra k-regular.*

Proof. Let $a \in C_k(S)$. Since $C_k(S)$ forms a subsemiring of S, we can deduce that $a^3 \in C_k(S)$. We can rewrite the equation $a + a \cdot a^3 = a \cdot a^3$ as $a + a \cdot a^2 \cdot a = a \cdot a^2 \cdot a$. Therefore, $a + \sum_{i=1}^{n} a_i z^2 c_i = \sum_{i=1}^{n} c_j z^2 d_j$ for all $z, a_i, b_i, c_j, d_j \in S$. Therefore, we can conclude that $C_k(S)$ is intra k-regular. □

Proposition 5.3.6. *If S is a semiring, then $C_k(S)$ is rectangular almost idempotent semiring.*

Proof. Considering an element $a \in C_k(S)$, we observe that $a^4 \in C_k(S)$, as $C_k(S)$ forms a subsemiring of S. Therefore, we can deduce that $a + a \cdot a^4 = a \cdot a^4$, which can be further simplified as $a + a \cdot a \cdot a \cdot a \cdot a = a \cdot a \cdot a \cdot a \cdot a$. This implies that $C_k(S)$ qualifies as a rectangular almost idempotent semiring. □

Proposition 5.3.7. *If S is a semiring, then $C_k(S)$ is zero almost idempotent semiring.*

Proof. Suppose $a \in C_k(S)$. Consequently, $a^2 \in C_k(S)$ as well. Furthermore, we have $a + a \cdot a^2 = a \cdot a^2 \implies a + a \cdot a \cdot a = a \cdot a \cdot a$. Consequently, we can conclude that $C_k(S)$ forms a zero almost idempotent semiring. □

Proposition 5.3.8. *If S is a multiplicatively band, then $C_k(S)$ is completely regular.*

Proof. Let a be an element of $C_k(S)$. Consequently, a^2 also belongs to $C_k(S)$ due to the fact that $C_k(S)$ is a subsemiring of S. Additionally, since every element of $C_k(S)$ is almost idempotent, we have $a + a^2 + a = a^2 + a = a^2 = a$, using the fact that S is a multiplicatively band. Moreover, we can observe that $a(a^2 + a) = a \cdot a^2 = a \cdot a = a^2 = a^2 + a$. Thus, we can conclude that $C_k(S)$ is completely regular. □

Theorem 5.3.9. *Let S be a simple semiring. If $C_k(S)$ is a simple semiring if and only if $S = \{0, 1\}$*

Proof. Let S be a simple semiring and $C_k(S)$ be also a simple semiring. Then $1+a = 1$ for all $a \in C_k(S)$ (i).
Now, if $1 \in C_k(S)$, then $1 + 1 \cdot a = 1 \cdot a$ for all $a \in S\backslash\{0\} \Longrightarrow 1 + a = a$ (ii).

By combining (i) and (ii), we get $a = 1$ for all $a \in S$. Again we know that $0 \in C_k(S)$. So, $S = \{0, 1\}$.

Conversely, suppose that $S = \{0, 1\}$. Again S is a semiring with identity element 1, from Corollary 5.2.9, $C_k(S)$ is an additively idempotent semiring i.e. $a + a = a$ for all $a \in C_k(S)$. So, $1 + a = 1$ for all $a \in C_k(S)$. Accordingly, $C_k(S)$ is a simple semiring. □

5.4 Characteristics and Attributes of $C_k(S)$

In the following section, we outline a number of properties pertaining to the k-center of a semiring.

Theorem 5.4.1. *If φ is an epimorphism from S to S' then $\varphi(C_k(S)) \subseteq C_k(S')$.*

Proof. Let us consider S be an semiring and $\varphi : S \longrightarrow S'$ is an epimorphism of S'. Suppose $\varphi(C_k(S)) = \{\varphi(s) : s \in C_k(S)\}$. Our aim is to provide evidence that $\varphi(C_k(S)) \subseteq C_k(S')$. Let $\varphi(s_1) \in \varphi(C_k(S))$; where $s_1 \in S'\backslash\{0\}$. Since φ is onto, there exist $y \in S$ such that y has a preimage x of $S'\backslash\{0\}$ i.e. $\varphi(y) = x$. Since $s_1 \in C_k(S)$, we have $s_1 + s_1y = s_1y$ and $s_1y = ys_1$. Now for any $x \in S'\backslash\{0\}$, we can observe the following equalities : $\varphi(s_1) + \varphi(s_1)x = \varphi(s_1) + \varphi(s_1)\varphi(y) = \varphi(s_1) + \varphi(s_1y) = \varphi(s_1+s_1y) = \varphi(s_1y) = \varphi(s_1)\varphi(y) = \varphi(s_1)x$. Moreover, we have $\varphi(s_1)x = \varphi(s_1)\varphi(y) = \varphi(s_1y) = \varphi(ys_1) = \varphi(y)\varphi(s_1) = x\varphi(s_1)$, since φ is an epimorphism. Consequently, we can conclude that $\varphi(s_1) \in C_k(S')$, and thus, $\varphi(C_k(S)) \subseteq C_k(S')$. □

Theorem 5.4.2. *Let S and S' be two semirings. If $f : S \longrightarrow S'$ is a monomorphism, then $f(C_k(S)) = C_k(f(S))$.*

Proof. Let us assume $x \in f(C_k(S))$. This implies that $x = f(y)$ for some $y \in C_k(S)$. Our goal is to demonstrate that $f(y) + f(y)s = f(y)s$ and $f(y)s = sf(y)$ for all $s \in f(S)\backslash\{0\}$. Now for any $s \in f(S)\backslash\{0\}$, we have $f(y) + f(y)s = f(y) + f(y)f(r) = f(y + yr) = f(yr) = f(y)f(r) = f(y)s$ and $f(y)s = f(y)f(r) = f(yr) = f(ry) = f(r)f(y) = sf(y)$. Therefore, $x = f(y) \in C_k(f(S))$. Hence, $f(C_k(S)) \subseteq C_k(f(S))$. Now, let's consider $x' \in C_k(f(S))$. Then $x' = f(r')$ for some $r' \in S$. Our objective is to prove that $r' \in C_k(S)$. Since $x' \in C_k(f(S))$, for any $f(s) \in f(S)\backslash\{0\}$, it follows that $x'+x'f(s) = x'f(s) \Longrightarrow f(r')+f(r')f(s) = f(r')f(s) \Longrightarrow f(r'+r's) = f(r's) \Longrightarrow r' + r's = r's$ and $x'f(s) = f(s)x' \Longrightarrow f(r')f(s) = f(s)f(r') \Longrightarrow f(r's) = f(sr') \Longrightarrow r's = sr'$, since f is a monomorphism. Therefore, $r' \in C_k(S)$. Consequently, we can establish that $C_k(f(S)) \subseteq f(C_k(S))$, leading to the conclusion that $f(C_k(S)) = C_k(f(S))$. □

Theorem 5.4.3. *If two semirings S_1 and S_2 are isomorphic, then their k-centers $C_k(S_1)$ and $C_k(S_2)$ are isomorphic.*

Proof. Suppose we have two isomorphic semirings S_1 and S_2. In this case, there is an isomorphism $f : S_1 \longrightarrow S_2$. Let's consider an element $x \in C_k(S_1)$. For any non-zero element $s_1 \in S_1$, we have $x + xs_1 = xs_1$ and $xs_1 = s_1x$. Let's define $f(x) = y$; where $y \in S_2$. Since f is an isomorphism, for any non-zero element $s_2 \in S_2$, there exists $s_1 \in S_1\backslash\{0\}$ such that $f(s_1) = s_2$. Consequently, we have $y + ys_2 = f(x) + f(x)f(s_1) = f(x + xs_1) = f(xs_1) = f(x)f(s_1) = ys_2$ and $ys_2 = f(x)f(s_1) = f(xs_1) = f(s_1x) = f(s_1)f(x) = s_2y$, considering that $x \in C_k(S)$. As a result, we can conclude that $y \in C_k(S_2)$. Thus, we have $f(C_k(S_1)) \subseteq C_k(S_2)$. Now, let's consider an element $b \in C_k(S_2)$. We can express $b = f(a)$; where $a \in S_1$. Since f is an isomorphism, for any $y \in S_2\backslash\{0\}$, there exists $x \in S_1$ such that $y = f(x)$. Since $b \in C_k(S_2)$, it follows that $b + by = by$ and $by = yb$. By examining these equations, we can deduce that $b + by = by \Longrightarrow f(a) + f(a)f(x) = f(a)f(x) \Longrightarrow f(a + ax) = f(ax) \Longrightarrow a + ax = ax$ and $by = yb \Longrightarrow f(a)f(x) = f(x)f(a) \Longrightarrow f(ax) = f(xa) \Longrightarrow ax = xa$, assuming that f is an isomorphism. These conclusions lead us to the fact that $a \in C_k(S_1)$. Consequently, we can state that $b = f(a) \in f(C_k(S_1))$. Therefore,

$C_k(S_2) \subseteq f(C_k(S_1))$, which implies that $C_k(S_2) = f(C_k(S_1))$. Hence, we can assert that $g = f_{|_{C_k(S_1)}} : C_k(S_1) \longrightarrow C_k(S_2)$ is an isomorphism from $C_k(S_1)$ onto $C_k(S_2)$. □

Usually, the converse of Theorem 5.4.3 is invalid. This can be observed by considering the following counter example.

Example 5.4.4. *It should be noted that* $C_k(\mathbb{Z}_0^+) = \{0\}$ *and* $C_k(\mathbb{R}_0^+) = \{0\}$. *Hence,* $C_k(\mathbb{Z}_0^+)$ *and* $C_k(\mathbb{R}_0^+)$ *are isomorphic but* $\mathbb{Z}_0^+$ *and* $\mathbb{R}_0^+$ *are not isomorphic.*

Proposition 5.4.5. *Let* S *be a c-semiring. If* $x \in C_k(S)$, *then* $x = xa$ *for all* $a \in S\backslash\{0\}$.

Proof. Consider a c-semiring S and let x belong to the set $C_k(S)$. Given $a \in S\backslash\{0\}$, it follows from $x \in C_k(S)$ that $x+x\cdot a = x\cdot a$. This equation can be further simplified as $x\cdot 1+x\cdot a = x\cdot a$, which leads to $x(1+a) = x\cdot a$. Since 1 acts as an absorbing element with respect to addition, we deduce $x \cdot 1 = x \cdot a$, ultimately yielding $x = xa$. □

Remark 5.4.6. *If* $C_k(S) = S$, *then* S *is commutative. The converse may not hold, as demonstrated in example 5.2.5; where* S *is commutative despite* $C_k(S) \neq S$. *Given this scenario, we can observe that* $B(3,1) = \{0,1,2\}$ *and* $C_k(B(3,1)) = \{0,2\}$. *Consequently, it is evident that* $C_k(B(3,1))$ *does not equal* $B(3,1)$.

Proposition 5.4.7. *If* S *is a semiring, then* $C_k(S)$ *is a semidomain.*

Proof. Considering a and b as nonzero elements in $C_k(S)$, let $ab = 0$. Since a belongs to $C_k(S)$, it satisfies the condition $a + ax = ax$ and $xa = ax$ for all x in S (except 0). By substituting $x = b$ into the previous equations, we find that $a + ab = ab$ and $ab = ba$. Consequently, we conclude that $a + 0 = 0$, which implies $a = 0$. Hence, $C_k(S)$ can be identified as a semidomain. □

A widely acknowledged fact is that if S is a semiring, the center of the matrix semiring $M_n(S)$ can be expressed as $Z(M_n(S)) = \{aI_n \in M_n(S) : a \in Z(S)\}$; where I_n represents the $n \times n$ identity matrix.

Regarding the k-center of the matrix semiring, we have the following notable outcome :

Theorem 5.4.8. *Let S be a semiring with identity. Then $a \in C_k(S)$ if and only if $aI_n \in C_k(M_n(S))$.*

Proof. Let $a \in C_k(S)$. Then $a + ab = ab$ for all $b \in S \setminus \{0\}$.

Now $aI_n + (aI_n)B$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & ab_{22} & \cdots & ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & ab_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a + ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & a + ab_{22} & \cdots & ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & a + ab_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & ab_{22} & \cdots & ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & ab_{nn} \end{pmatrix} = (aI_n)B, \text{ since } a \in C_k(S).$$

Again $$(aI_n)B = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & ab_{22} & \cdots & ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & ab_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}a & b_{12}a & \cdots & b_{1n}a \\ b_{21}a & b_{22}a & \cdots & b_{2n}a \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}a & b_{n2}a & \cdots & b_{nn}a \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} = B(aI_n),$$

since $a \in C_k(S)$. This implies that $aI_n \in C_k(M_n(S))$.

Conversely, suppose that $aI_n \in C_k(M_n(S))$. We want to show that $a \in C_k(S)$.

For any $x \in S\backslash\{0\}$, let B $= \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}$. Since $aI_n \in C_k(M_n(S))$, it follows that $aI_n + (aI_n)B = (aI_n)B$ and $(aI_n)B = B(aI_n)$. Comparing both sides, we find that $a + ax = ax$ and $ax = xa$. Consequently, we can conclude that $a \in C_k(S)$. □

By employing the aforementioned Theorem 5.4.8, we derive the subsequent result:

Corollary 5.4.9. *Let S be a semiring with identity. Then the k-center $C_k(M_n(S)) = \{aI_n \in M_n(S) : a \in C_k(S)\}$; where I_n is the $n \times n$ identity matrix.*

The k-center of a polynomial semiring over a semiring yields the following result :

Theorem 5.4.10. *Let $S[x]$ be a polynomial semiring over a semiring S. Then $f(x) = a_0 + a_1x + ... + a_nx^n \in C_k(S[x])$ if and only if $a_i \in C_k(S); i = 0, 1, 2, ..., n$.*

Proof. Consider the function $f(x) = a_0 + a_1x + ... + a_nx^n \in C_k(S[x])$. Let $d \in S\backslash\{0\}$. We can deduce that $d \in S \setminus \{0\}$ implies that $d \in S[x] \setminus \{0\}$. Since $f(x) \in C_k(S[x])$, we have $f(x) + f(x)d = f(x)d$ and $f(x)d = df(x)$. This implies that $(a_0 + a_1x + ... + a_nx^n) + (a_0 + a_1x + ... + a_nx^n)d = (a_0 + a_1x + ... + a_nx^n)d$ and $d(a_0 + a_1x + ... + a_nx^n) = (a_0 + a_1x + ... + a_nx^n)d$. Thus, we have $(a_0 + a_0d) + (a_1 + a_1d)x + ... + (a_n + a_nd)x^n = a_0d + a_1dx + ... + a_ndx^n$ and $da_0 + da_1x + ... + da_nx^n = a_0d + a_1dx + ... + a_ndx^n$. Comparing both sides, it follows that $a_0 + a_0d = a_0d$, $a_1 + a_1d = a_1d, ..., a_n + a_nd = a_nd$ and $a_0d = da_0$, $a_1d = da_1, ..., a_nd = da_n$. As d is arbitrary, we conclude that $a_i \in C_k(S)$, $i = 0, 1, ..., n$.

