A STUDY OF CERTAIN GENERALIZED KINDS OF CHARACTERIZED SUBGROUPS AND THEIR IMPLICATIONS

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**A STUDY OF CERTAIN GENERALIZED KINDS OF CHARACTERIZED SUBGROUPS AND THEIR IMPLICATIONS**" submitted by **Mr. Ayan Ghosh** who got his name registered on **22nd March, 2021 (Index No: 30/21/Maths/27)** for the award of Ph.D. (Science) degree of Jadavpur University, Kolkata is absolutely based upon his own work under the supervision of **Prof. Pratulananda Das** and that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

P. Das 29.03.2023

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Dedicated to my parents

Jadab Ghosh
and
Shubhra Ghosh

and to my Chotdi

Late Kanchan Lata Nayek

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Notation

Throughout this thesis, we adopt the following notations. In some cases, we have mentioned the page numbers where the corresponding terms have been discussed explicitly.

- $\mathbb{C}$: Set of complex numbers.
- $\mathbb{R}$: Set of real numbers.
- $\mathbb{Q}$: Set of rational numbers.
- $\mathbb{Z}$: Set of integers.
- $\mathbb{N}$: Set of natural numbers (note that we do not consider zero as a natural number).
- $\mathbb{T}$: Circle group.
- $\{x\}$: Fractional part of x (page no. xxiv).
- $\|x\|$: Distance from the integers, i.e., $\min\big\{\{x\}, 1-\{x\}\big\}$.
- F : Set of all unbounded modulus functions.
- $\mathbb{F}$: Set of all strictly increasing unbounded modulus functions.
- $\mathbb{G}$: Set of weight functions (page no. xxi).
- $d(A)$: Natural density of $A \subseteq \mathbb{N}$.
- $d_\alpha(A)$: Natural density of order α of $A \subseteq \mathbb{N}$.
- $\mathcal{I}$: Ideal.
- $\mathcal{I}^*$: Dual filter of the ideal $\mathcal{I}$.
- Fin : Collection of all finite subsets of $\mathbb{N}$.
- $\mathcal{I}_d$: Natural density ideal.
- $\mathcal{I}_{d_\alpha}$: Natural density ideal of order α.
- $\mathcal{I}_g$: Simple density ideal.
- $\mathcal{I}_g(f)$: Moduler simple density ideal.
- $t_{(a_n)}(\mathbb{T})$: Characterized subgroup.

- $t^s_{(a_n)}(\mathbb{T})$: s-Characterized subgroup.
- $t^\alpha_{(a_n)}(\mathbb{T})$: α-Characterized subgroup.
- $t^{f,g}_{(a_n)}(\mathbb{T})$: f^g-Characterized subgroup.
- $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$: $\mathcal{I}$-Characterized subgroup.
- **D**-set : Dirichlet set (page no. 74).
- **A**-set : Arbault set (page no. 74).
- **N**-set : Set of absolute convergence (page no. 74).
- s**A**-set : Statistical Arbault set (page no. 78).

List of Definitions

Throughout this thesis, we frequently use some terms. The definition of each of them can be found in the corresponding page number mentioned below.

Abstract

The thesis is concerned with certain kinds of generalized characterized subgroups of the Circle group $\mathbb{T}$ and their applications in Trigonometric Series Theory or Fourier Analysis. Due to its strong relation with the torsion subgroup, topologically torsion subgroup, uniform distribution of sequence mod 1 and trigonometric thin set, the characterized subgroup has deep roots in different branches of Mathematics as Topological Algebra, Number Theory and Harmonic Analysis. However the study of generalized characterized subgroups has recently gained attention of researchers due to its ability to provide more general view in this context. Many problems on generalized characterized subgroups as well as characterized subgroups are still open. In this thesis, some of these open problems of the literature are considered and more general solutions are provided.

The thesis is divided into three parts. Part I of the thesis deals with some generalized notions of convergence where we have discussed many interesting results related to moduler simple density functions and corresponding ideals. These results enable us to construct various generalized characterized subgroups of $\mathbb{T}$ in Part II. In this part we have provided complete description of these generalized characterized subgroups for arithmetic sequences and solved many open problems from literature. In Part III, as an application, we have presented a new class of trigonometric thin sets namely statistical Arbault sets properly containing the class of classical Arbault sets as well as a large subfamily of $\mathbf{N}$-sets. In particular this class happens to properly contain the types of $\mathbf{N}$-sets which have been extensively used in the literature. It is worthwhile to note that this class provides uncountably many $F_{\sigma\delta}$ subgroups which cannot be characterized.

The aim of this thesis is to present an elaborate description of the topic alongside all the new results which would hopefully be useful to not only the experts working in this field, but also a starting point for those who wish to enter this field.

Publications

A list of publications resulted from the work of this thesis is appended below.

- P. Das, A. Ghosh, Generating subgroups of the circle using a generalized class of density functions, **Indagationes Mathematicae**, 32(3) (2021), 598–618.
- P. Das, A. Ghosh, Solution of a general version of Armacost's problem on topologically torsion elements, **Acta Mathematica Hungarica**, 164(1) (2021), 243–264.
- A. Ghosh, Some further remarks on characterized subgroups generated by modular simple density, **Quaestiones Mathematicae** (2022), 1–11, doi = 10.2989/16073606.2022.2058436.
- P. Das, A. Ghosh, On a new class of trigonometric thin sets extending Arbault sets, **Bulletin des Sciences Mathematiques**, 179 (2022), 103–157.
- A. Ghosh, Topologically $\mathcal{I}$-torsion elements of the circle, **Ricerche di Matematica** (2022), doi = 10.1007/s11587-022-00751-z.

Contents

PREFACE

In this chapter we recall some classical results and notions which will be needed frequently in this thesis.

0.1 Density functions and corresponding ideals

A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is called an ideal on $\mathbb{N}$ whenever

- $\mathbb{N} \notin \mathcal{I}$,
- if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,
- if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called a proper admissible ideal if $\mathcal{I}$ is not $\mathcal{P}(\mathbb{N})$ or ϕ and $\{n\} \in \mathcal{I}$ for all $n \in \mathbb{N}$. For an ideal $\mathcal{I}$ on $\mathbb{N}$, the dual filter $\mathcal{I}^*$ of $\mathcal{I}$ is defined as

$$\mathcal{I}^* = \{A \in \mathcal{P}(\mathbb{N}) : \mathbb{N} \setminus A \in \mathcal{I}\},$$

whereas the coideal $\mathcal{I}^+$ of $\mathcal{I}$ is defined as

$$\mathcal{I}^+ = \{B \subseteq \mathbb{N} : B \notin \mathcal{I}\}.$$

The simplest example of proper admissible ideal is Fin which is the collection of all finite subsets of $\mathbb{N}$. A brief but useful account on ideals can be found in [32].

For two subsets A, B of $\mathbb{N}$ and an ideal $\mathcal{I}$, we will write

- $A \subseteq^{\mathcal{I}} B$ if $A \setminus B \in \mathcal{I}$,
- $A \subseteq_{\mathcal{I}} B$ if $A \subseteq B$ and $B \setminus A \in \mathcal{I}$,
- $A =^{\mathcal{I}} B$ if $A \triangle B \in \mathcal{I}$.
- $A \subset \mathbb{N}$ is called $\mathcal{I}$-translation invariant if $A+n = \{m+n \in \mathbb{N} : \ m \in A\}$ belongs to $\mathcal{I}$ for all $n \in \mathbb{Z}$.

Note that for any $A \subseteq_{\mathcal{I}} B$ and $B \in \mathcal{I}^+$ implies $A \in \mathcal{I}^+$.

Definition 0.1.1. *An ideal $\mathcal{I}$ is called translation invariant if every $A \in \mathcal{I}$ is $\mathcal{I}$-translation invariant.*

Definition 0.1.2. *An ideal $\mathcal{I}$ is called dense if for every infinite $A \subset \mathbb{N}$ there is an infinite $B \in \mathcal{I}$ such that $B \subset A$.*

Definition 0.1.3. *An ideal $\mathcal{I}$ on $\mathbb{N}$ is called a P-ideal if for each sequence (A_n) of sets in $\mathcal{I}$ there exists a set A in $\mathcal{I}$ such that $A_n \subseteq^* A$ for all $n \in \mathbb{N}$ (by $A_n \subseteq^* A$ we mean $A_n \setminus A \in Fin$).*

Next we will see a strong relation between a P-ideal and a *submeasure*. Recall that a *submeasures* on $\mathbb{N}$ is a function $\varphi\colon 2^{\mathbb{N}} \to [0, \infty]$ such that:

- $\varphi(\emptyset) = 0$,
- if $A \subset B$ then $\varphi(A) \leq \varphi(B)$,
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,
- $\varphi(\{n\}) < \infty$ for all $n \in \mathbb{N}$.

Definition 0.1.4. *([32]) A submeasure φ is called a lower semicontinuos submeasure (in short, lscsm) if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [1, n])$ for all $A \subset \mathbb{N}$.*

Note that the above condition is equivalent to the classical lower semicontinuity of the function $\varphi\colon 2^{\mathbb{N}} \to [0, \infty]$. For any lscsm φ, let us consider the exhaustive ideal $Exh(\varphi)$ defined as

$$Exh(\varphi) = \{A \subset \mathbb{N} \colon \lim_{n\to\infty} \varphi(A \setminus [1, n]) = 0\}.$$

Our next theorem is an important observation in this direction which was provided by I. Farah in [50, Lemma 1.2.2].

Theorem 0.1.5. *For every lscsm φ on $\mathbb{N}$, $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal.*

Every ideal $\mathcal{I}$ on $\mathbb{N}$ can be treated as a subset of the Cantor space $2^{\mathbb{N}}$ in view of the fact that $\mathcal{P}(\mathbb{N})$ and $2^{\mathbb{N}}$ can be identified via the characteristic functions.

Definition 0.1.6. *An ideal $\mathcal{I}$ on $\mathbb{N}$ is called analytic if it corresponds to an analytic subset of the Cantor space $2^{\mathbb{N}}$.*

In the literature about ideal convergence, a very prominent role has been played by the class of analytic P-ideals (see for example [65], [69]) while these ideals had long been topics of much interest in set theory (see [50, 58, 81] where more references can be found).

Following theorem is a highly nontrivial result of Solecki [80] which gives a characterization of analytic P-ideals on $\mathbb{N}$.

Theorem 0.1.7. *If $\mathcal{I}$ is an analytic P-ideal on $\mathbb{N}$ then it is of the form $Exh(\varphi)$ for some lscsm φ on $\mathbb{N}$.*

Let us now recall some of the density functions and their corresponding ideals which will play crucial role in our following chapters. For $n, m \in \mathbb{N}$ with $n < m$, let $[n, m]$ denote the set $\{n, n+1, n+2, \ldots, m\}$.

Definition 0.1.8. *(see [24],[25]) The lower and the upper natural densities of $A \subset \mathbb{N}$ are defined by*

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n}$$

and

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, we say that the natural density of A exists and it is denoted by $d(A)$.

As usual,

$$\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$$

denotes the ideal of "natural density zero" sets and $\mathcal{I}_d^*$ is the dual filter, i.e., $\mathcal{I}_d^* = \{A \subset \mathbb{N} : d(A) = 1\}$.

In [13] a natural extension of the notion of natural density was introduced. Here the authors replaced the term n with a non linear term n^α in the definition of natural density where $0 < \alpha \leq 1$. The motivation of this extension came from the urge to investigate different kinds of densities and the problem of comparing them with the natural density.

Definition 0.1.9. *(see [13]) The lower and the upper natural densities of order α of $A \subset \mathbb{N}$ are defined by*

$$\underline{d}_\alpha(A) = \liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n^\alpha}$$

and

$$\overline{d}_\alpha(A) = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n^\alpha}.$$

If $\underline{d}_\alpha(A) = \overline{d}_\alpha(A)$, we say that the natural density of order α of A exists and it is denoted by $d_\alpha(A)$.

Similarly one can define the corresponding ideal by

$$\mathcal{I}_{d_\alpha} = \{A \subset \mathbb{N} : d_\alpha(A) = 0\}.$$

On the other hand, in 2015 in [6] the authors defined a new class of densities using weight functions which again extended the concept of natural density of order α. Though the main motivation of considering this new class was not just a mere extension. It was actually originated to construct a large number of non-comparable analytic P-ideals. Eventually in [6], it was shown that one can construct uncountably many non-comparable P-ideals corresponding to different choices of the weight function g, all different from the natural density ideal $\mathcal{I}_d$.

Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n \to \infty} g(n) = \infty$. The upper density of weight g was defined in [6] by the formula

$$\overline{d}_g(A) = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{g(n)}$$

for $A \subset \mathbb{N}$. The lower density of weight g, $\underline{d}_g(A)$ is defined in a similar way. Then the family

$$\mathcal{I}_g = \{A \subset \mathbb{N} : \overline{d}_g(A) = 0\}$$

also forms an ideal. It was observed in [6] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$. So we additionally assume that $n/g(n) \nrightarrow 0$ and denote the set of all such weight functions g by $\mathbb{G}$.

Lately there have been certain other as also more general versions of density functions which we now recall. The modulus functions are defined as functions $f : [0, \infty) \to [0, \infty)$ which satisfy the following properties.

(i) $f(x) = 0 \Leftrightarrow x = 0$

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in (0, \infty)$ [Triangle inequality]

(iii) f is non-decreasing

(iv) f is right continuous at 0.

In 2014, a notion of density function was introduced using modulus functions. Precisely, the upper f density function [2] was defined in the following way:

$$\overline{d}^f(A) = \limsup_{n\to\infty} \frac{f(|A \cap [0, n-1]|)}{f(n)}.$$

Similarly the lower f density function $\underline{d}^f$ is defined. As a natural consequence, the family

$$\mathcal{I}(f) = \{A \subset \mathbb{N} : \overline{d}^f(A) = 0\}$$

forms an ideal on $\mathbb{N}$.

The approaches of [2], [6] and [13] were unified in [18]. Let us now recall the definition of the density function d_g^f, henceforth called "moduler simple density function" (where f is an unbounded modulus function) introduced in [18] which plays key role in Chapter 4.

Definition 0.1.10. *For $A \subseteq \mathbb{N}$, the lower and upper moduler simple density function is defined as*

$$\underline{d}_g^f(A) = \liminf_{n\to\infty} \frac{f(|A \cap [1, n]|)}{f(g(n))} \text{ and } \overline{d}_g^f(A) = \limsup_{n\to\infty} \frac{f(|A \cap [1, n]|)}{f(g(n))}.$$

If $\underline{d}_g^f(A) = \overline{d}_g^f(A)$, we say that $d_g^f(A)$ exists.

Following the nomenclature of [18], the collection

$$\mathcal{I}_g(f) = \{A \subset \mathbb{N} : d_g^f(A) = 0\}$$

denotes the corresponding ideal (henceforth called moduler simple density ideal) and $\mathcal{I}_g^*(f)$ is the dual filter, i.e.,

$$\mathcal{I}_g^*(f) = \{A \subset \mathbb{N} : d_g^f(\mathbb{N} \setminus A) = 0\}.$$

Further in view of our next result we can assume $\mathbb{G}$ to consist of only non-decreasing weight functions $g : \mathbb{N} \to [0, \infty)$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)} \nrightarrow 0$ without any loss of generality.

Lemma 0.1.11. *[18] Let f be an unbounded modulus function as specified. For each function $g \in \mathbb{G}$ there exists a non-decreasing function $g' \in \mathbb{G}$ such that $\mathcal{I}_{g'}(f) = \mathcal{I}_g(f)$. Moreover, $f(g'(n)) \leq f(g(n))$ for all $n \in \mathbb{N}$.*

0.2 Some general notions of convergence

The idea of natural density was later used to define the notion of statistical convergence ([83], see also [53, 76]).

Definition 0.2.1. *A sequence of real numbers (x_n) is said to converge to a real number x_0 statistically if for any $\varepsilon > 0$, $d(\{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\}) = 0$.*

Though the concrete notion of statistical convergence is quite new but the idea of statistical convergence has been used extensively in different areas of mathematics for a long period of time. Applications of statistical convergence to Number Theory and Fourier Analysis can be found in [24, 25, 74]. Later on, statistical convergence was further investigated from the sequence space point of view and linked with summability theory.

Theorem 0.2.2. *[76] A sequence of real numbers (x_n) converges to a real number x_0 statistically if and only if there exists a set $M \in \mathcal{I}_d^*$ such that $(x_n)_{n \in M}$ usually converges to x_0.*

The following notion first appeared in the work of the celebrated mathematician Henry Cartan [28] and then again reappeared in 2000.

Definition 0.2.3. *[65] In a topological space X, given an ideal $\mathcal{I}$, we say that a sequence $(x_n)_{n \in \mathbb{N}}$ is $\mathcal{I}$-convergent to $x \in X$ whenever for every open set U containing x, the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ (we will write $x_n \to x$ w.r.t $\mathcal{I}$).*

Note that for the ideal Fin, corresponding ideal convergence coincides with the usual convergence. For the ideals $\mathcal{I}_d$, $\mathcal{I}_{d_\alpha}$, $\mathcal{I}_g$, $\mathcal{I}(f)$ and $\mathcal{I}_g(f)$ corresponding ideal convergence are called statistical convergence, statistical convergence of order α, statistical convergence of weight g, f-statistical convergence and f^g-statistical convergence respectively.

Definition 0.2.4. *[65] In a topological space X, given an ideal $\mathcal{I}$, we say that a sequence $(x_n)_{n \in \mathbb{N}}$ is $\mathcal{I}^*$-convergent to $x_0 \in X$ if there exists a set $M \in \mathcal{I}^*$ such that $(x_n)_{n \in M}$ usually converges to x_0.*

In view of our above definitions and results it is straight forward to see that the concept of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are nothing but a generalization of statistical convergence and its equivalent form. We have already seen that both these notions are equivalent for the natural density ideal $\mathcal{I}_d$. However for arbitrary ideals this is not in general true. The following is one of the most important result in this direction.

Theorem 0.2.5. *[65] For a proper admissible ideal $\mathcal{I}$, $\mathcal{I}$-convergence coincides with the $\mathcal{I}^*$-convergence if and only if $\mathcal{I}$ is a P-ideal.*

In [18] it was shown that for any unbounded modulus function f and $g \in \mathbb{G}$ the moduler simple density ideal $\mathcal{I}_g(f)$ is an analytic P-ideal. Therefore for all these ideals, i.e., $\mathcal{I}_d$, $\mathcal{I}_{d_\alpha}$, $\mathcal{I}_g$, $\mathcal{I}(f)$ and $\mathcal{I}_g(f)$, the $\mathcal{I}$-convergence coincides with the corresponding $\mathcal{I}^*$-convergence.

0.3 The circle group

In this section we recall some of the basic definitions and facts related to circle group which is one of the main object studied in this thesis.

The circle group $\mathbb{T}$ is a subgroup of the complex numbers $\mathbb{C}$ consists of all complex numbers on the unit circle, i.e.,

$$\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}.$$

Equivalently one can also define

$$\mathbb{T} = \{e^{ix} \ : \ x \in \mathbb{R}\}.$$

Consider the epimorphism $\phi : \ \mathbb{R} \to \mathbb{T}$ defined as $\phi(x) = e^{2\pi ix}$. Now it is easy to check that $Ker(\phi) = \mathbb{Z}$. Therefore ϕ induces an isomorphism ψ between $\mathbb{R}/\mathbb{Z}$ and $\mathbb{T}$.

Note that $\mathbb{T}$ with the subspace topology induced from the euclidean metric of $\mathbb{C}$ is a topological group as well as $\mathbb{R}/\mathbb{Z}$ with the quotient topology induced from the euclidean metric of $\mathbb{R}$. Moreover, ψ (defined above) is a topological isomorphism from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{T}$. In this thesis, it is preferable to deal with the additive notation of the circle group, i.e., $\mathbb{R}/\mathbb{Z}$ which we denote by $\mathbb{T}$.

In the circle group while considering the usual metric, the distance between two points is the length of the minimal arc connecting them. However this metric is compatible with the usual topology, i.e., the quotient topology from the euclidean one in $\mathbb{R}$.

Sometimes, it is preferable to deal with elements of $\mathbb{R}$ rather than $\mathbb{R}/\mathbb{Z}$. So we identify $\mathbb{T}$ with the interval [0,1] identifying 0 and 1. Any real valued function f defined on $\mathbb{T}$ can be identified with a periodic function defined on the whole real line $\mathbb{R}$ with period 1, i.e., $f(x+1) = f(x)$ for every real x.

When referring to a set $X \subseteq \mathbb{T}$ we assume that $X \subseteq [0,1]$ and $0 \in X$ if and only if $1 \in X$. For a real x, we denote its fractional part by $\{x\}$. In this way one can define the norm in $\mathbb{T}$ in the following way. For $x \in \mathbb{T}$ we define

$$\|x\| = \min\big\{\{x\}, 1 - \{x\}\big\}.$$

Observe that this norm $\|.\|$ of $\mathbb{T}$ is nothing but the integer norm $\|.\|_{\mathbb{Z}}$ of $\mathbb{R}$.

Chapter 1

INTRODUCTION

1.1 Motivation and Background

For a sequence of integers (v_n) and $x \in [0,1]$, the behavior of the sequence $(v_n x)$ mod 1 has deep roots in Harmonic Analysis, Dynamical System, Number Theory and Topology (see [3, 38, 46, 48, 66, 86]).

Recall that a sequence of real numbers (x_n) is said to be *uniformly distributed mod 1*, if for every $[c,d] \subseteq [0,1)$ one has

$$\lim_{n\to\infty} \frac{|\{i : 0 \leq i < n,\ \{x_i\} \in [c,d]\}|}{n} = d - c.$$

For a sequence of integers (v_n), it is obvious that the set

$$W_{\mathbf{v}} = \{x \in [0,1] : (v_n x) \text{ is uniformly distributed mod } 1\}$$

is contained in $[0,1] \setminus \mathbb{Q}$. In his celebrated result [86], H. Weyl proved that the set $W_{\mathbf{v}} = \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ if $v_n = P(n)$ for some $P(x) \in \mathbb{Z}[x]$. On the other hand, for every number $\alpha \in [0,1] \setminus \mathbb{Q}$, there exists a sequence $\mathbf{v} = (v_n)$ such that $\alpha \notin W_{\mathbf{v}}$.

Now consider an irrational number α with the regular continued fraction approximation $\alpha = [u_0; u_1, u_2, ...]$ with sequence of convergents $\frac{r_n}{b_n} = [u_0; u_1, u_2, ..., u_n]$. From Theorem 4.3 [66] it follows that the sequence $(b_n \beta)$ is uniformly distributed mod 1 for almost all $\beta \in \mathbb{R}$ in the sense of Lebesgue measure. However note that as $||b_n\alpha||_{\mathbb{Z}} \to 0$, so $\alpha \notin W_{\mathbf{b}}$. Moreover, Larcher [70] proved in 1988 that "if the continued fraction expansion of an irrational number α is bounded" then the set

$$\{\beta \in \mathbb{R} : ||b_n\beta||_{\mathbb{Z}} \to 0\}$$

is nothing but the subgroup $\langle\alpha\rangle + \mathbb{Z}$ of $\mathbb{R}$. This aspect of the distribution of the sequence $(v_n x)$, notoriously "complementary" to the uniform distribution mod 1, paves the path towards the study of characterized subgroups which is the backbone of the present Thesis.

Instead of using the fractional part $\{x_j\}$ or working modulo 1 in $\mathbb{R}$, one can also work in the circle group $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ by using the canonical epimorphism (projection) $\pi : \mathbb{R} \to \mathbb{T}$ (i.e., instead of considering $||b_n\beta||_{\mathbb{Z}} \to 0$ in $\mathbb{R}$, one can equivalently consider

$b_n\pi(\beta) \to 0$ in $\mathbb{T}$).

The behavior of the sequence $(v_n x)$ mod 1, is related to different trigonometric thin sets which play interesting role in the convergence of a trigonometric series in Harmonic Analysis (see Chapter 6 for more details). For $x \in \mathbb{T}$, the behavior of the sequence $(v_n x)$ is also related to Hausdorff group topologies with or without non-trivial convergent sequences.

1.2 Characterized Subgroups

The sequence $(v_n x)$ in a topological abelian group is used to generalize the notion of torsion subgroups. Consequently, it is closely related to the notion of topologically torsion subgroups (later came to be known as characterized subgroups) which is fundamental in the study of locally compact abelian groups.

Recall that an element x of an abelian group X is torsion if there exists an integer $k > 0$ such that $kx = 0$ (more specifically called k-torsion in this case). An element x of an abelian topological group X is [20]:

(i) *topologically torsion* if $n!x \to 0$;

(ii) *topologically p-torsion*, for a prime p, if $p^n x \to 0$.

It is obvious that any p-torsion element is topologically p-torsion. Armacost [5] defined the subgroups

$$X_p = \{x \in X : p^n x \to 0\} \text{ and } X! = \{x \in X : n!x \to 0\}$$

of an abelian topological group X, and started to describe the elements of these subgroups. Note that the above two notions are just special cases of the following general notion considered in (Section 4.4.2, [46]).

Definition 1.2.1. *Let (a_n) be a sequence of integers. An element x in an abelian topological group X is called* (topologically) $\underline{a}$-torsion element *if $a_n x \to 0$.*

When there is no confusion regarding the sequence (a_n), these elements are simply called topologically torsion elements.

Definition 1.2.2. *For a sequence of integers (a_n), the set*

$$t_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \textit{ in } \mathbb{T}\}$$

is called a "characterized" (by (a_n)) "subgroup" of $\mathbb{T}$.

Even if the notion was inspired by the various (earlier) instances mentioned above, the term characterized appeared much later, coined in [15] and since then been of much interest in different areas of Mathematics (one must see the excellent survey article [78] where the rich history along with the results and references can be found). Again coming back to Armacost,

(a) obviously $t_{(p^n)}(\mathbb{T})$ contains the Prüfer group $\mathbb{Z}(p^\infty)$. Armacost [5] proved that $t_{(p^n)}(\mathbb{T})$ simply coincides with $\mathbb{Z}(p^\infty)$ and x is a topologically p-torsion element if and only if $supp(x)$ (defined after Fact 3.1.2) is finite.

(b) at the same time, he [5] posed the problem to describe the group $\mathbb{T}! = t_{(n!)}(\mathbb{T})$ which was much later resolved independently and almost simultaneously in [46, Chap. 4] and by J.-P. Borel [17].

In particular, in both the above mentioned instances, the sequences of integers, concerned are arithmetic sequences. Recall that a sequence of positive integers (a_n) is called an arithmetic sequence if

$$1 < a_1 < a_2 < a_3 < \ldots < a_n < \ldots \quad \text{and } a_n | a_{n+1} \quad \text{for every } n \in \mathbb{N}.$$

For any sequence of integers (a_n), the sequence of ratios (q_n) is defined as

$$q_1 = a_1 \text{ and } q_n = \frac{a_n}{a_{n-1}} \text{ for } n \geq 2.$$

Observe that (q_n) becomes a sequence of integers when (a_n) is an arithmetic sequence. For $A \subseteq \mathbb{N}$, we say that A is

(i) q-bounded if the sequence of ratios $(q_n)_{n\in A}$ is bounded.

(ii) q-divergent if the sequence of ratios $(q_n)_{n\in A}$ diverges to ∞.

We say that the sequence (a_n) is q-bounded (q-divergent) if the set of naturals, i.e., $\mathbb{N}$ is q-bounded (q-divergent).

As has been seen, some of the most interesting cases studied are the topologically torsion elements characterized by arithmetic sequences. The results of Armacost [5] and Borel [17] were considered in full general settings with arbitrary arithmetic sequences in [77] and then with more clarity in [41] where topologically torsion elements were completely described for the class of general arithmetic sequences.

Another important result in this direction is Eggleston's dichotomy [47], where Eggleston observed (see [9]) that the asymptotic behavior of the sequence $q_n := \frac{a_n}{a_{n-1}}$ of ratios has a strong impact on the size of $t_{(a_n)}(\mathbb{T})$:

(E1) $t_{(a_n)}(\mathbb{T})$ is countable if (q_n) is bounded;

(E2) $|t_{(a_n)}(\mathbb{T})| = c$ if $q_n \to \infty$.

Bíró, Deshouillers and Sós [15] established the important fact that every countable subgroup of $\mathbb{T}$ is characterized. The whole history concerning these investigations along with relevant references can be found in the surveys [37, 78] as also the recent article [38].

1.3 Generalized characterized subgroups

Recently in 2019 these characterized subgroups were considered in more general settings where different models of convergence came into the picture. Even if the correspondence $(a_n) \mapsto t_{(a_n)}(\mathbb{T})$ is monotone decreasing (with respect to inclusion), in many cases (for example consider any q-bounded arithmetic sequence) the subgroup $t_{(a_n)}(\mathbb{T})$ is rather small, even if the sequence (a_n) is not too dense (any arithmetic sequence is a geometric progression, so has exponential growth). This suggests that asking $a_nx \to 0$ maybe somewhat too restrictive. A very natural instinct should be to consider modes of convergence which are more general than the notion of usual convergence and here the idea of natural density came into picture, as motivated by the above mentioned observation, Dikranjan, Das and Bose [38] introduced the notion of statistically characterized subgroups of $\mathbb{T}$ by relaxing the condition $a_nx \to 0$ with the condition $a_nx \to 0$ statistically.

Definition 1.3.1. *[38] For a sequence of integers (a_n) the subgroup*

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_nx \to 0 \ \textit{statistically in} \ \mathbb{T}\}$$

of $\mathbb{T}$ is called a statistically characterized (shortly, an s-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

Definition 1.3.2. *Let (a_n) be a sequence of integers. An element $x \in \mathbb{T}$ is called topologically s-torsion if $x \in t^s_{(a_n)}(\mathbb{T})$.*

In [38], it was observed that the topologically s-torsion elements of $\mathbb{T}$ form a proper Borel subgroup of $\mathbb{T}$ of cardinality $\mathfrak{c}$ for any arithmetic sequence (a_n).

In [19] another attempt was made to generate nice subgroups of $\mathbb{T}$ using certain kinds of density functions when the notion of natural density of order α was used to generate corresponding characterized subgroups.

Definition 1.3.3. *[19] For a sequence of integers (a_n) the subgroup*

$$t^\alpha_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_nx \to 0 \ \alpha\textit{-statistically in} \ \mathbb{T}\}$$

of $\mathbb{T}$ is called an α-statistically characterized (shortly, an α-characterized) (by (a_n)) subgroup of $\mathbb{T}$.

In [19, 38] these subgroups were studied in great details. In fact it was seen that they indeed generate, for a given arithmetic sequence (a_n), new nontrivial subgroups different from $t_{(a_n)}(\mathbb{T})$ as well as $t^s_{(a_n)}(\mathbb{T})$.

1.4 Main results and Contribution

This thesis contains seven chapters. The first chapter is the Introduction and the last chapter (i.e., Chapter 7) is devoted to open problems and future work where Chapter 2 to Chapter 6 includes the main contributions. Also, we divide this thesis into three parts. The first part consists of a single chapter namely Chapter 2 where we have discussed several modes of convergence and corresponding ideals with particular attention to the modular simple density ideal. The second part contains three chapters namely Chapter 3, Chapter 4 and Chapter 5 where various generalized characterized subgroups are investigated in great details. The third part consists of two chapters namely Chapter 6 and Chapter 7 where applications related to generalized characterized subgroups are given.

Part-I : f^g-DENSITY FUNCTIONS AND CORRESPONDING IDEALS

In Chapter 2, we continue our investigation of the ideal $\mathcal{I}_g(f)$ from [18]. This chapter deals with some properties of the ideal $\mathcal{I}_g(f)$ and many comparative results between the ideals $\mathcal{I}_g(f)$, $\mathcal{I}_g$ and $\mathcal{I}(f)$. The first main result regarding the ideal $\mathcal{I}_g(f)$ is the following:

Lemma 1.4.1. *For any unbounded modulus function f there exists a strictly increasing unbounded modulus function f' such that for any $g \in \mathbb{G}$, $\mathcal{I}_g(f) = \mathcal{I}_g(f')$ and $f'(n) \leq f(n)$ for all $n \in \mathbb{N}$.*

Due to this lemma, rather considering f to be an unbounded modulus function, without loss of any generality one can additionally consider f to be a strictly increasing open continuous bijection. Using this strong property of the modulus function f, in our next result we are able to construct a chain of ideals between Fin and $\mathcal{I}(f)$ which plays key role to generate an uncountable tower of Borel subgroups between corresponding generalized characterized subgroups.

Theorem 1.4.2. *For any unbounded modulus function f there exists a $\mathbb{G}_0 = \{g_\alpha \in \mathbb{G} : \alpha \in (0,1)\} \subseteq \mathbb{G}$ such that the followings hold:*

(a) $\mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}_{g_\beta}(f)$ *whenever* $\alpha < \beta, \alpha, \beta \in (0,1)$.

(b) $\bigcap_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f) \supsetneq Fin$.

(c) $\bigcup_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}(f)$.

Our next result is the most important observation regarding the ideals $\mathcal{I}_g(f)$ in this chapter (which was left out in [18]) where we show that there exists an antichain of cardinality $\mathfrak{c}$ of such ideals.

Theorem 1.4.3. *For any two unbounded modulus functions f_1, f_2, there exists a family $\mathbb{G}_0 \subseteq \mathbb{G}$ of cardinality $\mathfrak{c}$ such that $\mathcal{I}_g(f_i)$ is incomparable with $\mathcal{I}(f_j)$ for each $g \in \mathbb{G}_0$ and $i, j \in \{1,2\}$. Also $\mathcal{I}_{g_1}(f_i)$, $\mathcal{I}_{g_2}(f_j)$ are incomparable for $i, j \in \{1,2\}$ and any two distinct $g_1, g_2 \in \mathbb{G}_0$.*

Part-II : GENERALIZED CHARACTERIZED SUBGROUPS

In Chapter 3, we provide a complete characterization of topologically s-torsion as well as topologically α-torsion elements of $\mathbb{T}$ for a general arithmetic sequence.

Theorem 1.4.4. *Let, (a_n) be an arithmetic sequence and $x \in [0,1)$. Then x is a topological α-torsion element (i.e. $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$) if and only if either $d_\alpha(supp(x)) = 0$ or if $\overline{d}_\alpha(supp(x)) > 0$, then for all $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$ the following holds:*

(a) If A is q-bounded, then:

(a1) *If* $A \subseteq^{\alpha} supp(x)$*, then* $A+1 \subseteq^{\alpha} supp(x)$*,* $A \subseteq^{\alpha} supp_q(x)$ *and there exists* $A' \subseteq A$ *with* $d_{\alpha}(A \setminus A') = 0$ *such that* $\lim_{n\in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ *in* $\mathbb{R}$*.*

Moreover, if $A+1$ *is q-bounded, then* $A+1 \subseteq^{\alpha} supp_q(x)$*.*

(a2) *If* $d_{\alpha}(A \cap supp(x)) = 0$*, then there exists* $A' \subseteq A$ *with* $d_{\alpha}(A \setminus A') = 0$ *such that* $\lim_{n\in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ *in* $\mathbb{R}$*.*

Moreover, if $A+1$ *is q-bounded, then* $d_{\alpha}((A+1) \cap supp(x)) = 0$ *as well.*

(b) *If* A *is q-divergent, then* $\lim_{n\in B} \varphi(\frac{c_n}{q_n}) = 0$ *for some* $B \subseteq A$ *with* $d_{\alpha}(A \setminus B) = 0$*.*

Due to complex nature of the above theorem sometimes it is not helpful to use this characterization to determine whether an element $x \in \mathbb{T}$ is a topologically α-torsion or not. In other words, for some sequences (α-splitting sequence) one can obtain simpler characterizations of these elements.

Theorem 1.4.5. *Let* (a_n) *be an arithmetic sequence and* $x \in [0,1)$ *has canonical representation (3.1). If the sequence of ratios* (q_n) *has the* α*-splitting property, then* $\varphi(x)$ *is a topological* α*-torsion element i.e.* $\varphi(x) \in t^{\alpha}_{(a_n)}(\mathbb{T})$ *if and only if the following conditions hold:*

(i) $B^S(x)+1 \subseteq^{\alpha} supp(x)$*,* $B^S(x) \subseteq^{\alpha} supp_q(x)$*, and if* $\overline{d}_{\alpha}(B^S(x)) > 0$ *then* $\lim_{n\in B_1^S(x)} \frac{c_{n+1}+1}{q_{n+1}} = 1$ *in* $\mathbb{R}$*, where* $B_1 \subseteq B$ *with* $d_{\alpha}(B \setminus B_1) = 0$*.*

(ii) *If* $\overline{d}_{\alpha}(B^N(x)) > 0$ *then* $\lim_{n\in B_1^N(x)} \frac{c_{n+1}}{q_{n+1}} = 0$ *in* $\mathbb{R}$*, where* $B_1 \subseteq B$ *with* $d_{\alpha}(B \setminus B_1) = 0$*.*

(iii) *If* $\overline{d}_{\alpha}(D^S(x)) > 0$ *then* $\lim_{n\in D_1^S(x)} \varphi\left(\frac{c_n}{q_n}\right) = 0$*, where* $D_1 \subseteq D$ *with* $d_{\alpha}(D \setminus D_1) = 0$*.*

For notations and terminology we refer to Section 3.4.