Conversely, suppose that $a_i \in C_k(S)$. Since $a_i \in C_k(S)$, $a_i + a_ib = a_ib$ and $a_ib = ba_i$ for all $b \in S\backslash\{0\}$. We have to show that $f(x) \in C_k(S[x])$. For any $g(x) = b_0 + b_1x + ... + b_kx^k \in S[x]\backslash\{0\}$, $f(x) + f(x)g(x) = (a_0 + a_1x + ... + a_nx^n) + (a_0 + a_1x + ... + a_nx^n)(b_0 + b_1x + ... + b_kx^k) = (a_0 + a_1x + ... + a_nx^n) + \big(a_0b_0 + \sum_{i+j=1} a_ib_jx + \sum_{i+j=2} a_ib_jx^2 + ... + \sum_{i+j=n} a_ib_jx^n + ... + a_nb_kx^{n+k}\big) = a_0 + a_0b_0 + (a_1 + \sum_{i+j=1} a_ib_j)x + (a_2 + \sum_{i+j=2} a_ib_j)x^2 + ... + (a_n + \sum_{i+j=n} a_ib_j)x^n + ... + a_nb_kx^{n+k} = a_0b_0 + \sum_{i+j=1} a_ib_jx + \sum_{i+j=2} a_ib_jx^2 +$

$... + \sum_{i+j=n} a_ib_jx^n + ... + a_nb_kx^{n+k} = (a_0 + a_1x + ... + a_nx^n)(b_0 + b_1x + ... + b_kx^k) = f(x)g(x)$. Again $f(x)g(x) = (a_0 + a_1x + ... + a_nx^n)(b_0 + b_1x + ... + b_kx^k) = a_0b_0 + \sum_{i+j=1} a_ib_jx + \sum_{i+j=2} a_ib_jx^2 + ... + \sum_{i+j=n} a_ib_jx^n + ... + a_nb_kx^{n+k} = b_0a_0 + \sum_{i+j=1} b_ja_ix + \sum_{i+j=2} b_ja_ix^2 + ... + \sum_{i+j=n} b_ja_ix^n + ... + b_ka_nx^{n+k} = (b_0 + b_1x + ... + b_kx^k)(a_0 + a_1x + ... + a_nx^n) = g(x)f(x)$. Thus, $f(x) + f(x)g(x) = f(x)g(x)$ and $f(x)g(x) = g(x)f(x)$. Therefore, $f(x) \in C_k(S[x])$. □

The corollary stated below is a direct consequence of Theorem 5.4.10.

Corollary 5.4.11. *Let S be a semiring and $S[x]$ be the polynomial semiring over S. Then the k-center $C_k(S[x]) = \{a_0 + a_1x + a_2x^2 + ... + a_nx^n \in S[x] : a_i \in C_k(S),\ i = 0, 1, 2, ..., n\}$.*

We would like to share an extremely useful Theorem concerning the cartesian product of the k-center of two semirings.

Theorem 5.4.12. *If S_1 and S_2 are two semirings, then $C_k(S_1 \times S_2) = C_k(S_1) \times C_k(S_2)$.*

Proof. Let S_1 and S_2 be two semirings with zero elements 0_{S_1} and 0_{S_2} respectively. Let $z \in C_k(S_1 \times S_2)$. Then $z = (x, y) \in S_1 \times S_2$ and for any $(a, b) \in (S_1 \times S_2)\backslash\{0_{S_1}, 0_{S_2}\}$, $(x, y) + (x, y)(a, b) = (x, y)(a, b)$ and $(x, y)(a, b) = (a, b)(x, y) \Longrightarrow (x + xa, y + yb) = (xa, yb)$ and $(xa, yb) = (ax, by)$. Comparing both sides, we get $x + xa = xa$, $y + yb = yb$, $xa = ax$, $yb = by$. As a result, we can conclude that $x + xa = xa$, $xa = ax$ for all $a \in S_1\backslash\{0_{S_1}\}$ and $y + yb = yb$, $yb = by$ for all $b \in S_2\backslash\{0_{S_2}\}$. Consequently, we can deduce that $x \in C_k(S_1)$ and $y \in C_k(S_2)$. Therefore, it can be concluded that $z = (x, y) \in C_k(S_1) \times C_k(S_2)$, which implies $C_k(S_1 \times S_2) \subseteq C_k(S_1) \times C_k(S_2)$ (i).

For reverse part, let $(a, b) \in C_k(S_1) \times C_k(S_2)$. This means that $a \in C_k(S_1)$ and $b \in C_k(S_2)$. For any $x \in S_1\backslash\{0_{S_1}\}$, we observe that $a + ax = ax$, $ax = xa$ and $y \in S_2\backslash\{0_{S_2}\}$, we have $b + by = by$, $by = yb$. Now $(a, b) + (a, b)(x, y) = (a + ax, b + by) = (ax, by) = (a, b)(x, y)$. Likewise, we can show that $(a, b)(x, y) = (ax, by) =$

$(xa, yb) = (x, y)(a, b)$, using the fact that $a \in C_k(S_1)$ and $b \in C_k(S_2)$. This implies that $(a, b) \in C_k(S_1 \times S_2)$ and hence $C_k(S_1) \times C_k(S_2) \subseteq C_k(S_1 \times S_2)$ (ii).

From (i) and (ii), we can conclude that $C_k(S_1 \times S_2) = C_k(S_1) \times C_k(S_2)$. □

We will now demonstrate the practical use of Theorem 5.4.12 through the following example.

Example 5.4.13. *Consider two semirings* $(\mathbb{N}, \oplus, \odot)$ *with* $a \oplus b = max\{a, b\} = b$ *as addition on* $\mathbb{N}$*; where* $b > a$ *and* $a \odot b$ *for the usual multiplication of natural numbers and* $(\mathbb{Z}_0^+, +, \cdot)$*, where "+" is usual addition and "·" is usual multiplication. Now we take a semiring* $(\mathbb{N} \times \mathbb{Z}_0^+, +, \cdot)$ *with component-wise addition and multiplication. Then* $C_k(\mathbb{N} \times \mathbb{Z}_0^+) = C_k(\mathbb{N}) \times C_k(\mathbb{Z}_0^+) = \mathbb{N} \times \{0\}$.

In the subsequent theorem, our primary objective is to establish that the k-center of a semiring, which is a rectangular band under multiplication, indeed constitutes a b-lattice semiring.

Theorem 5.4.14. *Let* S *be a semiring with identity* 1*. If* $(S, \cdot)$ *is a rectangular band, then* $C_k(S)$ *is a b-lattice semiring.*

Proof. In the case where S is a semiring with an identity element 1, it is known that $C_k(S)$ forms an additively idempotent semiring and possesses additive commutativity. Consequently, the structure $(C_k(S), +)$ can be recognized as a semilattice. Let us consider an element $a \in C_k(S)$. For any nonzero element $b \in S$, the equation $a + ab = ab$ holds. By substituting $b = a$ into this equation, we obtain $a + a \cdot a = a \cdot a$, which can be simplified as $a + a^2 = a^2$ (i).

Additionally, due to the property of $(S, \cdot)$ being a rectangular band, we have $a + ab = ab \Longrightarrow (a + ab) \cdot a = ab \cdot a \Longrightarrow a \cdot a + aba = aba \Longrightarrow a^2 + a = a$ (ii).

Combining equations (i) and (ii), we conclude that $a^2 = a$. Thus, we establish that $(C_k(S), .)$ forms a band. Consequently, $C_k(S)$ can be classified as a b-lattice semiring. □

The next aim on our agenda is to establish that $C_k(S)$ forms a semilattice. To accomplish this, we initiate with the technical finding.

Theorem 5.4.15. *Let S be a semiring with identity element 1. Define a binary relation "$\leq_S$" on $C_k(S)$ by "$a \leq_S b$" if and only if $a+b=a$ for all a, b in $C_k(S)$. Then $(C_k(S), \leq_S)$ forms a partial ordered set.*

Proof. Now $a \leq_S b \Longleftrightarrow a+b=a$ for all $a, b \in C_k(S)$. Take an element $a \in C_k(S)$. This implies that $a+ax=ax$ holds for all $x \in S\backslash\{0\}$. Setting $x=1$, we get $a+a1=a1$, which simplifies to $a+a=a$. This equation holds for all $a \in C_k(S)$, showing that $a \leq_S a$. Thus, the relation "$\leq_S$" is reflexive.

Now, assume that $a \leq_S b$ and $b \leq_S a$ hold for elements $a, b \in C_k(S)$. According to the definitions, we have $a \leq_S b \Longleftrightarrow a+b=a$ for $a, b \in C_k(S)$, and $b \leq_S a \Longleftrightarrow b+a=b$ for $a, b \in C_k(S)$. This implies that $a=b$ for $a, b \in C_k(S)$. Hence, the relation"$\leq_S$" is antisymmetric.

Suppose $a \leq_S b$ and $b \leq_S c$ hold for elements $a, b, c \in C_k(S)$. Using the definitions, we find that $a \leq_S b \Longleftrightarrow a+b=a$ for $a, b \in C_k(S)$, and $b \leq_S c \Longleftrightarrow b+c=b$ for $b, c \in C_k(S)$. From these two equations, we have $a+c=a+b+c=a+b=a$. Thus, $a \leq_S c$ holds for all $a, c \in C_k(S)$. Consequently, the relation "$\leq_S$" is transitive. Therefore, $(C_k(S), \leq_S)$ forms a partial ordered set. □

Theorem 5.4.16. *If S is a semiring with identity then $C_k(S)$ forms a semilattice.*

Proof. Let S be a semiring with identity element 1. Then from Corollary 5.2.9, it follows that $C_k(S)$ is an additively idempotent semiring. Define a binary relation "$\leq_S$" on $C_k(S)$ by "$a \leq_S b$" if and only if $a+b=a$ for all a, b in $C_k(S)$. We can establish from Theorem 5.4.15 that $(C_k(S), \leq_S)$ forms a partial ordered set. Utilizing the fact that $C_k(S)$ is an additive idempotent semiring, we observe that $a+b+a=a+a+b=a+b$, demonstrating that $a+b \leq_S a$ for all elements a and b in $C_k(S)$. Again $a+b+b=a+b$, since $C_k(S)$ is an additive idempotent semiring. This implies that $a+b \leq_S b$ for all a, b in $C_k(S)$. So, $a+b$ is a lower bound of a and b. We now show that $a+b$ is the greatest lower bound of a and b. Let g be another lower bound of a and b. Then $g \leq_S a \Longleftrightarrow g+a=g$ for all $g, a \in C_k(S)$ and $g \leq_S b \Longleftrightarrow g+b=g$ for all $g, b \in C_k(S)$. Now $g+a=g \Longrightarrow g+a+b=g+b \Longrightarrow g+a+b=g$, since $g+b=g$. Thus, $g \leq_S a+b$ holds for all $a, b \in C_k(S)$. So, $a+b$ is the greatest lower bound of a

and b. Hence, $C_k(S)$ forms a semilattice. □

Theorem 5.4.17. *[28] A commutative semiring is a bounded distributive lattice if and only if it is a simple multiplicatively idempotent semirng.*

Theorem 5.4.18. *If $C_k(S)$ is bounded distributive lattice, then it is a trivial semiring.*

Proof. According to the definition of $C_k(S)$, it can be identified as a commutative semiring. Assuming that $C_k(S)$ is a simple semiring, we can deduce that $1 + a = 1(i)$ for all $a \in C_k(S)$. Furthermore, if $1 \in C_k(S)$ and for any $a \in S\backslash\{0\}$, then we can conclude that $1 + 1 \cdot a = 1 \cdot a$, which simplifies to $1 + a = a$ (ii).

Combining equations (i) and (ii), we arrive at the conclusion that $a = 1$ for all elements $a \in S$. Additionally, we know that $0 \in C_k(S)$. Consequently, it follows that $C_k(S)$ is a trivial semiring. □

Remark 5.4.19. *If S is a semiring with identity, then it follows that $C_k(S)$ constitutes a semilattice. However, the above mentioned Theorem 5.4.18 demonstrates that $C_k(S)$ fails to constitute a bounded distributive lattice. In the event that $C_k(S)$ were to be considered a bounded distributive lattice, it would consequently be reduced to a trivial semiring.*

5.5 k-Central Semiring

This section introduces the concept of a k-central semiring, followed by an examination of its key properties. Subsequently, we delve into establishing various characterizations of this semiring. To begin, we provide the definition of a k-central semiring.

Definition 5.5.1. A semiring S is said to be a k-central semiring if $C_k(S) = S$.

In order to illustrate k-central semiring, we now generate some instances.

Example 5.5.2. *Consider the set of integers $\mathbb{Z}^+$ with the operations $a+b = lcm\{a, b\}$ and $a \cdot b = ab$. Then $(\mathbb{Z}^+, +, \cdot)$ is a semiring with zero element 1. Then $C_k(\mathbb{Z}^+) = \mathbb{Z}^+$. Therefore, $(\mathbb{Z}^+, +, \cdot)$ is a k-central semiring.*

Example 5.5.3. *Consider $S = \{0, x, y, 1\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	y	1
0	0	x	y	1
x	x	x	y	x
y	y	y	y	y
1	1	x	y	1

.	0	x	y	1
0	0	0	0	0
x	0	x	y	x
y	0	y	y	y
1	0	x	y	1

Consequently, $(S, +, \cdot)$ forms a k-central semiring due to the equality $C_k(S) = \{0, x, y, 1\} = S$.

Note that $Z(S)$ is a subsemiring of S but generally, $Z(S) \neq S$ for a semiring S.

Remark 5.5.4. *If S is a k-central semiring, then S is a central semiring but the converse is not necessarily true. Note that commutative semirings are central semirings but not k-central semirings. In particular, semifields are central semirings but not k-central semirings.*

We are now determining the relationship between the k-central semiring and the k-regular semiring.

Theorem 5.5.5. *Every k-central semiring is a k-regular semiring.*

Proof. Let S be a k-central semiring. Then $C_k(S) = S$. Let $a \in C_k(S)$. Then $a^2 \in C_k(S)$, since $C_k(S)$ is a subsemiring of S. Furthermore, we observe that $a + a \cdot a^2 = a \cdot a^2$, implying that $a + a \cdot a \cdot a = a \cdot a \cdot a$. As a result, we conclude that $C_k(S)$ is a k-regular semiring. □

The subsequent pair of illustrations demonstrate that the converse of the aforementioned Theorem 5.5.5 does not hold, meaning that every k-regular semiring is not a k-central semiring.

Example 5.5.6. *The semiring $(\mathbb{Z}_0^+, \oplus, \odot)$ is k-regular; where $a \oplus b = max\{a, b\}$ and $a \odot b = min\{a, b\}$ for all $a, b \in \mathbb{Z}_0^+$. But $C_k(\mathbb{Z}_0^+) = \{0, 1\} \neq S$. Therefore, $(\mathbb{Z}_0^+, \oplus, \odot)$ is not a k-central semiring.*

Example 5.5.7. *Consider $S = \{0, x, 1\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	1	1

Then $(S, +, \cdot)$ is a k-regular semiring. But $C_k(S) = \{0\} \neq S$ i.e. $(S, +, \cdot)$ is not a k-central semiring.

We now present a characterization Theorem of k-central semiring in the context of c-semiring.

Theorem 5.5.8. *Let S be a c-semiring with zero element 0 and identity element 1. Then S is a k-central semiring if and only if $|S| = 2$.*

Proof. If we let $|S| = 2$, then $S = \{0, 1\}$. It is evident that $0, 1 \in C_k(S)$, thus establishing $C_k(S) = S$. As a result, S can be classified as a k-central semiring.

Conversely, if we assume that S is a k-central semiring, denoted as $C_k(S) = S$, then it follows that both 0 and 1 exist within S. This implies that 0 and 1 are elements of $C_k(S)$. Based on Proposition 5.2.12, we can conclude that $|S| = 2$. □

By virtue of Corollary 5.2.9, we can deduce the following result :

Proposition 5.5.9. *If S is a k-central semiring with identity, then S is an additively idempotent semiring.*

Proof. Assume S is a k-central semiring. Then $C_k(S) = S$. Consider an arbitrary element $a \in S = C_k(S)$. For any $b \in S \setminus \{0\}$, we have $a + ab = ab$. In particular, when we take b as the identity element 1, we obtain $a + a \cdot 1 = a \cdot 1$. Consequently, we can deduce that $a + a = a$ for all $a \in S$. As a result, S is an additively idempotent semiring. □

Proposition 5.5.10. *If S is an additively idempotent mono-semiring, then S is a k-central semiring.*

Proof. Suppose S is an additively idempotent mono-semiring. It is evident that $C_k(S) \subseteq S$. Consider $a(\neq 0)$ be an arbitrary element in S. For any b belonging to S excluding 0, we have $a+ab = a+a+b = a+b = ab$ and $ab = a+b = b+a = ba$ as S is an additively idempotent mono-semiring. Consequently, for any b in S (excluding 0), we have $a + ab = ab$ and $ab = ba$. Additionally, $0 \in C_k(S)$. Hence, we can conclude that $S \subseteq C_k(S)$, which implies $S = C_k(S)$. Consequently, S is a k-central semiring. $\square$

Proposition 5.5.11. *Any additively idempotent mono-semiring S with identity element 1 is a k-central semiring if and only if it satisfies the quasi identity $x + 1 = y + 1 \Longrightarrow x = y$ for all $x, y \in S$.*

Proof. Let us prove the necessity. Given that $x+1 = y+1$. As S is a mono-semiring, we find that $x + 1 = y + 1 \Longrightarrow x \cdot 1 = y \cdot 1 \Longrightarrow x = y$. Hence, we conclude that it satisfies the quasi identity.

We aim to demonstrate sufficiently. It is evident that $C_k(S) \subseteq S$. Our task is to establish that $S \subseteq C_k(S)$. Suppose we take an arbitrary element $x(\neq 0)$ from S. Since S is an additively idempotent semiring, we have $x+1 = x+1+1$ ($1+1 = 1$, as S is an additively idempotent semiring) $= (x+1)+1$ for all $x \in S$. This implies that $x+1 = x$ for all $x \in S$ based on the given quasi identity. Furthermore, for any $x \in S \setminus \{0\}$, we can observe that $x + xy = x(1 + y) = xy$. Additionally, $xy = x + y = y + x = yx$, as S is mono-semiring. Consequently, for any $x \in S \setminus \{0\}$, $x + xy = xy$ and $xy = yx$. Thus, it follows that $x(\neq 0) \in C_k(S)$. We can also include $0 \in C_k(S)$. As a result, we can conclude that $S \subseteq C_k(S)$, leading to $S = C_k(S)$. Hence, we establish that S is a k-central semiring. $\square$

Let us now introduce the most important characterization theorem for a k-central semiring.

Theorem 5.5.12. *Let S be a commutative semiring with identity 1. Then S is a k-central semiring if and only if $1 \in C_k(S)$.*

Proof. Assuming S is a k-central semiring, we can establish $C_k(S) = S$. This implies that $1 \in S = C_k(S)$.

Conversely, suppose that $1 \in C_k(S)$. Clearly, $C_k(S) \subseteq S$. Let $a(\neq 0)$ be an arbitrary element in S. For any b in S excluding 0, we have $a + ab = a1 + a1b = a(1 + 1b) = ab$, due to the fact that $1 \in C_k(S)$. Therefore, for any b in S excluding 0, $a + ab = ab$, and since S is commutative, $ab = ba$. Consequently, a (not equal to 0) belongs to $C_k(S)$. Additionally, $0 \in C_k(S)$. Hence, $S \subseteq C_k(S)$ and consequently, $S = C_k(S)$. Therefore, S is a k-central semiring. □

Based on Theorem 5.5.12, the subsequent Corollary can be derived as follows :

Corollary 5.5.13. *Let S be a semiring with identity 1. A central semiring S is a k-central semiring if and only if $1 \in C_k(S)$.*

The subsequent two theorems showcase various properties of a k-central semiring.

Theorem 5.5.14. *Let S be a k- central semiring. If $(S, \cdot)$ is a rectangular band, then the following are true.*

(a) $(S, \cdot)$ is a band.

(b) $(S, +)$ is a band.

Proof. (a) Let S be a k- central semiring. Suppose S is a k-central semiring. Let a and $b \neq 0$ be elements of S such that $a + ab = ab$, which implies $(a + ab)a = aba$. Considering the fact that $(S, \cdot)$ is a rectangular band, we can further deduce that $a^2 + aba = aba$, leading to $a^2 + a = a$ (i).

Now, consider setting $b = a$ in the equation $a + ab = ab$. This yields $a + aa = aa$, which can be rewritten as $a + a^2 = a^2$ (ii).

From (i) and (ii), we get $a^2 = a$. Hence, $(S, \cdot)$ is a band.

(b) Let's consider an element a belonging to the set $C_k(S)$. This means that for all a in S and all non-zero elements b in S, the equation $a + ab = ab$ holds true. If we substitute b with a, we get $a + a^2 = a^2$, which further simplifies to $a + a = a$, since $(S, \cdot)$ is a band. As a result, we can conclude that $(S, +)$ is also a band. □

Theorem 5.5.15. *If S be a k- central semiring, then the following are true.*

(i) If $(S, \cdot)$ has a left zero, then $(S, +)$ is a band.

(ii) If $(S, \cdot)$ has a left identity, then $(S, +)$ has a left identity.

Proof. (i) Consider a k-central semiring S. Let a and b be elements of S; where b is nonzero. If $a + ab = ab$, then it follows that $a + a = a$ due to the presence of a left zero element in $(S, \cdot)$. Therefore, we can conclude that $(S, +)$ forms a band.

(ii) Let $a \in C_k(S)$. Consequently, for all $a \in S$ and $b \in S \backslash \{0\}$, we have $a+ab = ab$. Given that $(S, \cdot)$ possesses a left identity, we can deduce that $ab = b$ for any a and b belonging to S. This, in turn, implies that $a + b = b$. Hence, we can conclude that $(S, +)$ also has a left identity. □

Proposition 5.5.16. *Let S be k- central semiring. If $(S, \cdot)$ has a left zero, then $(S, +)$ is E-inversive semigroup.*

Proof. By hypothesis, $(S, \cdot)$ has a left zero 'a' i.e. $ab = a$ for all $b \in S$. Consider S is a k- central semiring and $a \in C_k(S)$. Then for any $b \in S \backslash \{0\}$, $a+ab = ab$. This implies that $ab+ab = ab$ for all ab in $E[+]$ (since $(S, \cdot)$ has a left zero 'a',); where $E[+]$ is the of all additive idempotents in $(S, +)$. In other words, there exists an element $a \in S$ such that $ab + ab = ab$. From this, we can conclude that b is an E-inversive element, thus establishing that $(S, +)$ is an E-inversive semigroup. □

We bring this chapter to a close by analyzing the behavior of the k-central semiring within the framework of PRD.

Proposition 5.5.17. *If S is both a PRD and a k-central semiring, then S must also be a simple semiring.*

Proof. Assuming S is a k-central semiring and $a \in C_k(S)$. Then $a + ab = ab$ for all $b \in S \backslash \{0\}$. Since $(S, \cdot)$ is an abelian group, then a^{-1} exists for any $a \in S$. Taking $b = a^{-1}$, we find that $a + aa^{-1} = aa^{-1}$. From this equation, we deduce that $a + 1 = 1$ for all $a \in S$. As a consequence, we can conclude that S is a simple semiring. □

Proposition 5.5.18. *If S is both a PRD and a k-central semiring, then S must also be a multiplicatively subidempotent.*

Proof. Assuming S is a PRD, according to proposition 5.5.17, if S is a k-central semiring, then the equation $1 + a = 1$ holds for all $a \in S$. Consequently, we can deduce that $a(1 + a) = a \cdot 1$, which simplifies to $a + a^2 = a$. Thus, the equation $a + a^2 = a$ holds for all elements $a \in S$. As a result, it can be concluded that S is a multiplicative subidempotent. □

Proposition 5.5.19. *Let S be a PRD. If S is a k-central semiring, then S has a left zero.*

Proof. Assuming S is a PRD, by referencing proposition 5.5.17, we deduce that if S is a k-central semiring, it must also be a simple semiring. Let us consider an element $a \in C_k(S)$. For any non-zero element $b \in S$, we observe that $a + ab = ab$, which leads us to the equation $a(1 + b) = ab$. Simplifying further, we find $a \cdot 1 = ab$, and subsequently $a = ab$, given that S is a simple semiring. As a consequence, we conclude that S possesses a left zero. □

Chapter 6

Generalized Center of a Semiring

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Generalized Center of a Semiring

6.1 Introduction

In the field of abstract algebra, the center of a group G, is the set of elements that commute with every element of G. It is denoted $Z(G)$, from German Zentrum, meaning center. In set-builder notation, $Z(G) = \{a \in G : ab = ba \text{ for all } b \in S\}$. The notion of center of ring is analogous to the center of group. Again the notion of center of semiring is analogous to the center of ring. However, to delve further into this topic and explore novel findings, we introduce a distinct variation known as the generalized center of a semiring. The generalized center, denoted as $C_G(S)$, is a specific center constructed to establish new results concerning semirings. The generalized center of semiring is defined by $C_G(S) = \{a \in S : a + ab = a + ba \text{ for all } b \in S\}$. Interestingly, in additively cancellative semirings, the generalized center $C_G(S)$ coincides with the conventional center $Z(S)$ of the semiring. This intriguing observation serves as motivation for defining and exploring the properties of the generalized center of semiring. So, the generlaized center of semiring is precisely generalization of the center of semiring. Consequently, this chapter focuses on discussing the characteristics of this substructure within semirings and delves into the algebraic structure of the center in various classes of semirings.

The primary objective of this chapter is to provide an in-depth understanding of these topics, particularly emphasizing the class of semirings in which a semiring S

aligns with its generalized center. To accomplish this goal, we introduce a new type of semiring known as the generalized central semiring and thoroughly investigate its properties in conjunction with the generalized center of a semiring.

6.2 $C_G(S)$ of a Semiring S

In this section, we introduce the concept of a generalized center of a semiring, along with a variety of examples and fundamental findings that have relevance for upcoming results and subsequent sections.

Definition 6.2.1. *Let S be a semiring. A subset $C_G(S)$ of a semiring S is called generalized center of S which is defined by $C_G(S) = \{a \in S : a + ab = a + ba$ for all $b \in S\}$.*

Proposition 6.2.2. *If S is a semiring with zero element 0, then $0 \in C_G(S)$.*

Proof. Given that 0 is the zero element of S, we can observe that $0 \cdot x = 0 = x \cdot 0$ for all $x \in S$. Now for any $x \in S$, $0 + 0 \cdot x = 0 + x \cdot 0$, as 0 serves as the zero element of S. As a consequence, we can conclude that $0 \in C_G(S)$. □

Proposition 6.2.3. *If S is a semiring with identity element 1, then $1 \in C_G(S)$.*

Proof. Given that 1 serves as the identity element in S, we can deduce that $1 \cdot x = x \cdot 1 = x$ for all $x \in S$. Moreover, for any $x \in S$, we have $1 + 1 \cdot x = 1 + x \cdot 1$, due to the fact that 1 acts as the identity element of S. Consequently, it follows that $1 \in C_G(S)$. □

Here, we present a number of examples illustrating the concept of the generalized center in a semiring.

Example 6.2.4. *Consider $(\mathbb{N}, \oplus, \odot)$ is a semiring; where $a \oplus b = max\{a, b\}$ and $a \odot b = min\{a, b\}$. Then $C_G(\mathbb{N}) = \mathbb{N}$.*

Example 6.2.5. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S,+,\cdot)$ *is a semiring and* $C_G(S) = \{0, x, 1\} = S$.

Example 6.2.6. *Consider* $S = \{0, a, b, c\}$. *Define the operations "+" and "·" on* S *by means of the following tables :*

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	c	c

By using these operations, we can establish that $(S,+,\cdot)$ *forms a non-commutative semiring. In this case,* $Z(S) = \{0, a\}$. *However,* $C_G(S) = \{0, a, c\}$. *Therefore, we can conclude that* $Z(S) \neq C_G(S)$

The following example is taken from the source [[28], Example 1.10].

Example 6.2.7. *We consider a semigroup* $(M,\cdot)$ *with multiplication table*

.	0	1	a	b	c
0	0	0	0	0	c
1	0	1	a	b	c
a	0	a	a	a	c
b	0	b	b	b	c
c	0	c	c	c	c

Let $S = Sub(M)$ *be the set of all subsets of the semigroup* M. *Let us define "+" and "·" in* S *as :* $A + B = A \cup B$ *and* $A \cdot B = \{ab \mid a \in A, b \in B\}$ *for all* $A, B \in S$. *Then* $(S,+,\cdot)$ *is a semiring with zero element* ϕ *and the identity element* $1 = 1_M$. *We have* $|S| = 2^5 = 32$. *In addition,* S *is additively idempotent and* *multiplicatively idempotent non-commutative semiring. The center* $Z(S)$ *as the form* $Z(S) = \{\{\phi\}, \{0\}, \{1\}, \{0,1\}, \{c\}, \{0,c\}, \{0,1,c\}, \{1,c\}\}$.

The generalized center $C_G(S)$ *as of the form* $C_G(S) = \{\{\phi\}, \{0\}, \{1\}, \{0,1\}, \{c\}, \{0,c\}, \{0,1,c\}, \{1,c\}, \{1,a\}, \{1,b\}, \{a,b\}, \{1,a,b\}, \{a,b,c\}, \{1,a,c\}, \{1,b,c\}, \{1,a,b,c\}, \{0,1,a\}, \{0,1,b\}, \{0,a,b\}, \{0,1,a,b\}, \{0,a,b,c\}, \{0,1,a,c\}, \{0,1,b,c\}, \{0,1,a,b,c\}\}$.

Again $C_G(S) \backslash Z(S) = \{\{1,a\}, \{1,b\}, \{a,b\}, \{1,a,b\}, \{a,b,c\}, \{1,a,c\}, \{1,b,c\}, \{1,a,b,c\}, \{0,1,a\}, \{0,1,b\}, \{0,a,b\}, \{0,1,a,b\}, \{0,a,b,c\}, \{0,1,a,c\}, \{0,1,b,c\}, \{0,1,a,b,c\}\}$. *Those elements of* S *which are not the elements of* $C_G(S)$ *as well as* $Z(S)$ *are* $\{a\}$, $\{b\}$, $\{0,a\}$, $\{0,b\}$, $\{a,c\}$, $\{b,c\}$, $\{0,a,c\}$, $\{0,b,c\}$. *In this case,* $C_G(S) \neq Z(S)$ *and* $Z(S) \subset C_G(S)$. *Again* $C_G(S) \neq S$.

Proposition 6.2.8. *Let* S *be a semiring. Then* $Z(S) \subseteq C_G(S)$.

Proof. Let $a \in Z(S)$. Then $ab = ba$ for all $b \in S$. Consequently, for all $b \in S$, we also have $a + ab = a + ba$. This, in turn, implies that $a \in C_G(S)$. Hence, we can conclude that $Z(S) \subseteq C_G(S)$. □

Example 6.2.9. *Let* D *be a distributive lattice with identity element* 1. *Let* $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \in M_2(D)$. *For any* $\begin{pmatrix} b & c \\ 0 & d \end{pmatrix} \in M_2(D)$, $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix}\begin{pmatrix} b & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} ab & ac+ad \\ 0 & ad \end{pmatrix}$. *Again* $\begin{pmatrix} b & c \\ 0 & d \end{pmatrix}\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} = \begin{pmatrix} ba & ba+ca \\ 0 & da \end{pmatrix}$.

So, $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix}\begin{pmatrix} b & c \\ 0 & d \end{pmatrix} \neq \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}\begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$.

Therefore, $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \notin Z(M_2(D))$.

But $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} + \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}\begin{pmatrix} b & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} a+ab & a+ac+ad \\ 0 & a+ad \end{pmatrix} = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$.

Again $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} = \begin{pmatrix} a+ba & a+ba+ca \\ 0 & a+da \end{pmatrix} = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$.

So, $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} + \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}\begin{pmatrix} b & c \\ 0 & d \end{pmatrix} \neq \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}\begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$.

Thus, $\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \in C_G(M_2(D))$.

Similarly, we can show that $\{ a \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \} \in C_G(M_2(D))$.

But $\{ a \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \} \notin Z(M_2(D))$.

Also, $a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in Z(M_2(D))$, *and consequently,* $a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in C_G(M_2(D))$.

Subsequently, we analyze the behavior of the generalized center in the context of matrix semiring.

Theorem 6.2.10. *Let S be a semiring with identity. Then $a \in C_G(S)$ if and only if $aI_n \in C_G(M_n^d(S))$; where $M_n^d(S)$ is the set of all $n \times n$ diagonal matrices i.e.,*

$$M_n^d(S) = \{ \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} : a_{ii} \in S \}$$

Proof. Let $a \in C_G(S)$. Then $a + ab = a + ba$ for all $b \in S$.

Now $aI_n + (aI_n)B$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} ab_1 & 0 & \cdots & 0 \\ 0 & ab_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ab_n \end{pmatrix}$$

$$= \begin{pmatrix} a+ab_1 & 0 & \cdots & 0 \\ 0 & a+ab_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a+ab_n \end{pmatrix}$$

$$= \begin{pmatrix} a+b_1a & 0 & \cdots & 0 \\ 0 & a+b_2a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a+b_na \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} b_1a & 0 & \cdots & 0 \\ 0 & b_2a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_na \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$$

$= a + B(aI_n)$, since $a \in C_G(S))$

This implies that $aI_n \in C_G(M_n^d(S))$.