In Chapter 4, using the notion of modular simple density function we define the f^g-characterized subgroups of $\mathbb{T}$. Now it is easy to observe that the f^g-characterized subgroups generalize all such notion of generalized characterized subgroups that exist in the literature. Consequently not only the main results of [38] and [19] follow as special cases of our results but at the same time, the questions about simple density and f-density are resolved. Our next results show that the f^g-characterized subgroups indeed form new nontrivial Borel subgroups of $\mathbb{T}$.

Theorem 1.4.6. *For any sequence of integers* (a_n)*,* $t^{f,g}_{(a_n)}(\mathbb{T})$ *is an* $F_{\sigma\delta}$ *(hence, Borel) subgroup of* $\mathbb{T}$ *containing* $t_{(a_n)}(\mathbb{T})$*.*

Theorem 1.4.7. *For any arithmetic sequence* (a_n)*, we have* $|t^{f,g}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$*.*

Theorem 1.4.8. $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$ *for any arithmetic sequence* (a_n)*.*

A complete characterization of elements of $t^{f,g}_{(a_n)}(\mathbb{T})$ is not described in this chapter (which will be given in the next chapter in a more general setting). Next we provide a sufficient condition in this direction.

Theorem 1.4.9. *Let (a_n) be an arithmetic sequence and $x \in \mathbb{T}$. If $d_g^f(supp(x)) = 0$, then $x \in t_{(a_n)}^{f,g}(\mathbb{T})$.*

After that we focus on the fact that whether this newly obtained characterized subgroups are really new compared to already investigated s-characterized or α-characterized subgroups of $\mathbb{T}$. Thanks to our next two theorems which show that for suitable modulus function f and weight function g it is indeed possible to construct new generalized characterized subgroups.

Theorem 1.4.10. *For any unbounded modulus function f, there exists $g \in \mathbb{G}$ such that $t_{(a_n)}^{f,g}(\mathbb{T}) \subsetneq t_{(a_n)}^{\alpha}(\mathbb{T})$ and $t_{(a_n)}^{f,g}(\mathbb{T}) \subsetneq t_{(a_n)}^{s}(\mathbb{T})$.*

Theorem 1.4.11. *There exists an unbounded modulus function f such that for any $g \in \mathbb{G}$, $t_{(a_n)}^{f,g}(\mathbb{T}) \neq t_{(a_n)}^{\alpha}(\mathbb{T})$ and $t_{(a_n)}^{f,g}(\mathbb{T}) \neq t_{(a_n)}^{s}(\mathbb{T})$.*

Finally in our next theorem we construct an uncountable tower of generalized characterized subgroups which subsequently gives the solution of Problem 2.15 [19].

Theorem 1.4.12. *For each arithmetic sequence (a_n) and for any unbounded modulus function f there exists a family $\{B_\alpha : \alpha \in (0,1)\}$ of Borel subgroups of $\mathbb{T}$ such that the following statements hold:*

(i) Each B_α is f^{g_α}-characterized by the same arithmetic sequence (a_n).

(ii) $|B_\alpha| = \mathfrak{c}$ for all $\alpha \in (0,1)$.

(iii) $B_\alpha \subsetneq B_\beta$ whenever $\alpha < \beta$ for all $\alpha, \beta \in (0,1)$.

(iv) For every $\alpha \in (0,1)$, the group B_α properly contains the characterized subgroup $t_{(a_n)}(\mathbb{T})$.

(v) For every $\alpha \in (0,1)$, the group B_α is properly contained in the f-characterized subgroup $t_{(a_n)}^{f}(\mathbb{T})$.

(vi) Further $\bigcap\limits_{\alpha\in(0,1)} B_\alpha \supsetneq t_{(a_n)}(\mathbb{T})$ and $\bigcup\limits_{\alpha\in(0,1)} B_\alpha \subsetneq t_{(a_n)}^{f}(\mathbb{T})$.

In Chapter 5, we provide a complete characterization of topologically $\mathcal{I}$-torsion elements of $\mathbb{T}$ for a general arithmetic sequence and for a fairly large class of ideals, namely, all translation invariant analytic P-ideals. In particular our next theorem answers the open problem [38, Problem 6.10.] in the most general form.

Theorem 1.4.13. *(see also [41, Theorem 2.3]) Let $x \in \mathbb{T}$ and $\mathcal{I} \in \Im$. Then x is a topologically $\mathcal{I}$-torsion element (i.e., $x \in t_{(a_n)}^{\mathcal{I}}(\mathbb{T})$) if and only if either $supp(x) \in \mathcal{I}$ or if $supp(x) \in \mathcal{I}^+$, then for all $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}^+$ the following holds:*

(a) If A is q-bounded, then:

(a1) If $A \subseteq^{\mathcal{I}} supp(x)$, then $A+1 \subseteq^{\mathcal{I}} supp(x)$, $A \subseteq^{\mathcal{I}} supp_q(x)$ and there exists $A' \subseteq_{\mathcal{I}} A$ such that $\lim\limits_{n\in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$.

Moreover, if $A+1$ is q-bounded, then $A+1 \subseteq^{\mathcal{I}} supp_q(x)$.

(a2) If $A \cap supp(x) \in \mathcal{I}$, then there exists $A' \subseteq_{\mathcal{I}} A$ such that $\lim\limits_{n \in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$. Moreover, if $A+1$ is q-bounded, then $(A+1) \cap supp(x) \in \mathcal{I}$ as well.

(b) If A is q-divergent, then $\lim\limits_{n \in B} \frac{c_n}{q_n} = 0$ in $\mathbb{T}$ for some $B \subseteq_{\mathcal{I}} A$.

After that our next two corollaries show that for any arithmetic sequence (a_n) and $0 < \alpha_1 < \alpha_2 < 1$ we must have $t^{\alpha_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\alpha_2}_{(a_n)}(\mathbb{T})$ which solves Problem 2.14. posed in [19] in the most general form.

Corollary 1.4.14. *For $\mathcal{I} \in \Im$ and a subset $B \subset \mathbb{N}$ there exists $x \in \mathbb{T}$ with $supp(x) \subseteq B$ such that $x \notin t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ if and only if $B \in \mathcal{I}^+$.*

Corollary 1.4.15. *For any two $\mathcal{I}_1, \mathcal{I}_2 \in \Im$, if $\mathcal{I}_1 \subsetneq \mathcal{I}_2$ then $t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$.*

Part-III : APPLICATIONS AND OPEN QUESTIONS

In Chapter 6, we show that there are statistically characterized subgroups which can't be characterized by any sequence of integers establishing the "novelty" of the notion which was missing when the notion of statistically characterized subgroups was introduced in [38].

Theorem 1.4.16. *For any arithmetic sequence (u_n), the subgroup $t^s_{(u_n)}(\mathbb{T})$ is not an* **A***-set.*

This naturally paves the way for a new class of sets generated by the class of statistically characterized subgroups as basis namely statistical Arbault sets. These sets are introduced in Section 6.3 and some basic properties are established. Finally the last section is devoted to the comparison of this new class with the existing classes of trigonometric thin sets. Some of the main comparison results are presented below (for notations see Chapter 6).

Theorem 1.4.17. $s\mathcal{A} \cap \mathcal{N} \nsubseteq \mathcal{A}$.

Theorem 1.4.18. $s\mathcal{A} \nsubseteq \mathcal{N} \cup \mathcal{A}$.

Chapter 7 is the last chapter of this thesis which contains final comments and open questions regarding generalized characterized subgroups and statistical Arbault sets.

Part I

f^g-DENSITY FUNCTIONS AND CORRESPONDING IDEALS

Chapter 2

MODULAR SIMPLE DENSITY IDEAL*

2.1 Introduction

In this chapter we primarily continue the investigation of the ideal $\mathcal{I}_g(f)$ from [18] and present several new observations all of which have interesting applications in Chapter 4. To get an overview on the general development of the ideal $\mathcal{I}_g(f)$ we refer to the Section 0.1 of Preface. Before proceeding further we recall some basic definitions related to ideals on $\mathbb{N}$.

In general ideals generated by some kind of density function are called density ideals. However here we will use the following definition in the sense of Farah ([50], Definition 1.13.1, p 42). For a positive measure μ defined on subsets of $\mathbb{N}$, the support of μ is the set $\{n \in \mathbb{N} : \mu(\{n\}) > 0\}$. We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ is a density ideal if

$$\mathcal{I} = Exh(\varphi) = \{A \subset \mathbb{N} \colon \lim_{n\to\infty} \varphi(A \setminus [1,n]) = 0\}$$

where $\varphi := \sup_{i\in\mathbb{N}} \mu_i$ and μ_i are positive measures with pairwise disjoint supports being finite subsets of $\mathbb{N}$. We then say that $\mathcal{I}$ is a density ideal generated by the sequence $(\mu_i)_{i\in\mathbb{N}}$ (note that the set $Exh(\varphi)$ may not be an ideal in general).

The modulus functions are defined as functions $f : [0,\infty) \to [0,\infty)$ which satisfy the following properties.

(i) $f(x) = 0 \Leftrightarrow x = 0$

(ii) $f(x+y) \leq f(x) + f(y)$ for all $x, y \in (0,\infty)$ [Triangle inequality]

(iii) f is non-decreasing

(iv) f is right continuous at 0.

Note that the conditions (i)-(iv) imply the continuity of the modulus functions which will be useful in certain proofs. Some examples of such modulus functions [2] are given by

*Content of this chapter has been published in "**Indagationes Mathematicae** and **Quaestiones Mathematicae**".

1. $f(x) = x,\ x \in [0, \infty)$.

2. $f(x) = \frac{x}{1+x},\ x \in [0, \infty)$.

3. For any $\alpha \in (0, 1)$, $f(x) = x^\alpha$ for $x \in [0, \infty)$.

4. $f(x) = \log(1 + x),\ x \in [0, \infty)$.

Now one can easily observe that all the notions of density functions, precisely, natural density d [24], natural density d_α of order α [13], their generalization with respect to weight function g, d_g [6] and natural density with respect to an unbounded modulus function f, f-density d^f [2] are special cases of "f density of weight g", i.e., d_g^f [18]. Therefore,

- for $f(x) = x$ and $g(n) = n$ the moduler simple density ideal $\mathcal{I}_g(f)$ coincides with the ideal $\mathcal{I}_d$;
- for $f(x) = x$ and $g(n) = n^\alpha$ the moduler simple density ideal $\mathcal{I}_g(f)$ coincides with the ideal $\mathcal{I}_{d_\alpha}$;
- for any unbounded modulus function f and $g(n) = n$ the moduler simple density ideal $\mathcal{I}_g(f)$ coincides with the ideal $\mathcal{I}(f)$;
- for $f(x) = x$ and any weight function $g\mathbb{G}$ the moduler simple density ideal $\mathcal{I}_g(f)$ coincides with the ideal $\mathcal{I}_g$.

Let us now recall two main observations from [18] which provide the motivation for further investigation of the ideal $\mathcal{I}_g(f)$.

Proposition 2.1.1. *[18] For a modulus function f and $g \in \mathbb{G}$, the ideal $\mathbb{Z}_g(f)$ is a P-ideal. In fact $\mathbb{Z}_g(f)$ is equal to $Exh(\varphi)$ where*

$$\varphi(A) = \sup_{n \in \omega} \frac{f(|A \cap [1, n])}{f(g(n))} \quad \textit{for } A \subset \omega,$$

and φ is a lower semicontinuous submeasure on ω.

Proposition 2.1.2. *[18] There exists an unbounded modulus function f and $g' \in \mathbb{G}$ such that $\mathcal{I}_{g'}(f) \neq \mathcal{I}_g$ for each $g \in \mathbb{G}$.*

So the class $\mathcal{I}_g(f)$ properly contains the class $\mathcal{I}_g$. As a consequence the properties of the class $\mathcal{I}_g$ does not hold necessarily for the class $\mathcal{I}_g(f)$. Another important observation is that one can actually generate uncountably many analytic P-ideals using different choice of weight functions and modulus functions.

2.2 Certain properties of the ideal $\mathcal{I}_g(f)$

We start with the observation that for any unbounded modulus function f and $g \in \mathbb{G}$, the generated ideal $\mathcal{I}_g(f)$ contains at least one infinite subset of $\mathbb{N}$. From the next result mentioned below one can easily verify that $\mathcal{I}_g(f) \supsetneq Fin$.

Proposition 2.2.1. *For any unbounded modulus function f and $g \in \mathbb{G}$, $\mathcal{I}_g(f) \supsetneq Fin$.*

Proof. Let f be an unbounded modulus function and let $g \in \mathbb{G}$. Further let us define a sequence (n_k) inductively such that $n_0 := 1$ and $n_{k+1} := \min\{n \in \mathbb{N} : f(g(n)) \geq 2f(g(n_k))\}$. Then from [18, Lemma 3.6], we have

$$\mathcal{I}_g(f) = \{A \subset \mathbb{N} : \lim_{k\to\infty} \frac{f(|A \cap [n_k, n_{k+1})|)}{f(g(n_k))} = 0\}.$$

Now it is obvious that $\mathcal{I}_g(f) \supseteq Fin$. Set $A = \{m_k \in \mathbb{N} : m_k \in [n_k, n_{k+1})\}$ and note that

$$d_g^f(A) = \lim_{k\to\infty} \frac{f(|A \cap [n_k, n_{k+1})|)}{f(g(n_k))} = \lim_{k\to\infty} \frac{f(1)}{f(g(n_k))} = 0.$$

Therefore there always exists an infinite $A \subset \mathbb{N}$ such that $A \in \mathcal{I}_g(f) \setminus Fin$. □

Proposition 2.2.2. *For any unbounded modulus function f and for any $g \in \mathbb{G}$, the ideal $\mathcal{I}_g(f)$ is tall or dense.*

Proof. Let f be an unbounded modulus function and $g \in \mathbb{G}$. Then, from [18, Theorem 3.7] the density ideal $\mathcal{I}_g(f)$ is generated by the sequence of measures (μ_k) given by

$$\mu_k(A) = \frac{f(|A \cap [n_k, n_{k+1})|)}{f(g(n_k))}.$$

Therefore

$$\lim_{k\to\infty} \sup_{i\in\mathbb{N}} \mu_k(\{i\}) = \lim_{k\to\infty} \sup_{i\in\mathbb{N}} \frac{f(|\{i\} \cap [n_k, n_{k+1})|)}{f(g(n_k))} = \lim_{k\to\infty} \frac{f(1)}{f(g(n_k))} = 0.$$

Thus, from [6, Proposition 3.4], it follows that the density ideal $\mathcal{I}_g(f)$ is tall. □

Let F denote the set of all unbounded modulus functions. We define

$$\mathbb{F} = \{f \in F : f \text{ is strictly increasing}\}.$$

Our next proposition is a modified version of Proposition 2.3 [18] whose proof is omitted.

Proposition 2.2.3. *If $f, f' \in F$ are such that there exist $c_1, c_2 > 0$, $k \in \mathbb{N}$ for which $c_1 \leq f(x)/f'(x) \leq c_2$ for all $x \geq k$ then $\mathcal{I}_g(f) = \mathcal{I}_g(f')$ for every $g \in \mathbb{G}$.*

Lemma 2.2.4. *For any $f \in F$ there exists an $f' \in \mathbb{F}$ such that for any $g \in \mathbb{G}$, $\mathcal{I}_g(f) = \mathcal{I}_g(f')$ and $f'(n) \leq f(n)$ for all $n \in \mathbb{N}$.*

Proof. Let $f \in F$. We construct a sequence (n_k) recursively in the following way:

$$n_1 = 1 \text{ and } n_{k+1} = \min\{n \in \mathbb{N} : f(n) > f(n_k)\} \text{ for all } k \in \mathbb{N}.$$

Note that if $r_1, r_2 \in [n_k, n_{k+1} - 1]$ then $f(r_1) = f(r_2)$ and $f(n_{k+1}) > f(n_k)$ for all $k \in \mathbb{N}$. Now we define $f' : [0, \infty) \to [0, \infty)$ by

$$f'(x) = f(n_k - 1) + (x - n_k + 1) \cdot \frac{f(n_{k+1} - 1) - f(n_k - 1)}{n_{k+1} - n_k}$$

when $x \in [n_k - 1, n_{k+1} - 1)$. From the construction it is easy to check that f' is a strictly increasing modulus function (i.e. $f' \in \mathbb{F}$) such that $f'(n) \le f(n)$ for all $n \in \mathbb{N}$ and $f'(n_k - 1) = f(n_k - 1)$ for all $k \in \mathbb{N}$.

Now for all $x \ge n_2$ we have $x \in [n_{k+1} - 1, n_{k+2} - 1)$ for some $k \in \mathbb{N}$. Therefore, we can see that

$$\frac{f(x)}{f'(x)} \le \frac{f(n_{k+2}-1)}{f'(n_{k+1}-1)} = \frac{f(n_{k+1})}{f(n_{k+1}-1)} \le \frac{f(n_{k+1}-1)+f(1)}{f(n_{k+1}-1)} \le 1 + \frac{f(1)}{f(n_k)} \le 2,$$

and at the same time we have

$$\begin{aligned}\frac{f(x)}{f'(x)} \ge \frac{f(n_{k+1}-1)}{f'(n_{k+2}-1)} &= \frac{f(n_{k+1}-1)}{f(n_{k+2}-1)} = \frac{f(n_{k+1}-1)}{f(n_{k+1})} \\ &\ge \frac{f(n_{k+1})-f(1)}{f(n_{k+1})} \\ &\ge 1 - \frac{f(1)}{f(n_{k+1})} \ge 1 - \frac{f(1)}{f(n_2)}.\end{aligned}$$

In view of Proposition 2.2.3 it then follows that for any $g \in \mathbb{G}$, $\mathcal{I}_g(f) = \mathcal{I}_g(f')$ as desired. □

The above lemma is the most powerful result in this section. Due to this lemma, rather considering f to be an unbounded modulus function, without loss of any generality one can additionally consider f to be a strictly increasing homeomorphism.

Remark 2.2.5. *For any $f \in \mathbb{F}$ the following conditions hold.*
(1) $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$ *i.e. f is injective.*
(2) *f is continuous over* $(0, \infty)$.
(3) $f(n) \to \infty$ *as* $n \to \infty$.
(4) *f is surjective.*
(5) *There exists a strictly increasing continuous function $h : [0,\infty) \to [0,\infty)$ such that* $f(h(x)) = x = h(f(x))$.
(6) $h(x+y) \ge h(x) + h(y)$ *for all* $x, y \in (0, \infty)$.

2.3 Some comparative results concerning the ideal $\mathcal{I}_g(f)$

In [18] comparisons were made between the ideals $\mathcal{I}_g$ and $\mathcal{I}_g(f)$ showing that for suitable choice of the modulus function f, $\mathcal{I}_g \ne \mathcal{I}_g(f)$ (see Proposition 2.5 and Remark 2.6 [18]). However no comparative study was carried out between the ideals $\mathcal{I}(f)$ and $\mathcal{I}_g(f)$. So in this section our particular interest is the question whether there are instances where these two ideals would be different.

Lemma 2.3.1. *[18, Lemma 3.3] Let f be an unbounded modulus function and let $g \in \mathbb{G}$ be such that $\frac{f(n)}{f(g(n))} \to \infty$. Then there exists a set $A \subset \mathbb{N}$ such that the sequence $\left(\frac{f(|A\cap[1,n]|)}{f(g(n))}\right)$ is bounded but not convergent to 0.*

Lemma 2.3.2. *[18, Lemma 3.4] If f_1 and f_2 are two modulus functions and $g_1, g_2 \in \mathbb{G}$ are such that there exist $c_1, c_2 > 0$, $k \in \mathbb{N}$ for which $\frac{f_1(x)}{f_2(x)} \geq c_1$ for all $x \neq 0$ and $\frac{f_1(g_1(n))}{f_2(g_2(n))} \leq c_2$ for all $n \geq k$, then $\mathcal{I}_{g_1}(f_1) \subseteq \mathcal{I}_{g_2}(f_2)$.*

Proposition 2.3.3. *Let f be an unbounded modulus function. If $g_1, g_2 \in \mathbb{G}$ are such that $\frac{f(n)}{f(g_2(n))} \geq a > 0$ and $\frac{f(g_2(n))}{f(g_1(n))} \to \infty$ then $\mathcal{I}_{g_1}(f) \subsetneq \mathcal{I}_{g_2}(f)$.*

Proof. Taking $f_1 = f_2 = f$ in Lemma 2.3.2, we get $\mathcal{I}_{g_1}(f) \subseteq \mathcal{I}_{g_2}(f)$. We choose a function $g_3 : \mathbb{N} \to [0, \infty]$ in such a way that $f(g_3(n)) := \sqrt{f(g_1(n)) \cdot f(g_2(n))}$ holds for all $n \in \mathbb{N}$. The existence of such a function g_3 is assured as also this function is well-defined as the function f is non-decreasing. Now

$$\lim_{n\to\infty} \frac{f(g_1(n))}{f(g_3(n))} = \lim_{n\to\infty} \sqrt{\frac{f(g_1(n))}{f(g_2(n))}} = \lim_{n\to\infty} \frac{f(g_3(n))}{f(g_2(n))} = 0. \tag{2.1}$$

Since $\frac{f(n)}{f(g_2(n))} \geq a$ for some $a \in (0, \infty)$ and $\frac{f(g_2(n))}{f(g_1(n))} \to \infty$, we have

$$\begin{aligned} \frac{f(n)}{f(g_1(n))} &= \frac{f(n)}{f(g_2(n))} \cdot \frac{f(g_2(n))}{f(g_1(n))} \to \infty \\ \Rightarrow \frac{f(n)}{f(g_3(n))} &= \sqrt{\frac{f(n)^2}{f(g_1(n))f(g_2(n))}} \to \infty. \end{aligned}$$

Therefore, from Lemma 2.3.1, there exists $A \subset \mathbb{N}$ such that $(\frac{f(|A\cap[1,n]|)}{f(g_3(n))})_{n\in\mathbb{N}}$ is bounded but not convergent to zero. We claim that $A \in \mathcal{I}_{g_2}(f) \setminus \mathcal{I}_{g_1}(f)$. Indeed from (2.1) we have

$$\frac{f(|A \cap [1, n]|)}{f(g_2(n))} = \frac{f(|A \cap [1, n]|)}{f(g_3(n))} \cdot \frac{f(g_3(n))}{f(g_2(n))} \to 0$$

whereas

$$\frac{f(|A \cap [1, n]|)}{f(g_1(n))} = \frac{f(|A \cap [1, n]|)}{f(g_3(n))} \cdot \frac{f(g_3(n))}{f(g_1(n))} \not\to 0.$$

This shows that $A \in \mathcal{I}_{g_2}(f) \setminus \mathcal{I}_{g_1}(f)$. □

Corollary 2.3.4. *For any unbounded modulus function f and $g \in \mathbb{G}$ if $f(n)/f(g(n)) \to \infty$ then $\mathcal{I}_g(f) \subsetneq \mathcal{I}(f)$.*

Proof. Taking $g_2(n) = n$ for all $n \in \mathbb{N}$ and $g_1 = g$ in Proposition 2.3.3, we obtain $\mathcal{I}_g(f) \subsetneq \mathcal{I}(f)$. □

Proposition 2.3.5. *For any unbounded modulus function f, there exists $g \in \mathbb{G}$ such that $\mathcal{I}(f) \subsetneq \mathcal{I}_g(f)$.*

Proof. Let f be an unbounded modulus function. Since f is non-decreasing, there exists a strictly increasing sequence of natural numbers (a_n) such that $f(a_{n+1}) > nf(a_n)$. Set $b_n = a_{4n-2}$ and $d_n = a_{4n}$ for all $n \in \mathbb{N}$. Now we define a function g in the following way

$$g(n) = \begin{cases} d_k & \text{for } b_k < n \leq d_k \\ n & \text{for } d_k < n \leq b_{k+1}. \end{cases}$$

From the construction, it is evident that g is non-decreasing. Since $g(n) \to \infty$ and $\frac{d_k}{g(d_k)} = 1$ for all $k \in \mathbb{N}$ we have $\frac{n}{g(n)} \nrightarrow 0$ and therefore $g \in \mathbb{G}$. Note that $g(n) \geq n$ for all $n \in \mathbb{N}$. As f is non-decreasing, $f(g(n)) \geq f(n)$ i.e. $\frac{f(n)}{f(g(n))} \leq 1$ for all $n \in \mathbb{N}$. Therefore, in view of Lemma 2.3.2, we have $\mathcal{I}(f) \subseteq \mathcal{I}_g(f)$.

Let us define $A = \bigcup_{k=1}^{\infty} (b_k, c_k]$ where $c_k = a_{4k-1} < d_k$. Consider any $m \in \mathbb{N}$ and set $n_m = b_m$. Now for any $n > n_m$ we have either $n \in (b_{m_0}, c_{m_0}]$ or $n \in (c_{m_0}, d_{m_0}]$ or $n \in (d_{m_0}, b_{m_0+1}]$ for some natural number $m_0 \geq m$.
For $n \in (b_{m_0}, c_{m_0}]$, we have

$$\frac{f(|A \cap [1,n]|)}{f(g(n))} \leq \frac{f(|A \cap [1,c_{m_0}]|)}{f(d_{m_0})} \leq \frac{f(c_{m_0})}{f(d_{m_0})} < \frac{1}{4m_0 - 1} \leq \frac{1}{m}.$$

For $n \in (c_{m_0}, d_{m_0}]$, we have

$$\begin{aligned}\frac{f(|A \cap [1,n]|)}{f(g(n))} \leq \frac{f(|A \cap [1,d_{m_0}]|)}{f(d_{m_0})} &= \frac{f(|A \cap [1,c_{m_0}]|)}{f(d_{m_0})} \\ &\leq \frac{f(c_{m_0})}{f(d_{m_0})} < \frac{1}{4m_0 - 1} \leq \frac{1}{m}.\end{aligned}$$

And, for $n \in (d_{m_0}, b_{m_0+1}]$ we also have

$$\begin{aligned}\frac{f(|A \cap [1,n]|)}{f(g(n))} \leq \frac{f(|A \cap [1,b_{m_0+1}]|)}{f(n)} &= \frac{f(|A \cap [1,c_{m_0}]|)}{f(n)} \\ &\leq \frac{f(c_{m_0})}{f(d_{m_0})} < \frac{1}{4m_0 - 1} \leq \frac{1}{m}.\end{aligned}$$

As a result we can conclude that $\frac{f(|A \cap [1,n]|)}{f(g(n))} < \frac{1}{m}$ for all $n > n_m$. Since $m \in \mathbb{N}$ was chosen arbitrarily we obtain $\frac{f(|A \cap [1,n]|)}{f(g(n))} \to 0$ i.e. $A \in \mathcal{I}_g(f)$. But $A \notin \mathcal{I}(f)$, because for all $k \in \mathbb{N}$, observe that

$$\frac{f(|A \cap [1,c_k]|)}{f(c_k)} \geq \frac{f(c_k - b_k)}{f(c_k)} \geq \frac{f(c_k) - f(b_k)}{f(c_k)} \to 1 \text{ (since, } \lim_{k \to \infty} \tfrac{f(a_{4k-2})}{f(a_{4k-1})} = 0).$$

Thus we conclude that $\mathcal{I}(f) \subsetneq \mathcal{I}_g(f)$. □

Remark 2.3.6. *For each $A = \{n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq \mathbb{N}$ we can define*

$$g_A(n) = \begin{cases} d_{n_k} & \text{for } b_{n_k} < n \leq d_{n_k} \\ n & \text{otherwise} \end{cases}.$$

It is easy to verify that $g_A \in \mathbb{G}$ and $\mathcal{I}(f) \subsetneq \mathcal{I}_{g_A}(f)$. Therefore, there exists $\mathfrak{c}$ many choice of $g \in \mathbb{G}$ for which $\mathcal{I}(f) \subsetneq \mathcal{I}_g(f)$.

Our next theorem provides a chain of ideals between Fin and $\mathcal{I}(f)$ for any unbounded modulus function f.

Theorem 2.3.7. *For any $f \in \mathbb{F}$ there exists a $\mathbb{G}_0 = \{g_\alpha \in \mathbb{G} : \alpha \in (0,1)\} \subseteq \mathbb{G}$ such that the followings hold:*

(a) $\mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}_{g_\beta}(f)$ *whenever* $\alpha < \beta, \alpha, \beta \in (0,1)$.

(b) $\bigcap_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f) \supsetneq Fin$.

(c) $\bigcup_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}(f)$.

Proof. Let $f \in \mathbb{F}$. In view of Remark 2.2.5 (5), there exists a strictly increasing continuous function $h : [0,\infty) \to [0,\infty)$ such that $f(h(x)) = x = h(f(x))$. Now for any $\alpha \in (0,1)$ we define $g_\alpha : \mathbb{N} \to [0,\infty)$ by

$$g_\alpha(n) = h(r^\alpha) \text{ when } \lfloor h(r) \rfloor + 1 \leq n < \lfloor h(r+1) \rfloor + 1 \text{ for some } r \in \mathbb{N}.$$

Since $|h(r+1) - h(r)| \geq 1$, the intervals $[\lfloor h(r) \rfloor + 1, \lfloor h(r+1) \rfloor + 1)$ are well defined. Therefore g_α is well defined for each $\alpha \in (0,1)$. Clearly g_α is nondecreasing and $\lim_{n\to\infty} g_\alpha(n) = \infty$. Now, observe that $g_\alpha(n) \leq n$ for all $n \in \mathbb{N}$ which implies $\lim_{n\to\infty} \frac{n}{g_\alpha(n)} \nrightarrow 0$. So we conclude that $g_\alpha \in \mathbb{G}$ for each $\alpha \in (0,1)$ and denote the set of all such functions g_α by $\mathbb{G}_0$.

(a) Note that for any $n \in \mathbb{N}$, $\exists$ a $r \in \mathbb{N}$ such that $n \in [\lfloor h(r) \rfloor + 1, \lfloor h(r+1) \rfloor + 1)$. Subsequently for any $\beta \in (0,1)$

$$\lim_{n\to\infty} \frac{f(n)}{f(g_\beta(n))} \geq \lim_{r\to\infty} \frac{f(h(r))}{f(h(r^\beta))} = \lim_{r\to\infty} \frac{r}{r^\beta} = \infty \tag{2.2}$$

and whenever $\alpha < \beta$, we have

$$\lim_{n\to\infty} \frac{f(g_\beta(n))}{f(g_\alpha(n))} = \lim_{r\to\infty} \frac{f(h(r^\beta))}{f(h(r^\alpha))} = \lim_{r\to\infty} \frac{r^\beta}{r^\alpha} = \infty.$$

Thus in view of Proposition 2.3.3 it follows that $\mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}_{g_\beta}(f)$ whenever $\alpha < \beta$.

(b) It is obvious that $Fin \subseteq \bigcap_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f)$. Consider the set $A = \{\lfloor h(2^n) \rfloor + 1 : n \in \mathbb{N}\}$. Now for any $\alpha \in (0,1)$ we have

$$d^f_{g_\alpha}(A) = \lim_{n\to\infty} \frac{f(|A \cap [1,n])}{f(g_\alpha(n))} = \lim_{n\to\infty} \frac{f(n)}{f(h(2^{\alpha n}))} \leq \lim_{n\to\infty} \frac{n}{2^{\alpha n}} \to 0.$$

As this is true for all $\alpha \in (0,1)$, $A \in \bigcap_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f)$. As A is infinite, we can conclude that $Fin \subsetneq \bigcap_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f)$.

(c) From equation (2.2) and Corollary 2.3.4, we have $\mathcal{I}_{g_\alpha}(f) \subseteq \mathcal{I}(f)$ for all $\alpha \in (0,1)$ i.e. $\bigcup_{\alpha \in (0,1)} \mathcal{I}_{g_\alpha}(f) \subseteq \mathcal{I}(f)$.
Now let us define $B = \{\lfloor h(\lfloor f(n) \log f(n) \rfloor) \rfloor + 1 : n \in \mathbb{N}\}$. Observe that for any

$\alpha \in (0,1)$ we have

$$
\begin{aligned}
d^f_{g_\alpha}(B) = \lim_{n\to\infty} \frac{f(|B\cap[1,n])}{f(g_\alpha(n))} &= \lim_{n\to\infty} \frac{f(n)}{f(g_\alpha(\lfloor h(\lfloor f(n)\log f(n)\rfloor)\rfloor+1))} \\
&= \lim_{n\to\infty} \frac{f(n)}{f(h((\lfloor f(n)\log f(n)\rfloor)^\alpha))} \\
&\geq \lim_{n\to\infty} \frac{f(n)}{f(h((f(n)\log f(n))^\alpha))} \\
&= \lim_{n\to\infty} \frac{f(n)}{(f(n)\log f(n))^\alpha} \ (\tfrac{\infty}{\infty} \text{ form}) \\
&= (\frac{1-\alpha}{\alpha}) \cdot \lim_{n\to\infty} (f(n)\log f(n))^{(1-\alpha)} \\
&= \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
d^f(B) = \lim_{n\to\infty} \frac{f(|B\cap[1,n])}{f(n)} &= \lim_{n\to\infty} \frac{f(n)}{f(\lfloor h(\lfloor f(n)\log f(n)\rfloor)\rfloor+1)} \\
&\leq \lim_{n\to\infty} \frac{f(n)}{f(h(\lfloor f(n)\log f(n)\rfloor))} \\
&= \lim_{n\to\infty} \frac{f(n)}{\lfloor f(n)\log f(n)\rfloor} \\
&\leq \lim_{n\to\infty} \frac{f(n)}{f(n)\log f(n)-1} \\
&= \lim_{n\to\infty} \frac{1}{\log f(n) - \frac{1}{f(n)}} \\
&= 0.
\end{aligned}
$$

Therefore $B \notin \bigcup_{\alpha\in(0,1)} \mathcal{I}_{g_\alpha}(f)$ but $B \in \mathcal{I}(f)$. It now immediately follows that $\bigcup_{\alpha\in(0,1)} \mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}(f)$.

□

We end this section with some more comparative results which are interesting in their own right and also come to our use in Chapter 4. The following lemma plays the most vital role hereafter.

Lemma 2.3.8. *For any two unbounded modulus functions f_1, f_2, there exists a strictly increasing sequence (a_n) of natural numbers such that $\frac{f_1(a_{n+1})}{f_1(a_n)} > n$, $\frac{f_2(a_{n+1})}{f_2(a_n)} > n$ and $a_{n+1} > 2a_n$.*

Proof. Set a_1=1. Choose $a_2 \in \mathbb{N}$ in such a way that $a_2 > 2a_1$ while $f_1(a_2) > f_1(1), f_1(a_2) > f_2(1)$ and $f_2(a_2) > f_1(1), f_2(a_2) > f_2(1)$ are satisfied. Now inductively we define

$$a_{n+1} = \{r \in \mathbb{N} :\ \min\{f_1(r), f_2(r)\} > n \max\{f_1(a_n), f_2(a_n)\} \text{ and } r > 2a_n\}$$

for all $n \in \mathbb{N}$. Since $f_1(n) \to \infty$ and $f_2(n) \to \infty$, so a_{n+1} is well defined for all $n \in \mathbb{N}$. From the construction, it is clear that (a_n) is strictly increasing and $\frac{f_1(a_{n+1})}{f_1(a_n)} > n$, $\frac{f_2(a_{n+1})}{f_2(a_n)} > n$ and $a_{n+1} > 2a_n$ for all $n \in \mathbb{N}$. □

Our next result is the most important observation regarding the ideals $\mathcal{I}_g(f)$ in this section (which was left out in [18]). Here we show that there exists an antichain of cardinality $\mathfrak{c}$ of such ideals in line of Theorem 2.7 [6]. In order to prove the result, we need the following: Two sets $P, Q \subseteq \mathbb{N}$ are said to be almost disjoint if $P \cap Q$ is finite. However [26, Theorem 5.35] assures the existence of a family $\mathcal{J}$ of infinite pairwise almost disjoint subsets of $\mathbb{N}$ with $|\mathcal{J}| = \mathfrak{c}$.