Conversely, suppose that $aI_n \in C_G(M_n^d(S))$. Now we show that $a \in C_G(S)$. For any $x \in S$, let B $= \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}$. Since $aI_n \in C_G(M_n^d(S))$, it follows that $aI_n + (aI_n)B = a + B(aI_n)$. Comparing both sides, we find that $a + ax = a + xa$. Therefore, $a \in C_G(S)$. □

Theorem 6.2.11. *Let S be a commutative semiring with identity. Then $aI_n \in C_G(M_n(S))$ for all $a \in S$.*

Proof. Let $a \in C_G(S)$. Then $a + ab = a + ba$ for all $b \in S$.

Now $aI_n + (aI_n)B$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & ab_{22} & \cdots & ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & ab_{n} \end{pmatrix}$$

$$= \begin{pmatrix} a+ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & a+ab_{22} & \cdots & ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & a+ab_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a+b_{11}a & b_{12}a & \cdots & b_{1n}a \\ b_{21}a & a+b_{22}a & \cdots & b_{2n}a \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}a & b_{n2}a & \cdots & a+b_{nn}a \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} b_{11}a & b_{12}a & \cdots & b_{1n}a \\ b_{21}a & b_{22}a & \cdots & b_{2n}a \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}a & b_{n2}a & \cdots & b_{n}a \end{pmatrix} = aI_n + B(aI_n),$$ since S is commutative semiring. This conveys that $aI_n \in C_G(M_n(S))$. □

Theorem 6.2.12. *The generalized center of a semiring S is a subsemiring of S.*

Proof. Since S is a semiring with zero element 0, then $0 \in C_G(S)$. Consequently, the generalized center $C_G(S)$ of a semiring S is non-empty. Let S be a semiring and $x, y \in C_G(S)$. We will now demonstrate that $x + y \in C_G(S)$ and $xy \in C_G(S)$. For any $b \in S, x + xb = x + bx$, since $x \in C_G(S)$ and $y + yb = y + by$, since $y \in C_G(S)$. Now, by combining these equations, we obtain $(x+y)+(x+y)b = x+y+xb+yb = x + xb + y + yb = x + bx + y + by = (x + y) + b(x + y)$. Thus, we conclude that $x + y \in C_G(S)$.

Furthermore, we have $xy + xyb = x(y + yb) = x(y + by) = xy + xby = (x + xb)y = (x + bx)y = xy + bxy$. Therefore, $xy \in C_G(S)$. Consequently, $C_G(S)$ forms a subsemiring of S. □

Remark 6.2.13. *The Example 6.2.6 shows that the generalized center of a semiring S is not an ideal of S, since $b \cdot c = b \notin C_G(S) = \{0, a, c\}$.*

Theorem 6.2.14. *If S is a mono-semiring, then $S = C_G(S)$.*

Proof. Indeed, $C_G(S) \subseteq S$ is evident. To establish $S \subseteq C_G(S)$, we need to demonstrate that every element a in S is also an element of $C_G(S)$. Let $a \in S$. Given that S is a mono-semiring, it follows that $a + b = ab$ for all $a, b \in S$. Now, consider any $b \in S$. We can observe that $a + ab = a + a + b = a + b + a = a + ba$. Consequently, we deduce that $a \in C_G(S)$. Consequently, $S \subseteq C_G(S)$. Thus, we can conclude that $S = C_G(S)$. □

6.3 Analysis of $C_G(S)$'s Properties

In this segment, we elucidate a set of fundamental properties concerning the generalized center of a semiring, which are commonly assumed as given in conventional semiring theory.

Theorem 6.3.1. *If φ is an epimorphism from S to S', then $\varphi(C_G(S)) \subseteq C_G(S')$.*

Proof. Let S be an semiring and $\varphi : S \longrightarrow S'$ is an epimorphism of S'. Consider $\varphi(C_G(S)) = \{\varphi(s) : s \in C_G(S)\}$. We have to show that $\varphi(C_G(S)) \subseteq C_G(S')$. Let $\varphi(s_1) \in \varphi(C_G(S))$ and $x \in S'$. Since φ is onto, there exist $y \in S$ such that y has a preimage x of S' i.e. $\varphi(y) = x$. Since $s_1 \in C_G(S)$, $s_1 + s_1y = s_1 + ys_1$. Now for any $x \in S'$, $\varphi(s_1) + \varphi(s_1)x = \varphi(s_1) + \varphi(s_1)\varphi(y) = \varphi(s_1) + \varphi(s_1y) = \varphi(s_1 + s_1y) = \varphi(s_1 + ys_1) = \varphi(s_1) + \varphi(ys_1) = \varphi(s_1) + \varphi(y)\varphi(s_1) = \varphi(s_1) + x\varphi(s_1)$. Therefore, $\varphi(s_1) \in C_G(S')$. Hence, $\varphi(C_G(S)) \subseteq C_G(S')$. □

Theorem 6.3.2. *Let S and S' be two semirings. If $f : S \to S'$ is a monomorphism, then $f(C_G(S)) = C_G(f(S))$.*

Proof. Suppose $x \in f(C_G(S))$. Then $x = f(y)$ for some $y \in C_G(S)$. Our goal is to prove that $f(y) + f(y)s = f(y) + sf(y)$ for all $s \in f(S)$. For any $s \in f(S)$, we can observe the following chain of equalities : $f(y) + f(y)s = f(y) + f(y)f(r) = f(y) + f(yr) = f(y+yr) = f(y+ry) = f(y)+f(ry) = f(y)+f(r)f(y) = f(y)+sf(y)$. Hence, we can conclude that $x = f(y) \in C_G(f(S))$. Consequently, $f(C_G(S)) \subseteq C_G(f(S))$. Now, let's consider $x' \in C_G(f(S))$. Then $x' = f(r')$ for some $r' \in S$. Our objective is to demonstrate that $r' \in C_G(S)$. Since $x' \in C_G(f(S))$, we can deduce that for any $f(s) \in f(S)$, the following equation holds : $x' + x'f(s) = x' + f(s)x' \implies f(r')+f(r')f(s) = f(r')+f(s)f(r') \implies f(r')+f(r's) = f(r')+f(sr') \implies f(r'+r's) = f(r' + sr')$. Since f is a monomorphism, we can conclude that $r' + r's = r' + sr'$. Therefore, $r' = f(r') \in f(C_G(S))$. Consequently, $C_G(f(S)) \subseteq f(C_G(S))$. Thus, we have demonstrated that $C_G(f(S)) = f(C_G(S))$ as desired. □

Theorem 6.3.3. *If two semirings S_1 and S_2 are isomorphic, then their generalized centers $C_G(S_1)$ and $C_G(S_2)$ are isomorphic.*

Proof. Consider two semirings S_1 and S_2 which are isomorphic. Then there is an isomorphism $f : S_1 \longrightarrow S_2$. Let $x \in C_G(S_1)$. Then for any $s_1 \in S_1$, $x+xs_1 = x+s_1x$; where $y \in S_2$. Since f is an isomorphism, for any $s_2 \in S_2$, there exists $s_1 \in S_1$ such that $f(s_1) = s_2$. Thus, $y + ys_2 = f(x) + f(x)f(s_1) = f(x + xs_1) = f(x + s_1x) = f(x)+f(s_1x) = f(x)+f(s_1)f(x) = y+s_2y$, since $x \in C_G(S_1)$. Therefore, $y \in C_G(S_2)$. Consequently, $f(C_G(S_1)) \subseteq C_G(S_2)$. Again let $b \in C_G(S_2)$. Then $b = f(a)$; where $a \in S_1$. Since f is an isomorphism, for any $y \in S_2$, there exists $x \in S_1$ such that $y = f(x)$. Since $b \in C_G(S_2)$, it follows that $b + by = b + yb$. Now $b + by = b + yb \implies f(a) + f(a)f(x) = f(a) + f(x)f(a) \implies f(a) + f(ax) = f(a) + f(xa) \implies f(a + ax) = f(a + xa) \implies a + ax = a + xa$, since f is an isomorphism. This implies that $a \in C_G(S_1)$. Therefore, $b = f(a) \in f(C_G(S_1))$. Thus, $C_G(S_2)) \subseteq f(C_G(S_1))$ and hence $C_G(S_2) = f(C_G(S_1))$. So, $g = f_{|C_G(S_1)} : C_G(S_1) \longrightarrow C_G(S_2)$ is well defined and it is an isomorphism from $C_G(S_1)$ onto $C_G(S_2)$. □

The following example makes it evident that the converse of theorem 6.3.3 is not valid.

Example 6.3.4. *Consider $S_1 = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S_1, +, \cdot)$ is a semiring and $C_G(S_1) = \{0, x, 1\}$.

Consider $S_2 = \{0, 1, 2, 3\}$. Define the operations "+" and "." on S by means of the following tables :

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	3	3

Then $(S_2, +, \cdot)$ is a non-commutative semiring. Here, $C_G(S_2) = \{0, 1, 3\}$.

Now, $|C_G(S_1)| = |C_G(S_2)| = 3$. So, $C_G(S_1)$ and $C_G(S_2)$ are isomorphic. But $|S_1| = 3$ and $|S_2| = 4$. Therefore, S_1 and S_2 are not isomorphic.

Theorem 6.3.5. *If S is a simple semiring, then the generalized center $C_G(S)$ is also a simple semiring of S.*

Proof. Since S is a simple semiring, then $1 + a = 1$ for all $a \in S$. This implies that $1 + a = 1$ for all $a \in C_G(S)$, since $C_G(S)$ is a subsemiring of S. So, $C_G(S)$ is also a simple semiring of S. □

Theorem 6.3.6. *If S is a semiring, then $C_G(S)$ is an antisimple semiring of S.*

Proof. Consider a semiring S. Let $r \in C_G(S)$. For any $x \in S$, it follows that $r + rx = r + xr$. Furthermore, for any $x \in S$, we can observe that $(r+1)+(r+1)x = r+1+rx+x = r+rx+1+x = r+xr+1+x = r+1+xr+x\cdot 1 = (r+1)+x(r+1)$. From this, we conclude that $r + 1 \in C_G(S)$. Additionally, we know that $0 \in C_G(S)$. Hence, $P'(C_G(S)) = \{0\} \cup \{r + 1 : r \in C_G(S)\}$. Accordingly, $C_G(S)$ itself forms an antisimple semiring within S. □

Theorem 6.3.7. *Let D be a semiring with unity 1. If D is a division semiring, then $C_G(D)$ is a division subsemiring of D.*

Proof. Let D be a division semiring and $C_G(D)$ be the generalized center of D. Take $a \neq 0$ as an arbitrary element of $C_G(D)$. Consequently, Da represents a nonzero left ideal of D. Given that D is a division semiring, we can deduce that $Da = D$. Thus, there exists $d \in D$ such that $da = 1$. Since $a \in C_G(D)$, for any $d \in D$, $a + ad = a + da$. Since D is a division semiring, for a non zero element a in D, there exists an inverse element a' such that $aa' = a'a = 1$. Now, we can establish that $d = d \cdot 1 \implies d = d(aa') \implies d = (da)a' \implies d = 1 \cdot a' \implies d = a'$. Additionally, $aa' = 1 \implies ad = 1$. The only remaining task is to demonstrate that $d \in C_G(D)$. As $a \in C_G(D)$, for any $x \in D$, we can observe that $a + ax = a + xa \implies da + dax = da + dxa \implies 1 + x = 1 + dxa \implies 1 \cdot d + xd = 1 \cdot d + dxad \implies d + xd = d + dx \cdot 1 \implies d + xd = d + dx$, since $da = ad = 1$. This implies that $d \in C_G(D)$. Consequently, we find that $ad = da = 1$ for some $d \in C_G(D)$. Hence, we conclude that 'a' is a unit in $C_G(D)$. Therefore, $C_G(D)$ serves as a division subsemiring of D. □

Theorem 6.3.8. *Let $S[x]$ be a polynomial semiring over a semiring S. If $f(x) = a_0 + a_1x + ... + a_nx^n \in C_G(S[x])$, then $a_i \in C_G(S)$; $i = 0, 1, 2, ..., n$. The converse is true if $a_i \in Z(S)$; $i = 0, 1, 2, ..., n$.*

Proof. Consider $f(x) = a_0 + a_1x + ... + a_nx^n \in C_G(S[x])$. Let $d \in S$. Now $d \in S$ implies that $d \in S[x]$. Given that $f(x) \in C_G(S[x])$, we can deduce that $f(x) + f(x)d = f(x) + df(x)$. This implies that $(a_0 + a_1x + ... + a_nx^n) + (a_0 + a_1x + ... + a_nx^n)d = (a_0 + a_1x + ... + a_nx^n) + d(a_0 + a_1x + ... + a_nx^n)$. Thus, we have $(a_0 + a_0d) + (a_1 + a_1d)x + ... + (a_n + a_nd)x^n = (a_0 + da_0) + (a_1 + da_1)x + ... + (a_n + da_n)x^n$. By comparing both sides, we can conclude that $a_0 + a_0d = a_0 + da_0$, $a_1 + a_1d = a_1 + da_1, ..., a_n + a_nd = a_n + da_n$. Since d is arbitrary, it follows that $a_i \in C_G(S)$ for $i = 0, 1, ..., n$.

Conversely, suppose that $a_i \in Z(S)$. Since $a_i \in Z(S)$, $a_ib = ba_i$ for all $b \in S$. Our goal is to demonstrate that $f(x) \in C_G(S[x])$. For any $g(x) = b_0 + b_1x + ... + b_kx^k \in S[x]$, we have $f(x) + f(x)g(x) = (a_0 + a_1x + ... + a_nx^n) + (a_0 + a_1x + ... + a_nx^n)(b_0 + b_1x + ... + b_kx^k)$. This expression simplifies to $(a_0 + a_1x + ... + a_nx^n) + \big(a_0b_0 + \sum_{i+j=1} a_ib_jx +$

$\sum_{i+j=2} a_i b_j x^2 + ... + \sum_{i+j=n} a_i b_j x^n + ... + a_n b_k x^{n+k}) = a_0 + a_0 b_0 + (a_1 + \sum_{i+j=1} a_i b_j)x + (a_2 + \sum_{i+j=2} a_i b_j)x^2 + ... + (a_n + \sum_{i+j=n} a_i b_j)x^n + ... + a_n b_k x^{n+k} = a_0 + b_0 a_0 + (a_1 + \sum_{i+j=1} b_j a_i)x + (a_2 + \sum_{i+j=2} b_j a_i)x^2 + ... + (a_n + \sum_{i+j=n} b_j a_i)x^n + ... + b_k a_n x^{n+k}$. Hence, we can conclude that $f(x) + f(x)g(x) = f(x) + g(x)f(x)$. Consequently, we can establish that $f(x) \in C_G(S[x])$. □

Theorem 6.3.9. *If S_1 and S_2 are two semirings, then $C_G(S_1 \times S_2) = C_G(S_1) \times C_G(S_2)$.*

Proof. Let S_1 and S_2 be two semirings. Suppose $z \in C_G(S_1 \times S_2)$. Then $z = (x, y) \in S_1 \times S_2$ and for any $(a, b) \in S_1 \times S_2$, we have $(x, y) + (x, y)(a, b) = (x, y) + (a, b)(x, y)$. Comparing both sides, we get $x + xa = x + ax$ and $y + yb = y + by$. This implies that $x + xa = x + ax$ for all $a \in S_1$ and $y + yb = y + by$ for all $b \in S_2$. Thus, it follows that $x \in C_G(S_1)$ and $y \in C_G(S_2)$. Therefore, $z = (x, y) \in C_G(S_1) \times C_G(S_2)$ and hence $C_G(S_1 \times S_2) \subseteq C_G(S_1) \times C_G(S_2)$ (*i*).

Considering the reverse case, let $(a, b) \in C_G(S_1) \times C_G(S_2)$. This indicates that $a \in C_G(S_1)$ and $b \in C_G(S_2)$. Consequently, for any $x \in S_1$, we observe that $a + ax = a + xa$ and for any $y \in S_2$, we have $b + by = b + yb$. Now, by examining $(a, b) + (a, b)(x, y) = (a + ax, b + by) = (a + xa, b + yb) = (a, b) + (x, y)(a, b)$, since $a \in C_G(S_1)$ and $b \in C_G(S_2)$. We can conclude that $(a, b) \in C_G(S_1 \times S_2)$, affirming $C_G(S_1) \times C_G(S_2) \subseteq C_G(S_1 \times S_2)$ (*ii*).

Consequently, based on (*i*) and (*ii*), we can deduce that $C_G(S_1 \times S_2)$ equals $C_G(S_1) \times C_G(S_2)$. □

Applying the principles outlined in Theorem 6.3.9, we formulate the following example.

Example 6.3.10. *Consider two semiring $(\mathbb{N}, \oplus, \odot)$; where $a \oplus b = min\{a, b\}$ and $\odot$ is usual multiplication and $(\mathbb{Z}_0^+, +, \cdot)$, where "+" is usual addition and "·" is usual multiplication. Now we take a semiring $(\mathbb{N} \times \mathbb{Z}_0^+, +, \cdot)$ with component-wise addition and multiplication. Then $C_G(\mathbb{N} \times \mathbb{Z}_0^+) = \mathbb{N} \times \mathbb{Z}_0^+$.*

6.4 Generalized Central Semiring

In the following section, we outline the concept of a generalized central semiring and analyze several key properties related to this semiring.

Definition 6.4.1. A semiring S is called a generalized central semiring if $C_G(S) = S$.

In order to provide a clearer understanding of generalized central semirings, we present the following examples.

Example 6.4.2. *Let's consider a distributive lattice D. In this case, it can be observed that for any $a, b \in D$, both equations $a+ab = a$ and $a+ba = a$ hold true. Consequently, it follows that $a + ab = a + ba$ for all $a, b \in D$. Based on this, we can deduce that $C_G(D) = D$. Therefore, we can conclude that every distributive lattice can be classified as a generalized central semiring.*

Example 6.4.3. *Let R be a commutative ring and Ω_R be the set of all ideals of R. Define $\oplus$ and $\odot$ by $I_1 \oplus I_2 = \{a_1 + b_1 / a_1 \in I_1, b_1 \in I_2\}$ and $I_1 \odot I_2 = \{\sum_{i=1}^{n} a_i b_i / a_1 \in I_1, b_1 \in I_2\}$ for all $I_1, I_2 \in \Omega_R$; where n is a fnite number but not fixed. Then the semiring $(\Omega_R, \oplus, \odot)$ is a generalized central semiring.*

Evidently, $C_G(\Omega_R) \subseteq \Omega_R$. We have to show that $\Omega_R \subseteq C_G(\Omega_R)$. Let $I \in \Omega_R$. To demonstrate that $I \in C_G(\Omega_R)$, which means showing that $I \oplus (I \odot I_1) = I \oplus (I_1 \odot I)$ for all $I_1 \in \Omega_R$. By virtue of R being commutative, we have $I \odot I_1 = I_1 \odot I$. Consequently, $I \oplus (I \odot I_1) = I \oplus (I_1 \odot I)$ for all $I_1 \in \Omega_R$. Therefore, $I \in C_G(\Omega_R)$. Since I is an arbitrary element of Ω_R, it follows that $\Omega_R \subseteq C_G(\Omega_R)$ and hence $C_G(\Omega_R) = \Omega_R$. Hence, Ω_R qualifies as a generalized central semiring.

Example 6.4.4. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	1
1	0	1	1

Then $(S, +, \cdot)$ *is a semiring and* $C_G(S) = \{0, x, 1\} = S$*. Therefore,* $(S, +, \cdot)$ *is a generalized central semiring.*

Example 6.4.5. *Consider* $S = \{0, x, y, 1\}$*. Define the operations "+" and "." on* S *by means of the following tables :*

+	0	x	y	1
0	0	x	y	1
x	x	x	y	1
y	y	y	y	1
1	1	1	1	1

.	0	x	y	1
0	0	0	0	0
x	0	x	x	x
y	0	x	y	y
1	0	x	y	1

Then $(S, +, \cdot)$ *is a semiring and* $C_G(S) = \{0, x, y, 1\} = S$*. Therefore,* $(S, +, \cdot)$ *is a generalized central semiring.*

Example 6.4.6. $\{$*A class of finite semiring*$\}$ *: Let* n, i *be integers such that* $2 \leq n$*,* $0 \leq i < n$*, and* $B(n, i) = \{0, 1, 2, ..., n-1\}$*. We define addition and multiplication in* $B(n, i)$ *by the following equations (let* $m = n - i$*) :*

$$x + y = \begin{cases} x + y, & \text{if } x + y \leq n - 1 \\ l, & \text{if } x + y \geq n \text{ ; where } l \equiv (x+y) mod m \text{ and } i \leq l \leq n - 1. \end{cases}$$

$$x \cdot y = \begin{cases} xy, & \text{if } xy \leq n - 1 \\ l, & \text{if } xy \geq n \text{ ; where } l \equiv (xy) mod m \text{ and } i \leq l \leq n - 1. \end{cases}$$

Then the set $B(n, i)$ *is a commutative semiring with zero* 0 *and identity* 1 *under addition and multiplication. Then* $C_G(B(n, i)) = B(n, i)$*. Therefore,* $B(n, i)$ *is a generalized central semiring.*

Note that $Z(S)$ is a subsemiring of S but generally, $Z(S) \neq S$ for a semiring S.

Remark 6.4.7. *If* S *is a commutative semiring, then the generalized central semiring coincides with the central semiring.*

Remark 6.4.8. *If* S *is an additively cancellative semiring, then the generalized central semiring coincides with the central semiring.*

Remark 6.4.9. *If S is a mono-semiring, then the generalized central semiring coincides with the central semiring.*

We conclude this section by proving that the class of all generalized central semirings forms a variety. In order to do so, we begin by demonstrating the following Lemmas.

Lemma 6.4.10. *Let S be a generalized central semiring and S' be an subsemiring of S. Then S' is a generalized central semiring.*

Proof. Let S be a generalized central semiring i.e. $C_G(S) = S$. We have to show that S' is an generalized central semiring i.e. $C_G(S') = S'$. Let $a \in S' \subseteq S = C_G(S)$. Consequently, $a + ab = a + ba$ for all $a,\ b \in S$. This implies that $a + ab = a + ba$ for all $a,\ b \in S'$, since S' is a subsemiring of S. Consequently, $a \in C_G(S')$. This establishes that $S' \subseteq C_G(S')$ and hence $C_G(S') = S'$. As a result, S' is a generalized central semiring. □

Lemma 6.4.11. *Every homomorphic image of a generalized central semiring is a generalized central semiring.*

Proof. Let S be a generalized central semiring with identity 1_S. Let $f : S \to S'$ be onto homomorphism. Then S' is the homomorphic image of the generalized central semiring S. Our goal is to demonstrate that S' is a generalized central semiring. As f is an onto homomorphism, we can express $S' = \{f(a) : a \in S\}$. Now $f(1_S) = 1_{S'}$ is the identity element of S'. Let $s' \in S'$. Then there exists $a \in S$ such that $f(a) = s'$. Since S is a generalized central semiring, $a + ab = a + ba$ for all $b \in S$. Consequently, for any $f(b) \in S'$, we have $s' + s'f(b) = f(a) + f(a)f(b) = f(a) + f(ab) = f(a + ab) = f(a + ba) = f(a) + f(ba) = f(a) + f(b)f(a) = s' + f(b)s'$. Thus, we can conclude that $s' = f(a) \in C_G(S') = S'$. Since s' is an arbitrary element of S', it follows that S' is a generalized central semiring. □

Lemma 6.4.12. *Let $\{S_i : i = 1, 2, ..., n\}$ be a finite family of semirings. Then the direct product of semirings $S = \prod_{i=1}^{n} S_i$ is generalized central semiring if and only if each semiring S_i is generalized central semiring.*

Proof. Suppose that each semiring S_i of the family $\{S_i : i = 1, 2, ..., n\}$ is generalized central semiring. Let $(x_1, x_2, ..., x_n) \in S$; where each $x_i \in S_i$. Since each S_i is generalized central semiring, for any $y_i \in S_i$, we have $x_i + x_i y_i = x_i + y_i x_i$ for all $i = 1, 2, ..., n$. Thus it follows that

$(x_1, x_2, ..., x_n) + (x_1, x_2, ..., x_n)(y_1, y_2, ..., y_n) = (x_1, x_2, ..., x_n) + (x_1y_1, x_2y_2, ..., x_ny_n) =$
$(x_1 + x_1y_1, x_2 + x_2y_2, ..., x_n + x_ny_n) = (x_1 + y_1x_1, x_2 + y_2x_2, ..., x_n + y_nx_n)$
$= (x_1, x_2, ..., x_n) + (y_1x_1, y_2x_2, ..., y_nx_n)$
$= (x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n)(x_1, x_2, ..., x_n)$. Cosequently, S is a generalized central semiring.

Conversely, suppose that $S = \prod_{i=1}^{n} S_i$ is generalized central semiring. We have to show that each semiring S_i is generalized central semiring. Let us consider the mapping $\pi_i : S \to S_i$ defined by $\pi((x_1, x_2, ..., x_n)) = x_i$ for all $(x_1, x_2, ..., x_n) \in S$. Then π is an onto homomorphism from $S = \prod_{i=1}^{n} S_i$ to S_i. Thus by Theorem 6.4.10, S_i is generalized central semiring for all $i = 1, 2, ..., n$ and hence the proof. □

Theorem 6.4.13. *The class of all generalized central semiringas is a variety.*

Proof. By establishing the validity of Lemma 6.4.10, 6.4.11, and 6.4.12, we have demonstrated the closure of the class of generalized central semirings under subsemirings, homomorphic images, and direct products. As a consequence, we can conclude that the class of all generalized central semirings forms a variety. □

Chapter 7

On the Hypercenter of a Semiring

Chapter 7

On the Hypercenter of a Semiring

7.1 Introduction

The notion of the hypercenter of a finite group has a long history and plays a vital role in group theory. Consider a finite group G, and let $1 \leq Z_1(G) \leq Z_2(G) \leq ...$ be a series of subgroups of G; where $Z_1(G) = Z(G)$ is the center of G and $Z_{i+1}(G)$, for $i \geq 1$, is defined by $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Let $H(G) = \bigcup_i Z_i(G)$. The subgroup $H(G)$ is called the hypercenter of G. Clearly $H(G)$ is nilpotent in G. In 1949, Reinhold Baer [4] introduced the notion of hypercenter of a finite group. In [5] he extensively studied the properties of $H(G)$ and provided various characterizations of $H(G)$. T. A. Peng, in [51], established some necessary and sufficient conditions for an element or a subgroup of a finite group G to lie in $H(G)$. In 1976, R. K. Agrawal [2] also introduced the notion of the generalized hypercenter of a finite group, proving that it is supersolvable. Furthermore, the generalized hypercenter is contained within the intersection of maximal supersolvable subgroups. I.N. Herstein [34] defined the hypercenter of a ring R to be the set $T(R) = \{a \in R : ax^n = x^n a, n = n(x,a) \geq 1, \text{ all } x \in R\}$ and demonstrated that, for a ring R with no non-zero nil ideals the hypercenter $T(R)$ coincides with the center $Z(R)$ of R. The hypercenter of rings has been extensively studied by various authors, and relevant results can be found in [27], [37], [48]. Motivated by Herstein's hypercenter, we extend this concept to semirings and discuss its properties. We then focus on the algebraic structure of this center for different classes of semirings. Additionally, we introduce the notion of a hypercentral semiring and explore its properties.

7.2 $T(S)$ of a Semiring S

Within this section, we lay out the definition of the hypercenter in a semiring, accompanied by pertinent examples and fundamental results that have implications for both subsequent outcomes and upcoming sections.

Definition 7.2.1. *Let S be a semiring. A subset $T(S)$ of a semiring S is called hypercenter of S which is defined by $T(S) = \{a \in S : ax^n = x^n a, n = n(x, a) \geq 1,$ all $x \in S\}$.*

Proposition 7.2.2. *If S is a semiring with zero element 0, then $0 \in T(S)$.*

Proof. Suppose $a \in T(S)$. Given that $a \in T(S)$, it satisfies the condition $ax^n = x^n a$ $x \in S$; where $n = n(x, a) \geq 1$. Consequently, for every $x \in S$, we have $0 \cdot x^n = 0 = x^n \cdot 0$. As a result, we can conclude that $0 \in T(S)$. □

Proposition 7.2.3. *If S is a semiring with multiplicative identity element 1, then $1 \in T(S)$.*

Proof. Let $a \in T(S)$. Since $a \in T(S)$, then $ax^n = x^n a$, $n = n(x, a) \geq 1$ for all $x \in S$. Moreover, for every x in S, we observe that $1 \cdot x^n = x^n = x^n \cdot 1$. Consequently, we can conclude that $1 \in T(S)$. □

Let's explore some concrete illustrations of hypercenters in semirings.

Example 7.2.4. $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{Z}_0^+ = \{x \in \mathbb{Z} : x \geq 0\}$ *and* $\mathbb{Q}_0^+ = \{x \in \mathbb{Q} : x \geq 0\}$ *are commutative proper semirings with zero. In this case,* $T(\mathbb{R}_0^+) = \mathbb{R}_0^+$, $T(\mathbb{Z}_0^+) = \mathbb{Z}_0^+$ *and* $T(\mathbb{Q}_0^+) = \mathbb{Q}_0^+$.

Example 7.2.5. *Consider $(\mathbb{N}, \oplus, \odot)$ is a semiring; where $a \oplus b = max\{a, b\}$ and $a \odot b = min\{a, b\}$. Then $T(\mathbb{N}) = \mathbb{N}$.*

Example 7.2.6. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S, +, \cdot)$ is a semiring and $T(S) = \{0, x, 1\} = S$.

Example 7.2.7. *Consider $S = \{0, a, b, c\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	c	c

Then $(S, +, \cdot)$ is a non-commutative semiring. In this case, $Z(S) = T(S) = \{0, a\}$.

Example 7.2.8. *We consider a semigroup $(M, \cdot)$ with multiplication table*

+	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Let $S = Sub(M)$ be the set of all subsets of the semigroup M. Let us define "+" and "·" in S as : $A + B = A \cup B$ and $A \cdot B = \{ab \mid a \in A, b \in B\}$ for all $A, B \in S$. Then $(S, +, \cdot)$ is a semiring with zero element ϕ ; see [[28], Example 1.10] . We have $|S| = 2^3 = 8$. In addition, S is additively idempotent and multiplicatively idempotent non-commutative semiring. The center $Z(S)$ as the form $Z(S) = \{\{\phi\}, \{0\}\}$. The hypercenter $T(S)$ as of the form $T(S) = \{\{\phi\}, \{0\}\}$. In this case, $T(S) = Z(S)$. Again $T(S) \neq S$. This is the non- trivial example of $T(S)$.

Theorem 7.2.9. *The hypercenter of a semiring S is a subsemiring of S.*

Proof. Assuming S is a semiring with the zero element 0, it follows that $0 \in T(S)$ and hence forth $T(S)$ is a non-empty set. Let S be a semiring and $p, q \in T(S)$. We aim to prove that $p + q \in T(S)$ and $pq \in T(S)$. Since $p \in T(S)$, we have $px^n = x^n p$, $n = n(x, p) \geq 1$ for all $x \in S$. Similarly, since $q \in T(S)$, we have $qx^m = x^m q$, $m = m(x, q) \geq 1$ for all $x \in S$. Now, let's examine $px^{mn} = px^n ... x^n$(repeated m-times) $= x^n px^n ... x^n$ (repeated $m - 1$-times) $= x^n x^n px^n ... x^n$(repeated $m - 2$-times) $= ... = x^n ... x^n p$ (repeated m-times) $= x^{mn} p$. Similarly, let's analyze $qx^{mn} = qx^m ... x^m$(repeated n-times) $= x^m qx^m ... x^m$ (repeated $n - 1$-times) $= x^m x^m qx^m ... x^m$(repeated $n - 2$-times) $= ... = x^m ... x^m q$ (repeated n-times) $= x^{mn} q$. Now, for any $x \in S$, we consider $(p + q)x^{mn} = px^{mn} + qx^{mn} = x^{mn} p + x^{mn} q$ (since $p, q \in T(S)$) $= x^{mn}(p + q)$. Hence, we have established that $p + q \in T(S)$. Similarly, for any $x \in S$, we consider $(pq)x^{mn} = p(qx^{mn}) = p(x^{mn} q)$ (since $q \in T(S)$) $= (px^{mn})q = (x^{mn} p)q = x^{mn}(pq)$, since $p \in T(S)$. Thus, we have shown that $pq \in T(S)$. Consequently, we can conclude that $T(S)$ forms a subsemiring of S. □

7.3 Foundational Aspects of $T(S)$ in Semiring S

This section is dedicated to exploring the fundamental properties of $T(S)$. To begin, we will focus on exploring the isomorphism property of $T(S)$ in relation to S.