Theorem 2.3.9. *For any two unbounded modulus functions f_1, f_2, there exists a family $\mathbb{G}_0 \subseteq \mathbb{G}$ of cardinality $\mathfrak{c}$ such that $\mathcal{I}_g(f_i)$ is incomparable with $\mathcal{I}(f_j)$ for each $g \in \mathbb{G}_0$ and $i, j \in \{1, 2\}$. Also $\mathcal{I}_{g_1}(f_i)$, $\mathcal{I}_{g_2}(f_j)$ are incomparable for $i, j \in \{1, 2\}$ and any two distinct $g_1, g_2 \in \mathbb{G}_0$.*

Proof. Let f_1, f_2 be two unbounded modulus functions. In view of Lemma 2.3.8, we can find a strictly increasing sequence of natural numbers (a_n) such that $\frac{f_i(a_{n+1})}{f_i(a_n)} > n$ for $i \in \{1, 2\}$ and $a_{n+1} > 2a_n$. Set $b_n = a_{4n-2}$ and $d_n = a_{4n}$ for all $n \in \mathbb{N}$. Now for any $P = \{p_1 < p_2 < \ldots < p_k < \ldots\} \in \mathcal{J}$, we define

$$g_P(n) = \begin{cases} d_{p_k} & \text{for } b_{p_k} < n \leq b_{p_k+1} \\ n & \text{otherwise.} \end{cases}$$

Note that $g_P \in \mathbb{G}$. Consider the set

$$A_P = \bigcup_{k=1}^{\infty} (b_{p_k}, c'_{p_k}] \text{ where, } c'_{p_k} = a_{4p_k-1} < d_{p_k}.$$

We take any $m \in \mathbb{N}$ and set $n_m = b_{p_m}$. Now for any $n > n_m$ we have either $n \in (b_{p_{m_0}}, b_{p_{m_0}+1}]$ or $n \in (b_{p_{m_0}+1}, b_{p_{(m_0+1)}}]$ for some natural number $m_0 \geq m$.
For $n \in (b_{p_{m_0}}, b_{p_{m_0}+1}]$ and $i \in \{1, 2\}$, we have

$$\begin{aligned} \frac{f_i(|A_P \cap [1, n]|)}{f_i(g_P(n))} \leq \frac{f_i(|A_P \cap [1, b_{p_{m_0}+1}]|)}{f_i(d_{p_{m_0}})} &= \frac{f_i(|A_P \cap [1, c'_{p_{m_0}}]|)}{f_i(d_{p_{m_0}})} \\ &\leq \frac{f_i(c'_{p_{m_0}})}{f_i(d_{p_{m_0}})} < \frac{1}{4m_0 - 1} \leq \frac{1}{m}. \end{aligned}$$

On the other hand, for $n \in (b_{p_{m_0}+1}, b_{p_{(m_0+1)}}]$ and $i \in \{1, 2\}$ we also have

$$\begin{aligned} \frac{f_i(|A_P \cap [1, n]|)}{f_i(g_P(n))} \leq \frac{f_i(|A_P \cap [1, b_{p_{(m_0+1)}}]|)}{f_i(n)} &= \frac{f_i(|A_P \cap [1, c'_{p_{m_0}}]|)}{f_i(n)} \\ &\leq \frac{f_i(c'_{p_{m_0}})}{f_i(d_{p_{m_0}})} < \frac{1}{4m_0 - 1} \leq \frac{1}{m}. \end{aligned}$$

As a result we can conclude that $\frac{f_i(|A_P \cap [1,n]|)}{f_i(g_P(n))} < \frac{1}{m}$ for all $n > n_m$. Since $m \in \mathbb{N}$ was chosen arbitrarily we obtain $\frac{f_i(|A_P\cap[1,n]|)}{f_i(g_P(n))} \to 0$, i.e., $A_P \in \mathcal{I}_{g_P}(f_i)$ for $i \in \{1,2\}$. But $A_P \notin \mathcal{I}(f_j)$ for $j \in \{1,2\}$, because for all $k \in \mathbb{N}$, observe that

$$\begin{aligned}\frac{f_j(|A_P \cap [1, c'_{p_k}]|)}{f_j(c'_{p_k})} &\geq \frac{f_j(c'_{p_k} - b_{p_k})}{f_j(c'_{p_k})} \\ &\geq \frac{f_j(c'_{p_k}) - f_j(b_{p_k})}{f_j(c'_{p_k})} \to 1 \text{ (since, } \lim_{k\to\infty} \tfrac{f_j(a_{4p_k-2})}{f_j(a_{4p_k-1})} = 0\text{).}\end{aligned}$$

Thus we conclude that $A_P \in \mathcal{I}_{g_P}(f_i) \setminus \mathcal{I}(f_j)$ where $i, j \in \{1,2\}$.

Now consider the set

$$B_P = \bigcup_{k=1}^{\infty} (c_{p_k+1}, b_{p_k+1}] \text{ where, } c_{p_k+1} = b_{p_k+1} - d_{p_k}.$$

Observe that $c_{p_k+1} = b_{p_k+1} - d_{p_k} = a_{4p_k+2} - a_{4p_k} > 3a_{4p_k} > d_{p_k}$. We take any $m \in \mathbb{N}$ and set $n_m = b_{p_m}$. Now for any $n > n_m$ we have either $n \in (b_{p_{m_0}}, c_{p_{m_0}+1}]$ or $n \in (c_{p_{m_0}+1}, b_{p_{(m_0+1)}}]$ for some natural number $m_0 \geq m$.
For $n \in (c_{p_{m_0}+1}, b_{p_{(m_0+1)}}]$ and $j \in \{1,2\}$, we have

$$\begin{aligned}\frac{f_j(|B_P \cap [1,n]|)}{f_j(n)} &\leq \frac{f_j(|B_P \cap [1, b_{p_{(m_0+1)}}]|)}{f_j(c_{p_{m_0}+1})} \\ &= \frac{f_j(|B_P \cap [1, b_{p_{m_0}+1}]|)}{f_j(c_{p_{m_0}+1})} \\ &\leq \frac{f_j(p_{m_0} d_{p_{m_0}})}{f_j(c_{p_{m_0}+1})} \leq \frac{p_{m_0} f_j(d_{p_{m_0}})}{f_j(a_{4p_{m_0}+2} - a_{4p_{m_0}})} \\ &< \frac{f_j(a_{4p_{m_0}+1})}{f_j(a_{4p_{m_0}+2}) - f_j(a_{4p_{m_0}})} \leq \frac{1}{4p_{m_0}} \leq \frac{1}{4m_0} \leq \frac{1}{m}\end{aligned}$$

On the other hand, for $n \in (b_{p_{m_0}}, c_{p_{m_0}+1}]$ and $j \in \{1,2\}$ we also have

$$\begin{aligned}\frac{f_j(|B_P \cap [1,n]|)}{f_j(n)} &\leq \frac{f_j(|B_P \cap [1, c_{p_{m_0}+1}]|)}{f_j(b_{p_{m_0}})} \\ &= \frac{f_j(|B_P \cap [1, b_{p_{(m_0-1)}+1}]|)}{f_j(b_{p_{m_0}})} \\ &\leq \frac{f_j(p_{(m_0-1)} d_{p_{(m_0-1)}})}{f_j(b_{p_{m_0}})} \leq \frac{p_{(m_0-1)} f_j(d_{p_{(m_0-1)}})}{f_j(a_{4p_{m_0}-2})} \\ &< \frac{f_j(a_{4p_{(m_0-1)}+1})}{f_j(a_{4p_{m_0}-2})} \leq \frac{f_j(a_{4p_{m_0}-3})}{f_j(a_{4p_{m_0}-2})} \leq \frac{1}{4m_0 - 3} \leq \frac{1}{m}\end{aligned}$$

As a result we can conclude that $\frac{f_j(|B_P\cap[1,n]|)}{f_j(g_P(n))} < \frac{1}{m}$ for all $n > n_m$. Since $m \in \mathbb{N}$ was chosen arbitrarily we obtain $\frac{f_j(|B_P\cap[1,n]|)}{f_j(g_P(n))} \to 0$ i.e. $B_P \in \mathcal{I}(f_j)$ where $j \in \{1,2\}$. But

$B_P \notin \mathcal{I}_{g_P}(f_i)$ for $i \in \{1, 2\}$, since for all $k \in \mathbb{N}$ we must have

$$\frac{f_i(|B_P \cap [1, b_{p_k+1}]|)}{f_i(g_P(b_{p_k+1}))} \geq \frac{f_i(b_{p_k+1} - c_{p_k+1})}{f_i(d_{p_k})} = \frac{f_i(d_{p_k})}{f_i(d_{p_k})} = 1.$$

Thus $B_P \in \mathcal{I}(f_j) \setminus \mathcal{I}_{g_P}(f_i)$ where $i, j \in \{1, 2\}$.

Let us next define $\mathbb{G}_0 = \{g_P \in \mathbb{G} : \ P \in \mathcal{J}\}$. It has already been observed that $\mathcal{I}_g(f_i)$ is incomparable with $\mathcal{I}(f_j)$ for any $g \in \mathbb{G}_0$ and $i, j \in \{1, 2\}$. Now, consider any two distinct sets $P = \{p_1 < p_2 < \ldots < p_k < \ldots\}$, $Q = \{q_1 < q_2 < \ldots < q_k < \ldots\} \in \mathcal{J}$. We intend to show that $\mathcal{I}_{g_P}(f_i)$, $\mathcal{I}_{g_Q}(f_j)$ are incomparable where $i, j \in \{1, 2\}$. We already have $A_P \in \mathcal{I}_{g_P}(f_i)$ and $A_Q \in \mathcal{I}_{g_Q}(f_j)$ where $i, j \in \{1, 2\}$ and A_Q, g_Q are similarly constructed as A_P, g_P respectively.

In view of the fact that P, Q are infinite and almost disjoint there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ we have

$$\begin{aligned} q_{j(k)} &< p_k < q_{j(k)+1} \\ \Rightarrow \quad q_{j(k)} + 1 &\leq p_k < q_{j(k)+1} \\ \Rightarrow \quad b_{q_{j(k)}+1} &\leq b_{p_k} < b_{q_{j(k)+1}} \\ \Rightarrow \quad b_{q_{j(k)}+1} &\leq b_{p_k} < c_{p_k} < b_{q_{j(k)+1}}. \end{aligned}$$

From the construction of g_Q, it follows that $g_Q(c_{p_k}) = c_{p_k}$ and consequently for all $k \in \mathbb{N}$ and $j \in \{1, 2\}$ we have,

$$\frac{f_j(|A_P \cap [1, c_{p_k}]|)}{f_j(g_Q(c_{p_k}))} \geq \frac{f_j(c_{p_k} - b_{p_k})}{f_j(c_{p_k})} \geq \frac{f_j(c_{p_k}) - f_j(b_{p_k})}{f_j(c_{p_k})} \to 1.$$

This shows that $A_P \notin \mathcal{I}_{g_Q}(f_j)$ for $j \in \{1, 2\}$. Similarly we can also show that $A_Q \notin \mathcal{I}_{g_P}(f_i)$ for $i \in \{1, 2\}$. Thus, we can conclude that $\mathcal{I}_{g_1}(f_i)$ and $\mathcal{I}_{g_2}(f_j)$ are incomparable for $i, j \in \{1, 2\}$ and any two distinct $g_1, g_2 \in \mathbb{G}_0$. □

Corollary 2.3.10. *For any unbounded modulus function f, there exists $g \in \mathbb{G}$ such that $\mathcal{I}_g(f)$ is not comparable with $\mathcal{I}(f)$.*

Proposition 2.3.11. *For any unbounded modulus function f, there exist $g_1, g_2 \in \mathbb{G}$ such that $\mathcal{I}_{g_1}(f) \subsetneq \mathcal{I}(f) \subsetneq \mathcal{I}_{g_2}(f)$.*

Proof. Let f be an unbounded modulus function. Therefore, from Proposition 2.3.5, there exists a $g_2 \in \mathbb{G}$ such that $\mathcal{I}(f) \subsetneq \mathcal{I}_{g_2}(f)$.

As $\mathcal{I}(f) \neq Fin$, choose a set $A = \{n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq \mathbb{N}$ such that $A \in \mathcal{I}(f)$ (i.e. $d^f(A) = 0$). Consequently we have $\lim_{k\to\infty} \frac{f(k)}{f(n_k)} = 0$. Next define,

$$g_1(n) = \begin{cases} 1 & \text{for } 1 \leq n < n_1 \\ k & \text{for } \ n_k \leq n < n_{k+1}. \end{cases}$$

It is easy to observe that g_1 is non-decreasing while $\lim_{n\to\infty} \frac{n}{g_1(n)} \nrightarrow 0$ follows from the fact that $\frac{n_k}{g_1(n_k)} = \frac{n_k}{k} > 1$ for all $k \in \mathbb{N}$ i.e. $g_1 \in \mathbb{G}$. Note that

$$\lim_{n\to\infty} \frac{f(n)}{f(g(n))} \geq \lim_{k\to\infty} \frac{f(n_k)}{f(k)} = \infty.$$

Hence from Corollary 2.6, we have $\mathcal{I}_{g_1}(f) \subsetneq \mathcal{I}(f)$ and we are done. $\square$

2.4 Conclusion

In this chapter we continue our investigation of the ideal $\mathcal{I}_g(f)$ in the line of direction of the article [18]. In Section 2.2, certain properties of these ideals are observed. Among them the most crucial result is Lemma 2.2.4 which is fundamental to prove most of the main results in the next section. In [18] the ideal $\mathcal{I}_g(f)$ was compared with the ideal $\mathcal{I}_g$ but no comparison was done between the ideals $\mathcal{I}_g(f)$ and $\mathcal{I}(f)$. In Section 2.3, we provide many comparison results between all these ideals which will play interesting role in Chapter 4. Lastly we construct a chain (Theorem 2.3.7) as well as an antichain (Theorem 2.3.9) of these ideals.

Part II

GENERALIZED CHARACTERIZED SUBGROUPS

Chapter 3

α-CHARACTERIZED SUBGROUPS OF THE CIRCLE*

3.1 Introduction

In [19, 38] the s-characterized and α-characterized subgroups were introduced and studied in great details. In fact it was seen that they indeed generate topologically nice Borel subgroups of the Circle for any sequence of integers. Also, for a given arithmetic sequence (a_n), the α-characterized subgroups are new nontrivial subgroups different from $t_{(a_n)}(\mathbb{T})$ as well as $t^s_{(a_n)}(\mathbb{T})$. To get an overview on these subgroups we refer to the Section 1.3 of Chapter 1.

Definition 3.1.1. *Let (a_n) be a sequence of integers. An element $x \in \mathbb{T}$ is called topologically α-torsion if $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$.*

The following fact is essential to represent an arbitrary element of $\mathbb{T}$ only in terms of its support.

Fact 3.1.2. *[41] For any arithmetic sequence (a_n) and $x \in [0, 1)$, we can build a unique sequence of integers (c_n), where $0 \leq c_n < q_n$, such that*

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \tag{3.1}$$

and $c_n < q_n - 1$ for infinitely many n.

Proof. For better clarity we recall the construction of the sequence (c_n). Consider $c_1 = \lfloor a_1 x \rfloor$, where $\lfloor\ \rfloor$ denotes the integer part. Therefore, $x - \frac{c_1}{a_1} < \frac{1}{a_1}$.

Suppose, $c_1, c_2, \ldots, c_k$ are defined for some $k \geq 1$ with $x_k = \sum\limits_{n=1}^{k} \frac{c_n}{a_n}$ and $x - x_k < \frac{1}{a_k}$. Then the $(k+1)$-th element is defined as $c_{k+1} = \lfloor a_{k+1}(x - x_k) \rfloor$. □

For $x \in [0, 1)$ with canonical representation (3.1), we define

$$supp(x) = \{n \in \mathbb{N} \ : \ c_n \neq 0\}.$$

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And,

$$supp_q(x) = \{n \in \mathbb{N} \,:\, c_n = q_n - 1\}.$$

One can easily observe that $supp_q(x) \subseteq supp(x)$.

Remark 3.1.3. *For $\alpha = 1$, the statistical convergence of order α simply coincides with the usual statistical convergence. For our convenience, instead of considering $\alpha \in (0,1)$, we would take $\alpha \in (0,1]$ from which we can infer that all the results presented in the next sections hold for both statistical convergence and α-statistical convergence.*

In [38] several sufficient conditions are given to show whether an element is topologically s-torsion or not but a complete characterization of topologically s-torsion element was missing. So the authors posed the following open problem:

Problem 3.1.4. *[38, Problem 6.10.] Let (a_n) be an arithmetic sequence such that $q_n > 2$ for infinitely many n. Does there exist a characterization of the elements of the subgroup $t^s_{(a_n)}(\mathbb{T})$ only in terms of the support.*

In this chapter we are going to present a complete description of topologically s-torsion as well as topologically α-torsion elements of $\mathbb{T}$ in a single frame (Theorem 3.3.1). And as a consequence we are able to provide a positive answer in the general case for the next open problem considered in [19].

Problem 3.1.5. *[35, Problem 2.14.] For any arithmetic sequence (a_n) and $0 < \alpha_1 < \alpha_2 < 1$, is $t^{\alpha_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\alpha_2}_{(a_n)}(\mathbb{T})$?*

Note that, the above problem was answered positively for the arithmetic sequence (2^n) in the very recent paper [34].

3.2 Basic definitions, notations and results

Before proceeding to our main result we present below certain basic definitions, notations and results which will be needed in the next section.

For two subsets A, B of $\mathbb{N}$, we will write $A \subseteq^\alpha B$ if $d_\alpha(A \setminus B) = 0$ and $A =^\alpha B$ if $d_\alpha(A \triangle B) = 0$. The following two characterizations of α-statistical convergence would be of much help in our investigations. The following observation, though not explicitly stated anywhere, follows from the fact that $\mathcal{I}_{d_\alpha} = \{A \subset \mathbb{N} : d_\alpha(A) = 0\}$ is a P-ideal [6] and Theorem 3.2 [65].

Theorem 3.2.1. *For a sequence of real numbers (x_n), $x_n \to x_0$ α-statistically if and only if there exists a subset A of $\mathbb{N}$ with $d_\alpha(\mathbb{N} \setminus A) = 0$, such that $\lim\limits_{n \in A} x_n = x_0$.*

It is known (folklore) that a sequence (x_n) converges to $\xi \in \mathbb{R}$ *α-statistically* if and only if for any $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$, there exists an infinite $A' \subseteq A$ such that $\lim\limits_{n \in A'} x_n = \xi$. We will make use of a similar result repeatedly which is stated below without any proof.

Lemma 3.2.2. *A sequence (x_n) converges to $\xi \in \mathbb{T}$* α-statistically *if and only if for any $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$, there exists an infinite $A' \subseteq A$ such that $\lim\limits_{n \in A'} x_n = \xi$.*

Now we are going to introduce some equations which will be used repeatedly in our next section. Let (a_n) be an arithmetic sequence and $x \in [0,1)$ has canonical representation (3.1). Then we have, for all non-negative integer k,

$$q_n \cdot q_{n+1} \cdots q_{n+k} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n+1}}{a_n} \cdots \frac{a_{n+k}}{a_{n+k-1}} = \frac{a_{n+k}}{a_{n-1}}. \tag{3.2}$$

Observe that for any natural $m > n$, we have

$$\sum_{i=n}^{m} \frac{c_i}{a_i} \cdot a_{n-1} \leq \sum_{i=n}^{m} \frac{q_i - 1}{a_i} \cdot a_{n-1} = 1 - \frac{a_{n-1}}{a_m} < 1.$$

Therefore,

$$\{a_{n-1}x\} = \sum_{i=n}^{\infty} \frac{c_i}{a_i} \cdot a_{n-1} = \left(c_n \cdot \frac{a_{n-1}}{a_n} + c_{n+1} \cdot \frac{a_{n-1}}{a_{n+1}} + \dots \right)$$

$$= \frac{c_n}{q_n} + \frac{c_{n+1}}{q_n q_{n+1}} + \dots + \frac{c_{n+k}}{q_n \cdot q_{n+1} \cdots q_{n+k}} + \sum_{i=k+1}^{\infty} \frac{c_{n+i}}{q_n \cdot q_{n+1} \cdots q_{n+i}}. \tag{3.3}$$

Similarly,

$$\begin{aligned} \{a_{n+k}x\} &= \{(q_n q_{n+1} \cdots q_{n+k}) a_{n-1} x\} \\ &= q_n \cdot q_{n+1} \cdots q_{n+k} \left(\frac{c_n}{q_n} + \dots + \sum_{i=k+1}^{\infty} \frac{c_{n+i}}{q_n \cdot q_{n+1} \cdots q_{n+i}} \right) \end{aligned}$$

$$= q_n \cdot q_{n+1} \cdots q_{n+k} \sum_{i=k+1}^{\infty} \frac{c_{n+i}}{q_n \cdot q_{n+1} \cdots q_{n+i}}. \tag{3.4}$$

From equations (3.4) and equations (3.3), we get

$$\{a_{n-1}x\} = \frac{c_n}{q_n} + \frac{c_{n+1}}{q_n \cdot q_{n+1}} + \dots + \frac{c_{n+k}}{q_n \cdot q_{n+1} \cdots q_{n+k}} + \frac{\{a_{n+k}x\}}{q_n \cdot q_{n+1} \cdots q_{n+k}}. \tag{3.5}$$

For all $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we define

$$\sigma_{n,k} = \frac{c_n}{q_n} + \frac{c_{n+1}}{q_n \cdot q_{n+1}} + \dots + \frac{c_{n+k}}{q_n \cdot q_{n+1} \cdots q_{n+k}}. \tag{3.6}$$

Therefore, equation (3.5) becomes,

$$\{a_{n-1}x\} = \sigma_{n,k} + \frac{\{a_{n+k}x\}}{q_n \cdot q_{n+1} \cdots q_{n+k}}. \tag{3.7}$$

Further putting $k = 0$ in equation (3.5), we finally obtain

$$\{a_{n-1}x\} = \frac{c_n}{q_n} + \frac{\{a_n x\}}{q_n}. \tag{3.8}$$

Let $a = (a_n)$ be a given arithmetic sequence. Now for any $B \subseteq \mathbb{N}$ with $\overline{d}_\alpha(B) > 0$ consider

$$t_{(a_B)}(\mathbb{T}) = \{x \in \mathbb{T} : \lim_{n \in B} a_n x = 0 \text{ in } \mathbb{T}\}$$

and

$$t^\alpha_{(a_B)}(\mathbb{T}) = \{x \in \mathbb{T} : \lim_{n \in B'} a_n x = 0 \text{ in } \mathbb{T} \text{ for some } B' \subseteq B \text{ with } d_\alpha(B \setminus B') = 0\}.$$

(Note that we only consider $B \subseteq \mathbb{N}$ with $\overline{d}_\alpha(B) > 0$ as otherwise for any $B \subseteq \mathbb{N}$ with $d(B) = 0$ we will have $t^\alpha_{(a_B)}(\mathbb{T}) = \mathbb{T}$ –which does not play any further roll in this context). Therefore, for all $B \subseteq \mathbb{N}$ with $\overline{d}_\alpha(B) > 0$, we have $t^\alpha_{(a_n)}(\mathbb{T}) \subseteq t^\alpha_{(a_B)}(\mathbb{T})$ and $t^\alpha_{(a_n)}(\mathbb{T}) = \bigcap_{B \in [\mathbb{N}]^{\aleph_0} \ \& \ \overline{d}_\alpha(B) > 0} t^\alpha_{(a_B)}(\mathbb{T})$.

Lemma 3.2.3. *If $B \subseteq \mathbb{N}$ with $\overline{d}_\alpha(B) > 0$ and $x \in [0, 1)$ with $x \in t_{(a_{B-1})}(\mathbb{T})$ (where $B - 1 = \{k - 1 : k \in B\}$) then the following hold:*

i) If $B \subseteq^\alpha supp(x)$ and q-bounded, then $B \subseteq^\alpha supp_q(x)$ and there exists $B' \subseteq B$ with $d_\alpha(B \setminus B') = 0$ such that $\lim_{n \in B'} \{a_{n-1}x\} = 1$ in $\mathbb{R}$.

ii) If $d_\alpha(B \cap supp(x)) = 0$, then there exists $B' \subseteq B$ with $d_\alpha(B \setminus B') = 0$ such that $\lim_{n \in B'} \{a_{n-1}x\} = 0$ in $\mathbb{R}$.

Proof. i) Let $q = 1 + \max_{n \in B}\{q_n\}$ and $B' = B \cap supp(x)$. Since $B' \subseteq B$ and $d_\alpha(B \setminus supp(x)) = 0$, we get $d_\alpha(B \setminus B') = d_\alpha(B \setminus (B \cap supp(x))) = d_\alpha(B \setminus supp(x)) = 0$. Therefore,

$$\{a_{n-1}x\} \geq \frac{c_n}{q_n} > \frac{1}{q} \quad \forall\, n \in B' \text{ (since } c_n \geq 1 \text{ for all } n \in B') .$$

But as $x \in t_{(a_{B-1})}(\mathbb{T})$, thus we can conclude that $\lim_{n \in B'} \{a_{n-1}x\} = 1$ in $\mathbb{R}$.
Therefore,

$$\begin{aligned} 1 - \frac{1}{q_n} &< 1 - \frac{1}{q} < \{a_{n-1}x\} = \frac{c_n}{q_n} + \frac{\{a_n x\}}{q_n} \\ &< \frac{c_n + 1}{q_n} \quad \text{for almost all } n \in B' \text{ (from equation (3.8))}. \end{aligned}$$

$$\Rightarrow q_n - 1 < c_n + 1 \text{ i.e. } c_n > q_n - 2 \text{ for almost all } n \in B'.$$

Hence, $c_n = q_n - 1$ for almost all $n \in B'$ i.e. $B' \subseteq^* supp_q(x)$, which implies $B \subseteq^\alpha supp_q(x)$.

ii) Let $B' = B \setminus supp(x)$. Observe that $B' \subseteq B$ and $d_\alpha(B \setminus B') = d_\alpha(B \setminus (B \setminus supp(x))) = d_\alpha(B \cap supp(x)) = 0$. Now, from equation (3.8), we have

$$\{a_{n-1}x\} = 0 + \frac{\{a_n x\}}{q_n} < \frac{1}{2}\ \forall\, n \in B' \text{ (since } c_n = 0\ \forall\, n \in B').$$

Then in view of the fact that $x \in t_{(a_{B-1})}(\mathbb{T})$, we must have, $\lim_{n\in B'}\{a_{n-1}x\} = 0$ in $\mathbb{R}$. $\square$

Our next lemma creates a bridge between a q-bounded and a q-divergent subsets of $\mathbb{N}$ which is essential to prove our main result, i.e., Theorem 3.3.1.

Lemma 3.2.4. *Let (a_n) be an arithmetic sequence. Consider $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$ where A is not q-bounded. If there does not exists any q-bounded subset $A' \subseteq A$ with $\overline{d}_\alpha(A') > 0$ then there exists a q-divergent set $B \subseteq A$ with $d_\alpha(A \setminus B) = 0$.*

Proof. Let $A_m = \{n \in A \ : \ q_n = m\}$ for some $m \in \mathbb{N} \setminus \{1\}$. Since there does not exists any q-bounded set $A' \subseteq A$ with $\overline{d}_\alpha(A') > 0$, we conclude that $d_\alpha(A_m) = 0$ for all $m \in \mathbb{N} \setminus \{1\}$. If there exists $k \in \mathbb{N}$ such that A_m is finite for all $m > k$, then setting $B = \bigcup_{m=k+1}^{\infty} A_m$, it is easy to observe that B is q-divergent with $d_\alpha(A \setminus B) = 0$ (since $d_\alpha(\bigcup_{m=2}^{k} A_m) = 0$).

Otherwise without any loss of generality, we can assume that A_m is infinite for all $m \in \mathbb{N} \setminus \{1\}$. Now, considering the sequence $(A_m)_{m\in\mathbb{N}}$ of α-density zero sets, one can find $C \subset \mathbb{N}$ with $d_\alpha(C) = 0$ such that $A_m \setminus C$ is finite for all $m \in \mathbb{N} \setminus \{1\}$ (for explicit construction of such a set, see [76], also the existence follows from the fact that $\mathcal{I}_{d_\alpha}$ is a P-ideal, see [6]).

Let, $B = A \setminus C$. Since $d_\alpha(C) = 0$, we conclude that $d_\alpha(A \setminus B) = 0$. Consider any $l \in \mathbb{N} \setminus \{1\}$. Let, $n_l = \ max \ \{n \ : \ n \in A_m \setminus C$ and $m \leq l\}$. Now, for all $n \in B$ with $n > n_l$, we have $q_n > l$. Since l was taken arbitrarily, it follows that B is q-divergent. $\square$

3.3 Main results

Theorem 3.3.1. *Let, (a_n) be an arithmetic sequence and $x \in [0, 1)$. Then x is a topological α-torsion element (i.e. $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$) if and only if either $d_\alpha(supp(x)) = 0$ or if $\overline{d}_\alpha(supp(x)) > 0$, then for all $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$ the following holds:*

(a) If A is q-bounded, then:

(a1) If $A \subseteq^\alpha supp(x)$, then $A + 1 \subseteq^\alpha supp(x)$, $A \subseteq^\alpha supp_q(x)$ and there exists $A' \subseteq A$ with $d_\alpha(A \setminus A') = 0$ such that $\lim_{n\in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$.
Moreover, if $A + 1$ is q-bounded, then $A + 1 \subseteq^\alpha supp_q(x)$.

(a2) If $d_\alpha(A \cap supp(x)) = 0$, then there exists $A' \subseteq A$ with $d_\alpha(A \setminus A') = 0$ such that $\lim_{n\in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$.
Moreover, if $A + 1$ is q-bounded, then $d_\alpha((A + 1) \cap supp(x)) = 0$ as well.

(b) If A is q-divergent, then $\lim_{n\in B} \varphi(\frac{c_n}{q_n}) = 0$ for some $B \subseteq A$ with $d_\alpha(A \setminus B) = 0$.

Proof. **Necessity**: Suppose $\overline{d}_\alpha(supp(x)) > 0$ and $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$. Therefore, there exists $M \subseteq \mathbb{N}$ with $d_\alpha(\mathbb{N} \setminus M) = 0$ such that

$$\lim_{n\in M}\{a_{n-1}x\} = 0 \text{ in } \mathbb{T}. \tag{3.9}$$

Consider any $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$. We take $B = M \cap A$. Then $B \subseteq A$ and $d_\alpha(A \setminus B) = d_\alpha(A \cap (\mathbb{N} \setminus M)) = 0$. As $B \subseteq M$, from equation (3.9), we get $\lim\limits_{n \in B}\{a_{n-1}x\} = 0$ in $\mathbb{T}$. Consequently, there exists $B \subseteq A$ with $d_\alpha(A \setminus B) = 0$ such that $x \in t_{(a_{B-1})}(\mathbb{T})$.
(a) Suppose first that A is q-bounded. The following two cases can arise:

(a1) First, suppose $A \subseteq^\alpha supp(x)$. Then $B \subseteq A$ is q-bounded and $B \subseteq^\alpha supp(x)$. Since, $x \in t_{(a_{B-1})}(\mathbb{T})$ and B is q-bounded, from Lemma 3.2.3 we conclude that $B \subseteq^\alpha supp_q(x)$ and $\lim\limits_{n \in A'}\{a_{n-1}x\} = 1$ in $\mathbb{R}$, where $A' \subseteq B$ with $d_\alpha(B \setminus A') = 0$.
Therefore, from equation (3.8)

$$\begin{aligned} 1 &= \lim_{n \in A'}\left(\frac{c_n}{q_n} + \frac{\{a_nx\}}{q_n}\right) = \lim_{n \in A'}\left(\frac{q_n - 1 + \{a_nx\}}{q_n}\right) \\ &= \lim_{n \in A'}\left(1 - \frac{1-\{a_nx\}}{q_n}\right) \Rightarrow \lim_{n \in A'} \frac{1-\{a_nx\}}{q_n} = 0. \end{aligned}$$

Hence, we get

$$\lim_{n \in A'}\{a_nx\} = 1 \text{ (since, } A' \subseteq B \text{ is } q\text{-bounded).} \tag{3.10}$$

Now from the definition of canonical representation (3.1), $c_{n+1} \leq q_{n+1} - 1$ for all $n \in \mathbb{N}$. Again from equation (3.8), we have

$$\{a_nx\} = \frac{c_{n+1}}{q_{n+1}} + \frac{\{a_{n+1}x\}}{q_{n+1}} < \frac{c_{n+1}+1}{q_{n+1}} \leq 1.$$

Hence from equation (3.10), it follows that

$$1 = \lim_{n \in A'}\{a_nx\} \leq \lim_{n \in A'} \frac{c_{n+1}+1}{q_{n+1}} \leq 1 \qquad \text{i.e. } \lim_{n \in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1 \tag{3.11}$$

Now, $q_{n+1} \geq 2$ for all $n \in \mathbb{N}$. From equation (3.11), we can observe that $c_{n+1} + 1 > 1$ (i.e. $c_{n+1} \neq 0$) for almost all $n \in A'$. Which implies $A' + 1 \subseteq^* supp(x)$. Since, $d_\alpha(B \setminus A') = 0$, we obtain $B + 1 \subseteq^\alpha supp(x)$.

As $B \subseteq A$ and $d_\alpha(A \setminus B) = 0$, we must have $A + 1 \subseteq^\alpha supp(x)$, $A \subseteq^\alpha supp_q(x)$ and $\lim\limits_{n \in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ for some $A' \subseteq A$ where $d_\alpha(A \setminus A') = 0$. If $A + 1$ is q-bounded, proceeding as in the first part of the proof, we get $A + 1 \subseteq^\alpha supp_q(x)$.

(a2) Now let $d_\alpha(A \cap supp(x)) = 0$. Since $B \subseteq A$, we must have $d_\alpha(B \cap supp(x)) = 0$. Then from Lemma 3.2.3, we can conclude that $\lim\limits_{n \in A'}\{a_{n-1}x\} = 0$ in $\mathbb{R}$ for some $A' \subseteq B$ with $d_\alpha(B \setminus A') = 0$. Therefore putting $k = 1$ in equation (3.6) and equation (3.7), we get

$$\lim_{n \in A'}\left(\frac{c_n}{q_n} + \frac{c_{n+1}}{q_nq_{n+1}} + \frac{\{a_{n+1}x\}}{q_nq_{n+1}}\right) = \lim_{n \in A'}\{a_{n-1}x\} = 0$$

$$\Rightarrow \lim_{n \in A'} \frac{c_{n+1}}{q_nq_{n+1}} = \lim_{n \in A'} \frac{\{a_{n+1}x\}}{q_nq_{n+1}} = 0 \text{ (Since } c_n, \{a_nx\} \geq 0 \text{ and } q_n > 0 \text{).} \tag{3.12}$$

Now as $A' \subseteq B$ is q-bounded, equation (3.12) implies that $\lim\limits_{n \in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where

$A' \subseteq B \subseteq A$ and $d_\alpha(A \setminus A') = 0$.

Moreover, if $A + 1$ is q-bounded, then vanishing of the last limit implies that $(A' + 1) \cap supp(x)$ is finite. Thus $d_\alpha((A+1) \cap supp(x)) = 0$ (Since, $d_\alpha((A+1) \setminus (A'+1)) = d_\alpha(A \setminus A') = 0$).

(b) Suppose A is q-divergent i.e. $\lim_{n \in A} q_n = \infty$. Then from equation (3.8), we get

$$\lim_{n \in B} \varphi\left(\frac{c_n}{q_n} + \frac{\{a_n x\}}{q_n}\right) = \lim_{n \in B} \varphi(\{a_{n-1}x\}) = 0 \text{ in } \mathbb{T} \text{ for some } B \subseteq A \text{ with } d_\alpha(A \setminus B) = 0$$

$$\Rightarrow \lim_{n \in B} \varphi\left(\frac{c_n}{q_n}\right) = 0 \text{ in } \mathbb{T} \ \ (\text{Since, } \{a_n x\} < 1 \text{ and } \lim_{n \in B} q_n = \infty).$$

Before proving the sufficiency of the conditions, we need to reformulate the necessary conditions in a stronger iterated version. For any $A \in [\mathbb{N}]^{\aleph_0}$ and $k \in \mathbb{N} \cup \{0\}$, we define $L_k(A) = \bigcup_{i=0}^{k} (A + i)$. Now putting $k = k + 1$ in equation (3.6), we obtain

$$\sigma_{n,k+1} = \sigma_{n,k} + \frac{c_{n+k+1}}{q_n q_{n+1} \cdots q_{n+k+1}}. \tag{3.13}$$

Therefore, from equation (3.7) and equation (3.13), it follows that

$$\{a_{n-1}x\} = \sigma_{n,k+1} + \frac{\{a_{n+k+1}x\}}{q_n q_{n+1} \cdots q_{n+k+1}} = \sigma_{n,k} + \frac{c_{n+k+1}}{q_n q_{n+1} \cdots q_{n+k+1}} + \frac{\{a_{n+k+1}x\}}{q_n q_{n+1} \cdots q_{n+k+1}} \tag{3.14}$$

$$\Rightarrow \sigma_{n,k} \leq \{a_{n-1}x\} < \sigma_{n,k} + \frac{c_{n+k+1}}{q_n q_{n+1} \cdots q_{n+k+1}} + \frac{1}{2^{(k+2)}}. \tag{3.15}$$

Claim 3.3.2. *Let $x \in [0, 1)$ has canonical representation (3.1) such that (a) and (b) of Theorem 3.3.1 hold. Let $A \subseteq \mathbb{N}$ be q-bounded with $\overline{d}_\alpha(A) > 0$. If $L_k(A)$ is q-bounded for some $k \in \mathbb{N} \cup \{0\}$, then the following hold:*

(i) If $A \subseteq^\alpha supp(x)$, then $L_k(A) \subseteq^\alpha supp_q(x)$ and $\lim_{n \in A'+k+1} \frac{c_n+1}{q_n} = 1$ in $\mathbb{R}$ for some $A' \subseteq A$ with $d_\alpha(A \setminus A') = 0$. Therefore there exists $n_k \in \mathbb{N}$ such that for all $n \in A'$ with $n \geq n_k$,

$$\sigma_{n,k} = 1 - \frac{1}{q_n q_{n+1} \cdots q_{n+k}} \geq 1 - \frac{1}{2^{k+1}}. \tag{3.16}$$

Moreover if $A + k + 1$ is q-divergent, then

$$\lim_{n \in A+k+1} \frac{c_n}{q_n} = \lim_{n \in A} \frac{c_{n+k+1}}{q_{n+k+1}} = 1 \textit{ in } \mathbb{R}. \tag{3.17}$$

(ii) If $d_\alpha(A \cap supp(x)) = 0$, then $d_\alpha(L_k(A) \cap supp(x)) = 0$ and $\lim_{n \in A'} \frac{c_{n+k+1}}{q_{n+k+1}} = 0$ in $\mathbb{R}$ for some $A' \subseteq A$ and $d_\alpha(A \setminus A') = 0$.