Theorem 7.3.1. *If φ is an automorphism of S then $\varphi(T) \subset T$.*

Proof. Suppose S is a semiring and $\varphi : S \longrightarrow S$ is an automorphism of S. Let $\varphi(T) = \{\varphi(t) : t \in T\}$. Our goal is to prove that $\varphi(T) \subset T$. Consider an element $\varphi(t_1) \in \varphi(T)$. Since φ is onto, there exist $y \in S$ such that y has an preimage x of S such that $\varphi(y) = x$. Let $t_1 \in T$. For any $y \in S$ such that $t_1 y^k = y^k t_1$; $k = k(t_1, y) \geq 1$, we can proceed with the following steps : Now, take an arbitrary $x \in S$, we observe that $\varphi(t_1)x^k = \varphi(t_1)\{\varphi(y)\}^k = \varphi(t_1)\varphi(y^k) = \varphi(t_1 y^k) = \varphi(y^k t_1)$ (since $t_1 \in T$) $= \varphi(y^k)\varphi(t_1) = (\varphi(y))^k \varphi(t_1) = x^k \varphi(t_1)$, since φ is an automorphism. Therefore, we conclude that $\varphi(t_1) \in T$, thereby establishing $\varphi(T) \subset T$. □

Theorem 7.3.2. *If two semirings S_1 and S_2 are isomorphic, then their hypercenters $T(S_1)$ and $T(S_2)$ are isomorphic.*

Proof. Consider two isomorphic semirings, denoted as S_1 and S_2. Then there is an isomorphism $f : S_1 \longrightarrow S_2$. Let $x \in T(S_1)$. Then for any $s_1 \in S_1$, $xs_1^n = s_1^n x$, $n = n(x, s_1) \geq 1$. Let $f(x) = y$; where $y \in S_2$. Since f is an isomorphism, for any $s_2 \in S_2$, there exists $s_1 \in S_1$ such that $f(s_1) = s_2$. Thus, $ys_2^n = f(x)f(s_1)^n = f(x)f(s_1^n)$(since f is an isomorphism, $f(s_1)^n = f(s_1^n)$) $= f(xs_1^n) = f(s_1^n x) = f(s_1^n)f(x) = f(s_1))^n f(x) = s_2^n y$, since $x \in T(S_1)$. Therefore, $y \in T(S_2)$. Thus, $f(T(S_1)) \subseteq T(S_2)$. Again, let $b \in T(S_2)$. Then $b = f(a)$; where $a \in S_1$. Since f is an isomorphism, for any $y \in S_2$, there exists $x \in S_1$ such that $y = f(x)$. Since $b \in T(S_2)$, it follows that $by^n = y^n b \Longrightarrow f(a)(f(x))^n = (f(x))^n f(a) \Longrightarrow f(a)f(x^n) = f(x^n)f(a) \Longrightarrow f(ax^n) = f(x^n a) \Longrightarrow ax^n = x^n a$, since f is an isomorphism. This implies that $a \in T(S_1)$. Hence, we conclude that $b = f(a) \in f(T(S_1))$. Consequently, $T(S_2)) \subseteq f(T(S_1))$ and hence $T(S_2) = f(T(S_1))$. As a result, $g = f_{|T(S_1)} : T(S_1) \longrightarrow T(S_2)$ is well defined and it is an isomorphism from $T(S_1)$ onto $T(S_2)$. □

Theorem 7.3.3. *Let S and S' be two semiring. If $f : S \to S'$ is a monomorphism, then $f(T(S)) = T(f(S))$.*

Proof. Suppose $x \in f(T(S))$. Then $x = f(y)$ for some $y \in T(S)$. Our objective is to prove that $f(y)s^n = s^n f(y)$ for all $s \in f(S)$. Now for any $s \in f(S)$, we have $f(y)s^n = f(y)(f(r))^n = f(y)f(r^n) = f(yr^n) = f(r^n y) = f(r^n)f(y) = (f(r))^n f(y) = s^n f(y)$, since $y \in T(S)$. Consequently, $x = f(y) \in T(f(S))$. Thus, we establish that $f(T(S)) \subseteq T(f(S))$. Now, suppose x' is an element of $T(f(S))$. This implies that $x' = f(r')$ for some r' in S. Our aim is to demonstrate that r' belongs to $T(S)$. Since x' is an element of $T(f(S))$, we can deduce that for any $f(s)$ in $f(S)$, the equation $x'(f(s))^n = (f(s))^n x'$ holds. Consequently, $f(r')f(s^n) = f(s^n)f(r')$, which further simplifies to $f(r's^n) = f(s^n r')$. Considering that f is a monomorphism, we can derive $r's^n = s^n r'$. Therefore, $r' \in T(S)$. As a result, we establish that $T(f(S)) \subseteq f(T(S))$. Hence, we can conclude that $T(f(S)) = f(T(S))$. □

Theorem 7.3.4. *If S_1 and S_2 are two semirings, then $T(S_1 \times S_2) = T(S_1) \times T(S_2)$.*

Proof. Let S_1 and S_2 be two semirings with zero elements 0_{S_1} and 0_{S_2} respectively. Suppose $z \in T(S_1 \times S_2)$. Then $z = (x, y) \in S_1 \times S_2$. Additionally, for any $(a, b) \in S_1 \times S_2$, the following condition holds : $(x, y)(a, b)^n = (a, b)^n(x, y)$. This condition can be further simplified as $(x, y)(a^n, b^n) = (a^n, b^n)(x, y)$, which implies $(xa^n, yb^n) = (a^nx, b^ny)$. By comparing both sides, we can deduce that $xa^n = a^nx$ and $yb^n = b^ny$. This implies that $xa^n = a^nx$ for all $a \in S_1$ and $yb^n = b^ny$ for all $b \in S_2$. Thus, it follows that $x \in T(S_1)$ and $y \in T(S_2)$. Consequently, we can conclude that $z = (x, y) \in T(S_1) \times T(S_2)$, leading to the inclusion $T(S_1 \times S_2) \subseteq T(S_1) \times (S_2) \quad (i)$.

For reverse part, let $(a, b) \in T(S_1) \times T(S_2)$. This means that $a \in T(S_1)$ and $b \in T(S_2)$. Consequently, for any $x \in S_1$, we have $ax^n = x^na$ and $y \in S_2$, we have $by^n = y^nb$. Now, $(a, b)(x, y)^n = (a, b)(x^n, y^n) = (ax^n, by^n) = (x^na, y^nb) = (x^n, y^n)(a, b) = (x, y)^n(a, b)$, since $a \in T(S_1)$ and $b \in T(S_2)$ for all $(x, y) \in S_1 \times S_2$. This implies that $(a, b) \in T(S_1 \times S_2)$ and hence $T(S_1) \times T(S_2) \subseteq T(S_1 \times S_2) \qquad (ii)$.

From (i) and (ii), we can conclude that $T(S_1 \times S_2) = T(S_1) \times T(S_2)$. □

Theorem 7.3.5. *Let S be a semiring with identity. Then $a \in T(S)$ if and only if $aI_n \in T(M_n^d(S))$; where $M_n^d(S)$ is the set of all $n \times n$ diagonal matrices of the form :*

$$M_n^d(S) = \left\{ \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} : \ a_{ii} \in S \right\}$$

Proof. Let $a \in T(S)$. Then $ax^n = x^na$; $n = n(x, a)$ for all $x \in S$. Since $a \in T(S)$, we have $ab_1^{n_1} = b_1^{n_1}a$, $ab_2^{n_2} = b_2^{n_2}a$,...,$ab_n^{n_n} = b_n^{n_n}a$. Let $n = \text{lcm}\{n_1, n_2, ..., n_n\}$. Now, $ab_1^n = ab_1^{n_1k} = ab_1^{n_1}b_1^{n_1}...b_1^{n_1}$ (k-times) $= b_1^{n_1}ab_1^{n_1}...b_1^{n_1}$ ($k-1$-times) $= b_1^{n_1}b_1^{n_1}a...b_1^{n_1}$ ($k-2$-times) $= ... = b_1^{n_1}...b_1^{n_1}a$ (k- times) $= b_1^{n_1k}a = b_1^na$. Similarly, we get $ab_2^n = b_2^na$,...,$ab_n^n = b_n^na$. Now, for any $B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$,

$$(aI_n)B^n = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \cdots \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \text{ (n-times)}$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \cdots \begin{pmatrix} b_1^2 & 0 & \cdots & 0 \\ 0 & b_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^2 \end{pmatrix} = \cdots$$

$$= \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b_1^n & 0 & \cdots & 0 \\ 0 & b_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^n \end{pmatrix} = \begin{pmatrix} ab_1^n & 0 & \cdots & 0 \\ 0 & ab_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ab_n^n \end{pmatrix}$$

$$= \begin{pmatrix} b_1^n a & 0 & \cdots & 0 \\ 0 & b_2^n a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^n a \end{pmatrix} = \begin{pmatrix} b_1^n & 0 & \cdots & 0 \\ 0 & b_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^n \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}^n \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} = B^n(aI_n).$$

This implies that $aI_n \in T(M_n^d(D))$.

Conversely, suppose that $aI_n \in T(M_n^d(D))$. Now we show that $a \in T(S)$. For any $x \in S$, let $B = \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}$. Since $aI_n \in T(M_n^d(D))$, it follows that $(aI_n)B^n = B^n(aI_n)$. Comparing both sides, we find that $ax^n = x^n a$. Consequently, $a \in T(S)$. □

Theorem 7.3.6. *If S is a nil semiring, then $T(S) = S$*

Proof. Assume S is a nil semiring and $a \in S$. It is evident that $T(S) \subseteq S$. Let b be any element in S. Our objective is to demonstrate that a belongs to $T(S)$. Since S is a nil semiring, there exists a natural number n such that $b^n = 0$. Consequently, $ab^n = 0$ and $b^n a = 0$. As a result, we can deduce that $ab^n = b^n a$, and this holds true for $n = n(b, a) \geq 1$ for all $b \in S$. Thus, we can conclude that a belongs to $T(S)$, leading to the inclusion $S \subseteq T(S)$. Therefore, we can infer that $T(S) = S$. □

Theorem 7.3.7. *If S is a semiring, then $T(S)$ is an antisimple semiring of S.*

Proof. Suppose S is a semiring. Let $r \in T(S)$. Then $rx^n = x^n r$; $n(x, r) \geq 1$, for all $x \in S$. Moreover, we observe that $(r + 1)x^n = rx^n + x^n = x^n r + x^n$ (since $r \in T(S)$) $= x^n(r + 1)$; $n(x, r) \geq 1$, for all $x \in S$. Consequently, we conclude that $r + 1 \in T(S)$. Therefore, we can deduce that $P'(T(S)) = \{0\} \cup \{r + 1 : r \in T(S)\}$. Thus, it follows that $T(S)$ is also an antisimple semiring of S. □

Theorem 7.3.8. *If S is a simple semiring, then $T(S)$ is also a simple semiring of S.*

Proof. Given that S is a simple semiring, it follows that $1 + a = 1$ for every a in S. Consequently, this condition holds true for all elements a belonging to the subsemiring $T(S)$, as $T(S)$ is a subset of S. Therefore, $T(S)$ can also be regarded as a simple semiring of S. □

Theorem 7.3.9. *If D is a division semiring, the hypercenter $T(D)$ of D is a division semiring.*

Proof. Consider the division semiring D and its hypercenter $T(D)$. It is known that $1 \in D$. Let $a(\neq 0)$ be any element of $T(D)$. Consequently, Da becomes a nonzero left ideal of D. As D is a division semiring, we deduce that $Da = D$. Thus there exists $d \in D$ such that $da = 1$. Since $a \in T(D)$, for any $d \in D$, we have $ad^n = d^n a = d^{n-1}da = d^{n-1} \cdot 1$ (due to $da = 1$) $= d^{n-1}$. Continuing this process, we obtain $ad^n = d^{n-1} \Longrightarrow ad^n a = d^{n-1}a \Longrightarrow ad^{n-1}da = d^{n-2}da \Longrightarrow ad^{n-1} = d^{n-2}$. By applying the same reasoning repeatedly, we arrive at $ad^2 = d \Longrightarrow ad^2 a = da \Longrightarrow ad \cdot da = da \Longrightarrow ad = 1$. Thus, we conclude that $ad = da = 1$. The remaining task is to demonstrate that $d \in T(D)$. Since $a \in T(D)$, for any $x \in D$, $ax^m = x^m a$;

$m = m(x, a) \geq 1$. Now, $a(x^m d) = (ax^m)d = (x^m a)d$ (since $a \in T(D)$) $= x^m ad = 1{\cdot}x^m$ (since $ad = 1$) $= adx^m$(since $ad = 1$) $= a(dx^m)$. By applying the cancellation law, we obtain $x^m d = dx^m$. Therefore, we can conclude that $d \in T(D)$. Consequently, $T(D)$ can be classified as a division semiring. □

Theorem 7.3.10. *Let S be any semiring. If $a \in T(S)$ is nilpotent then aS is a nil right ideal of S.*

Proof. Suppose $a \neq 0$ is a nilpotent element in T, which means there exists some positive integer $n > 1$ such that $a^n = 0$ and $a^{n-1} \neq 0$. Consider an element $x \in S$. Since $a^{n-1} \in T$, we have $(ax)^m a^{n-1} = a^{n-1}(ax)^m = 0$ for a suitable positive integer $m \geq 1$. Let's choose the smallest integer i such that $(ax)^u a^i = 0$ for some positive integer $u \geq 1$. If $i = 1$, then we would have $(ax)^{u+1} = 0$, implying that ax is nilpotent. If $i > 1$, then since $x(ax)^u a \cdot a^{i-1} = 0$, $(xa)^{u+1} a^{i-1} = 0$. Since $a^{i-1} \in T$, $a^{i-1}((xa)^{u+1})^s = ((xa)^{u+1})^s a^{i-1} = 0$ for some $s \geq 1$. Thus $a^{i-1}(xa)^r = 0$, where $r = (u+1)s$; hence $a^{i-2}(ax)^{r+1} = 0$. Since $a^{i-2} \in T$, $((ax)^{r+1})^v a^{i-2} = a^{i-2}((ax)^{r+1})^v = 0$. But this contradicts the minimal nature of i. In conclusion, we have $i = 1$, and therefore, ax is nilpotent for every $x \in S$. Consequently, aS is nil, establishing the validity of the theorem. □

7.4 Hypercentral Semiring

This section is dedicated to defining the concept of an hypercentral semiring and examining several properties relevant to this semiring.

Definition 7.4.1. *A semiring S is said to be hypercentral semiring if $T(S) = S$.*

Example 7.4.2. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	1
1	0	1	1

Then $(S,+,\cdot)$ *is a semiring and* $T(S)=\{0,x,1\}=S$.

Example 7.4.3. *Consider* $S=\{0,x,y,1\}$. *Define the operations "+" and "." on* S *by means of the following tables:*

+	0	x	y	1
0	0	x	y	1
x	x	x	y	1
y	y	y	y	1
1	1	1	1	1

.	0	x	y	1
0	0	0	0	0
x	0	x	x	x
y	0	x	y	y
1	0	x	y	1

Then $(S,+,\cdot)$ *is a semiring and* $T(S)=\{0,x,y,1\}=S$.