Proof. (i) Observe that $\overline{d}_\alpha(L_k(A)) > 0$ follows from $\overline{d}_\alpha(A) > 0$. Now the fact $L_k(A)$ is q-bounded implies that $A + i$ is q-bounded for all $i \in \mathbb{N} \cup \{0\}$. In view of (a1) of Theorem 3.3.1 recursively we get

$$A + i \subseteq^\alpha supp_q(x) \quad \Rightarrow \quad L_k(A) \subseteq^\alpha supp_q(x).$$

Since $A + k$ is q-bounded we also get $\lim_{n \in A'+k+1} \frac{c_n+1}{q_n} = 1$ in $\mathbb{R}$ for some $A' \subseteq A$ with $d_\alpha(A \setminus A') = 0$ and $L_k(A') \subseteq supp_q(x)$. Therefore there exists $n_k \in \mathbb{N}$ such that for all $n \in A'$ with $n \geq n_k$,

$$\sigma_{n,k} = 1 - \frac{1}{q_n q_{n+1} \cdots q_{n+k}} \geq 1 - \frac{1}{2^{k+1}}. \tag{3.18}$$

Moreover if $A+k+1$ is q-divergent, then Equation (3.18) and (b) of [41, Theorem 2.3] implies

$$\lim_{n \in A+k+1} \frac{c_n}{q_n} = \lim_{n \in A} \frac{c_{n+k+1}}{q_{n+k+1}} = 1 \text{ in } \mathbb{R}. \tag{3.19}$$

(ii) As in (i), we have $A + i$ is q-bounded for all $i \in \mathbb{N} \cup \{0\}$. In view of (a2) of Theorem 3.3.1 recursively we get

$$d_\alpha((A + i) \cap supp(x)) = 0 \quad \Rightarrow \quad d_\alpha(L_k(A) \cap supp(x)) = 0$$

Since $d_\alpha((A+k) \cap supp(x)) = 0$ then again from (a2) of Theorem 3.3.1 we finally obtain $\lim_{n \in A'} \frac{c_{n+k+1}}{q_{n+k+1}} = 0$ in $\mathbb{R}$ for some $A' \subseteq A$ and $d_\alpha(A \setminus A') = 0$.

□

Sufficiency: If $d_\alpha(supp(x)) = 0$, then from [35, Proposition 2.6] it readily follows that $x \in t^\alpha_{(a_n)}(\mathbb{T})$ (for $\alpha = 1$ see [38, Theorem 4.3]). So let $\overline{d}_\alpha(supp(x)) > 0$ and $supp(x)$ satisfy conditions (a) and (b). To show that $x \in t^\alpha_{(a_n)}(\mathbb{T})$, in view of Lemma 3.2.2 it is sufficient to check the convergence criterion: for all $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$, there exists $B' \subseteq A$ such that $\lim_{n \in B'} \varphi(\{a_{n-1}x\}) = 0$. Indeed without any loss of generality, we can assume that either $d_\alpha(A \cap supp(x)) = 0$ or $A \subseteq^\alpha supp(x)$.

Case (i): First let A be q-bounded.

Subcase (i$_a$): Let us first assume that $L_k(A)$ is q-bounded for all $k \in \mathbb{N} \cup \{0\}$. Let $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{2^{k+1}} < \varepsilon$.

$*$ Let $A \subseteq^\alpha supp(x)$. Then, from (i) of Claim 3.3.2, $L_k(A) \subseteq^\alpha supp_q(x)$. Therefore, there exists $B' \subseteq A$ such that for all $n \in B'$,

$$\sigma_{n,k} = 1 - \frac{1}{q_n q_{n+1} \cdots q_{n+k}} \geq 1 - \frac{1}{2^{k+1}} > 1 - \varepsilon$$

$$\Rightarrow 1 - \varepsilon < \sigma_{n,k} \leq \{a_{n-1}x\} < 1 \quad \forall\, n \in B' \text{ (from equation (3.15))}.$$

$*$ Let $d_\alpha(A \cap supp(x)) = 0$. Then, from (ii) of Claim 3.3.2, $d_\alpha(L_k(A) \cap supp(x)) = 0$ and $\lim_{n \in B} \frac{c_{n+k+1}}{q_{n+k+1}} = 0$ in $\mathbb{R}$ for some $B \subseteq A$ with $d_\alpha(A \setminus B) = 0$. So, there exists $B' \subseteq B$ such that $\sigma_{n,k} = 0$ and $\frac{c_{n+k+1}}{q_{n+k+1}} < \varepsilon$ for all $n \in B'$. Therefore, from equation (3.15), we get

$$\{a_{n-1}x\} < \ \sigma_{n,k} + \frac{c_{n+k+1}}{q_n q_{n+1}\cdots q_{n+k+1}} + \frac{1}{2^{(k+2)}} \ < 2\varepsilon \ \forall\, n \in B'.$$

Thus in both cases, we have $\lim\limits_{n\in B'} \varphi(\{a_{n-1}x\}) = 0$ for some $B' \subseteq A$, as required.

Subcase (i$_b$): We assume that there exists an integer $k \geq 0$ such that $A + k + 1$ is not q-bounded but $A + i$ is q-bounded for all $i = 0, 1, 2, \ldots, k$. If there exists an $A' \subseteq A$ such that $\overline{d}_\alpha(A') > 0$ and $A' + k + 1$ is q-bounded, then without any loss of generality we can start with A' in place of A. If this process does not terminate after finitely many steps then we can conclude that there exists $B \subseteq A$ with $\overline{d}_\alpha(B) > 0$ such that $L_k(B)$ is q-bounded for all $k \in \mathbb{N}$. Consequently, we can consider B in place of A and proceed as in Subcase (i$_a$).

Now let us consider the case when there does not exist any $A' \subseteq A$ such that $\overline{d}_\alpha(A') > 0$ and $A' + k + 1$ is q-bounded. Therefore from Lemma 3.2.4, there exists $B \subseteq A$ with $d_\alpha(A \setminus B) = 0$ such that $B + k + 1$ is q-divergent i.e. $\lim\limits_{n\in B} q_{n+k+1} = \infty$. Clearly $L_k(B)$ is q-bounded. Further more

$$\lim_{n\in B} \frac{\{a_{n+k+1}x\}}{q_n q_{n+1}\cdots q_{n+k+1}} \leq \lim_{n\in B} \frac{1}{q_{n+k+1}} = 0. \tag{3.20}$$

Therefore, from equation (3.14) and equation (3.20), we get

$$\begin{aligned}\lim_{n\in B}\{a_{n-1}x\} &= \lim_{n\in B}\sigma_{n,k} + \lim_{n\in B}\frac{c_{n+k+1}}{q_n q_{n+1}\cdots q_{n+k+1}} + \lim_{n\in B}\frac{\{a_{n+k+1}x\}}{q_n q_{n+1}\cdots q_{n+k+1}}\\ &= \lim_{n\in B}\sigma_{n,k} + \lim_{n\in B}\frac{c_{n+k+1}}{q_n q_{n+1}\cdots q_{n+k+1}}.\end{aligned}$$

$*$ Let $A \subseteq^\alpha supp(x)$. Therefore $B \subseteq^\alpha supp(x)$. Consequently from equation (3.18) of Claim 3.3.2 and equation (3.21), we get

$$\begin{aligned}\lim_{n\in B'}\{a_{n-1}x\} &= \lim_{n\in B'}(1 - \frac{1}{q_n q_{n+1}\cdots q_{n+k}} + \frac{c_{n+k+1}}{q_n q_{n+1}\cdots q_{n+k+1}})\\ &= \lim_{n\in B'}(1 + \frac{1}{q_n q_{n+1}\cdots q_{n+k}}\cdot(\frac{c_{n+k+1}}{q_{n+k+1}} - 1)) \ = \ 1\end{aligned}$$

for some $B' \subseteq B$ with $d_\alpha(B \setminus B') = 0$.

$*$ Next let $d_\alpha(A \cap supp(x)) = 0$. Then there exists $B \subseteq A$ such that $\sigma_{n,k} = 0$ for all $n \in B$. Subsequently from (ii) of Claim 3.3.2 and equation (3.21), we have

$$\lim_{n\in B'}\{a_{n-1}x\} = \lim_{n\in B'}\frac{c_{n+k+1}}{q_n q_{n+1}\cdots q_{n+k+1}} \leq \lim_{n\in B'}\frac{c_{n+k+1}}{q_{n+k+1}} = 0$$

for some $B' \subseteq B$ with $d_\alpha(B \setminus B') = 0$. Thus in both cases, we again obtain that $\lim\limits_{n\in B'} \varphi(\{a_{n-1}x\}) = 0$ for some $B' \subseteq A$.

Case (ii): We assume that A is not q-bounded. If there exists $A' \subseteq A$ such that $\overline{d}_\alpha(A') > 0$ and A' is q-bounded then we can proceed as in Case (i) and consider A' in place of A. So, let us assume that there does not exist any $A' \subseteq A$ such that $\overline{d}_\alpha(A') > 0$ and A' is

q-bounded. Then from Lemma 3.2.4, there exists $B \subseteq A$ with $d_\alpha(A \setminus B) = 0$ such that B is q-divergent i.e. $\lim_{n\in B} q_n = \infty$. From hypothesis, we have $\lim_{n\in B'} \frac{c_n}{q_n} = 0$ in $\mathbb{T}$ for some $B' \subseteq B$ with $d_\alpha(B \setminus B') = 0$. Therefore, from equation (3.8), we obtain

$$\lim_{n\in B'} \varphi(\{a_{n-1}x\}) = \lim_{n\in B'} \varphi(\frac{c_n}{q_n} + \frac{\{a_n x\}}{q_n}) = 0 \text{ (since } \lim_{n\in B'} \tfrac{\{a_n x\}}{q_n} < \lim_{n\in B'} \tfrac{1}{q_n} = 0 \text{).}$$

Hence in all cases, we can conclude that for any $A \subseteq \mathbb{N}$ with $\overline{d}_\alpha(A) > 0$, there exists an infinite set $B' \subseteq A$ such that $\lim_{n\in B'} \varphi(\{a_{n-1}x\}) = 0$. This shows that $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$ i.e. x is a topologically α-torsion element of $\mathbb{T}$. □

Remark 3.3.3. *Since, for all $n \notin supp(x)$, we have $c_n = 0$, it is sufficient to consider only subsets of $supp(x)$ in item (b) of Theorem 3.3.1.*

Now we are in a position to answer **Problem 7.2.7**.

Proof of Problem 7.2.7: We intend to show that for any arithmetic sequence (a_n) and $0 < \alpha_1 < \alpha_2 < 1$, $t^{\alpha_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\alpha_2}_{(a_n)}(\mathbb{T})$. Since, for any $A \subset \mathbb{N}$ with $d_{\alpha_1}(A) = 0$ implies $d_{\alpha_2}(A) = 0$, it is easy to observe that $t^{\alpha_1}_{(a_n)}(\mathbb{T}) \subseteq t^{\alpha_2}_{(a_n)}(\mathbb{T})$. So all we need to do is to find a $y \in t^{\alpha_2}_{(a_n)}(\mathbb{T}) \setminus t^{\alpha_1}_{(a_n)}(\mathbb{T})$.

We choose $\beta \in \mathbb{R}$ such that $\frac{1}{\alpha_2} < \beta < \frac{1}{\alpha_1}$. Consider $x \in [0, 1)$ with $x \in \mathbb{T}$ and $supp(x) = \{\lfloor n^\beta \rfloor : n \in \mathbb{N}\}$ and $c_n = \lfloor \frac{q_n}{2} \rfloor$. Since $\alpha_2 \cdot \beta > 1$, we get $d_{\alpha_2}(supp(x)) = \lim_{n\to\infty} \frac{|supp(x)\cap[1,n]|}{n^{\alpha_2}} = \lim_{n\to\infty} \frac{n}{(n^\beta)^{\alpha_2}} = 0$. Therefore, from [35, Proposition 2.6] it follows that $x \in t^{\alpha_2}_{(a_n)}(\mathbb{T})$.

Similarly using the fact $\alpha_1 \cdot \beta < 1$ we conclude that $d_{\alpha_1}(supp(x)) > 0$. Let us first assume that there exists $B \subseteq supp(x)$ with $\overline{d}_{\alpha_1}(B) > 0$ such that B is q-bounded. But from our construction of $supp(x)$, we get $B \nsubseteq supp_q(x)$. Therefore, item (a1) of Theorem 3.3.1 does not hold and we get $x \notin t^{\alpha_1}_{(a_n)}(\mathbb{T})$. If there does not exists such a $B \subseteq supp(x)$ then from Lemma 3.2.4 there must exists a $B' \subseteq supp(x)$ with $d_{\alpha_1}(supp(x) \setminus B') = 0$ such that B' is q-divergent. But observe that $\lim_{n\in C} \frac{c_n}{q_n} \neq 0$ for any $C \subseteq B'$ i.e. item (b) of Theorem 3.3.1 does not hold and again we get $x \notin t^{\alpha_1}_{(a_n)}(\mathbb{T})$. Thus we find an $x \in t^{\alpha_2}_{(a_n)}(\mathbb{T}) \setminus t^{\alpha_1}_{(a_n)}(\mathbb{T})$.

3.4 α-Splitting sequence

In the remaining part of the article we follow in the line of investigations of [41] which would show that in certain circumstances, one can obtain more simplified equivalent criterions for the α-torsion elements. Before proceeding further, let us recall the following notion of "splitting" sequences which were considered in [41].

Definition 3.4.1. *[41, Definition 3.10] A sequence (q_n) of natural numbers has the splitting property if there exists a partition $\mathbb{N} = B \cup I$, such that the following statements hold:*

(a) B and I are either empty or infinite;

(b) I is q-divergent, in case I is infinite;

(c) B *is* q*-bounded, in case* B *is infinite.*

Here, B *and* I *witness the splitting property for* (q_n)*, where* B *and* I *can be uniquely defined up to a finite set.*

Proposition 3.4.2. *[41, Proposition 3.11] A sequence* (q_n) *has the splitting property if and only if there exists a natural number* M *such that the set* $\{n \in \mathbb{N} : \ q_n \in [M, m]\}$ *is finite for every* $m > M$*.*

As a natural consequence, we can think of generalizing the idea of a splitting sequence using natural density of order α.

Definition 3.4.3. *We say that, a sequence* (q_n) *of natural numbers has the* α*-splitting property if there exists a partition* $\mathbb{N} = B \cup D$*, such that the following statements hold:*

(a) B *and* D *are either empty or* $\overline{d}_\alpha(B), \overline{d}_\alpha(D) > 0$*.*

(b) *If* $\overline{d}_\alpha(B) > 0$*, then there exists* $B' \subseteq \mathbb{N}$ *with* $d_\alpha(B \triangle B') = 0$ *such that* B' *is* q*-bounded.*

(b) *If* $\overline{d}_\alpha(D) > 0$*, then there exists* $D' \subseteq \mathbb{N}$ *with* $d_\alpha(D \triangle D') = 0$ *such that* D' *is* q*-divergent.*

Here, B *and* D *witness the* α*-splitting property for* (q_n)*, where* B *and* D *can be uniquely determined up to a zero* α*-density set (i.e. if* $B_1 \cup D_1$ *is another partition of* $\mathbb{N}$*, witnessing the* α*-splitting property for* (q_n)*, then* $B_1 =^\alpha B$ *and* $D_1 =^\alpha D$*).*

Proposition 3.4.4. *A sequence* (q_n) *has the* α*-splitting property if and only if there exists a natural number* M *such that* $d_\alpha(\{n \in \mathbb{N} : \ q_n \in [M, m]\}) = 0$ *for every* $m > M$*.*

Proof. We assume that (q_n) has the α-splitting property. Now two cases can arise:

* At first, we consider $B = \emptyset$. Then there exists a $D' \subseteq \mathbb{N}$ with $d_\alpha(\mathbb{N} \setminus D') = 0$ such that D' is q-divergent. Take any $m \in \mathbb{N}$. Since D' is q-divergent, there exists an $n_m \in \mathbb{N}$ such that $q_n > m$ for all $n > n_m$ and $n \in D'$. We set $M = 1$. Then it is evident that for all $m > M$

$$\begin{aligned} d_\alpha(\{n \in \mathbb{N} : q_n \in [M, m]\}) &\leq d_\alpha(\{n \in D' : q_n \in [M, m]\} + d_\alpha(\mathbb{N} \setminus D') \\ &\leq d_\alpha(\{n \in D' : n \leq n_m\}) = 0. \end{aligned}$$

* Let $B \neq \emptyset$. Then we have $\overline{d}_\alpha(B) > 0$ and consequently there exists a $B' \subseteq \mathbb{N}$ with $d_\alpha(B \triangle B') = 0$ such that B' is q-bounded. In this case, we set $M = 1 + \max_{n \in B'} \{q_n\}$. Therefore, for any $m > M$, we obtain

$$\begin{aligned} & d_\alpha(\{n \in \mathbb{N} : q_n \in [M, m]\}) \\ \leq\ & d_\alpha(\{n \in B' : q_n \in [M, m]\}) + d_\alpha(B \setminus B') \\ +\ & d_\alpha(\{n \in D' : q_n \in [M, m]\}) + d_\alpha(D \setminus D') \\ =\ & d_\alpha(\{n \in D' : q_n \in [M, m]\}) = 0 \text{ (from equation (3.21)}. \end{aligned}$$

Conversely, let there exists a natural number M such that

$$d_\alpha(\{n \in \mathbb{N} : q_n \in [M, m]\}) = 0 \text{ for all } m > M.$$

We set $B' = \{n \in \mathbb{N} : q_n \in [1, M-1]\}$ and $D' = \mathbb{N} \setminus B'$.

* If $\overline{d}_\alpha(B') > 0$ and $d_\alpha(D') = 0$, then we take $B = \mathbb{N}$ and $D = \emptyset$.
* If $d_\alpha(B') = 0$ and $\overline{d}_\alpha(D') > 0$, then we take $D = \mathbb{N}$ and $B = \emptyset$.
* If $\overline{d}_\alpha(B') > 0$ and $\overline{d}_\alpha(D') => 0$, then we take $B = B'$ and $D = D'$.

Clearly, B and D witness the α-splitting property for the sequence (q_n). □

From Proposition 3.4.2 and Proposition 3.4.4, it is obvious that every splitting sequence is a α-splitting sequence. However the converse is not necessarily true, nor it is true that every subset of $\mathbb{N}$ has the α-splitting property (an example not having the splitting property was given in [41, Example 3.12] but one must take into consideration that a non-splitting sequence can still be α-splitting).

Example 3.4.5. *For any $\beta > \frac{1}{\alpha}$ and $\beta \in \mathbb{N}$ let us define*

$$A_1 = \{n \in \mathbb{N} : \ n = k^\beta \text{ for some } k \in \mathbb{N}\},$$
$$A_2 = \{n \in \mathbb{N} : n = k^\beta + 1 \text{ for some } k \in \mathbb{N}\} \setminus A_1, \ldots,$$
$$A_{i+1} = \{n \in \mathbb{N} : n = k^\beta + i \text{ for some } k \in \mathbb{N}\} \setminus \bigcup_{j=1}^{i} A_j.$$

Take any $n \in \mathbb{N}$. One can find a $k \in \mathbb{N}$ such that $k^\beta \le n < (k+1)^\beta$. So we can write $n = k^\beta + i$ for some $i \in \mathbb{N} \cup \{0\}$ i.e. $n \in A_i$. Therefore, $\mathbb{N} = \bigcup_{i=1}^{\infty} A_i$ i.e. $(A_i)_{i \in \mathbb{N}}$ forms a partition of $\mathbb{N}$.

For each $i \in \mathbb{N}$, we now define $q_n = i + 1$ for all $n \in A_i$. Clearly, for $m, M \in \mathbb{N}$ and $m > M$, we have $\{n \in \mathbb{N} : q_n \in [M, m]\} = \bigcup_{i=M-1}^{m-1} A_i$. Since $d_\alpha(A_i) = 0$ for all $i \in \mathbb{N}$, we get $d_\alpha(\{n : q_n \in [M, m]\}) = 0$ for all $m, M \in \mathbb{N}$ and $m > M$. Therefore, from Proposition 3.4.4, (q_n) is an α-splitting sequence. But, we can observe that $\{n : q_n \in [M, m]\}$ cannot be finite for any $m, M \in \mathbb{N}$ and $m > M$ (since A_i is infinite for all $i \in \mathbb{N}$). Therefore, from Proposition 3.4.2, (q_n) is not a splitting sequence.

Example 3.4.6. *Let us define $q_n = \{i \in \mathbb{N} : n = 2^{i-2}(2k-1)$ for some $k \in \mathbb{N}\}$. For any $i \in \mathbb{N} \setminus \{1\}$, set $A_i = \{n \in \mathbb{N} : q_n = i\} = \{2^{i-1}k - 2^{i-2} : k \in \mathbb{N}\}$. From the construction it is evident that $\overline{d}_\alpha(A_i) > 0$ for all $i \in \mathbb{N} \setminus \{1\}$ and $\mathbb{N} = \bigcup_{i=2}^{\infty} A_i$. Now for any $m, M \in \mathbb{N}$ with $m > M$, observe that $\overline{d}_\alpha(\{n \in \mathbb{N} : q_n \in [M, m]\}) = \overline{d}_\alpha(\bigcup_{i=M}^{m} A_i) > 0$. Therefore, from Proposition 3.4.4, (q_n) is not an α-splitting sequence.*

Motivated by these two examples, we present below equivalent conditions for a sequence to be splitting or α-splitting (or in other words, equivalent formulations of Proposition 3.4.2 and Proposition 3.4.4).

Proposition 3.4.7. *Let (q_n) be a sequence of natural numbers. For all $i \in \mathbb{N}$, we define $A_i = \{n : q_n = i\}$. Then*

(i) (q_n) is a splitting sequence if and only if there does not exist a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of (A_n) such that A_{n_k} is infinite for all $k \in \mathbb{N}$:

(ii) (q_n) is an α-splitting sequence if and only if there does not exist a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of (A_n) such that $\overline{d}_\alpha(A_{n_k}) > 0$ for all $k \in \mathbb{N}$.

For the next result we will use the following notations. Let (a_n) be an arithmetic sequence and $x \in [0, 1)$ with canonical representation (3.1). Assume that the sequence of ratios (q_n) has the α-splitting property which means that there exists a partition $\mathbb{N} = B \cup D$ such that $(a), (b)$ and (c) of Definition 3.4.3 hold. We will write

- $B^S(x) = B \cap supp(x)$,
- $B^N(x) = B \cap (\mathbb{N} \setminus supp(x))$,
- $D^S(x) = D \cap supp(x)$.

From Remark 3.3.3, it follows that $D \cap (\mathbb{N} \setminus supp(x))$ does not play any role in Theorem 3.3.1. Note that if $B, D \neq \emptyset$, then there exists $B' \subseteq \mathbb{N}$ with $d_\alpha(B \triangle B') = 0$ and $D' \subseteq \mathbb{N}$ with $d_\alpha(D \triangle D') = 0$ such that $B'^S(x), B'^N(x)$ are q-bounded while $D'^S(x)$ is q-divergent. Our next result is a characterization of a topological α-torsion element, when the sequence of ratios (q_n) has the α-splitting property.

Theorem 3.4.8. *Let (a_n) be an arithmetic sequence and $x \in [0, 1)$ has canonical representation (3.1). If the sequence of ratios (q_n) has the α-splitting property, then x is a topological α-torsion element i.e. $x \in t^\alpha_{(a_n)}(\mathbb{T})$ if and only if the following conditions hold:*

(i) $B^S(x) + 1 \subseteq^\alpha supp(x)$, $B^S(x) \subseteq^\alpha supp_q(x)$, and if $\overline{d}_\alpha(B^S(x)) > 0$ then $\lim\limits_{n\in B_1^S(x)} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$, where $B_1 \subseteq B$ with $d_\alpha(B \setminus B_1) = 0$.

(ii) If $\overline{d}_\alpha(B^N(x)) > 0$ then $\lim\limits_{n\in B_1^N(x)} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where $B_1 \subseteq B$ with $d_\alpha(B \setminus B_1) = 0$.

(iii) If $\overline{d}_\alpha(D^S(x)) > 0$ then $\lim\limits_{n\in D_1^S(x)} \varphi\left(\frac{c_n}{q_n}\right) = 0$, where $D_1 \subseteq D$ with $d_\alpha(D \setminus D_1) = 0$.

Proof. **Necessity:** Let $x \in t^\alpha_{(a_n)}(\mathbb{T})$. Observe that (a) and (b) of Theorem 3.3.1 hold.

(i) If $d_\alpha(B^S(x)) = 0$, then there is nothing to prove. So, we consider the case when $\overline{d}_\alpha(B^S x) > 0$. Now, there exists a $B' \subseteq \mathbb{N}$ with $d_\alpha(B \triangle B') = 0$ such that B' is q-bounded. Since $d_\alpha(B \triangle B') = 0$, we get $B'^S(x) \subseteq^\alpha supp(x)$. Therefore, taking $A = B'^S(x)$ in Theorem 3.3.1 and applying (a1), we get $B'^S(x) + 1 \subseteq^\alpha supp(x)$, $B'^S(x) \subseteq^\alpha supp_q(x)$ and $\lim\limits_{n\in B_1'^S(x)} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$, where $B'_1 \subseteq B'$ and $d_\alpha(B' \setminus B'_1) = 0$.

Again, since $d_\alpha(B \triangle B') = 0$, we finally get $B^S(x) + 1 \subseteq^\alpha supp(x)$, $B^S(x) \subseteq^\alpha supp_q(x)$ and $\lim_{n \in B_1^S(x)} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$, where $B_1 = (B \cap B_1') \subseteq B$ with $d_\alpha(B \setminus B_1) = 0$.

(ii) Let $\overline{d}_\alpha(B^N(x)) > 0$. Since $d_\alpha(B^N(x) \cap supp(x)) = 0$, applying (a2) of Theorem 3.3.1 to $A = B'^N(x)$, we get $\lim_{n \in B_1^N(x)} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where $B_1 \subseteq B$ and $d_\alpha(B \setminus B_1) = 0$.

(iii) Let $\overline{d}_\alpha(D^S(x)) > 0$. Since there exists $D' \subseteq \mathbb{N}$ with $d_\alpha(D \triangle D') = 0$ such that D' is q-divergent, applying (b) of Theorem 3.3.1 to $A = D'^S(x)$ (As, $d_\alpha(D \triangle D') = 0 \Rightarrow \overline{d}_\alpha(D'^S(x)) = \overline{d}_\alpha(D^S(x)) > 0$), we get $\lim_{n \in D_1^S(x)} \varphi\left(\frac{c_n}{q_n}\right) = 0$ in $\mathbb{T}$, where $D_1 \subseteq D$ and $d_\alpha(D \setminus D_1) = 0$.

Sufficiency: Let the conditions hold. It suffices to show that the conditions of Theorem 3.3.1 hold. If $d_\alpha(supp(x)) = 0$ then there is nothing to prove. So let us assume that $\overline{d}_\alpha(supp(x)) > 0$. Consider any $A \subseteq N$ with $\overline{d}_\alpha(A) > 0$.

(a) First suppose that A is q-bounded.

(a1) Let $A \subseteq^\alpha supp(x)$. Since A is q-bounded, $A \subseteq^\alpha B$. Therefore, $A \subseteq^\alpha B^S(x)$ and we get $\overline{d}_\alpha(B^S(x) > 0$. By (i), we have $B^S(x)+1 \subseteq^\alpha supp(x)$, $B^S(x) \subseteq^\alpha supp_q(x)$ and $\lim_{n \in B_1^S(x)} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$, where $B_1 \subseteq B$ and $d_\alpha(B \setminus B_1) = 0$. Again, since $A \subseteq^\alpha B^S(x)$, we get $A + 1 \subseteq^\alpha supp(x)$, $A \subseteq^\alpha supp_q(x)$ and $\lim_{n \in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$, where $A' = A \cap B_1 \subseteq A$ and $d_\alpha(A \setminus A') = 0$.

(a2) Now, let $d_\alpha(A \cap supp(x)) = 0$. Since A is q-bounded, $A \subseteq^\alpha B$. Therefore, $A \subseteq^\alpha B^N(x)$ and we get $\overline{d}_\alpha(B^N(x)) > 0$. By (ii), we have $\lim_{n \in B_1^N(x)} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where $B_1 \subseteq B$ and $d_\alpha(B \setminus B_1) = 0$. Now, taking $A' = A \cap B_1$, we get $\lim_{n \in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where $A' \subseteq A$ and $d_\alpha(A \setminus A') = 0$ (Since, $d_\alpha(A \setminus A') = d_\alpha(A \setminus B_1) = d_\alpha(A \setminus B) = 0$).

(b) Let us now assume that A is q-divergent. Then we have $A \subseteq^\alpha D$. From Remark 3.3.3, without any loss of generality we can assume that $A \subseteq supp(x)$. Therefore, $A \subseteq^\alpha D^S(x)$ and we get $\overline{d}_\alpha(D^S(x)) > 0$. By (iii), we have $\lim_{n \in D_1^S(x)} \varphi\left(\frac{c_n}{q_n}\right) = 0$, where $D_1 \subseteq D$ and $d_\alpha(D \setminus D_1) = 0$. Now, taking $A' = A \cap D_1$, we get $\lim_{n \in A'} \varphi\left(\frac{c_n}{q_n}\right) = 0$, where $A' \subseteq A$ and $d_\alpha(A \setminus A') = 0$ (since, $d_\alpha(A \setminus A') = d_\alpha(A \setminus D_1) = d_\alpha(A \setminus D) = 0$). Therefore, from theorem 3.3.1, we can conclude that $x \in t^\alpha_{(a_n)}(\mathbb{T})$.

□

In particular, one can obtain simpler characterizations of topological α-torsion elements when $supp(x)$ is either q-bounded or q-divergent for the given arithmetic sequence.

Corollary 3.4.9. *If $supp(x)$ is q-bounded, then $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$ if and only if the following statements hold:*

(i) $d_{\alpha}((supp(x)+1) \setminus supp(x)) = 0$, *and*

(ii) $d_{\alpha}(supp(x) \setminus supp_q(x)) = 0$.

Proof. Let $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$. If $d_{\alpha}(supp(x)) = 0$ then there is nothing to prove. So, assume that $\overline{d}_{\alpha}(supp(x)) > 0$. Since $supp(x)$ is q-bounded and $\overline{d}_{\alpha}(supp(x)) > 0$, we set $A = supp(x)$. Therefore from item (a1) of Theorem 3.3.1, we get $supp(x) + 1 \subseteq^{\alpha} supp(x)$ and $supp(x) \subseteq^{\alpha} supp_q(x)$. Thus, we have (i) $d_{\alpha}((supp(x)+1) \setminus supp(x)) = 0$, and (ii) $d_{\alpha}(supp(x) \setminus supp_q(x)) = 0$.

In order to prove the sufficiency of the conditions, if possible, suppose that there is a $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$ for which (i) does not hold i.e $\overline{d}_{\alpha}((supp(x)+1) \setminus supp(x)) > 0$. Since, $\overline{d}_{\alpha}(supp(x)+1) = \overline{d}_{\alpha}(supp(x))$, we must have $\overline{d}_{\alpha}(supp(x)) > 0$. Now, taking $A = supp(x)$ and applying item (a1) of Theorem 3.3.1, we get $A + 1 \subseteq^{\alpha} supp(x)$ i.e. $d_{\alpha}(supp(x) + 1 \setminus supp(x)) = 0$ − which is a contradiction. Therefore (i) holds true.

Now, let us consider that (ii) does not hold i.e. $\overline{d}_{\alpha}(supp(x) \setminus supp_q(x)) > 0$ but $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$. Set $A = supp(x) \setminus supp_q(x)$ and $q = 1 + \max_{n \in supp(x)} \{q_n\}$. Consequently, from equation (3.8), we obtain

$$\frac{1}{q} < \lim_{n\in A} \frac{c_n}{q_n} \leq \lim_{n\in A} \{a_{n-1}x\} < \lim_{n\in A} \frac{c_n+1}{q_n} \leq \lim_{n\in A} \frac{q_n-1}{q_n} = 1 - \lim_{n\in A} \frac{1}{q_n} < 1 - \frac{1}{q}$$

$$\Rightarrow \lim_{n\in A} \{a_{n-1}x\} \neq 0 \text{ in } \mathbb{T} \text{ for some } \overline{d}_{\alpha}(A) > 0$$

− Which is a contradiction. Therefore (ii) holds true. □

Corollary 3.4.10. *If $supp(x)$ is q-divergent, then $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$ if and only if the following statements hold:*

(i) $\lim_{n\in D'} \varphi\left(\frac{c_n}{q_n}\right) = 0$ *for some* $D' \subseteq supp(x)$ *with* $d_{\alpha}(supp(x) \setminus D') = 0$*; and*

(ii) For every $D \subseteq^{\alpha} supp(x)$ *such that* $D - 1$ *is q-bounded,* $\lim_{n\in D'} \frac{c_n}{q_n} = 0$ *in* $\mathbb{R}$*, where* $D' \subseteq D$ *and* $d_{\alpha}(D \setminus D') = 0$.

Proof. First, let $x \in t^{\alpha}_{(a_n)}(\mathbb{T})$. If $d_{\alpha}(supp(x)) = 0$, then there is nothing to prove. So, let us assume that $\overline{d}_{\alpha}(supp(x)) > 0$. Now, taking $A = supp(x)$ and applying item (b) of Theorem 3.3.1, we can conclude that (i) holds true. Next let us suppose that $A = D - 1$ is q-bounded for some $D \subseteq supp(x)$. If $d_{\alpha}(A) = d_{\alpha}(D) = 0$, then there is nothing to prove. Therefore, we can assume $\overline{d}_{\alpha}(A) > 0$. Since, $supp(x)$ is q-divergent, we have $d_{\alpha}(A \cap supp(x)) = 0$. Now, applying item (a2) of Theorem 3.3.1, we have $\lim_{n\in A'} \frac{c_{n+1}}{q_{n+1}} = 0$

in $\mathbb{R}$ for some $A' \subseteq A$ with $d_\alpha(A \setminus A') = 0$. Putting $D' = A' + 1$, we get $\lim\limits_{n \in D'} \frac{c_n}{q_n} = 0$ in $\mathbb{R}$, where $D' \subseteq D$ and $d_\alpha(D \setminus D') = 0$.

Conversely, let us assume that the conditions hold. To prove that $x \in t^\alpha_{(a_n)}(\mathbb{T})$, we need to show (a) and (b) of Theorem 3.3.1 hold. Since (b) follows from (i), it is sufficient to show only (a). If $d_\alpha(supp(x)) = 0$, then $x \in t^\alpha_{(a_n)}(\mathbb{T})$. So, assume that $\overline{d}_\alpha(supp(x)) > 0$. Now, take any $A \subseteq N$ with $\overline{d}_\alpha(A) > 0$. If A is q-bounded, then $d_\alpha(supp(x) \cap A) = 0$. Therefore, we need to prove only (a2). If $d_\alpha((A+1) \cap supp(x)) = 0$, then taking $A' + 1 = (A+1) \setminus supp(x)$, we get $\lim\limits_{n \in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where $A' \subseteq A$ with $d_\alpha(A \setminus A') = 0$. Now considering the situation when $\overline{d}_\alpha((A+1) \cap supp(x)) > 0$, taking $D = (A+1) \cap supp(x)$ and applying (ii), we get $\lim\limits_{n \in D'} \frac{c_n}{q_n} = 0$ in $\mathbb{R}$ for some $D' \subseteq D$ with $d_\alpha(D \setminus D') = 0$. Thus, putting $A' = D' - 1$ in a similar manner, we obtain that $\lim\limits_{n \in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$ for some $A' \subseteq A$ with $d_\alpha(A \setminus A') = 0$. Therefore, (a2) holds and we finally have $x \in t^\alpha_{(a_n)}(\mathbb{T})$. $\square$

The following observations follow from our main results, giving certain particular cases of an element x of $\mathbb{T}$ being or not being a topologically α-torsion element.