Example 7.4.4. *Let's consider the set* $S=\{1,2,3,6\}$, *the subset of natural numbers* $\mathbb{N}$. *Now define two binary operations* $\wedge$ *and* $\vee$ *on* S *by* $a\wedge b=gcd\{a,b\}$ *and* $a\vee b=lcm\{a,b\}$ *for all* $a,b\in S$. *As a result, we can conclude that the structure* $(S,\vee,\wedge)$ *is a bounded distributive lattice. Consequently,* S *is a commutative semiring. Therefore, the semiring* S *possesses the property that* $T(S)=S$, *making it a hypercentral semiring.*

We aim to demonstrate that the class of all hypercentral semirings constitutes a variety. To accomplish this objective, we will now shift our focus to proving the following three Lemmas, which will lay the foundation for the desired Theorem.

Lemma 7.4.5. *Let* S *be a hypercentral semiring and* S' *be an subsemiring. Then* S' *is a hypercentral semiring.*

Proof. Assume S is a hypercentral semiring, where $T(S)=S$. Our goal is to demonstrate that S' is also a hypercentral semiring with $T(S')=S'$. Let $a\in S'\subseteq S=T(S)$. Then $ab^n=b^na$; $n=n(b,a)\geq 1$ for all $b\in S$. This implies that $ab^n=b^na$; $n=n(b,a)\geq 1$ for all $b\in S'$, since S' is a subsemiring of S. This implies that $a\in T(S')$. Therefore, $S'\subseteq T(S')$. Moreover, since $T(S')$ is a subsemiring of S', we also have $T(S')\subseteq S'$. Hence, $T(S')=S'$. Consequently, S' is a hypercentral semiring. ∎

Lemma 7.4.6. *Every homomorphic image of a hypercentral semiring is a hypercentral semiring.*

Proof. Suppose we have two hypercentral semirings: S with identity element 1_S and S' with identity element $1_{S'}$. Let $f : S \to S'$ be an onto homomorphism. Our goal is to show that S' is also a hypercentral semiring. Since f is an onto homomorphism, so $S' = \{f(a) : a \in S\}$. Furthermore, we know that $f(1_S) = 1_{S'}$, confirming that the identity element is preserved under f. Now, let's take an arbitrary element $s' \in S'$. By definition, there exists $a \in S$ such that $f(a) = s'$. Since S is a hypercentral semiring, $ab^n = b^n a$ for all $a, b \in S$. Consider any $f(b) \in S'$, we have : $s' f(b)^n = f(a)f(b)^n = f(a)f(b^n) = f(ab^n) = f(b^n a) = f(b^n)f(a) = f(b)^n f(a) = f(b)^n s'$. Therefore, $s' = f(a) \in T(S') = S'$. Since s' is an arbitrary element of S', we conclude that S' satisfies the hypercentral property as well, making it a hypercentral semiring. □

Lemma 7.4.7. *Let $\{S_i : i = 1, 2, ..., n\}$ be a finite family of semirings. Then the direct product of semirings $S = \prod_{i=1}^{n} S_i$ is hypercentral semiring if and only if each semiring S_i is hypercentral semiring.*

Proof. Suppose that each semiring S_i of the family $\{S_i : i = 1, 2, ..., n\}$ is hypercentral. Since each S_i is hypercentral semiring, for any $y_i \in S_i$, we have $x_i y_i^n = y_i^n x_i$ for all $i = 1, 2, ..., n$. Now, $(x_1, x_2, ..., x_n)(y_1, y_2, ..., y_n)^n$

$= (x_1, x_2, ..., x_n)(y_1, y_2, ..., y_n)...(y_1, y_2, ..., y_n)$(repeated n-times)

$= (x_1, x_2, ..., x_n)(y_1^n, y_2^n, ..., y_n^n) = (x_1 y_1^n, x_2 y_2^n, ..., x_n y_n^n)$

$= (y_1^n x_1, y_2^n x_2, ..., y_n^n x_n) = (y_1^n, y_2^n, ..., y_n^n)(x_1, x_2, ..., x_n)$

$= (y_1, y_2, ..., y_n)...(y_1, y_2, ..., y_n)$(repeated n-times) $(x_1, x_2, ..., x_n)$

$= (y_1, y_2, ..., y_n)^n (x_1, x_2, ..., x_n)$. Cosequently, S is a hypercentral semiring.

Conversely, suppose that $S = \prod_{i=1}^{n} S_i$ is hypercentral semiring. Our goal is to demonstrate that each individual semiring S_i is also hypercentral. To do this, we'll utilize the mapping $\pi_i : S \to S_i$ defined by $\pi((x_1, x_2, ..., x_n)) = x_i$ for all $(x_1, x_2, ..., x_n) \in S$. This mapping π serves as an onto homomorphism from $S = \prod_{i=1}^{n} S_i$ to S_i. By invoking Lemma 7.4.6, we can conclude that S_i is a hypercentral semiring for all $i = 1, 2, ..., n$. Thus, our objective is proven. □

Theorem 7.4.8. *The class of all hypercentral semirings is a variety.*

Proof. By establishing the results from Lemmas 7.4.5, 7.4.6, and 7.4.7, it is demonstrated that the class of hypercentral semirings exhibits closure under subsemirings, homomorphic images, and direct products. Hence, we can conclude that the class of all of hypercentral semirings forms a variety. □

Chapter 8

The Interrelation of Centers

Chapter 8

The Interrelation of Centers

8.1 Introduction

In earlier sections, we established various types of semiring centers and explored their properties in detail. Moreover, we utilized these centers to construct fascinating semirings. Additionally, we presented several characterizations for these semirings. Conclusively, this thesis concludes by examining the interrelationships among the centers of semiring and establishing connections between the intriguing semirings we have developed.

8.2 The Relationship among Centers

The initial objective in this chapter is to develop a correlation between the center $Z(S)$, $E_c(S)$, and $B(S)$ of a semiring S.

Theorem 8.2.1. *The almost idempotent center of a semiring S is a subsemiring of $Z(S)$.*

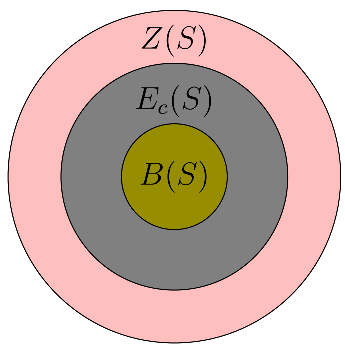
Proof. We know that $Z(S)$ is a subsemiring of S. Furthermore, Theorem 3.2.12 demonstrates that the almost idempotent center $E_c(S)$ of a semiring S also constitutes a subsemiring of S. Consequently, based on the definition, we get $E_c(S) \subseteq Z(S)$. Thus, we can conclude that $E_c(S)$ of S is indeed a subsemiring of $Z(S)$. □

Proposition 8.2.2. *If S is an idempotent semiring, then $E_c(S) = Z(S)$.*

Proof. It is evident that $E_c(S) \subseteq Z(S)$. Our goal is to establish that $Z(S) \subseteq E_c(S)$. Consider an element $a \in Z(S)$. Consequently, $ab = ba$ for all $b \in S$. Since S is an idempotent semiring, we have $a + a = a$ and $a^2 = a$ for all $a \in S$. This implies that $a + a^2 = a^2$. As $a \in Z(S)$, we have $ab = ba$. Hence, both $a + a^2 = a^2$ and $ab = ba$ hold true. Consequently, $a \in E_c(S)$. Thus, we can conclude that $Z(S) \subseteq E_c(S)$. Therefore, we can deduce that $E_c(S) = Z(S)$. □

Theorem 8.2.3. *[39] (Lemma 2.3.8) Let S be a c-semiring. Then $a \in B(S)$ if and only if $a \in E_c(S)$ and there exists a homomorphism $f_a : S \longrightarrow S_a$ such that $x \longmapsto (ax, f_a(x))$ is an isomorphism of S onto $aS \times S_a$ and f_a is identity on S_a.*

Remark 8.2.4. *If S is a c-semiring, then we have the following diagram :*



Furthermore, our attention now shifts to establishing the connection between the conventional center $Z(S)$, the h-center $C_h(S)$, and the almost idempotent center $E_c(S)$ of the semiring S. Additionally, we present the subsequent outcome :

Proposition 8.2.5. *Let S be a semiring. Then $C_h(S)$ is an ideal of $Z(S)$.*

Proof. Based on Theorem 4.3.1, we can conclude that $C_h(S)$ forms a subsemiring within S. By the definition of $C_h(S)$, it is evident that $C_h(S)$ is contained in $Z(S)$. Consequently, $C_h(S)$ can be identified as a subsemiring of $Z(S)$. Suppose we have $r \in Z(S)$ and $a \in C_h(S)$. Given that $r \in Z(S)$, it satisfies the condition $rs = sr$ for any $s \in S$. Moreover, since $a \in C_h(S)$, it fulfills the properties $a + ab = a$ and $ab = ba$ for all $b \in S$. For any $b \in S$, it follows that $arb = abr = bar$, as both $r \in Z(S)$ and

$a \in C_h(S)$. Additionally, we have $ar + arb = ar + abr = (a + ab)r = ar$ and $arb = bar$ for every $b \in S$. Therefore, we can conclude that $ar \in C_h(S)$. Similarly, we can demonstrate that $ra \in C_h(S)$. Consequently, we can deduce that $C_h(S)$ functions as an ideal of $Z(S)$. □

Theorem 8.2.6. *If S is a c-semiring, then $C_h(S) = Z(S)$.*

Proof. Initially, we observe that the set $C_h(S)$ is contained within $Z(S)$. Our task is to demonstrate that $Z(S)$ is also contained within $C_h(S)$. Let us consider an element $a \in Z(S)$. This implies that $ax = xa$ for all $x \in S$. As S is a c-semiring, we have the property $1 + x = 1$ for any $x \in S$. Now, for any $b \in S$, we can deduce that $a + ab = a(1 + b) = a1 = a$, as $1 + b = 1$ and $ab = ba$, since $a \in Z(S)$. Consequently, we can conclude that $a \in C_h(S)$. Hence, $Z(S)$ is contained within $C_h(S)$. Thus, we have established that $C_h(S) = Z(S)$. □

Proposition 8.2.7. *If S is a multiplicatively idempotent semiring with identity* 1, *then $C_h(S) \subseteq E_c(S)$.*

Proof. Suppose $a \in C_h(S)$. Consequently, we have $a + ab = a$ and $ab = ba$ for any $b \in S$. Specifically, if we take $b = a$, we obtain $a + a^2 = a$. As S is a multiplicatively idempotent semiring, we have $a + a^2 = a^2$. Moreover, since a satisfies $ab = ba$ for all a and b in S (as $a \in C_h(S)$), we can conclude that $a \in E_c(S)$. Therefore, we can establish that $C_h(S) \subseteq E_c(S)$. □

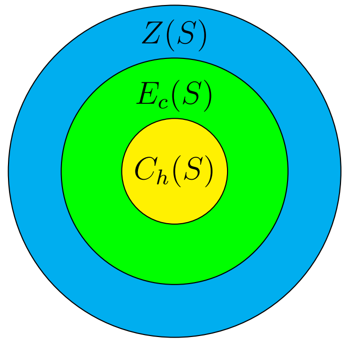
Example 8.2.8. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S, +, \cdot)$ is an idempotent semiring and $E_c(S) = \{0, x, 1\} = S$ and $C_h(S) = \{0, x\}$. $E_c(S) \nsubseteq C_h(S)$. This example shows that every element of $E_c(S)$ is not an element of $C_h(S)$.

Remark 8.2.9. *For multiplicatively idempotent semiring S, we have the following :*



In the subsequent analysis, we direct our attention towards establishing the correlation between the standard center $Z(S)$ and the k-center $C_k(S)$ of a semiring S. Furthermore, we have obtained the following result :

Proposition 8.2.10. *If S be a semiring, then $C_k(S)$ is an ideal of $Z(S)$.*

Proof. Consider $C_k(S)$ as a subsemiring of $Z(S)$. Let $r \in Z(S)$ and $a \in C_k(S)$. As $r \in Z(S)$, it satisfies the condition $rs = sr$ for all $s \in S$. Similarly, since $a \in C_k(S)$, it satisfies the conditions $a + ab = ab$ and $ab = ba$ for all $b \in S\backslash\{0\}$. Consequently, for any $b \in S\backslash\{0\}$, we have $arb = abr = bar$, owing to the commutativity of $r \in Z(S)$ and $a \in C_k(S)$. Moreover, $ar + arb = ar + abr = (a + ab)r = abr = arb$ and $arb = bar$ for all $b \in S\backslash\{0\}$. Therefore, $ar \in C_k(S)$. Similarly, we can demonstrate that $ra \in C_k(S)$. Thus, $C_k(S)$ qualifies as an ideal of $Z(S)$. □

Corollary 8.2.11. *If S is a semifield, then either $C_k(S) = Z(S)$ or $C_k(S) = \{0\}$.*

Theorem 8.2.12. *Let S be an additively idempotent mono-semiring. Then $C_k(S)$ coincides with $Z(S)$.*

Proof. Clearly, $C_k(S)$ is a subset of $Z(S)$. Our goal is to prove the reverse inclusion, that is, $Z(S) \subseteq C_k(S)$. Let a be an element of $Z(S)$. This means that $ax = xa$ for all $x \in S$. Since S is an additively idempotent mono-semiring, we know that $a + a = a$ and $a+b = ab$ for all $a, b \in S$. Now, for any $b \in S$ excluding the zero element, we have $a + ab = a + a + b = a + b = ab$, using the properties that $a + a = a$, $a + b = ab$, and

$ab = ba$ (since $a \in Z(S)$). This implies that a is also an element of $C_k(S)$. Therefore, we can conclude that $Z(S) \subseteq C_k(S)$. Hence, we have shown that $C_k(S) = Z(S)$. □

According to the subsequent theorem, there is no distinction between k-center and usual center when considering a c-semifield.

Theorem 8.2.13. *If S is a c-semifield, then $C_k(S) = Z(S)$.*

Proof. To establish the equality $C_k(S) = Z(S)$, we begin by observing that $C_k(S) \subseteq Z(S)$. To demonstrate the reverse inclusion, that is, $Z(S) \subseteq C_k(S)$, suppose $a \in Z(S)$. This implies that $ax = xa$ for all $x \in S$. Moreover, as S is a c-semifield, for any nonzero element $b \in S$, there exists $d \in S$ such that $bd = db = 1$. Furthermore, we note that $x + 1 = 1$ holds true for all $x \in S$. Consequently, for any $b \in S\backslash\{0\}$, we have $a + ab = abd + ab = ab(d + 1) = ab$; where $d + 1 = 1$ and $ab = ba$, due to $a \in Z(S)$. Hence, we deduce that $a \in C_k(S)$, demonstrating that $Z(S) \subseteq C_k(S)$. Consequently, we establish the equality $C_k(S) = Z(S)$. □

Our upcoming task involves investigating the connection among $C_k(S)$, $C_h(S)$, and $E_c(S)$. Additionally, we possess the subsequent finding :

Theorem 8.2.14. *Let S be a semiring. Then every element of $C_k(S)$ is left zero if and only if $C_k(S) = C_h(S)$.*

Proof. Let's take a belonging to $C_k(S)$ into consideration. Given that a is a left zero element, we have $ab = a$ for every $b \in C_k(S)$. Consequently, for any $b \in S$ excluding zero, we can deduce that $a + ab = ab$ and $ab = ba$. Since a is a left zero element, we have $a + ab = a$ and $ab = ba$ for all $b \in S$, demonstrating that $a \in C_h(S)$. Consequently, we can conclude that $C_k(S) \subseteq C_h(S)$. Similarly, let's assume a to be an element of $C_h(S)$. Following the same reasoning, for any $b \in S$ excluding zero, we can derive that $a + ab = a$ and $ab = ba$. Therefore, $a + ab = ab$ and $ab = ba$ hold true for all b in S excluding zero, given that a is a left zero. Consequently, we can conclude that $a \in C_k(S)$. Therefore, $C_h(S) \subseteq C_k(S)$ and accordingly, we can establish that $C_k(S) = C_h(S)$.