- If $supp(x)$ is q-divergent and $\lim\limits_{n \in A} \frac{c_n}{q_n} = 0$ in $\mathbb{R}$ for some $A \subseteq supp(x)$ with $d_\alpha(supp(x) \setminus A) = 0$, then x is a topologically α-torsion element of $\mathbb{T}$.
- Suppose $x \in [0, 1)$ has canonical representation (3.1) with q-divergent support. If $d_\alpha(supp(x) \setminus \{n \in supp(x) : (c_n) \text{ is bounded}\}) = 0$, then x is a topologically α-torsion element of $\mathbb{T}$.
- Suppose there exists an $A \subseteq \mathbb{N}$ such that A is q-divergent and $d_\alpha(\mathbb{N} \setminus A) = 0$. Then x is a topologically α-torsion element of $\mathbb{T}$ if and only if $\lim\limits_{n \in D'} \varphi\left(\frac{c_n}{q_n}\right) = 0$ for some $D' \subseteq supp(x)$ with $d_\alpha(supp(x) \setminus D') = 0$ (where $D' \subseteq A \cap supp(x)$ as in Corollary 3.4.10).
- Let (a_n) be an arithmetic sequence and $x \in [0, 1)$ be such that
  (i) $supp(x) = \bigcup\limits_{n=1}^{\infty} [p_n, r_n]$, $p_n, r_n \in \mathbb{N}$, $p_n < r_n + 1 < p_{n+1}$ for all $n \in \mathbb{N}$;
  (ii) there exist $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $|r_n - p_n| \leq l$ and $|p_{n+1} - r_n| \leq l$;
  (iii) $supp(x)$ is q-bounded.

  Then x is not a topologically α-torsion element of $\mathbb{T}$.
- Let (a_n) be an arithmetic sequence and $x \in [0, 1)$ be such that
  (i) $\overline{d}_\alpha(supp(x)) > 0$ and $supp(x)$ is q-divergent;
  (ii) for all $n \in supp(x)$, $\frac{c_n}{q_n} \in [r_1, r_2]$, where $0 < r_1, r_2 < 1$.

  Then x is not a topologically α-torsion element of $\mathbb{T}$.

3.5 Conclusion

In this chapter we have described complete characterization of topologically s-torsion as well as topologically α-torsion elements of $\mathbb{T}$ in our main result namely, Theorem 3.3.1. To prove this result several construction has to be done in Section 3.2 among them Lemma 3.2.4 is most powerful which creates the bridge between q-bounded and q-divergent sets. And as an outcome of our main result we are able to solve some open problems regarding α-characterized subgroups considered in [19]. Finally we have obtained a simpler characterization of topologically s-torsion as well as topologically α-torsion elements of $\mathbb{T}$ for α-splitting sequences in Theorem 3.4.8.

Chapter 4

f^g-CHARACTERIZED SUBGROUPS OF THE CIRCLE*

4.1 Introduction

In this chapter we consider a more unified approach to the recent development of characterized subgroups by considering the notion of moduler simple density function d_g^f. As mentioned already, all the notions of density functions, precisely, natural density d [24], natural density with respect to an unbounded modulus function f, f-density d^f [2], natural density d_α of order α [13] and their generalization with respect to weight function g, d_g [6] are special cases of "f density of weight g", i.e., d_g^f [18]. Consequently all the results presented in the articles [38] and [19] become special cases of our results that are presented in this chapter. To get an overview on the general development of the density function d_g^f we refer to the Section 0.1 of Preface.

One can naturally think of the following general notion of convergence corresponding to the density function d_g^f.

Definition 4.1.1. *A sequence of real numbers (x_n) is said to converge to a real number x_0 f^g-statistically if for any $\varepsilon > 0$, $d_g^f(\{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\}) = 0$.*

As a natural consequence we can introduce our main definition of this section.

Definition 4.1.2. *For a sequence of integers (a_n) the subgroup*

$$t_{(a_n)}^{f,g}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \ f^g\text{-statistically in } \mathbb{T}\} \tag{4.1}$$

of $\mathbb{T}$ is called an f^g-statistically characterized *(shortly,* an f^g-characterized*) (by (a_n))* subgroup *of $\mathbb{T}$.*

Theorem 4.1.3. *For any sequence of integers (a_n), $t_{(a_n)}^{f,g}(\mathbb{T})$ is an $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.*

Proof. As the proof follows the same line of arguments as Theorem A [38], we only provide a brief sketch. It is easy to check that $t_{(a_n)}^{f,g}(\mathbb{T})$ is a subgroup of the circle group

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$\mathbb{T}$. Let us set $U_{n,k} := \left\{x \in \mathbb{T} : \|a_n x\| > \frac{1}{k}\right\}$ for $n, k \in \mathbb{N}$. From Definition 3.2, one can write

$$\begin{aligned} t^{f,g}_{(a_n)}(\mathbb{T}) &= \{x \in \mathbb{T} : (\forall k \in \mathbb{N})\ d^f_g(\{n : x \in U_{n,k}\}) = 0\} \\ &= \bigcap_{k=1}^{\infty} \left\{x \in \mathbb{T} : d^f_g(\{n : x \in U_{n,k}\}) = 0\right\} \\ &= \bigcap_{k=1}^{\infty} \left\{x \in \mathbb{T} : \lim_{m\to\infty} \frac{f(|\{i \in \mathbb{N} : x \in U_{i,k}\} \cap [1,m]|)}{f(g(m))} = 0\right\} \\ &= \bigcap_{k=1}^{\infty} \Bigg\{x \in \mathbb{T} : (\forall j \in \mathbb{N})(\exists m \in \mathbb{N}) \text{ such that} \\ &\qquad\qquad \frac{f(|\{i \in \mathbb{N} : x \in U_{i,k}\} \cap [1,n]|)}{f(g(n))} \leq \frac{1}{j} \text{ for all } n \geq m\Bigg\}. \end{aligned}$$

Subsequently writing

$$V_{k,j,n} = \left\{x \in \mathbb{T} : \frac{f(|\{i \in \mathbb{N} : x \in U_{i,k}\} \cap [1,n]|)}{f(g(n))} \leq \frac{1}{j}\right\}$$

one can show that $V_{k,j,n}$ is closed in $\mathbb{T}$ for every fixed triple k, j and n. The assertion then follows from the equality

$$t^{f,g}_{(a_n)}(\mathbb{T}) = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n\geq m} V_{k,j,n}.$$

□

Clearly the non-triviality of the newly obtained subgroups $t^{f,g}_{(a_n)}(\mathbb{T})$ depends on

(i) whether $t^{f,g}_{(a_n)}(\mathbb{T})$ actually becomes the whole circle group $\mathbb{T}$ and

(ii) whether as subgroups of $\mathbb{T}$, they are really 'new' compared to the already studied characterized subgroups $t_{(a_n)}(\mathbb{T})$ or their versions $t^s_{(a_n)}(\mathbb{T})$ and $t^\alpha_{(a_n)}(\mathbb{T})$.

The study of the first question (i) is easy, as it is known that $t_{(a_n)}(\mathbb{T}) = \mathbb{T}$ precisely when $a_n = 0$ for almost all n [9, 46]. Using this fact one can conclude that $t^{f,g}_{(a_n)}(\mathbb{T}) = \mathbb{T}$ precisely when $d^f_g(\{n : a_n \neq 0\}) = 0$. Since no arithmetic sequence (a_n) satisfies $d^f_g(\{n : a_n \neq 0\}) = 0$, we deduce that $t^{f,g}_{(a_n)}(\mathbb{T}) \neq \mathbb{T}$ for such sequences.

The second question (ii) is far more complicated and seems worth studying. We thoroughly investigate this problem for general arithmetic sequences.

As the general case seem quite complicated, so as in [38] we begin with a special case providing a basic example considering the sequence (2^n) and then step-by step, generalize the idea.

4.2 Main observations

4.2.1 The f^g-characterized subgroup for the sequence $\mathbf{a_n = 2^n}$.

Note that $t_{(2^n)}(\mathbb{T})$ is simply the Prüfer group $\mathbb{Z}(2^\infty)$. So it remains only to check that $t^{f,g}_{(2^n)}(\mathbb{T})$ contains an element x that does not belong to $\mathbb{Z}(2^\infty)$. It is known that $x \in \mathbb{Z}(2^\infty)$ precisely when $supp(x)$ is finite (see [41]). Note that, for $a_n = 2^n$, c_n can only be 0 or 1. Below we take $f(x) = \log(1+x)$, $g(n) = n^{\frac{1}{2}}$ and construct an element which belongs to $t^{f,g}_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$.

Example 4.2.1. *Choose $x \in \mathbb{T}$ with*

$$\text{supp}_{(2^n)}(x) = \bigcup_{n=1}^{\infty} [(2n-1)^{(2n-1)}, (2n)^{(2n)}]. \tag{4.2}$$

We will show that $x \in t^{f,g}_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$. To check that $x \in t^{f,g}_{(2^n)}(\mathbb{T})$, pick an $m \in \mathbb{N}$ and define a subset A of $\mathbb{N}$ as follows:

First let $B_n := [(2n-1)^{(2n-1)}, (2n)^{(2n)}]$. Clearly length of B_n diverges to ∞. Consequently one can choose $n_0 \in \mathbb{N}$ such that $(2n_0)^{(2n_0)} - (2n_0-1)^{(2n_0-1)} > m$. Now let

$$A_0 := [(2n_0)^{2n_0} - m, (2n_0)^{2n_0}] \text{ and } A'_0 := [(2n_0+1)^{(2n_0+1)} - m, (2n_0+1)^{(2n_0+1)}].$$

Similarly, let

$$A_k := [(2(n_0+k))^{(2(n_0+k))} - m, (2(n_0+k))^{(2(n_0+k))}]$$

and

$$A'_k := [(2(n_0+k)+1)^{(2(n_0+k)+1)} - m, (2(n_0+k)+1)^{(2(n_0+k)+1)}].$$

Finally, put $B = \bigcup_{k=0}^{\infty} (A_k \cup A'_k)$ and $A = B \cup [1, (2n_0-1)^{(2n_0-1)}]$. Note that $|A_k| = |A'_k| = m+1$, and so

$$\begin{aligned}
\overline{d}^f_g(A) &= \limsup_{n\to\infty} \frac{f(|A \cap [1,n]|)}{f(g(n))} \\
&= \max\left\{ \lim_{k\to\infty} \frac{f(2(m+1)(k+1))}{f((2(n_0+k)+1)^{\frac{1}{2}\cdot(2(n_0+k)+1)})}, \right. \\
&\qquad \left. \lim_{k\to\infty} \frac{f(2(m+1)(k+1)-(m+1))}{f((2(n_0+k))^{\frac{1}{2}\cdot(2(n_0+k))})} \right\} \\
&= \max\left\{ \lim_{k\to\infty} \frac{\log(1+2(m+1)(k+1))}{\log(1+(2(n_0+k)+1)^{\frac{1}{2}\cdot(2(n_0+k)+1)})}, \right. \\
&\qquad \left. \lim_{k\to\infty} \frac{\log(1+(2(m+1)(k+1)-m))}{\log(1+(2(n_0+k))^{\frac{1}{2}\cdot(2(n_0+k))})} \right\} = 0.
\end{aligned}$$

We claim that $\|2^n x\| < 1/2^m$ *for all* $n \in \mathbb{N} \setminus A$. *As* $n \in \mathbb{N} \setminus A$, *by the choice of* A *and the definition of* $B_n := [(2n-1)^{(2n-1)}, (2n)^{(2n)}]$, *we can deduce that*

(a) either $n \in [(2r)^{(2r)}+1, (2r+1)^{(2r+1)}-m-1]$ *for some* $r \in \mathbb{N}$ *which automatically implies that* $n+1, n+2, \ldots, n+m \in [(2r)^{(2r)}+1, (2r+1)^{(2r+1)}-1]$ *i.e.* $n+1, n+2, \ldots, n+m \notin \mathrm{supp}_{(2^n)}(x)$, *or*

(b) $n \in [(2r+1)^{(2r+1)}+1, (2(r+1))^{(2(r+1))}-m-1]$ *for some* $r \in \mathbb{N}$ *i.e.* $n+1, n+2, \ldots, n+m \in [(2r+1)^{(2r+1)}, (2(r+1))^{(2(r+1))}] \subset \mathrm{supp}_{(2^n)}(x)$.

In both cases we have $c_{n+1} = c_{n+2} = \ldots = c_{n+m}$. *In case (a) this leads to* $c_{n+1} = c_{n+2} = \ldots = c_{n+m} = 0$ *which implies*

$$2^n x = \frac{c_{n+1}}{2} + \frac{c_{n+2}}{2^2} + \ldots + \frac{c_{n+m}}{2^m} + \frac{c_{n+m+1}}{2^{m+1}} + \ldots = \frac{c_{n+m+1}}{2^{m+1}} + \ldots.$$

Therefore $\|2^n x\| < 1/2^m$. *In case (b) this leads to* $c_{n+1} = c_{n+2} = \ldots = c_{n+m} = 1$, *and subsequently*

$$2^n x = \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^m} + \frac{c_{n+m+1}}{2^{m+1}} + \ldots = 1 - \frac{1}{2^m} + \frac{c_{n+m+1}}{2^{m+1}} + \ldots.$$

As a result we can again conclude that $\|2^n x\| < 1/2^m$. *Since* $m \in \mathbb{N}$ *was chosen arbitrarily and* $\mathbb{N} \setminus A \in \mathcal{I}_g^*(f)$, *we obtain that* $(2^n x)$ f^g*-statistically converges to 0 in* $\mathbb{T}$ *i.e.* $x \in t^{f,g}_{(2^n)}(\mathbb{T})$. *According to [41],* $x \notin t_{(2^n)}(\mathbb{T})$ *as* $supp(x)$ *is infinite.*

However, we can actually prove that the newly obtained subgroup $t^{f,g}_{(2^n)}(\mathbb{T})$ contains uncountably more elements compared to $t_{(2^n)}(\mathbb{T})$ as had been observed for $t^s_{(2^n)}(\mathbb{T})$ (Proposition 3.5 [38]). We prove that in Proposition 4.2.4.

Now we are in a position to see that the element $x \in \mathbb{T}$ in Example 4.2.1 can be replaced by a more generally defined element of $\mathbb{T}$ without any restriction on f or g. To explain the choice, we note that for every x as in (4.2) such that $x \notin \mathbb{Z}(2^\infty)$, the support can be presented as a disjoint union of infinitely many consecutive intervals $\bigcup\limits_n B_n$. Let us define

$$\mathbb{I}_g^f = \left\{ \bigcup_{r=1}^{\infty} B_r : B_r = [n_{(2r-1)}, n_{(2r)}], \text{ for some } A = \{n_r\}_{r\in\mathbb{N}} \subset \mathbb{N} \text{ with } d_g^f(A) = 0 \right\}. \tag{4.3}$$

In Example 4.2.1 we used the following specific member of $\mathbb{I}_g^f$

$$B = \bigcup_{r=1}^{\infty} B_r \in \mathbb{I}_g^f, \text{ with } B_r := [(2r-1)^{(2r-1)}, (2r)^{(2r)}]. \tag{4.4}$$

Now we intend to show that $\mathbb{I}_g^f \not\subseteq \mathcal{I}_g(f)$. If possible let us assume that $\mathbb{I}_g^f \subseteq \mathcal{I}_g(f)$ for some unbounded modulus function f and $g \in \mathbb{G}$. Note that for any unbounded modulus function f and for any $g \in \mathbb{G}$, we have $\mathcal{I}_g(f) \neq Fin$. Therefore, we can choose $A = \{n_1 < n_2 < n_3 < \ldots\} \subset \mathbb{N}$ such that $I_A = \bigcup\limits_{r=1}^{\infty} B_r \in \mathbb{I}_g^f \subseteq \mathcal{I}_g(f)$, where

$B_r = [n_{(2r-1)}, n_{(2r)}]$. Then, for $A' = (n_{r+1}) \subseteq A$, we have $I_{A'} = \bigcup_{r=1}^{\infty} B'_r \in \mathbb{I}_g^f \subseteq \mathcal{I}_g(f)$, where $B'_r = [n_{(2r)}, n_{(2r+1)}]$. But this implies that $\mathbb{N} \in \mathcal{I}_g(f)$ which is a contradiction. Our next observation is a result concerning both $\mathbb{I}_g^f$ and $\mathcal{I}_g(f)$ in line of [38, Lemma 3.3] that will be frequently used in the sequel.

Lemma 4.2.2. $|\mathbb{I}_g^f| = |\mathcal{I}_g(f)| = \mathfrak{c}$.

Proof. Fix a specific member $B = \bigcup_{r=1}^{\infty} B_r \in \mathbb{I}_g^f$, e.g., as in (4.4). Fix a sequence $\xi = (z_i) \in \{0,1\}^{\mathbb{N}}$ and define $B^\xi = \bigcup_{k=1}^{\infty} B_{2k+z_k}$. In other words, this subset B^ξ of B is obtained by taking at each stage k either B_{2k} of B_{2k+1} depending on the choice imposed by ξ. As obviously $B^\xi \neq B^\eta$ for distinct $\xi, \eta \in \{0,1\}^{\mathbb{N}}$, this provides an injective map given by

$$\{0,1\}^{\mathbb{N}} \ni \xi \to B^\xi \in \mathbb{I}_g^f,$$

Since $|\{0,1\}^{\mathbb{N}}| = \mathfrak{c}$, we are done.

A similar proof works for $\mathcal{I}_g(f)$. □

Let us note that the element $x \in \mathbb{T}$ in Example 4.2.1 has the property $supp(x) \in \mathbb{I}_g^f$. Now we see that the argument works with any element x of $\mathbb{T}$ with $supp(x) \in \mathbb{I}_g^f$ where f is an unbounded modulus function and $g \in \mathbb{G}$.

Lemma 4.2.3. *Let $x \in \mathbb{T}$ be such that $supp_{(2^n)}(x) \in \mathbb{I}_g^f$. Then $x \in t_{(2^n)}^{f,g}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$.*

Proof. The fact that $x \notin t_{(2^n)}(\mathbb{T})$ follows from the fact that $supp(x) \in \mathbb{I}_g^f$ implies $supp(x)$ is infinite.

We take $\text{supp}_{(2^n)}(x) = \bigcup_{r=1}^{\infty} [n_{(2r-1)}, n_{(2r)}]$, where $A' = (n_r) \in \mathcal{I}_g(f)$. Let us define $B_r = [n_{(2r-1)}, n_{(2r)}]$ and $G_r := [n_{(2r)} + 1, n_{(2r+1)} - 1]$. Consider any $m \in \mathbb{N}$. We choose $r_0 \in \mathbb{N}$ such that $n_{r_0} > m$. Now we define, $A_0 = \{n_r : r \in \mathbb{N} \text{ and } r \geq r_0\}$ and $A_i = \{n_r - i : r \in \mathbb{N} \text{ and } r \geq r_0\} \cap \mathbb{N}$. Consequently

$$\begin{aligned} d_g^f(A_i) &= \lim_{n\to\infty} \frac{f(|(A_0 - i) \cap [1,n]|)}{f(g(n))} \\ &\leq \lim_{n\to\infty} \frac{f(|A_0 \cap [1,n]| + i)}{f(g(n))} \leq d_g^f(A') + \lim_{n\to\infty} \frac{f(i)}{f(g(n))} = 0 \end{aligned}$$

Finally put $A = \bigcup_{i=0}^{m} A_i \cup [1, n_{r_0}]$. We can then show that this A witnesses the needed f^g-statistical convergence with respect to $\varepsilon = 1/2^m$ following the line of the proof of Example 4.2.1. □

Immediately we have the following result.

Proposition 4.2.4. $|t_{(2^n)}^{f,g}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})| = \mathfrak{c}$.

Proof. In Lemma 4.2.3 we have shown that $\{x : supp_{(2^n)}(x) \in \mathbb{I}_g^f\} \subset t_{(2^n)}^{f,g}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$. Now as $|\{x : supp_{(2^n)}(x) \in \mathbb{I}_g^f\}| = |\mathbb{I}_g^f|$, so Lemma 4.2.2 tells us that

$|\{x : supp_{(2^n)}(x) \in \mathbb{I}_g^f\}| = |\mathbb{I}_g^f| = \mathfrak{c}$. That is, $|t_{(2^n)}^{f,g}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})| \geq \mathfrak{c}$ which gives our result.

□

On the basis of the existing knowledge that $t_{(2^n)}(\mathbb{T})$ is countably infinite in size, an obvious but important consequence coming from Proposition 4.2.4 is the following.

Corollary 4.2.5. $|t_{(2^n)}^{f,g}(\mathbb{T})| = \mathfrak{c}$.

Proof. The observation immediately follows as

$$t_{(2^n)}^{f,g}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T}) \subseteq t_{(2^n)}^{f,g}(\mathbb{T}) \subseteq \mathbb{T}.$$

. □

4.2.2 The general case for arithmetic sequences

In this section, we generalize the whole idea of the last section for arbitrary arithmetic sequences and try to generalize Example 4.2.1 and Corollary 4.2.5 in this context.

First we prove a lemma analogous to Lemma 4.2.3 which gives a sufficient condition for some x to be in $t_{(a_n)}^{f,g}(\mathbb{T})$.

Lemma 4.2.6. *Let (a_n) be an arithmetic sequence and let $x \in \mathbb{T}$ be such that $supp(x) \in \mathbb{I}_g^f$ and $c_n = q_n - 1$ for all $n \in supp(x)$. Then $x \in t_{(a_n)}^{f,g}(\mathbb{T})$.*

Proof. Let $x = \sum\limits_{n=1}^{\infty} \frac{c_n}{a_n}$ be the canonical representation of $x \in \mathbb{T}$ where $c_1 = 0$, c_n is either 0 or $(q_n - 1)$ for any $n > 1$ and $\{n : c_n = q_n - 1\} = \bigcup\limits_{r=1}^{\infty} B_r \in \mathbb{I}_g^f$ where as in 4.2.3 $B_r = [n_{(2r-1)}, n_{(2r)}]$ and $G_r := [n_{(2r)} + 1, n_{(2r+1)} - 1]$ for some infinite $B = (n_r) \subseteq \mathbb{N}$ (i.e. the sets B_r and G_r, $r \in \mathbb{N}$ forming a partition of $\mathbb{N}$). To show that $x \in t_{(a_n)}^{f,g}(\mathbb{T})$ we proceed exactly as in Lemma 4.2.3. We take an arbitrary $m \in \mathbb{N}$ and get the same $r_0 \in \mathbb{N}$ and $A \subset \mathbb{N}$ with $d_g^f(A) = 0$ where as before $A = \bigcup\limits_{i=0}^{m} A_i \cup [1, n_{r_0}]$, $A_i = \{n_r - i : r \in \mathbb{N}$ and $r \geq r_0\} \cap \mathbb{N}$. What is required now is to show that $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}\setminus A}} \|a_n x\| = 0$. For $n \in \mathbb{N} \setminus A$, by the choice of A and the definition of B_r and G_r, we deduce that

(a) either $n \in B_r$ for some $r \in \mathbb{N}$, which actually means that $n \in [n_{2r-1} + 1, n_{2r} - m - 1]$ and consequently $n+1, n+2, \ldots, n+m \in B_r$, or

(b) $n \in G_r$ for some $r \in \mathbb{N}$, and by the same reasoning as above we can again conclude that $n+1, n+2, \ldots, n+m \in G_r$.

In case (b) this leads to $c_{n+1} = c_{n+2} = \ldots = c_{n+m} = 0$, and consequently

$$a_n x = \sum_{k=n+1+m}^{\infty} \frac{c_k}{a_k} \cdot a_n \leq \sum_{k=n+1+m}^{\infty} \frac{q_k - 1}{a_k} \cdot a_n = \sum_{k=n+1+m}^{\infty} \left(\frac{1}{a_{k-1}} - \frac{1}{a_k} \right) \cdot a_n \leq \frac{a_n}{a_{m+n}}.$$

In case (a) this leads to $c_k = q_k - 1$ for $k = n+1, n+2, ..., n+m$ which implies that

$$a_n x = \sum_{k=n+1}^{n+m} \frac{q_k - 1}{a_k} \cdot a_n + \sum_{k=n+1+m}^{\infty} \frac{c_k}{a_k} \cdot a_n.$$

Now the first part is

$$\sum_{k=n+1}^{n+m} \frac{q_k - 1}{a_k} \cdot a_n = \sum_{k=n+1}^{n+m} \left(\frac{1}{a_{k-1}} - \frac{1}{a_k} \right) \cdot a_n = 1 - \frac{a_n}{a_{n+m}}$$

and the second part

$$\sum_{k=n+1+m}^{\infty} \frac{c_k}{a_k} . a_n \leq \sum_{k=n+1+m}^{\infty} \frac{q_k - 1}{a_k} \cdot a_n = \sum_{k=n+1+m}^{\infty} \left(\frac{1}{a_{k-1}} - \frac{1}{a_k} \right) \cdot a_n \leq \frac{a_n}{a_{m+n}}.$$

Therefore, we obtain that $\|a_n x\| \leq \frac{a_n}{a_{n+m}} \leq \frac{1}{2^m}$. As $m \in \mathbb{N}$ was chosen arbitrarily, so we conclude that $\|a_n x\|$ converges f^g-statistically to 0. □

Now we are going to provide another, very natural, sufficient condition for $x \in t^{f,g}_{(a_n)}(\mathbb{T})$ in line of (Theorem 4.4 [38]).

Theorem 4.2.7. *Let (a_n) be an arithmetic sequence and $x \in \mathbb{T}$. If $d^f_g(supp(x)) = 0$, then $x \in t^{f,g}_{(a_n)}(\mathbb{T})$.*

Proof. Set $A = supp(x)$. Then $d^f_g(A) = 0$, by hypothesis. Pick a positive $k \in \mathbb{N}$ and note that for any $i \in \mathbb{N}$,

$$\begin{aligned} d^f_g(A - i) = \lim_{n\to\infty} \frac{f(|(A-i) \cap [1,n]|)}{f(g(n))} &\leq \lim_{n\to\infty} \frac{f(|A \cap [1,n]| + i)}{f(g(n))} \\ &\leq d^f_g(A) + \lim_{n\to\infty} \frac{f(i)}{f(g(n))} = 0. \end{aligned}$$

Therefore, $A^* = \bigcup_{i=0}^{k} (A-i) \cap \mathbb{N} \in \mathcal{I}_g(f)$. Hence, it is enough to check that $\|a_n x\| \leq \frac{1}{k}$ for all $n \in \mathbb{N} \backslash A^*$. Note that $n \in \mathbb{N} \backslash A^*$ precisely when $n + i \notin A$ for $i = 0, 1, \ldots k$. This means that in the canonical representation of x one has $c_n = c_{n+1} = \ldots = c_{n+k} = 0$ for all $n \in \mathbb{N} \setminus A^*$. Hence,

$$\begin{aligned} \{a_n x\} = a_n \cdot \sum_{i=n+k+1}^{\infty} \frac{c_i}{a_i} &\leq a_n \cdot \sum_{i=n+k+1}^{\infty} \frac{q_i - 1}{a_i} \\ &= a_n \cdot \sum_{i=n+k+1}^{\infty} \left(\frac{1}{a_{i-1}} - \frac{1}{a_i} \right) \leq \frac{a_n}{a_{n+k}} \leq \frac{1}{2^k} < \frac{1}{k}. \end{aligned}$$

□

We shall invert this theorem in Corollary 4.2.15. Note that the following is the more general version of Theorem B [38].

Theorem 4.2.8. *For any arithmetic sequence* (a_n)*, we have* $|t^{f,g}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.

Proof. Clearly $t^{f,g}_{(a_n)}(\mathbb{T}) \subset \mathbb{T}$ implies $|t^{f,g}_{(a_n)}(\mathbb{T})| \leq |\mathbb{T}| = \mathfrak{c}$.

To prove the inequality $|t^{f,g}_{(a_n)}(\mathbb{T})| \geq |\mathbb{T}| = \mathfrak{c}$ we use two alternative arguments.

Let $B \in \mathbb{I}^f_g$. Define $x_B \in \mathbb{T}$ with $supp(x_B) = B$ and $c_n = q_n - 1$ for all $n \in B$. According to Lemma 4.2.6, $x_B \in t^{f,g}_{(a_n)}(\mathbb{T})$. Since the map $\mathbb{I}^f_g \ni B \mapsto x_B \in t^{f,g}_{(a_n)}(\mathbb{T})$ is obviously injective, $|t^{f,g}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$ by Lemma 4.2.2.

The second argument uses the fact that $|\mathcal{I}_g(f)| = \mathfrak{c}$ as has been shown in Proposition 4.2.2. This provides $\mathfrak{c}$ many elements $\{x_i : i \in I\}$ in $\mathbb{T}$ with distinct supports of f^g-density 0 and as a result applying Theorem 4.2.7, we obtain that $x_i \in t^{f,g}_{(a_n)}(\mathbb{T})$ for every $i \in I$. □

Below we have the more general version of Theorem C [38].

Theorem 4.2.9. $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$ *for any arithmetic sequence* (a_n).

Proof. If (q_n) is bounded then $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$, as $t_{(a_n)}(\mathbb{T})$ is countable.

Therefore, we consider (q_n) is not bounded. Then there exists $B \subset \mathbb{N}$ such that $(q_n)_{n \in B}$ diverges to ∞. Now, in view of Proposition 2.2.2 there exists a $B' \subseteq B$ such that $d^f_g(B') = 0$. So, in addition we can assume that $d^f_g(B) = 0$. Take

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \in \mathbb{T} \text{ with } supp(x) = B \text{ and } c_n = \left\lfloor \frac{q_n}{2} \right\rfloor \text{ for all } n \in B.$$

Then $x \in t^{f,g}_{(a_n)}(\mathbb{T})$ by Theorem 4.2.7, while $x \notin t_{(a_n)}(\mathbb{T})$ (by [41, Theorem 2.3]). This proves $t^{f,g}_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$.

□

It has already been showed that for any arithmetic sequence (a_n), the condition in Theorem 4.2.7 is not necessary for some $x \in \mathbb{T}$ to be in $t^{f,g}_{(a_n)}(\mathbb{T})$ (see Example 4.5 [38]). More precisely we have the following.

Example 4.2.10. We have already shown that $\mathbb{I}^f_g \not\subseteq \mathcal{I}_g(f)$. Therefore, there exists a $B \in \mathbb{I}^f_g$ such that $\overline{d}^f_g(B) > 0$. Let $x = \sum\limits_{n=1}^{\infty} \frac{c_n}{a_n} \in \mathbb{T}$ be such that $c_n = 0$ whenever $n \notin B$ and for all $n \in B$, $c_n = q_n - 1$ as described in Lemma 4.2.6. Then applying Lemma 4.2.6 we can see that $x \in t^{f,g}_{(a_n)}(\mathbb{T})$. Since $d^f_g(supp(x)) \neq 0$, it follows that $supp(x)$ does not satisfy the criteria of Theorem 4.2.7 though $x \in t^{f,g}_{(a_n)}(\mathbb{T})$.

Finally, following [38], we address the natural question as to, for an arithmetic sequence (a_n), which specific elements of $\mathbb{T}$ would not surely belong to $t^{f,g}_{(a_n)}(\mathbb{T})$.

Proposition 4.2.11. *Let* (a_n) *be a q-bounded arithmetic sequence,* f *be an unbounded modulus function and* $g \in \mathbb{G}$*. Consider* $x \in \mathbb{T}$ *be such that*

(i) $supp(x) = \bigcup\limits_{n=1}^{\infty} [l_n, k_n]$*, where* $l_n, k_n \in \mathbb{N}$*,* $l_n \leq k_n < l_{n+1} - 1$ *for all* $n \in \mathbb{N}$*;*

(ii) $\overline{d}_g^f(A) > 0$, *where* $A = \{l_n :\ n \in \mathbb{N}\}$.

Then $x \notin t_{(a_n)}^{f,g}(\mathbb{T})$.

Proof. Let $q_n \leq M$ for some $M \in \mathbb{N} \setminus \{1\}$. We set $B = \{n-2 :\ n \in A\}$. Therefore, for any $n \in B$, we can observe that $n+1 \notin supp(x)$ but $n+2 \in supp(x)$. Here, $\overline{d}_g^f(B) > 0$ by hypothesis. Now, for all $n \in B$, we have

$$\{a_n x\} = a_n \sum_{i=n+1}^{\infty} \frac{c_i}{a_i} = a_n \sum_{i=n+2}^{\infty} \frac{c_i}{a_i} \leq \frac{a_n}{a_{n+1}} = \frac{1}{q_{n+1}} \leq \frac{1}{2}.$$

But, for all $n \in B$, we also have

$$\{a_n x\} = a_n \sum_{i=n+1}^{\infty} \frac{c_i}{a_i} = a_n \sum_{i=n+2}^{\infty} \frac{c_i}{a_i} \geq \frac{a_n}{a_{n+2}} = \frac{1}{q_{n+1}q_{n+2}} \geq \frac{1}{M^2}.$$

Hence, we find a set $B \subseteq \mathbb{N}$ with $\overline{d}_g^f(B) > 0$ such that for all $n \in B$, $\|a_n x\| \in [\frac{1}{M^2}, \frac{1}{2}]$ i.e. $x \notin t_{(a_n)}^{f,g}(\mathbb{T})$. □

Corollary 4.2.12. *Let* (a_n) *be a q-bounded arithmetic sequence,* f *be an unbounded modulus function and* $g \in \mathbb{G}$. *Consider* $x \in \mathbb{T}$ *be such that*

(i) $supp(x) = \bigcup_{n=1}^{\infty} [l_n, k_n]$, *where* $l_n, k_n \in \mathbb{N}$, $l_n \leq k_n < l_{n+1} - 1$ *for all* $n \in \mathbb{N}$*;*

(ii) there exist $m \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$, $|k_n - l_n| \leq m$ *and* $|l_{n+1} - k_n| \leq m$.

Then $x \notin t_{(a_n)}^{f,g}(\mathbb{T})$.

Proof. Let us consider $A = \{l_n :\ n \in \mathbb{N}\}$. If possible we assume that $d_g^f(A) = 0$. We have already seen that for each fixed $i \in \mathbb{N}$, $d_g^f(A_i) = 0$ where $A_i = \{n-i :\ n \in A\} \cap \mathbb{N}$. Since, $|l_{n+1} - l_n| \leq |l_{n+1} - k_n| + |k_n - l_n| \leq 2m$, it follows that $\mathbb{N} = \bigcup_{i=0}^{2m} A_i \in \mathcal{I}_g(f)$. Therefore our assumption was wrong and we finally get $\overline{d}_g^f(A) > 0$. Now, we can observe that x satisfies all the conditions of Proposition 4.2.11. Thus $x \notin t_{(a_n)}^{f,g}(\mathbb{T})$. □

Again consider the following example.

Example 4.2.13. *Consider any unbounded modulus function* f *and* $g \in \mathbb{G}$*.Let,* $x = \frac{1}{p^r - 1}$ *(where* $r \in \mathbb{N} \setminus \{1\}$ *and* p *is any prime) and* $a_n = p^n$. *Take* $l_n = k_n = rn$ *and* $m = r$. *Therefore from Proposition 4.2.11 we obtain,* $x = \sum_{n=1}^{\infty} \frac{1}{p^{mn}} = \frac{1}{p^m - 1} \notin t_{(p^n)}^{f,g}(\mathbb{T})$. *A particular example is* $\frac{1}{8} \notin t_{(3^n)}^{f,g}(\mathbb{T})$.

We will generalize the idea of the above example to construct $x \in \mathbb{T}$ in another way (different from Proposition 3.15) which will lie outside $t_{(a_n)}^{f,g}(\mathbb{T})$.

Proposition 4.2.14. *Let (a_n) be an arithmetic sequence of integers, f be an unbounded modulus function and $g \in \mathbb{G}$. Consider $x \in \mathbb{T}$ with $\overline{d}_g^f(supp(x)) > 0$. If there exists $m_1, m_2 \in \mathbb{R}$ with $0 < m_1 \leq m_2 < \frac{1}{2}$ and $\forall\ n \in supp(x)$, $\frac{c_n}{q_n} \in [m_1, m_2]$, then $x \notin t^{f,g}_{(a_n)}(\mathbb{T})$.*

Proof. Let $x = \sum_{i \in supp(x)} \frac{c_i}{a_i}$ be the canonical representation of x. We define, $B = \{(n-1) \in \mathbb{N} : n \in supp(x)\}$. Since $\overline{d}_g^f(supp(x)) > 0$, we must have $\overline{d}_g^f(B) > 0$.

Now $\forall n \in B$ one has

$$\begin{aligned}\{a_n x\} = a_n \cdot \sum_{\substack{i \in supp(x)\\ i>n}} \frac{c_i}{a_i} \quad &\geq \quad a_n \cdot \sum_{\substack{i \in supp(x)\\ i>n}} \frac{m_1 \cdot \frac{a_i}{a_{i-1}}}{a_i} \;=\; a_n \cdot \sum_{\substack{i \in supp(x)\\ i>n}} \frac{m_1}{a_{i-1}} \\ &= \quad a_n \cdot \sum_{\substack{i \in B\\ i \geq n}} \frac{m_1}{a_i} \;\geq\; a_n \cdot \frac{m_1}{a_n} \;=\; m_1\end{aligned}$$

and

$$\begin{aligned}\{a_n x\} = a_n \cdot \sum_{\substack{i \in supp(x)\\ i>n}} \frac{c_i}{a_i} \quad &\leq \quad a_n \cdot \sum_{\substack{i \in supp(x)\\ i>n}} \frac{m_2 \cdot \frac{a_i}{a_{i-1}}}{a_i} \;=\; a_n \cdot \sum_{\substack{i \in B\\ i \geq n}} \frac{m_2}{a_i} \\ &\leq \quad m_2\left(1 + \frac{a_n}{a_{n+1}} + \frac{a_n}{a_{n+2}} + \ldots\right) \;\leq m_2 \cdot \frac{1}{(1-\frac{1}{2})} \;=\; 2m_2.\end{aligned}$$

Therefore $\forall n \in B, \quad \{a_n x\} \in [m_1, 2m_2] \quad \& \ B \notin \mathcal{I}_g(f)$ which implies $\|a_n x\|$ cannot f^g-statistically converge to 0. Thus $x \notin t^{f,g}_{(a_n)}(\mathbb{T})$. ☐

Corollary 4.2.15. *Let (a_n) be an arithmetic sequence. Then for a subset $B \subseteq \mathbb{N}$ there exists $x \in \mathbb{T}$ with $supp_{(a_n)}(x) \subseteq B$ and $x \notin t^{f,g}_{(a_n)}(\mathbb{T})$ if and only if $\overline{d}_g^f(B) > 0$.*

Proof. The conjunction of $x \notin t^{f,g}_{(a_n)}(\mathbb{T})$ and $supp_{(a_n)}(x) \subseteq B$ implies $\overline{d}_g^f(B) > 0$, by Theorem 4.2.7. On the other hand, if $\overline{d}_g^f(B) > 0$, then Proposition 4.2.14 provides an element $x \in \mathbb{T}$ such that $supp_{(a_n)}(x) \subseteq B$ and $x \notin t^{f,g}_{(a_n)}(\mathbb{T})$. ☐

4.3 Non-triviality of f^g-characterized subgroups

In this section, our main aim is to check whether this newly obtained f^g-characterized subgroups are really new compared to the already studied s-characterized subgroups and α-characterized subgroups.

Theorem 4.3.1. *For any unbounded modulus function f, there exists $g \in \mathbb{G}$ such that $t^{f,g}_{(a_n)}(\mathbb{T}) \subsetneq t^{\alpha}_{(a_n)}(\mathbb{T})$ and $t^{f,g}_{(a_n)}(\mathbb{T}) \subsetneq t^{s}_{(a_n)}(\mathbb{T})$.*

Proof. Write the identity function as f_1 for brevity, take $g(n) = \log(1+n)$ for all $n \in \mathbb{N}$ and $g_1(n) = n^\alpha$ for all $n \in \mathbb{N}$, where $0 < \alpha < 1$. Observe that

$$\lim_{n\to\infty} \frac{f_1(g_1(n))}{f_1(g(n))} = \lim_{n\to\infty} \frac{n^\alpha}{\ln(1+n)} = \lim_{n\to\infty} \alpha n^\alpha \cdot \frac{n+1}{n} = \infty$$

and

$$\lim_{n\to\infty}\frac{f_1(n)}{f_1(g_1(n))}=\lim_{n\to\infty}\frac{n}{n^\alpha}\to\infty.$$

Therefore from Proposition 2.3.3, it follows that $\mathcal{I}_g \subsetneq \mathcal{I}_\alpha$. Now, in view of (Proposition 2.6 [18]), for any unbounded modulus function f we have $\mathcal{I}_g(f)\subseteq\mathcal{I}_g$. Consequently one can choose $A\in\mathcal{I}_\alpha\backslash\mathcal{I}_g(f)$. Clearly this means $\overline{d}_g^f(A)>0$ and from Corollary 4.2.15 it follows that $x\in\mathbb{T}$ with $supp(x)\subseteq A$ satisfies $x\notin t^{f,g}_{(a_n)}(\mathbb{T})$. As $supp(x)\subseteq A\in\mathcal{I}_\alpha$ i.e. $d_\alpha(supp(x))=0$, from Theorem 4.2.7 we can conclude that $x\in t^\alpha_{(a_n)}(\mathbb{T})$. Thus $t^{f,g}_{(a_n)}(\mathbb{T})\subsetneq t^\alpha_{(a_n)}(\mathbb{T})$. Similarly taking $\alpha=1$, and choosing an appropriate support we can show that $t^{f,g}_{(a_n)}(\mathbb{T})\subsetneq t^s_{(a_n)}(\mathbb{T})$. □

Theorem 4.3.2. *There exists an unbounded modulus function f such that for any $g\in\mathbb{G}$, $t^{f,g}_{(a_n)}(\mathbb{T})\neq t^\alpha_{(a_n)}(\mathbb{T})$ and $t^{f,g}_{(a_n)}(\mathbb{T})\neq t^s_{(a_n)}(\mathbb{T})$.*

Proof. We consider $f(x)=\log(1+x)$ and any $g\in\mathbb{G}$. Let $A\subset\mathbb{N}$ be such that $|A\cap[1,n]|=\lfloor n^\beta\rfloor$ (where $0<\beta<\alpha<1$).
Then

$$d_\alpha(A)=\lim_{n\to\infty}\frac{|(A\cap[1,n])|}{n^\alpha}\le\lim_{n\to\infty}\frac{n^\beta}{n^\alpha}=\lim_{n\to\infty}\frac{1}{n^{\alpha-\beta}}=0.$$

Since $\frac{n}{g(n)}\nrightarrow 0$, there exists $(n_k)\subseteq\mathbb{N}$ such that $1+g(n_k)<(1+n_k)^2$. Now, we observe that

$$\frac{f(|(A\cap[1,n_k])|)}{f(g(n_k))}=\frac{\log(1+|(A\cap[1,n_k])|)}{\log(1+g(n_k))}\ge\frac{\log(n_k^\beta)}{\log(1+n_k)^2}\to\frac{\beta}{2}>0.$$

Therefore $\overline{d}_g^f(A)>0$ and again in view of Corollary 4.2.15, $x\in\mathbb{T}$ with $supp(x)\subseteq A$ lies outside $t^{f,g}_{(a_n)}(\mathbb{T})$. On the other hand $supp(x)\subseteq A$ and $d_\alpha(A)=0$ implies $x\in t^\alpha_{(a_n)}(\mathbb{T})\subseteq t^s_{(a_n)}(\mathbb{T})$ in view of Theorem 4.2.7. Hence $t^{f,g}_{(a_n)}(\mathbb{T})\neq t^\alpha_{(a_n)}(\mathbb{T})$ and $t^{f,g}_{(a_n)}(\mathbb{T})\neq t^s_{(a_n)}(\mathbb{T})$.

But we can actually say more. For each $0<\alpha\le 1$, we can choose β from $(0,\alpha)$ in $\mathfrak{c}$ many ways. Therefore, we also have $|t^\alpha_{(a_n)}(\mathbb{T})\setminus t^{f,g}_{(a_n)}(\mathbb{T})|=\mathfrak{c}$. □

Theorem 4.3.3. *For any unbounded modulus function f, there exists $\mathfrak{c}$ many $g\in\mathbb{G}$ such that $t^f_{(a_n)}(\mathbb{T})\subsetneq t^{f,g}_{(a_n)}(\mathbb{T})$.*

Proof. Let f be an unbounded modulus function. From Remark 2.3.6, there exists $\mathfrak{c}$ many $g\in\mathbb{G}$ such that $\mathcal{I}(f)\subsetneq\mathcal{I}_g(f)$. Therefore, there exists $A\subseteq\mathbb{N}$ such that $A\in\mathcal{I}_g(f)\setminus\mathcal{I}(f)$ i.e. $d^f_g(A)=0$ while $\overline{d}^f(A)>0$. The result then follows considering $x\in\mathbb{T}$ with $supp(x)\subseteq A$ from Corollary 4.2.15 and Theorem 4.2.7. □

Finally we have the following observations about the relations between characterized subgroups generated by two modulus functions which provides a broad picture about these subgroups.

• For any two unbounded modulus functions f_1,f_2, there exists a family $\mathbb{G}_0\subseteq\mathbb{G}$ of cardinality $\mathfrak{c}$ such that $t^{f_i,g}_{(a_n)}(\mathbb{T})$ is incomparable with $t^{f_j}_{(a_n)}(\mathbb{T})$ for each $g\in\mathbb{G}_0$ and $i,j\in\{1,2\}$. Also $t^{f_i,g_1}_{(a_n)}(\mathbb{T})$, $t^{f_j,g_2}_{(a_n)}(\mathbb{T})$ are incomparable for $i,j\in\{1,2\}$ and any two distinct $g_1,g_2\in\mathbb{G}_0$.

The result follows from Theorem 2.3.9 following the line of the proof of Theorem 4.3.3.

• For any unbounded modulus function f, there exist $g_1, g_2 \in \mathbb{G}$ such that $t^{f,g_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{f}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g_2}_{(a_n)}(\mathbb{T})$.

The result follows from Proposition 2.3.11 with the proof being analogous to the proof of Theorem 4.3.3.

In the recent article [19] the following open problem was posed:

Problem 4.3.4. *For any arithmetic sequence (a_n) and $0 < \alpha_1 < \alpha_2 < 1$, is $t^{\alpha_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\alpha_2}_{(a_n)}(\mathbb{T})$?*

We end this section with the following result which gives a positive solution to the above problem in the most general form.

Proposition 4.3.5. *For any unbounded modulus function f, if $g_1, g_2 \in \mathbb{G}$ are such that $\frac{f(n)}{f(g_2(n))} \geq a > 0$ and $\frac{f(g_2(n))}{f(g_1(n))} \to \infty$, then $t^{f,g_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g_2}_{(a_n)}(\mathbb{T})$.*

Proof. As from Proposition 2.3.3, it follows that $\mathcal{I}_{g_1}(f) \subsetneq \mathcal{I}_{g_2}(f)$, the rest of the proof can be done following the method of the proof of Theorem 4.3.3. □

4.4 An uncountable tower of Borel subgroups

In this section our primary motivation is the question "whether one can construct a tower of Borel (preferably characterized in certain sense) subgroups between two such groups". For example it is well known in the literature on the circle group $\mathbb{T}$ that there exists an uncountable chain of polishable subgroups, each of size $\mathfrak{c}$ between the Prüfer group $\mathbb{Z}(2^\infty)$ and $\mathbb{T}$ (see [1] for details). Now the s-characterized group $t^s_{(2^n)}(\mathbb{T})$ (which is uncountable and has size $\mathfrak{c}$) properly contains $t_{(2^n)}(\mathbb{T}) = \mathbb{Z}(2^\infty)$ [38]. However this result does not necessarily imply the existence of such a chain between $t_{(2^n)}(\mathbb{T})$ and $t^s_{(2^n)}(\mathbb{T})$. This question was considered in [34] and the following result was established.

Theorem 4.4.1. *[34] There is a family $\{B_\alpha : \alpha \in (0,1)\}$ of Borel subgroups of the circle group $\mathbb{T}$ that satisfies the following properties:*

(1) each B_α is α-characterized by the same sequence $(2^n)_n$;

(2) $B_\alpha \subsetneq B_\beta$ whenever $\alpha < \beta$ for all $\alpha, \beta \in (0,1)$;

(3) $|B_\alpha| = \mathfrak{c}$ for every $\alpha \in (0,1)$;

(4) for every $\alpha \in (0,1)$, the group B_α contains the Prüfer group $\mathbb{Z}(2^\infty)$;

(5) for every $\alpha \in (0,1)$, the group B_α is properly contained in the s-characterized subgroup $t^s_{(2^n)}(\mathbb{T})$;

(6) further $\bigcap\limits_{\alpha\in(0,1)} B_\alpha \supsetneq \mathbb{Z}(2^\infty)$ *and* $\bigcup\limits_{\alpha\in(0,1)} B_\alpha \subsetneq t^s_{(2^n)}(\mathbb{T})$.

It is now natural to ask whether Theorem 4.4.1 holds for any arithmetic sequence or whether a more general version could be established. Before establishing a more general result, we start with an illustrative example taking the modulus function $f(x) = \log(1+x)$ and the arithmetic sequence (2^n).

Proposition 4.4.2. *Consider the modulus function $f(x) = \log(1 + x)$. There exists a family $\mathbb{G}_0 = \{g_\alpha \in \mathbb{G} : \alpha \in (0, 1)\} \subset \mathbb{G}$ such that $t^{f,g_\alpha}_{(2^n)}(\mathbb{T}) \subsetneq t^{f,g_\beta}_{(2^n)}(\mathbb{T})$ for any $\alpha < \beta$.*

Proof. For any $\alpha \in (0, 1)$ we define

$$g_\alpha(n) = e^{r^\alpha} - 1 \text{ where } r \in (0, \infty) \text{ satisfies } n + 1 = e^r.$$

Take $n_1, n_2 \in \mathbb{N}$ and let $n_1 + 1 = e^{r_1}$ and $n_2 + 1 = e^{r_2}$. Now $n_1 = n_2 \Rightarrow \log(1 + n_1) = \log(1+n_2) \Rightarrow r_1 = r_2 \Rightarrow e^{r_1^\alpha} - 1 = e^{r_2^\alpha} - 1 \Rightarrow g_\alpha(n_1) = g_\alpha(n_2)$ i.e. g_α is well defined for each $\alpha \in (0, 1)$. Clearly g_α is non-decreasing, $\lim_{n\to\infty} g_\alpha(n) = \infty$ and $g_\alpha(n) < n$ for all $n \in \mathbb{N}$ and for each $\alpha \in (0, 1)$. Therefore $\lim_{n\to\infty} \frac{n}{g_\alpha(n)} \nrightarrow 0$ and $g_\alpha \in \mathbb{G}$. Now observe that whenever $\alpha < \beta$, we have

$$\lim_{n\to\infty} \frac{f(g_\alpha(n))}{f(g_\beta(n))} = \lim_{r\to\infty} \frac{f(e^{r^\alpha} - 1)}{f(e^{r^\beta} - 1)} = \lim_{r\to\infty} \frac{\log(e^{r^\alpha})}{\log(e^{r^\beta})} = \lim_{r\to\infty} \frac{r^\alpha}{r^\beta} = 0.$$

Thus in view of [18, Lemma 3.4], it follows that $\mathcal{I}_{g_\alpha}(f) \subseteq \mathcal{I}_{g_\beta}(f)$. So we conclude that $t^{f,g_\alpha}_{(2^n)}(\mathbb{T}) \subseteq t^{f,g_\beta}_{(2^n)}(\mathbb{T})$ whenever $\alpha < \beta,\ \alpha, \beta \in (0, 1)$.

Now consider the set $A = \{m \in \mathbb{N} : m = \lfloor e^{(\log(1+n))^{\frac{1}{\gamma}}} \rfloor - 1$ for some $n \in \mathbb{N}$ and $\alpha < \gamma < \beta\}$ and define $x \in \mathbb{T}$ be such that $supp_{(2^n)}(x) = A$. Observe that

$$\begin{aligned} d^f_{g_\alpha}(A) = \lim_{n\to\infty} \frac{f(|A \cap [1, n]|)}{f(g_\alpha(n))} &= \lim_{n\to\infty} \frac{\log(1 + n)}{\log(1 + g_\alpha(\lfloor e^{(\log(1+n))^{\frac{1}{\gamma}}} \rfloor - 1))} \\ &\geq \lim_{n\to\infty} \frac{\log(1 + n)}{\log(1 + e^{(\log(1+n))^{\frac{\alpha}{\gamma}}} - 1)} \\ &= \lim_{n\to\infty} (\log(1 + n))^{(1-\frac{\alpha}{\gamma})} \\ &= \infty \end{aligned}$$

whereas

$$\begin{aligned} d^f_{g_\beta}(A) = \lim_{n\to\infty} \frac{f(|A \cap [1, n]|)}{f(g_\beta(n))} &= \lim_{n\to\infty} \frac{\log(1 + n)}{\log(1 + g_\beta(\lfloor e^{(\log(1+n))^{\frac{1}{\gamma}}} \rfloor - 1))} \\ &\leq \lim_{n\to\infty} \frac{\log(1 + n)}{\log(1 + e^{(\log(\frac{1+n}{2}))^{\frac{\beta}{\gamma}}} - 1)} \text{ (for large } n) \\ &= \lim_{n\to\infty} \frac{\log(1 + n)}{(\log(\frac{1+n}{2}))^{\frac{\beta}{\gamma}}} \\ &= 0. \end{aligned}$$

Since $d^f_{g_\beta}(A) = 0$, from Theorem 4.2.7 it follows that $x \in t^{f,g_\beta}_{(2^n)}(\mathbb{T})$. Now consider $B = ((A - 1) \setminus A) \cap \mathbb{N}$. Observe that $|e^{(\log(1+\overline{n+1}))^{\frac{1}{\gamma}}} - e^{(\log(1+n))^{\frac{1}{\gamma}}}| \to \infty$ implies $(A - 1) \cap A$ is finite. So we must have $d^f_{g_\alpha}(B - 1) = d^f_{g_\alpha}(B) = d^f_{g_\alpha}(A) > 0$. Note that for all $n \in B - 1$, $n + 1 \notin supp_{(2^n)}(x)$ but $n + 2 \in supp_{(2^n)}(x)$. Hence as in one hand we obtain

$$\{2^n x\} = 2^n \cdot \sum_{k=n+1}^{\infty} \frac{c_k}{2^k} = 2^n \cdot \sum_{k=n+2}^{\infty} \frac{c_k}{2^k} \leq 2^n \cdot \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{2^n}{2^{n+1}} = \frac{1}{2},$$

on the other hand we have

$$\{2^n x\} = 2^n \cdot \sum_{k=n+2}^{\infty} \frac{c_k}{2^k} \geq 2^n \frac{1}{2^{n+2}} = \frac{1}{4}.$$

Therefore we find a $B \subseteq \mathbb{N}$ with $d^f_{g_\alpha}(B) > 0$ such that for all $n \in B$ we have $\{a_n x\} \in [\frac{1}{4}, \frac{1}{2}]$ i.e. $x \notin t^{f,g_\alpha}_{(2^n)}(\mathbb{T})$. Consequently we can conclude that $t^{f,g_\alpha}_{(2^n)}(\mathbb{T}) \subsetneq t^{f,g_\beta}_{(2^n)}(\mathbb{T})$ whenever $\alpha < \beta,\ \alpha, \beta \in (0, 1)$. □

Proposition 4.4.3. *For any arithmetic sequence (a_n) and $f \in \mathbb{F}$ and $g_1, g_2 \in \mathbb{G}$ if $\mathcal{I}_{g_1}(f) \subsetneq \mathcal{I}_{g_2}(f)$ then $t^{f,g_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g_2}_{(a_n)}(\mathbb{T})$.*

Proof. Since $\mathcal{I}_{g_1}(f) \subsetneq \mathcal{I}_{g_2}(f)$ it is straight forward that $t^{f,g_1}_{(a_n)}(\mathbb{T}) \subseteq t^{f,g_2}_{(a_n)}(\mathbb{T})$. Consider $A \in \mathcal{I}_{g_2}(f) \setminus \mathcal{I}_{g_1}(f)$. Therefore $\overline{d}^f_{g_1}(A) > 0$ and in view of Corollary 4.2.15 there exists $x \in \mathbb{T}$ with $supp_{(a_n)}(x) \subseteq A$ and $x \notin t^{f,g_1}_{(a_n)}(\mathbb{T})$. Observe that $d^f_{g_2}(supp_{(a_n)}(x)) = 0$. So Theorem 4.2.7 ensures that $x \in t^{f,g_2}_{(a_n)}(\mathbb{T})$. Thus, we conclude that $t^{f,g_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g_2}_{(a_n)}(\mathbb{T})$. □

Theorem 4.4.4. *(cf. Theorem 4.4.1) For each arithmetic sequence (a_n) and for any $f \in F$ there exists a family $\{B_\alpha : \alpha \in (0, 1)\}$ of Borel subgroups of $\mathbb{T}$ such that the following statements hold:*

(i) Each B_α is f^{g_α}-characterized by the same arithmetic sequence (a_n).

(ii) $|B_\alpha| = \mathfrak{c}$ for all $\alpha \in (0, 1)$.

(iii) $B_\alpha \subsetneq B_\beta$ whenever $\alpha < \beta$ for all $\alpha, \beta \in (0, 1)$.

(iv) For every $\alpha \in (0, 1)$, the group B_α properly contains the characterized subgroup $t_{(a_n)}(\mathbb{T})$.

(v) For every $\alpha \in (0, 1)$, the group B_α is properly contained in the f-characterized subgroup $t^f_{(a_n)}(\mathbb{T})$.

(vi) Further $\bigcap_{\alpha \in (0,1)} B_\alpha \supsetneq t_{(a_n)}(\mathbb{T})$ and $\bigcup_{\alpha \in (0,1)} B_\alpha \subsetneq t^f_{(a_n)}(\mathbb{T})$.

Proof. Take $B_\alpha = t^{f,g_\alpha}_{(a_n)}(\mathbb{T})$. Then (i) and (ii) immediately follows from Theorem 4.1.3 and Theorem 4.2.8.

In view of Lemma 2.2.4 and Theorem 2.3.7, there exists a $\mathbb{G}_0 = \{g_\alpha \in \mathbb{G} : \alpha \in (0, 1)\} \subseteq \mathbb{G}$ such that;

(a) $\mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}_{g_\beta}(f)$ for any $\alpha < \beta$.

(b) $\bigcap_{\alpha\in(0,1)} \mathcal{I}_{g_\alpha}(f) \supsetneq Fin.$

(c) $\bigcup_{\alpha\in(0,1)} \mathcal{I}_{g_\alpha}(f) \subsetneq \mathcal{I}(f).$

Note that (iii) follows directly from Proposition 4.4.3. Let $\alpha, \beta \in (0,1)$ and $\alpha < \beta$. Therefore, $t^{f,g_\alpha}_{(a_n)}(\mathbb{T}) \subseteq t^{f,g_\beta}_{(a_n)}(\mathbb{T})$ follows from the fact that $\mathcal{I}_{g_\alpha}(f) \subseteq \mathcal{I}_{g_\beta}(f)$. Now with the help of (a) we can consider an infinite $A_1 \subseteq \mathbb{N}$ such that $A_1 \in \mathcal{I}_{g_\beta}(f) \setminus \mathcal{I}_{g_\alpha}(f)$. Since $\overline{d}^f_{g_\alpha}(A_1) > 0$, from Corollary 4.2.15 there is an $x \in \mathbb{T}$ with $supp(x) \subseteq A_1$ such that $x \notin t^{f,g_\alpha}_{(a_n)}(\mathbb{T})$. But $d^f_{g_\beta}(A_1) = 0$ and in view of Theorem 4.2.7 we get $x \in t^{f,g_\beta}_{(a_n)}(\mathbb{T})$. Thus, we obtain that $t^{f,g_\alpha}_{(a_n)}(\mathbb{T}) \subsetneq t^{f,g_\beta}_{(a_n)}(\mathbb{T})$ i.e. (ii) is satisfied.
Now, in view of (b) and (c) there exists $A_2, A_3 \subseteq \mathbb{N}$ such that $A_2 \in \bigcap_{\alpha\in(0,1)} \mathcal{I}_{g_\alpha}(f) \setminus Fin$ and $A_3 \in \mathcal{I}(f) \setminus \bigcup_{\alpha\in(0,1)} \mathcal{I}_{g_\alpha}(f)$. Then following the line of argument as in (ii), we conclude that (vi) holds. Lastly observe that (iv) and (v) are easy consequence of (vi). □

As a consequence, we obtain the solution of the following open problem.

Problem 4.4.5. *[19, Problem 2.15] For any arithmetic sequence (a_n), is it true that $t_{(a_n)}(\mathbb{T}) \neq \bigcap_\alpha t^\alpha_{(a_n)}(\mathbb{T})$ and $t^s_{(a_n)}(\mathbb{T}) \neq \bigcup_\alpha t^\alpha_{(a_n)}(\mathbb{T})$?*

Note that for the unbounded modulus function $f(x) = x$, $t^f_{(a^n)}(\mathbb{T})$ is nothing but $t^s_{(a^n)}(\mathbb{T})$ and the inverse of f is defined as $h(x) = x$. Therefore in Theorem 2.3.7, the weight function g_α becomes $g_\alpha = (n-1)^\alpha$ and we immediately get $t^{f,g_\alpha}_{(a_n)}(\mathbb{T}) = t^\alpha_{(a_n)}(\mathbb{T})$. The positive solution of Problem 7.2.6 follows as an easy consequence of Theorem 4.4.4 (vi).

4.5 Conclusion

As one can observe that the f^g-characterized subgroups generalize all such notion of generalized characterized subgroups that exists in the literature and the consequence is that, not only the main results of [38] and [19] follow as special cases of our results, namely, Theorem 4.1.3 (extends Theorem A [38]), Theorem 4.2.8 (extending Theorem B [38]) and Theorem 4.2.9 (extending Theorem C [38]), at the same time, the questions about simple density and f-density are resolved. Finally the justification for the investigation is assured by Theorem 4.3.1 and Theorem 4.3.2 (proved in Section 4.3) which shows that for a given arithmetic sequence, we can indeed construct non-trivial Borel subgroups of $\mathbb{T}$ different from $t^s_{(a_n)}(\mathbb{T})$ or $t^\alpha_{(a_n)}(\mathbb{T})$ for suitable choice of modulus function f or the weight function g. Finally we end this chapter with some interesting comparative results about the generated subgroups and a general construction of an uncountable tower of characterized subgroups (Theorem 4.4.4) which subsequently gives the solution of Problem 2.15 [19].

Chapter 5

$\mathcal{I}$-CHARACTERIZED SUBGROUPS OF THE CIRCLE*

5.1 Introduction

As mentioned before, the complete characterization of topologically s-torsion elements and topologically α-torsion elements are described in Theorem 3.3.1 for arithmetic sequences. Our main quest in this section is to find the same, for much more general, topologically $\mathcal{I}$-torsion elements. Our method of proof for the Theorem 5.2.8 follows exactly the same line of arguments used in Theorem 3.3.1 but as the situation for general ideals is much more complex than the cases in Theorem 3.3.1 where basically the natural density ideal of order α were used, so several suitable modifications have to be brought in to tackle the new difficulties that arise. Before moving forward we refer to the Section 2.1 of Chapter 2 and Section 0.2 of Preface for basic definitions connected with ideals on $\mathbb{N}$ (cf. [50]).

In our recent article [36] we have introduced the following ideal version of characterized subgroups to find out why "Eggleston's dichotomy" [47] breaks down for s-characterized subgroups.

Definition 5.1.1. *[36] For a sequence of integers* (a_n)*, the subgroup*

$$t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \textit{ in } \mathbb{T} \textit{ w.r.t } \mathcal{I}\} \tag{5.1}$$

of $\mathbb{T}$ *is called* an $\mathcal{I}$-characterized (*by* (a_n)) subgroup *of* $\mathbb{T}$.

For any two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathbb{N}$, it is easy to note that $\mathcal{I}_1 \subseteq \mathcal{I}_2$ implies $t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T}) \subseteq t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$, where (a_n) is a sequence of integers.

Definition 5.1.2. *Let* (a_n) *be a sequence of integers. An element* $x \in \mathbb{T}$ *is called topologically* $\mathcal{I}$*-torsion element if* $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$.

In this chapter we will primarily be focused on the class of analytic P-ideals as they are instrumental in generating topologically nice subgroups of $\mathbb{T}$ as can be observed from the following result.

*Content of this chapter has been published in "**Ricerche di Matematica**".

Theorem 5.1.3. *[36] For any analytic P-ideal $\mathcal{I}$, $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is a $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.*

Theorem 5.1.4. *[82] Let X be a Polish space. Then for every Borel set B in X there is a finer Polish topology τ_B on X such that B is closed in X with respect to τ_B.*

Remark 5.1.5. *Since $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is a Borel set of $\mathbb{T}$, it is measurable with respect to the Haar measure μ of $\mathbb{T}$. More precisely, $\mu(t^{\mathcal{I}}_{(a_n)}(\mathbb{T})) = 1$ when $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) = \mathbb{T}$. Otherwise, when $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \neq \mathbb{T}$, $\mu(t^{\mathcal{I}}_{(a_n)}(\mathbb{T})) = 0$ since in this case the subgroup $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ will have infinite index (as $\mathbb{T}/t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is divisible as a quotient of the divisible group $\mathbb{T}$) and $\mu(\mathbb{T}) = 1$.*

By Theorem 5.1.3, the subgroup $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is a $F_{\sigma\delta}$ set. In general, it may not be complete with respect to the usual norm $\|.\|$ prevalent in $\mathbb{T}$ as one can see by taking any proper infinite $\mathcal{I}$-characterized subgroup (for example $t^{\mathcal{I}}_{(2^n)}(\mathbb{T})$), where $\mathcal{I}$ is any analytic P-ideal, which is dense, so non-closed, hence cannot be complete. However taking the metric

$$\delta(x, y) = \sup_{n\in\mathbb{N}}\{\|x - y\|, \|a_n(x - y)\|\}.$$

it can be shown as in [38] that $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is closed in $(\mathbb{T}, \delta)$. Further in view of Theorem 5.1.4 we can conclude that for any sequence (a_n) of natural numbers, and analytic P-ideal $\mathcal{I}$, $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is Polishable.

5.2 Main results

Before proceeding to our main results we present below certain basic definitions, notations which will be needed.

Let $\Im$ denote the set of all translation invariant analytic P-ideals over $\mathbb{N}$ except Fin and in this section $\mathcal{I}$ will always stand for a translation invariant analytic P-ideal and (a_n) will always denote an arithmetic sequence unless otherwise stated.

We start with a sufficient condition for an element of $\mathbb{T}$ to be a topologically $\mathcal{I}$-torsion element which will be used later.

Lemma 5.2.1. *For any analytic P-ideal $\mathcal{I}$ and $x \in \mathbb{T}$, if $supp(x) \in \mathcal{I}$ and $supp(x)$ is $\mathcal{I}$-translation invariant, then $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$.*

Proof. We consider $A = supp(x) \in \mathcal{I}$. Pick any $k \in \mathbb{N}$ and define $B = \bigcup_{i=0}^{k}(A - i)$. As A is $\mathcal{I}$-translation invariant, $B \in \mathcal{I}$. It is enough to check that $\|a_n x\| < \frac{1}{k}$ for all $n \in \mathbb{N} \setminus B \in \mathcal{I}^*$. Note that $n \in \mathbb{N} \setminus B$ precisely when $n + i \notin A$ for $i = 0, 1, 2, \ldots, k$. Therefore, in the canonical representation of x one must have

$$c_{n+1} = c_{n+2} = \ldots = c_{n+k} = 0 \quad n \in \mathbb{N} \setminus B.$$

This implies that

$$\{a_n x\} = \sum_{i=n+k+1}^{\infty} \frac{c_i}{a_i} \cdot a_n \quad \leq \quad \sum_{i=n+k+1}^{\infty} \frac{a_i/a_{i-1} - 1}{a_i} \cdot a_n \;\leq\; \frac{a_n}{a_{n+k}} \;\leq\; \frac{1}{2^k} \;<\; \frac{1}{k}.$$

As a consequence, we obtain that

$$\|a_n x\| < \frac{1}{k} \quad \text{for all } \ n \in \mathbb{N} \setminus B.$$

Hence, $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$. □

Our next three results portrait a generalized view of [38, Theorem 4.3, Theorem B, Theorem C] respectively.

Lemma 5.2.2. *For any $\mathcal{I} \in \Im$ and $x \in \mathbb{T}$, if $supp(x) \in \mathcal{I}$ then $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$.*

Proof. Since $\mathcal{I} \in \Im$, every $supp(x) \in \mathcal{I}$ is $\mathcal{I}$-translation invariant. Therefore, Lemma 5.2.1 ensures that $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$. □

Corollary 5.2.3. *For any $\mathcal{I} \in \Im$, $|t^{\mathcal{I}}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.*

Proof. First observe that $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \subset \mathbb{T}$ implies $|t^{\mathcal{I}}_{(a_n)}(\mathbb{T})| \leq |\mathbb{T}| = \mathfrak{c}$.

Fix a specific member $B = (b_n) \in \mathcal{I}$. Also fix a sequence $\xi = (z_n) \in \{0,1\}^{\mathbb{N}}$ and define $B^{\xi} = (b_{2n+z_n})$. As obviously $B^{\xi} \neq B^{\eta}$ for distinct $\xi, \eta \in \{0,1\}^{\mathbb{N}}$ which provides an injective map given by

$$\{0,1\}^{\mathbb{N}} \ni \xi \to B^{\xi} \in \mathcal{I}.$$

Note that $|\{0,1\}^{\mathbb{N}}| = \mathfrak{c}$. Therefore, $|\mathcal{I}| = \mathfrak{c}$. This provides $\mathfrak{c}$ many elements $\{x_i : i \in \Lambda\}$ in $\mathbb{T}$ with distinct supports in $\mathcal{I}$. By Lemma 5.2.2, $x_i \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ for every $i \in \Lambda$. □

Corollary 5.2.4. *For any $\mathcal{I} \in \Im$, $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \supsetneq t_{(a_n)}(\mathbb{T})$.*

Proof. Consider an infinite subset B of $\mathbb{N}$ with $B \in \mathcal{I}$. Let us choose $x \in \mathbb{T}$ such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \text{ with } supp(x) = B \text{ and } c_n = \left\lfloor \frac{q_n}{2} \right\rfloor \text{ for all } n \in B.$$

Then $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ by Lemma 5.2.2, while $x \notin t_{(a_n)}(\mathbb{T})$ by [41, Theorem 2.3]. This proves $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \supsetneq t_{(a_n)}(\mathbb{T})$. □

The following folklore fact about ideal convergence would be frequently used from now on.

Lemma 5.2.5. *(Folklore) For a sequence (x_n) and a P-ideal $\mathcal{I}$, $x_n \to x$ w.r.t $\mathcal{I}$ if and only if for any $A \in \mathcal{I}^+$, there exists an infinite $A' \subset A$ such that $\lim\limits_{n \in A'} x_n = x$.*

Let $\mathcal{I} \in \Im$. Now for any $B \in \mathcal{I}^+$, let $t_{(a_B)}(\mathbb{T}) = \{x \in \mathbb{T} : \lim\limits_{n \in B} a_n x = 0 \text{ in } \mathbb{T}\}$ and $t^{\mathcal{I}}_{(a_B)}(\mathbb{T}) = \{x \in \mathbb{T} : \lim\limits_{n \in B'} a_n x = 0 \text{ in } \mathbb{T} \text{ for some } B' \subseteq_{\mathcal{I}} B\}$. Therefore, for all $B \in \mathcal{I}^+$, we have $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \subseteq t^{\mathcal{I}}_{(a_B)}(\mathbb{T})$.

Our next lemma is a suitable modification of [41, Lemma 2.6] which will play key role in the proof of Theorem 5.2.8.

Lemma 5.2.6. *For $B \in \mathcal{I}^+$ and $x \in t_{(a_{B-1})}(\mathbb{T})$ the following hold:*

i) If $B \subseteq^{\mathcal{I}} supp(x)$ and is q-bounded, then $B \subseteq^{\mathcal{I}} supp_q(x)$ and there exists $B' \subseteq_{\mathcal{I}} B$ such that $\lim_{n\in B'} \{a_{n-1}x\} = 1$ in $\mathbb{R}$.

ii) If $B \cap supp(x) \in \mathcal{I}$, then there exists $B' \subseteq_{\mathcal{I}} B$ such that $\lim_{n\in B'} \{a_{n-1}x\} = 0$ in $\mathbb{R}$.