Conversely, if we assume $C_k(S) = C_h(S)$, and let $a \in C_k(S) = C_h(S)$, we can deduce that $a + ab = ab$ and $ab = ba$ hold true for all $b \in S\backslash\{0\}$. Additionally, we find that $a + ab = a$ and $ab = ba$ are satisfied for all $b \in S$. From these conditions, we can conclude that $a = ab$ for every $b \in S$, leading to the conclusion that a is a left zero. □

Proposition 8.2.15. *Let S be a semiring. Then $C_k(S) \subseteq E_c(S)$.*

Proof. If $a \in C_k(S)$, then a satisfies the properties $a + ab = ab$ and $ab = ba$ for all $b \in S\backslash\{0\}$. Notably, when we let $b = a$, it follows that $a + a^2 = a^2$. Thus, we conclude that $a + a^2 = a^2$ and $ab = ba$ hold for all $a, b \in S$, implying that $a \in E_c(S)$. Consequently, we can establish the inclusion $C_k(S) \subseteq E_c(S)$. □

Note 8.2.16. *But the following example shows that every element of $E_c(S)$ is not an element of $C_k(S)$.*

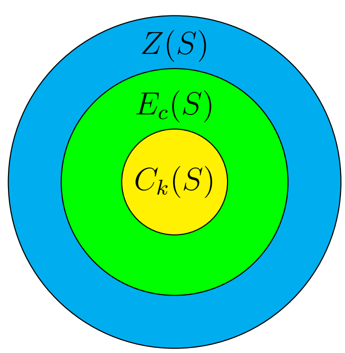
Example 8.2.17. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables :*

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S, +, \cdot)$ is an idempotent semiring and $E_c(S) = \{0, x, 1\} = S$ and $C_k(S) = \{0\}$. It follows that $E_c(S) \nsubseteq C_k(S)$.

Remark 8.2.18. *For any semiring S, we have the following :*



The subsequent theorem provides evidence that $C_k(S)$ and $E_c(S)$ coincide given a certain condition.

Theorem 8.2.19. *Let S be a semiring. If every element of $E_c(S)$ is left identity, then $C_k(S) = E_c(S)$.*

Proof. Suppose $a \in C_k(S)$. Consequently, we have $a + ab = ab$ and $ab = ba$ for all $b \in S\backslash\{0\}$. Specifically, when we choose $b = a$, we obtain $a + a^2 = a^2$. This implies that $a + a^2 = a^2$ and $ab = ba$ for any $a, b \in S$. As a result, we can conclude that $a \in E_c(S)$. Therefore, we can say that $C_k(S) \subseteq E_c(S)$.

Now, let's consider an element $a \in E_c(S)$. In this case, we have $a + a^2 = a^2$ and $ab = ba$ for all $b \in S$. Using these conditions, we can deduce that $a + a^2 + ab = a^2 + ab$, which further simplifies to $a + a(a + b) = a(a + b)$. Since a serves as the left identity of $E_c(S)$, we can conclude that $a + ab = ab$. Therefore, we have established that $a + ab = ab$ and $ab = ba$ for all $b \in S\backslash\{0\}$. Consequently, we can say that $E_c(S) \subseteq C_k(S)$. Hence, it follows that $C_k(S) = E_c(S)$. □

It's time to discover how the usual center $Z(S)$, the h-center $C_h(S)$, the k-center $C_k(S)$, and the generalized center $C_G(S)$ of a semiring S are interconnected.

Additionally, the following result holds true : We observe that if S is an additively cancellative semiring, the generalized center $C_G(S)$ coincides with the semiring's usual center denoted as $Z(S)$.

Proposition 8.2.20. *Let S be a commutative semiring. Then $Z(S) = C_G(S)$.*

Proof. In a commutative semiring S, it is known that $Z(S) = S$ (i).
It is also evident that $C_G(S) \subseteq S$. Consider an element $a \in S$. Since S is a commutative semiring, for any $a, b \in S$, $ab = ba$. Consequently, for all $a, b \in S$, we have $a + ab = a + ba$. Hence, $a \in C_G(S)$. This implies that $S \subseteq C_G(S)$. Therefore, we can conclude that $S = C_G(S)$ (ii).

From (i) and (ii), we can deduce that $Z(S) = C_G(S)$. □

Theorem 8.2.21. *If S is a mono-semiring, then $C_G(S) = Z(S)$.*

Proof. If S is a mono-semiring, then $S = C_G(S)$. Since S is a mono-semiring, for any $a, b \in S$, $ab = a + b = b + a = ba$. This indicates that S is a commutative semiring. We know that in a commutative semiring S, $Z(S) = S$. Accordingly, we have $C_G(S) = Z(S)$. □

Note 8.2.22. *As per the definition, it follows that* $C_h(S) \subseteq Z(S)$, $C_k(S) \subseteq Z(S)$ *and* $Z(S) \subseteq C_G(S)$. *Therefore, we can conclude that* $C_k(S) \subseteq C_G(S)$ *and* $C_h(S) \subseteq C_G(S)$.

Furthermore, we analyze the interplay between the standard center $Z(S)$, the h-center $C_h(S)$, the k-center $C_k(S)$, the generalized center $C_G(S)$, and the hypercenter $T(S)$ of a semiring S. Additionally, we derive the following finding :

Proposition 8.2.23. *Let S be a semiring. Then* $Z(S) \subseteq T(S)$.

Proof. Let $a \in Z(S)$. Then $ab = ba$ for all $b \in S$. Consequently, for all $b \in S$, we have $ab^n = b^n a$ when $n = 1$. This observation implies that $a \in T(S)$. Hence, we can conclude that $Z(S) \subseteq T(S)$. □

Theorem 8.2.24. *If S is a rectangular band with* 1, *then* $T(S) = Z(S)$.

Proof. Suppose S is a rectangular band. Then for any $a, b \in S$, we have the equality $aba = a$. Specifically, if we take $a1a = a$, we can deduce that $a^2 = a$. This observation leads to the conclusion that S is a multiplicatively band. We can further establish that $b = b^2 = bb = b^2b = b^3 = ... = b^n$; where $n(\geq 1)$ is an integer. Consider an element $a \in T(S)$. We can assert that $ab^n = b^n a$ for all $b \in S$; where $n = n(b, a) \geq 1$. This equality implies that $ab = ba$ for all $b \in S$, since $b^n = b$. Consequently, we can conclude that $a \in Z(S)$. Therefore, $T(S) \subseteq Z(S)$. By utilizing Proposition 8.2.23, we can deduce that $Z(S) \subseteq T(S)$. Hence, we can finally establish that $T(S) = Z(S)$. □

Theorem 8.2.25. *If S is a semiring, then* $C_h(S) \subseteq T(S)$ *and* $C_k(S) \subseteq T(S)$.

Proof. We have the inclusion $C_h(S) \subseteq Z(S)$, and also $Z(S) \subseteq T(S)$. Consequently, it follows that $C_h(S) \subseteq T(S)$. Similarly, considering $C_k(S) \subseteq Z(S)$, we can deduce that $C_k(S) \subseteq T(S)$. □

Theorem 8.2.26. *If S is an additively cancellative semiring, then* $C_G(S) \subseteq T(S)$.

Proof. Assume that a belongs to the generalized center of S denoted by $C_G(S)$. In such case, it holds true that $a+ab = a+ba$ for any $b \in S$. Given that S is an additively cancellative semiring, we can conclude that $ab = ba$ for all $a, b \in S$. Consequently, this implies that $a \in Z(S)$. Additionally, it is known that $Z(S) \subseteq T(S)$. Hence, we can infer that $a \in T(S)$. As a result, we can conclude that $C_G(S) \subseteq T(S)$. □

Note 8.2.27. *The generalized center in Example 7.2.8 is given by* $C_G(S) = \{\{\phi\}, \{0\}, \{a, b\}, \{0, a, b\}\}$*, while the hypercenter is* $T(S) = \{\{\phi\}, \{0\}\}$*. It is important to note that* $C_G(S) \neq T(S)$*.*

Theorem 8.2.28. *If S is a commutative semiring, then $T(S) = C_G(S)$.*

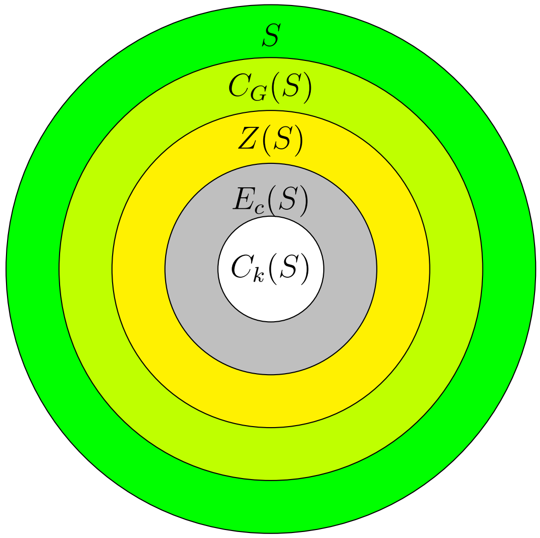
Proof. We know that if S is a commutative semiring, then $Z(S) = S$ (i).
By utilizing Proposition 8.2.23, we obtain $Z(S) \subseteq T(S)$ (ii).
In order to prove $T(S) = Z(S)$, we need to prove $T(S) \subseteq Z(S)$. As $T(S)$ constitutes a subsemiring of S, $T(S) \subseteq S$. Consequently, according to (i), we obtain $T(S) \subseteq S = Z(S)$. This implies that $T(S) \subseteq Z(S)$ (iii).

By combining (ii) and (iii), it follows that $T(S) = Z(S)$ (iv).
Moreover, based on Proposition 8.2.20, we have previously established that if S is a commutative semiring, then $Z(S) = C_G(S)$ (v).

Considering (iv) and (v), we can conclude that $T(S) = C_G(S)$. □

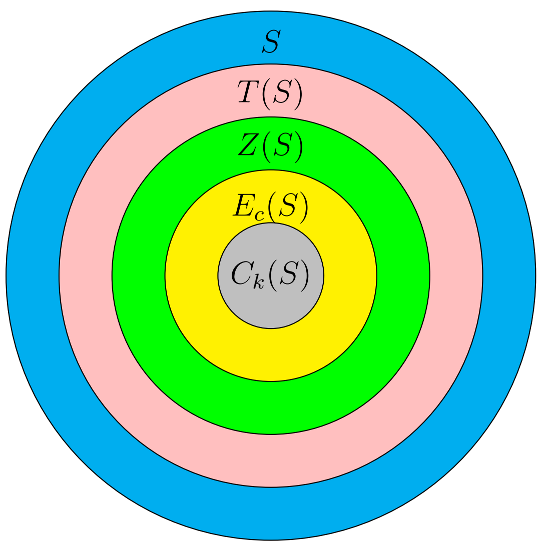
We now demonstrate the interrelation among the usual center, k-center, almost idempotent center and generalized center of a semiring.

$C_k(S) \subseteq E_c(S) \subseteq Z(S) \subseteq C_G(S) \subseteq S.$



We also delebrate the interrelation among the usual center, k-center, almost idempotent center and hypercenter of semiring.

$C_k(S) \subseteq E_c(S) \subseteq Z(S) \subseteq T(S) \subseteq S.$



8.3 Interactions within Central Semirings

Initially, our exploration focuses on the connection between h-central semiring and almost idempotent semiring. Additionally, we have obtained the subsequent finding :

Theorem 8.3.1. *In a PRD, every h-central semiring is an almost idempotent central semiring.*

Proof. Consider a semiring S that is both a PRD and an h-central semiring, denoted as $S = C_h(S)$. Since $E_c(S)$ is a subsemiring of S, we have $E_c(S) \subseteq S$. To demonstrate the reverse inclusion $S \subseteq E_c(S)$, let us take an arbitrary nonzero element $a \in S$. By applying the Proposition 4.6.19, we get in a PRD, if S is an h-central semiring, then for all a in S, $1 + a = a$ $\qquad (i)$.

Multiplying both sides of (i) by 'a', we find that $a(1 + a) = a \cdot a \Longrightarrow a + a^2 = a^2$. Furthermore, since S is a h-central semiring, we have $ab = ba$ for all $a, b \in S$. Hence, $a + a^2 = a^2$ and $ab = ba$ for all $b \in S$. Consequently, we can conclude that $a \in E_c(S)$. Additionally, we note that $0 \in E_c(S)$. Hence, we establish that $S \subseteq E_c(S)$, resulting in the equality $E_c(S) = S$. Therefore, S qualifies as an almost idempotent central semiring. □

In a particular scenario, we are currently establishing the connection between the k-central semiring and the h-central semiring.

Theorem 8.3.2. *In a PRD, every k-central semiring is an h-central semiring.*

Proof. Assume S to be a PRD and a k-central semiring. It is evident that $C_h(S) \subseteq S$. Take an arbitrary non-zero element a from S. Proposition 5.5.17 implies that in a PRD, every k-central semiring is a simple semiring. Hence, for any $b \in S$, we have $1 + b = 1$, which further leads to $a(1 + b) = a1$, resulting in $a + ab = a$ for all $a \in S$. Additionally, due to S being a k-central semiring, we know that $ab = ba$ for all $b \in S$. Therefore, both $a + ab = a$ and $ab = ba$ hold for all $b \in S$. Consequently, $a \in C_h(S)$. We can also observe that 0 is an element of $C_h(S)$. Thus, we conclude that $S \subseteq C_h(S)$ and, as a result, $C_h(S) = S$. Consequently, S is an h-central semiring □

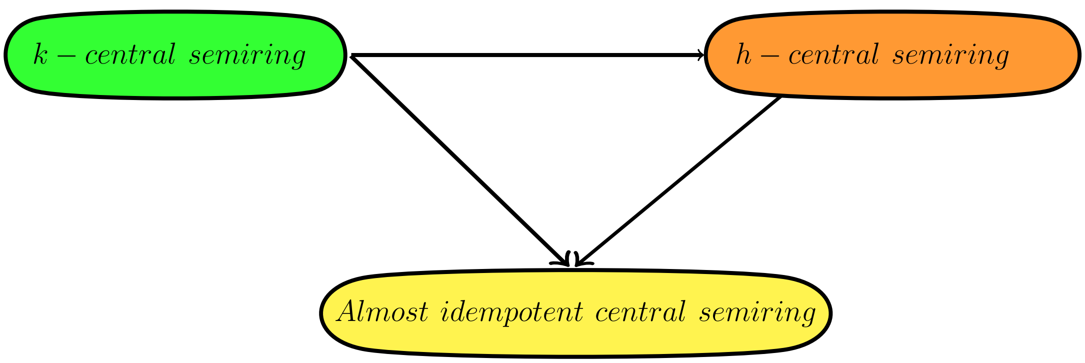
We conclude this chapter by exploring the connection between a k-central semiring and an almost idempotent central semiring.

Theorem 8.3.3. *Every k-central semiring is an almost idempotent central semiring.*

Proof. Let S be a k-central semiring. Clearly, $E_c(S) \subseteq S$. Consider an arbitrary non-zero element a in $S = C_k(S)$. For any b in S excluding zero, we have $a + ab = ab$. In particular, $a + a \cdot a = a \cdot a \Longrightarrow a + a^2 = a^2$. Moreover, $ab = ba$ holds $b \in S$, since $a \in C_k(S)$. Additionally, $0 \in E_c(S)$. Hence, we can conclude that $S \subseteq E_c(S)$, leading to $E_c(S) = S$. Consequently, S can be classified as an almost idempotent central semiring. □

The interrelation among k-central semiring , h-central semiring and almost idempotent central semiring can be shown through the following diagrammatic depiction:

k-central semiring $\Longrightarrow$ h-central semiring $\Longrightarrow$ almost idempotent central semiring and k-central semiring $\Longrightarrow$ almost idempotent central semiring.



List of Publications

1) S. Kar, S. Purkait and R. Sarkar : Birkhoff center of c-semiring; *Asian-European Journal of Mathematics*, Vol. 12, No. 1 (2019) 1950003 (11 pages).

2) R. Sarkar, S. Kar, B. Biswas and S.Purkait : Structure of Birkhoff center of c-semirings; *Asian-European Journal of Mathematics*, Vol. 14, No. 2 (2021) 2150023 (9 pages).

3) R. Sarkar, B. Biswas and S. Kar : On h-central semiring; Communicated for publication.

4) R. Sarkar, B. Biswas and S. Kar : k-center of a semiring; Communicated for publication.

5) R. Sarkar, B. Biswas and S. Kar : The structure of almost idempotent central semirings; Communicated for publication.

6) R. Sarkar, B. Biswas and S. Kar : Generalized center of a semiring; Communicated for publication.

7) R. Sarkar, B. Biswas and S. Kar : On the hypercenter of a semiring; Communicated for publication.

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