Proof. i) Let $q = 1+\max_{n\in B}\{q_n\}$ and $B' = B \cap supp(x)$. Since $B' \subseteq B$ and $B \setminus supp(x) \in \mathcal{I}$, we get $B \subseteq_{\mathcal{I}} B'$. Therefore

$$\{a_{n-1}x\} \geq \frac{c_n}{q_n} > \frac{1}{q} \quad \text{for all } \ n \in B' \text{ (Since } c_n \geq 1 \text{ for all } n \in B') .$$

But as $x \in t_{(a_{B-1})}(\mathbb{T})$, we can conclude that $\lim_{n\in B'} \{a_{n-1}x\} = 1$ in $\mathbb{R}$. Consequently

$$\begin{aligned} 1 - \frac{1}{q_n} &< 1 - \frac{1}{q} < \{a_{n-1}x\} = \frac{c_n}{q_n} + \frac{\{a_n x\}}{q_n} \\ &< \frac{c_n + 1}{q_n} \quad \text{for almost all } n \in B' \text{ (From equation (3.8)).} \end{aligned}$$

$$\Rightarrow q_n - 1 < c_n + 1, \text{ i.e., } c_n > q_n - 2 \text{ for almost all } \ n \in B'.$$

Hence, $c_n = q_n - 1$ for almost all $n \in B'$, i.e., $B' \subseteq^* supp_q(x)$, which implies $B \subseteq^{\mathcal{I}} supp_q(x)$.

ii) Let $B' = B \setminus supp(x)$. Observe that $B' \subseteq B$ and $B \setminus B' = B \setminus (B \setminus supp(x)) = B \cap supp(x) \in \mathcal{I}$, i.e., $B' \subseteq_{\mathcal{I}} B$. Now from equation (3.8) we have

$$\{a_{n-1}x\} = 0 + \frac{\{a_n x\}}{q_n} < \frac{1}{2} \quad \text{for all } \ n \in B' \text{ (Since } c_n = 0 \ \forall \, n \in B').$$

Then in view of the fact that $x \in t_{(a_{B-1})}(\mathbb{T})$, we must have $\lim_{n\in B'} \{a_{n-1}x\} = 0$ in $\mathbb{R}$. $\square$

Lemma 5.2.7. *Consider $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}^+$ where A is not q-bounded. If there does not exist any q-bounded subset $A' \subseteq A$ with $A' \in \mathcal{I}^+$ then there exists a q-divergent set $B \subseteq \mathbb{N}$ such that $B \subseteq_{\mathcal{I}} A$.*

Theorem 5.2.8. *(see also [41, Theorem 2.3]) Let $x \in \mathbb{T}$ and $\mathcal{I} \in \Im$. Then x is a topologically $\mathcal{I}$-torsion element (i.e., $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$) if and only if either $supp(x) \in \mathcal{I}$ or if $supp(x) \in \mathcal{I}^+$, then for all $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}^+$ the following holds:*

(a) If A is q-bounded, then:

(a1) If $A \subseteq^{\mathcal{I}} supp(x)$, then $A + 1 \subseteq^{\mathcal{I}} supp(x)$, $A \subseteq^{\mathcal{I}} supp_q(x)$ and there exists $A' \subseteq_{\mathcal{I}} A$ such that $\lim_{n\in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$.

Moreover, if $A + 1$ is q-bounded, then $A + 1 \subseteq^{\mathcal{I}} supp_q(x)$.

(a2) If $A \cap supp(x) \in \mathcal{I}$, then there exists $A' \subseteq_{\mathcal{I}} A$ such that $\lim_{n\in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$.

Moreover, if $A + 1$ is q-bounded, then $(A + 1) \cap supp(x) \in \mathcal{I}$ as well.

(b) If A is q-divergent, then $\lim\limits_{n\in B} \frac{c_n}{q_n} = 0$ in $\mathbb{T}$ for some $B \subseteq_{\mathcal{I}} A$.

Proof. **Necessity**: Suppose $supp(x) \in \mathcal{I}^+$ and $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$. Therefore there exists $M \subseteq \mathbb{N}$ with $M \in \mathcal{I}^*$ such that

$$\lim_{n\in M}\{a_{n-1}x\} = 0 \text{ in } \mathbb{T}. \tag{5.2}$$

Consider any $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}^+$. We take $B = M \cap A$. Then $B \subseteq A$ and $A \setminus B = A \cap (\mathbb{N} \setminus M) \in \mathcal{I}$, i.e., $B \subseteq_{\mathcal{I}} A$. As $B \subseteq M$, from equation (5.2) we get $\lim\limits_{n\in B}\{a_{n-1}x\} = 0$ in $\mathbb{T}$. Consequently, there exists $B \subseteq_{\mathcal{I}} A$ such that $x \in t_{(a_{B-1})}(\mathbb{T})$.

(a) Suppose first that A is q-bounded. The following two cases can arise:

(a1) First, suppose $A \subseteq^{\mathcal{I}} supp(x)$. Then $B \subseteq A$ is q-bounded and $B \subseteq^{\mathcal{I}} supp(x)$. Since, $x \in t_{(a_{B-1})}(\mathbb{T})$ and B is q-bounded, from Lemma 5.2.6 we conclude that $B \subseteq^{\mathcal{I}} supp_q(x)$ and $\lim\limits_{n\in A'}\{a_{n-1}x\} = 1$ in $\mathbb{R}$, where $A' \subseteq_{\mathcal{I}} B$. Subsequently from equation (3.8)

$$\begin{aligned} 1 &= \lim_{n\in A'}\left(\frac{c_n}{q_n} + \frac{\{a_n x\}}{q_n}\right) = \lim_{n\in A'}\left(\frac{q_n - 1 + \{a_n x\}}{q_n}\right) \\ &= \lim_{n\in A'}\left(1 - \frac{1-\{a_n x\}}{q_n}\right) \Rightarrow \lim_{n\in A'}\frac{1-\{a_n x\}}{q_n} = 0. \end{aligned}$$

Hence we have

$$\lim_{n\in A'}\{a_n x\} = 1 \text{ (Since, } A' \subseteq B \text{ is } q\text{-bounded).} \tag{5.3}$$

Now from the definition of canonical representation (3.1), $c_{n+1} \leq q_{n+1} - 1$ for all $n \in \mathbb{N}$. Again from equation (3.8), we have

$$\{a_n x\} = \frac{c_{n+1}}{q_{n+1}} + \frac{\{a_{n+1}x\}}{q_{n+1}} < \frac{c_{n+1}+1}{q_{n+1}} \leq 1.$$

From equation (5.3) it then follows that

$$1 = \lim_{n\in A'}\{a_n x\} \leq \lim_{n\in A'}\frac{c_{n+1}+1}{q_{n+1}} \leq 1, \qquad \text{i.e., } \lim_{n\in A'}\frac{c_{n+1}+1}{q_{n+1}} = 1. \tag{5.4}$$

Note that $q_{n+1} \geq 2$ for all $n \in \mathbb{N}$. From equation (5.4), we can observe that $c_{n+1}+1 > 1$ (i.e., $c_{n+1} \neq 0$) for almost all $n \in A'$. This implies $A'+1 \subseteq^* supp(x)$. Since $B \setminus A' \in \mathcal{I}$, we obtain $B+1 \subseteq^{\mathcal{I}} supp(x)$.

As $B \subseteq_{\mathcal{I}} A$, we must have $A+1 \subseteq^{\mathcal{I}} supp(x)$, $A \subseteq^{\mathcal{I}} supp_q(x)$ and $\lim\limits_{n\in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ for some $A' \subseteq_{\mathcal{I}} A$. If $A+1$ is q-bounded, proceeding as in the first part of the proof we can show that $A+1 \subseteq^{\mathcal{I}} supp_q(x)$.

(a2) Now let $A \cap supp(x) \in \mathcal{I}$. Since $B \subseteq A$, we must have $B \cap supp(x) \in \mathcal{I}$. Then from Lemma 5.2.6, we can conclude that $\lim\limits_{n\in A'}\{a_{n-1}x\} = 0$ in $\mathbb{R}$ for some $A' \subseteq_{\mathcal{I}} B$.

Therefore putting $k = 1$ in equation (3.6) and equation (3.7), we obtain

$$\lim_{n\in A'}(\frac{c_n}{q_n} + \frac{c_{n+1}}{q_nq_{n+1}} + \frac{\{a_{n+1}x\}}{q_nq_{n+1}}) = \lim_{n\in A'}\{a_{n-1}x\} = 0$$

$$\Rightarrow \lim_{n\in A'}\frac{c_{n+1}}{q_nq_{n+1}} = \lim_{n\in A'}\frac{\{a_{n+1}x\}}{q_nq_{n+1}} = 0 \text{ (Since } c_n, \{a_nx\} \geq 0 \text{ and } q_n > 0). \quad (5.5)$$

As $A' \subseteq B$ is q-bounded, equation (5.5) implies that $\lim_{n\in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where $A' \subseteq_{\mathcal{I}} B \subseteq_{\mathcal{I}} A$.

Moreover, if $A + 1$ is q-bounded, then vanishing of the last limit implies that $(A' + 1) \cap supp(x)$ is finite. Thus $(A + 1) \cap supp(x) \in \mathcal{I}$ (Since, $(A + 1) \setminus (A' + 1) \in \mathcal{I}$).

(b) Suppose A is q-divergent, i.e., $\lim_{n\in A} q_n = \infty$. Then from equation (3.8), we have

$$\lim_{n\in B}(\frac{c_n}{q_n} + \frac{\{a_nx\}}{q_n}) = \lim_{n\in B}\{a_{n-1}x\} = 0 \text{ in } \mathbb{T} \text{ for some } B \subseteq_{\mathcal{I}} A$$

$$\Rightarrow \lim_{n\in B}\frac{c_n}{q_n} = 0 \text{ in } \mathbb{T} \ \text{ (Since, } \{a_nx\} < 1 \text{ and } \lim_{n\in B} q_n = \infty).$$

Claim 5.2.9. *Before proving the sufficiency of the conditions, we need to reformulate the necessary conditions in a stronger iterated version. For an infinite subset A of $\mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we define $L_k(A) = \bigcup_{i=0}^{k}(A + i)$. Now putting $k = k + 1$ in equation (3.6), we obtain*

$$\sigma_{n,k+1} = \sigma_{n,k} + \frac{c_{n+k+1}}{q_nq_{n+1}\cdots q_{n+k+1}}. \quad (5.6)$$

Consequently from equation (3.7) and equation (5.6) it follows that

$$\{a_{n-1}x\} = \sigma_{n,k+1} + \frac{\{a_{n+k+1}x\}}{q_nq_{n+1}\cdots q_{n+k+1}} = \sigma_{n,k} + \frac{c_{n+k+1}}{q_nq_{n+1}\cdots q_{n+k+1}} + \frac{\{a_{n+k+1}x\}}{q_nq_{n+1}\cdots q_{n+k+1}} \quad (5.7)$$

$$\Rightarrow \sigma_{n,k} \leq \{a_{n-1}x\} < \sigma_{n,k} + \frac{c_{n+k+1}}{q_nq_{n+1}\cdots q_{n+k+1}} + \frac{1}{2^{(k+2)}}. \quad (5.8)$$

Let $x \in \mathbb{T}$ has canonical representation (3.1) such that (a) and (b) of Theorem 5.2.8 hold. Let $A \subseteq \mathbb{N}$ be q-bounded with $A \in \mathcal{I}^+$. If $L_k(A)$ is q-bounded for some $k \in \mathbb{N} \cup \{0\}$, then the following hold:

(i) If $A \subseteq^{\mathcal{I}} supp(x)$, then $L_k(A) \subseteq^{\mathcal{I}} supp_q(x)$ and $\lim_{n\in A'+k+1} \frac{c_n+1}{q_n} = 1$ in $\mathbb{R}$ for some $A' \subseteq_{\mathcal{I}} A$. Therefore there exists $n_k \in \mathbb{N}$ such that for all $n \in A'$ with $n \geq n_k$,

$$\sigma_{n,k} = 1 - \frac{1}{q_nq_{n+1}\cdots q_{n+k}} \geq 1 - \frac{1}{2^{k+1}}. \quad (5.9)$$

Moreover if $A + k + 1$ is q-divergent, then

$$\lim_{n\in A+k+1}\frac{c_n}{q_n} = \lim_{n\in A}\frac{c_{n+k+1}}{q_{n+k+1}} = 1 \text{ in } \mathbb{R}. \quad (5.10)$$

(ii) If $A \cap supp(x) \in \mathcal{I}$, then $L_k(A) \cap supp(x) \in \mathcal{I}$ and $\lim\limits_{n \in A'} \frac{c_{n+k+1}}{q_{n+k+1}} = 0$ in $\mathbb{R}$ for some $A' \subseteq_{\mathcal{I}} A$.

Sufficiency: If $supp(x) \in \mathcal{I}$, then from Lemma 5.2.1 it readily follows that $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$. So let $supp(x) \in \mathcal{I}^+$ and $supp(x)$ satisfy conditions (a) and (b). To show that $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$, in view of Lemma 5.2.5 it is sufficient to check the convergence criterion: for all $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}^+$, there exists an infinite set $B' \subseteq A$ such that $\lim\limits_{n \in B'} a_{n-1}x = 0$ in $\mathbb{T}$. Indeed without any loss of generality, we can assume that either $A \cap supp(x) \in \mathcal{I}$ or $A \subseteq^{\mathcal{I}} supp(x)$.

Case (i): First let A be q-bounded.

Subcase (i$_a$): Let us first assume that $L_k(A)$ is q-bounded for all $k \in \mathbb{N} \cup \{0\}$. Let $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{2^{k+1}} < \varepsilon$.

$*$ Let $A \subseteq^{\mathcal{I}} supp(x)$. Then from (i) of Claim 5.2.9, $L_k(A) \subseteq^{\mathcal{I}} supp_q(x)$. This implies the existence of $B' \subseteq A$ such that for all $n \in B'$

$$\sigma_{n,k} = 1 - \frac{1}{q_n q_{n+1} \cdots q_{n+k}} \geq 1 - \frac{1}{2^{k+1}} > 1 - \varepsilon$$

$$\Rightarrow\ 1 - \varepsilon < \sigma_{n,k} \leq \{a_{n-1}x\} < 1 \quad \text{for all } \ n \in B' \text{ (From equation (5.8)).}$$

$*$ Let $A \cap supp(x) \in \mathcal{I}$. Then from (ii) of Claim 5.2.9, $L_k(A) \cap supp(x) \in \mathcal{I}$ and $\lim\limits_{n \in B} \frac{c_{n+k+1}}{q_{n+k+1}} = 0$ in $\mathbb{R}$ for some $B \subseteq_{\mathcal{I}} A$. So there exists $B' \subseteq B$ such that $\sigma_{n,k} = 0$ and $\frac{c_{n+k+1}}{q_{n+k+1}} < \varepsilon$ for all $n \in B'$. Therefore from equation (5.8), it follows that

$$\{a_{n-1}x\} < \sigma_{n,k} + \frac{c_{n+k+1}}{q_n q_{n+1} \cdots q_{n+k+1}} + \frac{1}{2^{(k+2)}} < 2\varepsilon \ \text{ for all } \ n \in B'.$$

Thus in both cases, we have $\lim\limits_{n \in B'} \{a_{n-1}x\} = 0$ in $\mathbb{T}$ for some $B' \subseteq A$, as required.

Subcase (i$_b$): We assume that there exists an integer $k \geq 0$ such that $A + k + 1$ is not q-bounded but $A + i$ is q-bounded for all $i = 0, 1, 2, \ldots, k$. If there exists an $A' \subseteq A$ such that $A' \in \mathcal{I}^+$ and $A' + k + 1$ is q-bounded, then without any loss of generality we can start with A' in place of A. So let us consider the case when there does not exist any $A' \subseteq A$ such that $A' \in \mathcal{I}^+$ and $A' + k + 1$ is q-bounded. Therefore from Lemma 5.2.7, there exists $B \subseteq_{\mathcal{I}} A$ such that $B + k + 1$ is q-divergent, i.e., $\lim\limits_{n \in B} q_{n+k+1} = \infty$. Clearly $L_k(B)$ is q-bounded. Further more

$$\lim_{n \in B} \frac{\{a_{n+k+1}x\}}{q_n q_{n+1} \cdots q_{n+k+1}} \leq \lim_{n \in B} \frac{1}{q_{n+k+1}} = 0. \tag{5.11}$$

Subsequently from equation (5.7) and equation (5.11), we obtain

$$\begin{aligned} \lim_{n \in B} \{a_{n-1}x\} &= \lim_{n \in B} \sigma_{n,k} + \lim_{n \in B} \frac{c_{n+k+1}}{q_n q_{n+1} \cdots q_{n+k+1}} + \lim_{n \in B} \frac{\{a_{n+k+1}x\}}{q_n q_{n+1} \cdots q_{n+k+1}} \\ &= \lim_{n \in B} \sigma_{n,k} + \lim_{n \in B} \frac{c_{n+k+1}}{q_n q_{n+1} \cdots q_{n+k+1}}. \end{aligned} \tag{5.12}$$

$*$ Let $A \subseteq^{\mathcal{I}} supp(x)$. Therefore $B \subseteq^{\mathcal{I}} supp(x)$. Consequently in view of equation (5.9) of Claim 5.2.9 and equation (5.12), we get

$$\lim_{n\in B'}\{a_{n-1}x\} = \lim_{n\in B'}(1 - \frac{1}{q_nq_{n+1}\cdots q_{n+k}} + \frac{c_{n+k+1}}{q_nq_{n+1}\cdots q_{n+k+1}})$$
$$= \lim_{n\in B'}(1 + \frac{1}{q_nq_{n+1}\cdots q_{n+k}} \cdot (\frac{c_{n+k+1}}{q_{n+k+1}} - 1)) = 1$$

for some $B' \subseteq_{\mathcal{I}} B$.

$*$ Next let $A \cap supp(x) \in \mathcal{I}$. Then there exists $B \subseteq A$ such that $\sigma_{n,k} = 0$ for all $n \in B$. Subsequently from (ii) of Claim 5.2.9 and equation (5.12), we have

$$\lim_{n\in B'}\{a_{n-1}x\} = \lim_{n\in B'}\frac{c_{n+k+1}}{q_nq_{n+1}\cdots q_{n+k+1}} \le \lim_{n\in B'}\frac{c_{n+k+1}}{q_{n+k+1}} = 0$$

for some $B' \subseteq_{\mathcal{I}} B$. Thus in both cases, we again obtain that $\lim_{n\in B'}\{a_{n-1}x\} = 0$ in $\mathbb{T}$ for some $B' \subseteq A$.

Case (ii): We assume that A is not q-bounded. If there exists $A' \subseteq A$ such that $A' \in \mathcal{I}^+$ and A' is q-bounded then we can proceed as in Case (i) and consider A' in place of A. So, let us assume that there does not exist any $A' \subseteq A$ such that $A' \in \mathcal{I}^+$ and A' is q-bounded. Then from Lemma 5.2.7, there exists $B \subseteq_{\mathcal{I}} A$ such that B is q-divergent, i.e., $\lim_{n\in B} q_n = \infty$. From hypothesis, we have $\lim_{n\in B'}\frac{c_n}{q_n} = 0$ in $\mathbb{T}$ for some $B' \subseteq_{\mathcal{I}} B$. Therefore from equation (3.8), we obtain

$$\lim_{n\in B'}\{a_{n-1}x\} = \lim_{n\in B'}(\frac{c_n}{q_n} + \frac{\{a_nx\}}{q_n}) = 0 \text{ in } \mathbb{T} \text{ (Since } \lim_{n\in B'}\frac{\{a_nx\}}{q_n} < \lim_{n\in B'}\frac{1}{q_n} = 0 \text{).}$$

Hence in all cases, we can conclude that for any $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}^+$, there exists an infinite set $B' \subseteq A$ such that $\lim_{n\in B'}\{a_{n-1}x\} = 0$ in $\mathbb{T}$. This shows that $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$, i.e., x is a topologically $\mathcal{I}$-torsion element of $\mathbb{T}$. □

Remark 5.2.10. *Since for all $n \notin supp(x)$ we have $c_n = 0$, it is sufficient to consider only subsets of $supp(x)$ in item (b) of Theorem 5.2.8.*

The following two observations follow from our main result (i.e., Theorem 5.2.8) giving certain particular cases of an element x of $\mathbb{T}$ being or not being a topologically $\mathcal{I}$-torsion element. These observations play important role while proving the next two corollaries and solve the open problem [19, Problem 2.14.] in the most general form.

Corollary 5.2.11. *If $supp(x)$ is q-bounded, then $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ if and only if the following statements hold:*

(i) $(supp(x) + 1) \setminus supp(x) \in \mathcal{I}$, *and*

(ii) $supp(x) \setminus supp_q(x) \in \mathcal{I}$.

Proof. The proof follows from similar line of arguments as in Corollary 3.4.9 and so is omitted. □

Proposition 5.2.12. *Let $\mathcal{I} \in \mathfrak{I}$ and $x \in \mathbb{T}$ with $supp(x) \in \mathcal{I}^+$. If there exists $m_1, m_2 \in \mathbb{R}$ with $0 < m_1 \leq m_2 < \frac{1}{2}$ and for all $n \in supp(x)$, $\frac{c_n}{q_n} \in [m_1, m_2]$, then $x \notin t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$.*

Proof. The proof follows from similar line of arguments as in [38, Proposition 5.2] and so is omitted. □

Corollary 5.2.13. *For $\mathcal{I} \in \mathfrak{I}$ and a subset $B \subset \mathbb{N}$ there exists $x \in \mathbb{T}$ with $supp(x) \subseteq B$ such that $x \notin t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ if and only if $B \in \mathcal{I}^+$.*

Proof. Let, $x \notin t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ and $supp(x) \subseteq B$. Then, from Lemma 5.2.2 we obtain that $B \in \mathcal{I}^+$.

We consider $B = \{n_1 < n_2 < \ldots n_k < \ldots\} \in \mathcal{I}^+$. If B is q-bounded then take $x \in \mathbb{T}$ such that $supp(x) = C \subseteq B$ where $C + 1 \cap C = \emptyset$ and $C \in \mathcal{I}^+$ (In particular one can consider $C = (n_{2k})$ or $C = (n_{2k-1})$). Therefore, $C + 1 \setminus C = C + 1 \in \mathcal{I}^+$ and in view of Corollary 5.2.11 we get $x \notin t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$.

Now let us consider B is not q-bounded. If there exists an $A \subseteq B$ such that A is q-bounded and $A \in \mathcal{I}^+$ then by previous argument we are done here. So, we assume that there does not exist any $A \subseteq B$ such that A is q-bounded and $A \in \mathcal{I}^+$. Therefore, in view of Lemma 5.2.7 there exists $C \subseteq_{\mathcal{I}} B$ such that C is q-divergent. Consider $x \in \mathbb{T}$ such that $supp(x) = C$ and $c_n = \lfloor \frac{q_n}{3} \rfloor$. Sine $C \in \mathcal{I}^+$, in view of Proposition 5.2.12 we obtain that $x \notin t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$. □

Corollary 5.2.14. *For any two $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{I}$, if $\mathcal{I}_1 \subsetneq \mathcal{I}_2$ then $t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$.*

Proof. Note that $t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T}) \subseteq t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$ is obvious. Let $B \in \mathcal{I}_2 \setminus \mathcal{I}_1$. Therefore, from Corollary 5.2.13 there exists an $x \in \mathbb{T}$ with $supp(x) \subseteq B$ such that $x \notin t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T})$. But observe that $supp(x) \subseteq B \in \mathcal{I}_2$. Therefore, in view of Lemma 5.2.2 we obtain that $x \in t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$. Thus $t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$. □

5.3 $\mathcal{I}$-splitting sequence

In this section we follow the same line of investigations described in [41, Section 3.2] and show that in certain circumstances, one can obtain more simplified equivalent criterions for the topologically $\mathcal{I}$-torsion elements. Note that Definition 5.3.1, Proposition 5.3.2 and Theorem 5.3.6 are the counter parts of [41, Definition 3.10, Proposition 3.11, Corollary 3.13] respectively.

Definition 5.3.1. *We say that a sequence (q_n) of natural numbers has the $\mathcal{I}$-splitting property if there exists a partition $\mathbb{N} = B \cup D$ such that the following statements hold:*

(a) B and D are either empty or $B, D \in \mathcal{I}^+$.

(b) If $B \in \mathcal{I}^+$, then there exists $B' \subseteq \mathbb{N}$ with $B =^{\mathcal{I}} B'$ such that B' is q-bounded.

(c) If $D \in \mathcal{I}^+$, then there exists $D' \subseteq \mathbb{N}$ with $D =^{\mathcal{I}} D'$ such that D' is q-divergent.

Here, B and D witness the $\mathcal{I}$-splitting property for (q_n). Note that, if $B_1 \cup D_1$ is another partition of $\mathbb{N}$, witnessing the $\mathcal{I}$-splitting property for (q_n), then $B_1 =^{\mathcal{I}} B$ and $D_1 =^{\mathcal{I}} D$.

Proposition 5.3.2. *A sequence (q_n) has the $\mathcal{I}$-splitting property if and only if there exists a natural number M such that $\{n \in \mathbb{N} : \ q_n \in [M, m]\} \in \mathcal{I}$ for every $m > M$.*

Proof. We assume that (q_n) has the $\mathcal{I}$-splitting property. Now two cases can arise:

* At first, we consider $B = \emptyset$. Then there exists a $D' \subseteq_{\mathcal{I}} \mathbb{N}$ such that D' is q-divergent. Take any $m \in \mathbb{N}$. Since D' is q-divergent, there exists $n_m \in \mathbb{N}$ such that $q_n > m$ for all $n > n_m$ and $n \in D'$. We set $M = 1$. Then it is evident that for all $m > M$

$$\begin{aligned} \{n \in \mathbb{N} : q_n \in [M, m]\} &\subseteq \{n \in D' : q_n \in [M, m]\} \cup \mathbb{N} \setminus D' \qquad (5.13) \\ &= \{n \in D' : n \leq n_m\} \cup \mathbb{N} \setminus D' \in \mathcal{I}. \end{aligned}$$

* Let $B \neq \emptyset$. Then we have $B \in \mathcal{I}^+$ and consequently there exists a $B' \subseteq \mathbb{N}$ with $B \triangle B' \in \mathcal{I}$ such that B' is q-bounded. In this case, we set $M = 1 + \max_{n \in B'}\{q_n\}$. Therefore for any $m > M$, we obtain

$$\begin{aligned} & \{n \in \mathbb{N} : q_n \in [M, m]\} \\ \subseteq \ & \{n \in B' : q_n \in [M, m]\} \cup (B \setminus B') \cup \{n \in D' : q_n \in [M, m]\} \cup (D \setminus D') \\ = \ & \{n \in D' : q_n \in [M, m]\} \cup (B \setminus B') \cup (D \setminus D') \in \mathcal{I} \text{ (From equation (5.13))}. \end{aligned}$$

Conversely, let there exist a natural number M such that $\{n \in \mathbb{N} : q_n \in [M, m]\} \in \mathcal{I}$ for all $m > M$. We set $B' = \{n \in \mathbb{N} : q_n \in [1, M-1]\}$ and $D' = \mathbb{N} \setminus B'$.

* If $B' \in \mathcal{I}^+$ and $D' \in \mathcal{I}$, then we take $B = \mathbb{N}$ and $D = \emptyset$.
* If $B' \in \mathcal{I}$ and $D' \in \mathcal{I}^+$, then we take $D = \mathbb{N}$ and $B = \emptyset$.
* If $B' \in \mathcal{I}^+$ and $D' \in \mathcal{I}^+$, then we take $B = B'$ and $D = D'$.

Clearly, B and D witness the $\mathcal{I}$-splitting property for the sequence (q_n). □

Now we present below another equivalent condition for a sequence to be $\mathcal{I}$-splitting (or in other words, equivalent formulation of Proposition 5.3.2).

Proposition 5.3.3. *Let (q_n) be a sequence of natural numbers. For all $i \in \mathbb{N}$, we define $A_i = \{n : q_n = i\}$. Then (q_n) is an $\mathcal{I}$-splitting sequence if and only if there does not exist a subsequence $(A_{n_k})_{k \in \mathbb{N}}$ of (A_n) such that $A_{n_k} \in \mathcal{I}^+$ for all $k \in \mathbb{N}$.*

From Proposition 5.3.2, it is obvious that every splitting sequence is an $\mathcal{I}$-splitting sequence. However the converse is not necessarily true, nor it is true that every subset of $\mathbb{N}$ has the $\mathcal{I}$-splitting property (an example not having splitting property was given in [41, Example 3.12] but one must take into consideration that a non-splitting sequence can still be $\mathcal{I}$-splitting).

Example 5.3.4. *Consider an infinite $A = \{n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq \mathbb{N}$ such that $A \in \mathcal{I}$. Now let us define $A_1 = \{r \in \mathbb{N} : r \in [0, n_1 - 1]\} \cup \{n_k : k \in \mathbb{N}\}$, $A_2 = \{n_k + 1 : k \in \mathbb{N}\} \setminus A_1, \ldots, A_{i+1} = \{n_k + i : k \in \mathbb{N}\} \setminus \bigcup_{j=1}^{i} A_j$. Take any $r \in \mathbb{N}$.*

One can find a unique $k \in \mathbb{N}$ such that $n_k \leq r < n_{k+1}$. So we can write $r = n_k + i$ for some $i \in \mathbb{N} \cup \{0\}$, i.e., $n \in A_{i+1}$. Therefore, $\mathbb{N} = \bigcup_{i=1}^{\infty} A_i$, i.e., $(A_i)_{i \in \mathbb{N}}$ forms a partition of $\mathbb{N}$.

For each $i \in \mathbb{N}$, we now define $q_n = i+1$ for all $n \in A_i$. Clearly, for $m, M \in \mathbb{N}$ and $m > M$, we have $\{n \in \mathbb{N} : q_n \in [M, m]\} = \bigcup_{i=M-1}^{m-1} A_i$. Since $A_i \in \mathcal{I}$ for all $i \in \mathbb{N}$, we get $\{n : q_n \in [M, m]\} \in \mathcal{I}$ for all $m, M \in \mathbb{N}$ and $m > M$. Therefore, from Proposition 5.3.2, (q_n) is an $\mathcal{I}$-splitting sequence. But observe that $\{n : q_n \in [M, m]\}$ cannot be finite for any $m, M \in \mathbb{N}$ and $m > M$ (since A_i is infinite for all $i \in \mathbb{N}$). Therefore from [41] (q_n) is not a splitting sequence.

Example 5.3.5. *Let us define $q_n = \{i \in \mathbb{N} : n = 2^{i-2}(2k-1)$ for some $k \in \mathbb{N}\}$. Let $A_i = \{n \in \mathbb{N} : q_n = i\}$. If there exists an $i_0 \in \mathbb{N} \setminus \{1\}$ such that $A_{i_0} \in \mathcal{I}$ then observe that $\mathbb{N} = \bigcup_{r=0}^{2^{(i_0-1)}-1} A_{i_0} - r \in \mathcal{I}$ which is a contradiction. Therefore it is evident that $A_i \in \mathcal{I}^+$ for all $i \in \mathbb{N} \setminus \{1\}$ and $\mathbb{N} = \bigcup_{i=2}^{\infty} A_i$. Now for any $m, M \in \mathbb{N}$ with $m > M$, observe that $\{n \in \mathbb{N} : q_n \in [M, m]\} = \bigcup_{i=M}^{m} A_i \in \mathcal{I}^+$. Consequently from Proposition 5.3.2, it is evident that (q_n) is not an $\mathcal{I}$-splitting sequence.*

For the next result we use the following notations. Consider $x \in \mathbb{T}$ with canonical representation (3.1). Assume that the sequence of ratios (q_n) has the $\mathcal{I}$-splitting property which means that there exists a partition $\mathbb{N} = B \cup D$ such that $(a), (b)$ and (c) of Definition 5.3.1 hold. Note that if $B, D \neq \emptyset$, then there exists $B' \subseteq \mathbb{N}$ with $B \triangle B' \in \mathcal{I}$ and $D' \subseteq \mathbb{N}$ with $D \triangle D' \in \mathcal{I}$ such that $B'^S(x), B'^N(x)$ are q-bounded while $D'^S(x)$ is q-divergent. Our next result is a characterization of a topologically $\mathcal{I}$-torsion element, when the sequence of ratios (q_n) has the $\mathcal{I}$-splitting property.

Theorem 5.3.6. *Let $x \in \mathbb{T}$ has canonical representation (3.1). If the sequence of ratios (q_n) has the $\mathcal{I}$-splitting property, then x is a topologically $\mathcal{I}$-torsion element, i.e., $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ if and only if the following conditions hold:*

(i) $B^S(x) + 1 \subseteq^{\mathcal{I}} supp(x)$, $B^S(x) \subseteq^{\mathcal{I}} supp_q(x)$, and if $B^S(x) \in \mathcal{I}^+$ then $\lim_{n \in B_1^S(x)} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$, where $B_1 \subseteq_{\mathcal{I}} B$.

(ii) If $B^N(x) \in \mathcal{I}^+$, then $\lim_{n \in B_1^N(x)} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$, where $B_1 \subseteq_{\mathcal{I}} B$.

(iii) If $D^S(x) \in \mathcal{I}^+$, then $\lim_{n \in D_1^S(x)} \frac{c_n}{q_n} = 0$ in $\mathbb{T}$, where $D_1 \subseteq_{\mathcal{I}} D$.

Proof. The proof follows similar line of arguments as in Theorem 3.4.8 with suitable modifications and so is omitted. □

5.4 Conclusion

In this chapter we provide a complete characterization of topologically $\mathcal{I}$-torsion elements of $\mathbb{T}$ (Theorem 5.2.8) for a general arithmetic sequence and for a fairly large class of ideals, namely, all translation invariant analytic P-ideals. In particular Theorem 5.2.8 answers the open problem [38, Problem 6.10.] which is to give a characterization of the elements of the subgroup $t^s_{(a_n)}(\mathbb{T})$ only in terms of the support. After that our Corollary 5.2.14 shows that for any arithmetic sequence (a_n) and $0 < \alpha_1 < \alpha_2 < 1$ we must have $t^{\alpha_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\alpha_2}_{(a_n)}(\mathbb{T})$ which solves Problem 2.14. posed in [19] in the most general form. Finally in the last section we provide a simplified version of this characterization (i.e., Theorem 5.2.8).

Part III

APPLICATIONS AND OPEN QUESTIONS

Chapter 6

STATISTICAL ARBAULT SETS*

6.1 Introduction and the background

Our principal interest in this chapter is in trigonometric thin sets. A series of the form

$$\frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos 2\pi nx + d_n \sin 2\pi nx$$

is called a trigonometric series where $c_{n-1}, d_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. In [27], G. Cantor had shown the first uniqueness result:

If a trigonometric series converges to zero for all $x \in [0, 1]$ then all its coefficients get vanished.

After the initial progress, several modifications and generalizations have been done in this direction (see [72, 88] and also the survey articles [22, 71] for a general view) and historically trigonometric thin sets came into play while trying to understand the seemingly "bad" sets beyond which absolute convergence happens, precisely the families of exceptional sets considered in trigonometric series theory or Fourier analysis.

Of particular interest have been the so called "thin sets" like Arbault sets, Dirichlet sets, **N**-sets (also called a set of absolute convergence) which have been extensively investigated in the literature. One can think of an axiomatic approach to define trigonometric thin sets in general parlance. Following [22] we define a family $\mathcal{F}$ of subsets of $\mathbb{T}$ to be a family of thin sets if the next conditions hold:

(a) For each $x \in \mathbb{T}$, $\{x\} \in \mathcal{F}$,

(b) if $Y \subseteq X \in \mathcal{F}$ then $Y \in \mathcal{F}$,

(c) $\mathcal{F}$ does not contain any open interval.

From the family $\mathcal{F}$ we may construct a new family $\mathcal{F}_\sigma$ by

$$\mathcal{F}_\sigma = \{ \bigcup_{n=1}^{\infty} X_n : X_n \in \mathcal{F} \}.$$

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The typical small subsets of the unit interval $[0, 1]$ are the meager sets (i.e. the sets of the first Baire category), negligible sets (i.e. the sets of Lebesgue measure zero) and the porous sets (for a complete description see [89]). Since these sets are going to play some interesting roll in this article, we denote

$$\begin{aligned}\mathcal{M} &= \{X \subseteq [0,1] : X \text{ is meager}\},\\ \mathcal{L} &= \{X \subseteq [0,1] : X \text{ is negligible}\},\\ \mathcal{P} &= \{X \subseteq [0,1] : X \text{ is porous}\}.\end{aligned}$$

A set $X \subseteq [0, 1]$ is called an $\mathbf{H}$-set if there exists an increasing (by the word increasing it would always mean strictly increasing in this chapter) sequence of integers (k_n) and an interval I such that $k_n X \cap I = \emptyset$. We denote the family of all $\mathbf{H}$-sets by $\mathcal{H}$. In [22] the authors established that $\mathcal{H}_\sigma \subseteq \mathcal{M} \cap \mathcal{L}$. Also in [89], it was shown that $\mathcal{H} \subseteq \mathcal{P}$. Combining these results one can conclude that

$$\mathcal{H}_\sigma \subseteq \mathcal{M} \cap \mathcal{L} \cap \mathcal{P}_\sigma. \tag{6.1}$$

A subfamily $\mathcal{G} \subseteq \mathcal{F}$ is called a basis for $\mathcal{F}$ if for any $X \in \mathcal{F}$ there exists $Y \in \mathcal{G}$ such that $X \subseteq Y$. If the basis $\mathcal{G}$ consists of F_σ-sets, Borel sets, etc., then the basis is called F_σ basis, Borel basis, etc. It is well known that the families $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{P}_\sigma$ have a F_σ basis, a G_δ basis and a $G_{\delta\sigma}$ basis respectively.

The arithmetic difference of two sets $A, B \subseteq \mathbb{T}$ is defined as

$$A - B = \{z \in \mathbb{T} : z = x - y \text{ for some } x \in A,\ y \in B\}.$$

A family of thin sets $\mathcal{F}$ is called trigonometric if for any $A \in \mathcal{F}$, $A - A \in \mathcal{F}$.

An infinite sequence (a_n) (or, an infinite subset of $\mathbb{N}$) is called lacunary if

$$a_{n+1} - a_n \to \infty \ \forall\, n \in \mathbb{N}.$$

Note that any lacunary subset of $\mathbb{N}$ has natural density zero. For a real x it is well known that

$$2\|x\| \leq |\sin \pi x| \leq \pi\|x\|.$$

As is customary we will use the inequality by replacing $|\sin \pi x|$ and $\|x\|$ one by another, wherever required to serve our purpose. Let us now recall some definitions of classical thin sets have been of much interest (some equivalent definitions can be found in [22, 23, 49, 62]) and some of them will significantly influence our discoveries.

A set $X \subseteq [0, 1]$ is called

(1) a $\mathbf{D}$-set (Dirichlet set) if there exists an increasing sequence of naturals (a_n) such that $\|a_n x\|$ converges uniformly to 0 on X.

(2) an $\mathbf{A}$-set (Arbault set) if there exists an increasing sequence of naturals (a_n) such that $\|a_n x\|$ converges pointwise to 0 on X.

(3) an $\mathbf{R}$-set if there exists a sequence of naturals (a_n) and a sequence of reals (r_n) such that $r_n \nrightarrow 0$ and $\sum_{n=1}^{\infty} r_n \|a_n x\| < \infty$ for all $x \in X$.

(4) an $\mathbf{N}$-set if there exists a sequence of naturals (a_n) and a sequence of non-negative reals (r_n) such that $\sum_{n=1}^{\infty} r_n = \infty$ and $\sum_{n=1}^{\infty} r_n \|a_n x\| < \infty$ for all $x \in X$.

(5) an $\mathbf{N_0}$-set if there exists a sequence of naturals (a_n) such that $\sum_{n=1}^{\infty} \|a_n x\| < \infty$ for all $x \in X$.

(6) a $\mathbf{wD}$-set if X is universally measurable and for every positive Borel measure μ on $[0,1]$ there exists an increasing sequence of naturals (a_n) such that

$$\lim_{n\to\infty} \int_X |e^{2\pi i a_n x} - 1| d\mu(x) = 0.$$

The family of all $\mathbf{D}$-sets, $\mathbf{N}$-sets, $\mathbf{N_0}$-sets, $\mathbf{A}$-sets, $\mathbf{R}$-sets, $\mathbf{wD}$-sets will be denoted by $\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$, $\mathcal{R}$, $w\mathcal{D}$ respectively. It is known that $\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$ are the typical example of a family of trigonometric thin sets with a Borel basis (see [3]). Now we present some well known results about these classical families of trigonometric thin sets (for exact references and proofs we refer to the survey paper [22], see also [3, 12, 23, 63, 64, 75]):

$$
\begin{aligned}
&(R1) \quad \mathcal{D} \subsetneq \mathcal{N}_0 \subsetneq \mathcal{A} = \mathcal{R} \subsetneq w\mathcal{D},\\
&(R2) \quad \mathcal{D} \subsetneq \mathcal{N}_0 \subsetneq \mathcal{N} \subsetneq w\mathcal{D},\\
&(R3) \quad \mathcal{A} \subsetneq \mathcal{H}_\sigma \subsetneq \mathcal{P}_\sigma,\\
&(R4) \quad \mathcal{N} \nsubseteq \mathcal{P}_\sigma,\\
&(R5) \quad \mathcal{A} \nsubseteq \mathcal{N}_\sigma.
\end{aligned}
$$

Our main interest in this chapter is a class of trigonometric thin sets formed by statistically characterized subgroups as basis, which we call, statistical Arbault sets that does not seem to have been studied before. The first importance of this class is that it forms a new class of thin sets properly containing the class of classical Arbault sets. The other advantage is that unlike the class of Arbault sets, it is much bigger and actually contains a large subfamily of $\mathbf{N}$-sets, in particular, containing types of $\mathbf{N}$-sets which have been extensively used in Fourier analysis. In this particular respect this new class seems much more beneficial than the class of Arbault sets which is only known to contain $\mathbf{N_0}$-sets.

6.2 Background and the novelty of s-characterized subgroups

In [38] while developing the theory of s-characterized subgroups it was established that for any arithmetic sequence (a_n), the s-characterized subgroup $t^s_{(a_n)}(\mathbb{T})$ is always of size $\mathfrak{c}$ and strictly larger than the corresponding characterized subgroup $t_{(a_n)}(\mathbb{T})$ (even when $t_{(a_n)}(\mathbb{T})$ is uncountable). However this does not imply that using the notion of s-characterized subgroups one would actually obtain a new subgroup which can never be generated as in Definition 1.2.2, i.e., it can't be characterized by any sequence of naturals. So the natural question is whether there exists a s-characterized subgroup which is unequal with every possible characterized subgroup. The primary aim of this section is to fill that gap and show that we indeed can generate "new" subgroups following the process of [38].

The essence of the next lemma is to build a bridge between an arbitrary arithmetic sequence and an increasing sequence of naturals and subsequently it will play a very prominent role in the development of the main result of this section.

Lemma 6.2.1. *Let (u_n) be an arithmetic sequence and (a_n) be an increasing sequence of naturals. If $G = \{\frac{1}{u_n} : n \in \mathbb{N}\} \subseteq t_{(a_n)}(\mathbb{T})$ then a_n must be of the form $u_{k_n} v_n$ where $k_n \to \infty$ and q_{k_n+1} does not divide v_n for any $n \in \mathbb{N}$.*

Proof. Note that a_n can always be written in the form $u_{k_n} v_n$ (taking $u_{k_n} = u_1 = 1$ and $v_n = a_n$ for all $n \in \mathbb{N}$). So if we can show that (k_n) has no bounded subsequence then we are done.

If possible assume that there exists a subsequence (k_{n_i}) of (k_n) such that $k_{n_i} \leq m_0$ for all $i \in \mathbb{N}$. Without any loss of generality we can consider $a_{n_i} = u_{m_0} v_{n_i}$. Since $u_{n+1} = q_{n+1} u_n$, the condition q_{k_n+1} does not divide v_n for all $n \in \mathbb{N}$ is necessary to obtain a unique representation of a_n.

Now consider the element $x = \frac{1}{u_{m_0+1}} \in G$. Then observe that

$$a_{n_i} x = u_{m_0} v_{n_i} \frac{1}{u_{m_0+1}} = \frac{v_{n_i}}{q_{m_0+1}}.$$

Since $k_{n_i} = m_0$ for all $i \in \mathbb{N}$ (as assumed), we must have

$$q_{m_0+1} \nmid v_{n_i} \text{ for all } i \in \mathbb{N}$$

$$\Rightarrow \ \|a_{n_i} x\| \geq \frac{1}{q_{m_0+1}} \ \text{ for all } i \in \mathbb{N}.$$

This shows that $x \notin t_{(a_n)}(\mathbb{T})$ −which is a contradiction as $G \subseteq t_{(a_n)}(\mathbb{T})$ by our assumption. Thus we can conclude that $k_n \to \infty$. □

Corollary 6.2.2. *Let $G = \{\frac{1}{p^n} : n \in \mathbb{N}\}$ and (a_n) be an increasing sequence of naturals. If $G \subseteq t_{(a_n)}(\mathbb{T})$ then a_n must be of the form $p^{k_n} v_n$ where $k_n \to \infty$ and $p \nmid v_n$.*

We are now ready to prove the main result of this section, namely, Theorem 6.2.3. The importance of Theorem 6.2.3 is that we have been able to establish that every member of the class of s-characterized subgroups considered in [38] is essentially new i.e.

they can never be "characterized" by a sequence of integers. This observation not only vindicates the new approach taken in [38] but at the same time points to the possible existence of a new class of thin sets, not studied in the literature before. To be more precise, if one extends the notion of Arbault sets by replacing usual convergence by statistical convergence, which we will call "statistical Arbault sets" (officially given in the next section in Definition 6.3.1) then Theorem 6.2.3 can be reformulated as "the statistical Arbault set defined by any arithmetic sequence are not an Arbault set".

Theorem 6.2.3. *For any arithmetic sequence (u_n), the subgroup $t^s_{(u_n)}(\mathbb{T})$ is not an* **A**-*set.*

Proof. Let (u_n) be an arithmetic sequence. If possible assume that there exists an increasing sequence of naturals (a_n) such that

$$t^s_{(u_n)}(\mathbb{T}) \subseteq t_{(a_n)}(\mathbb{T}).$$

Observe that $t^s_{(u_n)}(\mathbb{T})$ contains the set $G = \{\frac{1}{u_n} : n \in \mathbb{N}\}$. From Lemma 6.2.1 it then follows that a_n must be of the form $u_{k_n} v_n$ where $k_n \to \infty$ and q_{k_n+1} does not divide v_n for all $n \in \mathbb{N}$.

Now, we choose a subsequence (a_{n_i}) of (a_n) (with $a_{n_1} = a_1$) which satisfies the following properties:

(i) $|k_{n_{(i+1)}} - k_{n_i}| \to \infty$,

(ii) $u_{k_{n_{(i+1)}}} \geq 4a_{n_i}$.

Let us define an element $x \in \mathbb{T}$ such that

$$\text{supp}_{(u_n)}(x) = \{k_{n_i} + 1 \ : \ i \in \mathbb{N}\} \ \text{ and } c_r = \left\lfloor \frac{q_r}{m_r} \right\rfloor \ \text{ for all } r \in \text{supp}_{(u_n)}(x). \tag{6.2}$$

(where $1 < m_r \leq q_r$). Then observe that

$$\begin{aligned} a_{n_i}x &\equiv_{\mathbb{Z}} v_{n_i}u_{k_{n_i}} \sum_{r=k_{n_i}+1}^{\infty} \frac{c_r}{u_r} \\ &\equiv_{\mathbb{Z}} v_{n_i}c_{k_{n_i}+1}\frac{u_{k_{n_i}}}{u_{k_{n_i}+1}} + v_{n_i}u_{k_{n_i}} \sum_{r=k_{n_{(i+1)}}+1}^{\infty} \frac{c_r}{u_r} \\ \Rightarrow a_{n_i}x &\equiv_{\mathbb{Z}} c_{k_{n_i}+1}\frac{v_{n_i}}{q_{k_{n_i}+1}} + a_{n_i} \sum_{r=k_{n_{(i+1)}}+1}^{\infty} \frac{c_r}{u_r}. \end{aligned}$$

Since $q_{k_{n_i}+1} \nmid v_{n_i}$ for all $i \in \mathbb{N}$, we must have

$$\|\frac{v_{n_i}}{q_{k_{n_i}+1}}\| = \frac{l_i}{q_{k_{n_i}+1}} \text{ for some } l_i \in \{1, 2, \ldots, \lfloor\frac{q_{k_{n_i}+1}}{2}\rfloor\}. \tag{6.3}$$

Set $m_{k_{n_i+1}} = 2l_i$. Then from Eq (6.2), we can write

$$c_{k_{n_i+1}} = \frac{q_{k_{n_i+1}} - e_i}{2l_i} = p_i \text{ (say) for some } e_i \in \{1, 2, \ldots, 2l_i - 1\}.$$

Clearly, $p_i \in \mathbb{N}$ and in view of Eq (6.3), we conclude that

$$\|c_{k_{n_i+1}} \frac{v_{n_i}}{q_{k_{n_i+1}}}\| = \|\frac{p_i l_i}{e_i + 2p_i l_i}\|$$

$$\Rightarrow \frac{1}{4} \leq \left\{c_{k_{n_i+1}} \frac{v_{n_i}}{q_{k_{n_i+1}}}\right\} \leq \frac{3}{4}.$$

From Property (ii), we also have

$$0 \leq a_{n_i} \sum_{r=k_{n_{(i+1)}}+1}^{\infty} \frac{c_r}{u_r} \leq \frac{a_{n_i}}{u_{k_{n_{(i+1)}}}} \leq \frac{1}{8}$$

$$\Rightarrow \frac{1}{4} \leq \{a_{n_i} x\} \leq \left\{c_{k_{n_i+1}} \frac{v_{n_i}}{q_{k_{n_i+1}}}\right\} + \left\{v_{n_i} u_{k_{n_i}} \sum_{r=k_{n_{(i+1)}}+1}^{\infty} \frac{c_r}{u_r}\right\} \leq \frac{7}{8}.$$

This shows that $x \notin t_{(a_n)}(\mathbb{T})$. As we also have $d(\text{supp}_{(u_n)}(x)) = 0$ so Theorem B [38] ensures that $x \in t^s_{(u_n)}(\mathbb{T})$ −which is a contradiction. Therefore we must have $t^s_{(u_n)}(\mathbb{T}) \nsubseteq t_{(a_n)}(\mathbb{T})$. Since the collection of all characterized subgroups form a basis of the family $\mathcal{A}$, we conclude that $t^s_{(u_n)}(\mathbb{T})$ is not an **A**-set. □

6.3 Statistical Arbault sets and basic properties of the family $s\mathcal{A}$

Now we are in a position to introduce the notion of a statistical Arbault set (in short **sA**-set) which is our prime interest in this chapter.

Definition 6.3.1. *A set $X \subseteq [0, 1]$ is called a statistical Arbault set (**sA**-set in short) if there exists an increasing sequence of naturals (a_n) such that $\|a_n x\|$ converges to 0 statistically for all $x \in X$.*

Throughout, the family of all **sA**-sets will be denoted by $s\mathcal{A}$. In this section we will primarily investigate certain basic properties of the family $s\mathcal{A}$ in line of the existing observations regarding the family $\mathcal{A}$.

We start with the known observation that the collection of all characterized subgroups form a $F_{\sigma\delta}$ basis of the family $\mathcal{A}$. Along the same line we have the following.

Proposition 6.3.2. *The family $s\mathcal{A}$ has an $F_{\sigma\delta}$ basis consisting of some subgroups of $\mathbb{T}$.*

Proof. Let $\mathcal{G}$ denote the family of all s-characterized subgroups of the circle i.e.

$$\mathcal{G} = \{t^s_{(a_n)}(\mathbb{T}) : \ (a_n) \text{ is an increasing sequence of naturals}\}.$$

Since every member $t^s_{(a_n)}(\mathbb{T})$ of the family $\mathcal{G}$ is an **sA**-set by definition of an s-characterized subgroup, it is obvious that $\mathcal{G}$ is a subfamily of $s\mathcal{A}$.

Now consider any $A \in s\mathcal{A}$. Then there exists an increasing sequence of naturals (b_n) such that $\|b_n x\| \to 0$ statistically for all $x \in A$. Therefore, $A \subseteq t^s_{(b_n)}(\mathbb{T})$ and we conclude that $\mathcal{G}$ is a basis for the family $s\mathcal{A}$. In [38] the authors have shown that every s-characterized subgroup is an $F_{\sigma\delta}$ subset of $\mathbb{T}$. Thus $\mathcal{G}$ is a $F_{\sigma\delta}$ basis for the family $s\mathcal{A}$. □

Proposition 6.3.3. *The family $s\mathcal{A}$ is a family of trigonometric thin sets.*

Proof. Consider any $x \in \mathbb{T}$. Since every countable set is an Arbault set (so a statistical Arbault set as well) by the main theorem of [17], we conclude that $\{x\} \in s\mathcal{A}$. If $A \subseteq B$ and $B \in s\mathcal{A}$, it is easy to observe that $A \in s\mathcal{A}$.

Take any $A \in s\mathcal{A}$. Then from Proposition 6.3.2 there exists a s-characterized subgroup H such that $A \subseteq H$. It is known that $\mu(H) = 0$ (see [38]) from which one can conclude that the family $s\mathcal{A}$ cannot contain an open interval. Also observe that

$$A - A \subseteq H - H = H \in s\mathcal{A}.$$

Thus $s\mathcal{A}$ is a family of trigonometric thin sets. □

Proposition 6.3.4. *If $A \in s\mathcal{A}$ and G is a subgroup of $\mathbb{T}$ generated by A, then $G \in s\mathcal{A}$.*

Proof. Let $A \in s\mathcal{A}$ and G be the subgroup of $\mathbb{T}$ generated by A. In view of Proposition 6.3.2 there exists a s-characterized subgroup $t^s_{(a_n)}(\mathbb{T})$ of $\mathbb{T}$ containing A. But from the definition of G, we must have $A \subseteq G \subseteq t^s_{(a_n)}(\mathbb{T})$. Since the family $s\mathcal{A}$ is a family of thin sets, we conclude that $G \in s\mathcal{A}$. □

Lemma 6.3.5. *[21] Let $\mathcal{F}$ be a family of trigonometric thin sets such that $\mathcal{D} \subseteq \mathcal{F}$. Then any base of $\mathcal{F}$ has cardinality at least $\mathfrak{c}$.*

Proposition 6.3.6. *For the family $s\mathcal{A}$, the following hold:*

(i) It cannot have a F_σ basis,

(ii) Every basis of $s\mathcal{A}$ has cardinality at least $\mathfrak{c}$.

Proof. (i) Since usual convergence implies statistical convergence it is easy to observe that every **A**-set is an **sA**-set i.e. $\mathcal{A} \subseteq s\mathcal{A}$. Note that $\mathcal{A}$ does not have a F_σ basis (for example the characterized subgroup $t_{2^{2^n}}(\mathbb{T})$ which is clearly an **A**-set, cannot be contained in a F_σ subset of $\mathbb{T}$ [3]). Therefore, the family $s\mathcal{A}$ cannot have a F_σ basis.

(ii) Since $\mathcal{D} \subseteq \mathcal{A} \subseteq s\mathcal{A}$, the result follows directly from Lemma 6.3.5 . □

Lemma 6.3.7. *[21] Let $\mathcal{F}$ be a family of trigonometric thin sets. Then every member of $\mathcal{F}$ is meager and has Lebesgue measure zero.*

Corollary 6.3.8. $s\mathcal{A} \subseteq \mathcal{M} \cap \mathcal{L}$.

Proposition 6.3.9. *The family $s\mathcal{A}$ is not an ideal.*

Proof. There are two perfect Dirichlet sets A, B such that $A + B = \mathbb{T}$. Note that $A, B \in s\mathcal{A}$. Now observe that if $A \cup B \in s\mathcal{A}$ then Proposition 6.3.4 ensures that $A + B \in s\mathcal{A}$; which is a contradiction. Thus the family $s\mathcal{A}$ is not an ideal. □

6.4 Inclusions between $s\mathcal{A}$ and other trigonometric thin families

In this section we will compare $s\mathcal{A}$ with classical trigonometric thin families. The following diagrams show the relationships between the class $s\mathcal{A}$ with other classes of trigonometric thin sets which we will discover in this section. Here an arrow indicates a proper inclusion and a crossed arrow indicates a non-inclusion.

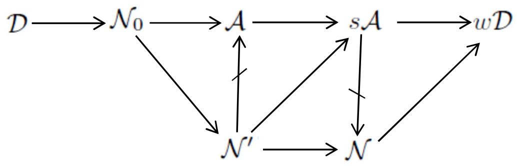

Figure-1

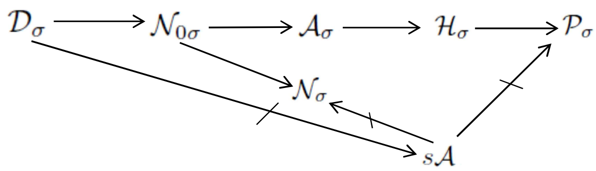

Figure-2

Our next two theorems ensure that this family $s\mathcal{A}$ is really new compared to already investigated classical trigonometric thin families (such as $\mathcal{D}, \mathcal{N}_0, \mathcal{A}, \mathcal{N}$ etc.) and provide a clear view regarding the families $s\mathcal{A}$, $\mathcal{A}$ and $\mathcal{N}$.

Theorem 6.4.1. $s\mathcal{A} \cap \mathcal{N} \nsubseteq \mathcal{A}$.

Proof. Let us define

$$A = \{x \in \mathbb{T} : d(\text{supp}_{(p^n)}(x)) = 0 \text{ and } \sum \frac{1}{n} |\sin p^n \pi x| < \infty\}.$$

Considering the sequence (r_n) where $r_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, from definition it follows that A is a **N**-set. In view of Theorem 4.3 [38], we also have $A \subseteq t^s_{(p^n)}(\mathbb{T})$. Consequently in view of Proposition 6.3.2, we have $A \in s\mathcal{A} \cap \mathcal{N}$.

Now it is left to be shown that $A \notin \mathcal{A}$. Since the class of characterized subgroups forms a basis of $\mathcal{A}$ [8] so if possible assume that $A \subseteq t_{(a_n)}(\mathbb{T})$ for some sequence of naturals (a_n). It is easy to observe that A contains all the p-adic numbers of certain rank. Then Corollary 6.2.2 ensures that the sequence (a_n) will be of the form $a_n = p^{k_n} u_n$ where $k_n \to \infty$ and $p \nmid u_n$. We are going to show that we can always construct an element $x \in A$ depending on the sequence (a_n) such that $x \notin t_{(a_n)}(\mathbb{T})$.

Consider an element x of $\mathbb{T}$ such that

$$\text{supp}_{(p^n)}(x) = \{k_{n_i} + 1 : \; n_1 = 1 \text{ and } k_{n_i} \geq k_{n_{(i-1)}} + (2n_{(i-1)} + 1)u_{n_{(i-1)}}\}$$

and $c_r(x) = 1$ for all $r \in \text{supp}_{(p^n)}(x)$ i.e. $x = \sum\limits_{r \in \text{supp}_{(p^n)}(x)} \frac{1}{p^r}$.

Since $(k_{n_{(i+1)}} - k_{n_i}) \geq (2n_i+1)u_{n_i}$ we have $B = (k_{n_i})$ is lacunary and consequently $d(\text{supp}_{(p^n)}(x)) = d(B) = 0$. Let j be any integer such that $k_{n_{(i-1)}} < j \leq k_{n_i}$. Note that

$$\{p^j x\} = p^j \sum_{r=i}^{\infty} \frac{1}{p^{k_{n_r}+1}} \leq \frac{1}{p^{k_{n_i}-j}}$$

$$\Rightarrow |\sin p^j \pi x| \leq \sin \frac{\pi}{p^{k_{n_i}-j}} \leq \frac{\pi}{p^{k_{n_i}-j}}$$

$$\Rightarrow \sum_{j=k_{n_{(i-1)}}+1}^{k_{n_i}} \frac{|\sin p^j \pi x|}{j} < \pi \sum_{j=k_{n_{(i-1)}}+1}^{k_{n_i}} \frac{1}{jp^{k_{n_i}-j}} < \frac{2\pi}{k_{n_{(i-1)}}}.$$

Consequently we have $\sum \frac{|\sin p^n \pi x|}{n} \leq \sum\limits_{i=1}^{\infty} \frac{2\pi}{k_{n_i}} \leq \sum\limits_{i=1}^{\infty} \frac{2\pi}{n_i^2} < \infty$ and so $x \in A$.

Now observe that

$$a_{n_i} x \equiv_{\mathbb{Z}} u_{n_i} \sum_{r=1}^{\infty} \frac{c_{k_{n_i}+r}(x)}{p^r}$$

$$\Rightarrow \{a_{n_i} x\} = \left\{\frac{u_{n_i}}{p}\right\} + u_{n_i} \sum_{r=b_i+1}^{\infty} \frac{c_{k_{n_i}+r}(x)}{p^r} \text{ where } b_i = k_{n_{(i+1)}} - k_{n_i}.$$

Since $b_i > n_i u_{n_i}$ implies $p^{b_i} > n_i u_{n_i}$, we must have $\frac{u_{n_i}}{p^{b_i}} < \frac{1}{p^2}$.

So it is evident that

$$\frac{1}{p} \leq \{a_{n_i} x\} \leq \frac{p-1}{p} + \frac{u_{n_i}}{p^{b_i}} < 1 - \frac{p-1}{p^2}$$

which implies $\|a_{n_i} x\| > \frac{p-1}{p^2}$. This shows that $x \notin t_{(a_n)}(\mathbb{T})$ —which is a contradiction since $A \subseteq t_{(a_n)}(\mathbb{T})$ and we have already shown that $x \in A$. Thus, there does not exist any increasing sequence (a_n) for which $A \subseteq t_{(a_n)}(\mathbb{T})$. Therefore, $A \notin \mathcal{A}$. □

Corollary 6.4.2. $\mathcal{A} \subsetneq s\mathcal{A}$.

Proof. The assertion follows directly from Theorem 6.4.1. □

Corollary 6.4.3. $s\mathcal{A} \nsubseteq \mathcal{N}_\sigma$.

Proof. Since $\mathcal{A} \nsubseteq \mathcal{N}_\sigma$ [63], the result follows directly from Corollary 6.4.2.

□

Our next result plays a crucial role in the proof of Theorem 6.4.5.

Lemma 6.4.4. *Let (p_n) be a lacunary sequence of naturals. Then for any increasing sequence of naturals (q_n), one of the following conditions hold:*

(i) *There exists a sequence of naturals (l_n) with the following properties.*

a) *(l_n) diverges to ∞.*

b) *$l_n < p_{(n+1)} - p_n$ for all $n \in \mathbb{N}$.*

c) *The set $L = \bigcup_{n=1}^{\infty} [p_n, p_n + l_n]$ does not contain at least one subsequence of (q_n).*

(**ii**) *There exists a $m \in \mathbb{N}$ such that $q_n \in L' = \bigcup_{r=1}^{\infty} [p_r, p_r + m]$ for all $n \in \mathbb{N}$.*

Proof. Since (p_n) is lacunary (i.e. $|p_{(n+1)} - p_n| \to \infty$) it is obvious that if (**ii**) holds then (**i**) cannot be true. Therefore, it is sufficient to show that if (**ii**) does not hold then (**i**) must hold.

So let us assume that there does not exist any $m \in \mathbb{N}$ for which $q_n \in L' = \bigcup_{r=1}^{\infty} [p_r, p_r + m]$ for all $n \in \mathbb{N}$. For all $i \in \mathbb{N}$ we define

$$n_i = \min\{n \in \mathbb{N} \ : \ q_n \notin L_i = \bigcup_{r=1}^{\infty} [p_r, p_r + i]\}.$$

It is evident that $n_{(i+1)} \geq n_i$. So instead of considering an increasing subsequence (n_{i_k}) of (n_i), without any loss of generality we simply assume that the sequence (n_i) is increasing. Now choose a subsequence (p_{r_i}) of (p_r) such that $q_{n_i} \in [p_{r_i} + i + 1, p_{r_i+1}]$. Let us now define

$$l_n = i \text{ when } r_i \leq n < r_{(i+1)}.$$

It can be readily checked that the set $L = \bigcup_{n=1}^{\infty} [p_n, p_n + l_n]$ does not contain the subsequence (q_{n_i}) of (q_n). By construction (l_n) is divergent and $l_n < p_{(n+1)} - p_n$ for all $n \in \mathbb{N}$. □

Theorem 6.4.5. $s\mathcal{A} \nsubseteq \mathcal{N} \cup \mathcal{A}$.

Proof. In order to prove the result we are going to find an $A \in s\mathcal{A} \setminus \mathcal{A}$ which cannot be contained in an F_σ set which in turn would imply that A cannot be an **N**-set since the family $\mathcal{N}$ has a F_σ-basis.

Consider $A = t^s_{(2^{2^n})}(\mathbb{T})$. Arbault have already shown that the set $t_{(2^{2^n})}(\mathbb{T})$ cannot be contained in an F_σ set [3]. Since $t_{(2^{2^n})}(\mathbb{T}) \subsetneq t^s_{(2^{2^n})}(\mathbb{T}) = A$ (by Theorem B [38]), we can conclude that A also cannot be contained in a F_σ set. Therefore, $A \in s\mathcal{A} \setminus \mathcal{N}$.

To show that $A \notin \mathcal{A}$ it is sufficient to show that there cannot exist an increasing sequence of naturals (b_n) such that $t_{(b_n)}(\mathbb{T}) \supseteq A$. On the contrary let us assume that such a sequence exists i.e. $A \subseteq t_{(b_n)}(\mathbb{T})$. Now observe that A contains all 2-adic numbers of certain ranks. In view of Corollary 6.2.2, consequently b_n has to be of the form $b_n = 2^{k_n} u_n$ where u_n must be odd and $k_n \to \infty$. Set $B = t_{(b_n)}(\mathbb{T})$.

Taking into account the two sequences (k_n) and (2^n) two possibilities can arise in view of Lemma 6.4.4.

(**i**) There exists an increasing sequence of naturals (l_n) satisfying the following.

a) (l_n) diverges to ∞.

b) $l_n < 2^n$ for all $n \in \mathbb{N}$.

c) The set $L = \bigcup_{n=1}^{\infty} [2^n, 2^n + l_n]$ does not contain at least one subsequence of (k_n).

So, there exists a subsequence (k_{n_i}) of (k_n) such that $k_{n_i} \notin L$ and $k_{n_i} > k_{n_{(i-1)}} + 4u_{n_{(i-1)}}$ for all $i \in \mathbb{N}$.
Let us now define

$$\text{supp}_{(2^n)}(x) = \{k_{n_i} + 1 : i \in \mathbb{N}\}.$$

Now proceeding exactly as in Theorem 6.4.1 we obtain that

$$\frac{1}{2} \leq \{2^{k_{n_i}} u_{n_i} x\} \leq \frac{3}{4}$$

$$\Rightarrow \lim_{i\to\infty} \|b_{n_i} x\| \neq 0 \text{ i.e. } x \notin t_{(b_n)}(\mathbb{T}).$$

Consider any arbitrary $\varepsilon > 0$. Since the sequence $l_n \to \infty$, there exists a $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{l_n}} < \varepsilon$ for all $n > n_0$. Therefore for all $n > n_0$ we have

$$\{2^{2^n} x\} = 2^{2^n} \sum_{r=2^n+1}^{\infty} \frac{c_r}{2^r} = 2^{2^n} \sum_{r=2^n+l_n+1}^{\infty} \frac{c_r}{2^r} \leq \frac{2^{2^n}}{2^{2^n+l_n}} = \frac{1}{2^{l_n}} < \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} \|2^{2^n} x\| = 0 \text{ i.e. } x \in t_{(2^{2^n})}(\mathbb{T}) \subseteq A$$

$-$which is a contradiction because $A \subseteq t_{(b_n)}(\mathbb{T})$.

(ii) On the other hand now suppose that there does not exist any sequence (l_n) with the properties **a**), **b**), **c**) described in (**i**). Then in view of Lemma 6.4.4 we can conclude that there exists a $m \in \mathbb{N}$ such that

$$(k_n) \subseteq L' = \bigcup_{r=1}^{\infty} I_r \text{ where } I_r = [2^r, 2^r + m].$$

Note that we can construct a subsequence (I_{r_i}) of (I_r) with the following properties:

- $I_{r_i} \cap (k_n) \neq \emptyset$ for each $i \in \mathbb{N}$,
- $2^{r_i} > 2^{r_{(i-1)}} + m + 4u_{n_{(i-1)}}$,
- $d(\{r_i : i \in \mathbb{N}\}) = 0$.

Now we choose a subsequence (k_{n_i}) of (k_n) such that $k_{n_i} \in I_{r_i}$ for each $i \in \mathbb{N}$.
Let us define

$$\text{supp}_{(2^n)}(x) = \{k_{n_i} + 1 \; : \; i \in \mathbb{N}\}$$

Now proceeding as in (**i**), we obtain that $x \notin B$. Again observe that for every

$n \in \mathbb{N} \setminus \{r_i : i \in \mathbb{N}\}$ we have $k_{n_i} \notin I_n$ and so

$$\{2^{2^n} x\} = 2^{2^n} \sum_{r=2^n+1}^{\infty} \frac{c_r}{2^r} = 2^{2^n} \sum_{r=2^{n+1}}^{\infty} \frac{c_r}{2^r} \leq \frac{2^{2^n}}{2^{2(n+1)-1}} < \frac{1}{2^n}.$$

Since $d(\{r_i : i \in \mathbb{N}\}) = 0$, from the above argument it follows that $x \in A$ $-$which is again a contradiction.

Therefore in either case we come to the conclusion that A cannot be contained in $t_{(b_n)}(\mathbb{T})$ for any increasing sequence of naturals (b_n). Thus $A \in s\mathcal{A} \setminus \mathcal{A}$ and this completes the proof. □

It is well known that a subfamily of $\mathcal{N}$, namely $\mathcal{N}_0$ is contained in $\mathcal{A}$. When it comes to the larger class $s\mathcal{A}$, one can again find a suitable subfamily of $\mathcal{N}$ which we denote by $\mathcal{N}'$ (containing $\mathcal{N}_0$) which is contained in $s\mathcal{A}$ but not in $\mathcal{A}$.

$X \subseteq [0,1]$ is in $\mathcal{N}'$ if there exists an increasing sequence of naturals (a_n) and a sequence of reals (r_n) with $nr_n \geq c$ for some $c > 0$ such that $\sum_{n=1}^{\infty} r_n \|a_n x\| < \infty$ for all $x \in X$. Clearly $\mathcal{N}' \subseteq \mathcal{N}$.

For an increasing sequence of naturals (a_n), let us define

$$O_{(a_n)}(\mathbb{T}) = \left\{ x \in \mathbb{T} : \sum_{n=1}^{\infty} \frac{\|a_n x\|}{n} < \infty \right\}.$$

Clearly $O_{(a_n)}(\mathbb{T}) \in \mathcal{N}'$. Then Equation 6.4 entails that the family

$$\mathbf{G} = \{O_{(a_n)}(\mathbb{T}) : (a_n) \text{ is an increasing sequence of naturals}\}$$

forms a F_σ basis for the family $\mathcal{N}'$. Since for any increasing sequence of naturals

$$\frac{\|a_n x\|}{n} \leq \|a_n x\| \text{ for all } n \in \mathbb{N},$$

each $\mathbf{N_0}$-set is contained in a set of the form $O_{(a_n)}(\mathbb{T})$. Therefore, we also have $\mathcal{N}_0 \subsetneq \mathcal{N}'$ (the strict inclusion follows from the fact that $O_{(2^{2^n})}(\mathbb{T})$ is not a $\mathbf{N_0}$-set [3]).

Remark 6.4.6. *The class $\mathcal{N}'$ have always found to play a special role among the members of the family $\mathcal{N}$, particularly in the instances of constructing $\mathbf{N}$-sets for counterexamples. In [3], in order to construct a $\mathbf{N}$-set which is not an $\mathbf{A}$-set, Arbault had chosen the $\mathbf{N}'$-set $O_{(2^n)}(\mathbb{T})$. Again the $\mathbf{N}'$-set $O_{(2^n)}(\mathbb{T})$ was used in [63] to give an example of a $\mathbf{N}$-set which is not a L_0^σ-set. The $\mathbf{N}'$-set $O_{(n!)}(\mathbb{T})$ is a $\mathbf{N}$-set which is not σ-porus [64] etc.*

Theorem 6.4.7. *$\mathcal{N}' \subsetneq s\mathcal{A}$.*

Proof. Let $A \in \mathcal{N}'$. Then there exists an increasing sequence of naturals (a_n) and a sequence of reals (r_n) with $nr_n \geq c$ for some $c > 0$ such that $\sum_{n=1}^{\infty} r_n \|a_n x\| < \infty$ for all $x \in A$. First we show that $A \subseteq t^s_{(a_n)}(\mathbb{T})$.

Take any $x \in A$. From the fact $\sum\limits_{n=1}^{\infty} r_n\|a_n x\| < \infty$ it readily follows that $\lim\limits_{n\to\infty} r_n\|a_n x\| = 0$. If $\lim\limits_{n\to\infty} \|a_n x\| = 0$ then $x \in t_{(a_n)}(\mathbb{T}) \subseteq t^s_{(a_n)}(\mathbb{T})$ and we are done. Now assume that $\lim\limits_{n\to\infty} \|a_n x\| \neq 0$. Consequently, there exists a subsequence (a_{n_k}) of (a_n) such that $\|a_{n_k} x\| \geq \epsilon_0$ for some $\epsilon_0 \in (0, \frac{1}{2}]$.

Observe that

$$\sum_{k=1}^{\infty} \frac{\epsilon_0}{n_k} \leq \sum_{n=1}^{\infty} \frac{\|a_n x\|}{n} \leq \frac{1}{c} \sum_{n=1}^{\infty} r_n\|a_n x\| < \infty \tag{6.4}$$

which shows that the series $\sum\limits_{k=1}^{\infty} \frac{1}{n_k}$ is convergent. Since $(\frac{1}{n_k})$ is a monotone decreasing sequence of positive reals, from Abel-Pringsheim's Theorem it follows that

$$\lim_{k\to\infty} \frac{k}{n_k} = 0 \text{ i.e. } d(\{n_k : k \in \mathbb{N}\}) = 0.$$

From the above we can conclude that there does not exist any subsequence (a_{n_k}) of (a_n) with $d(\{n_k : k \in \mathbb{N}\}) > 0$ for which $\|a_{n_k} x\| \geq \varepsilon$ for some $\varepsilon \in (0, \frac{1}{2}]$. Hence $(a_n x)$ must converge to 0 statistically and so $x \in t^s_{(a_n)}(\mathbb{T})$. Since x was chosen arbitrarily, we obtain $A \subseteq t^s_{(a_n)}(\mathbb{T})$. Therefore Proposition 6.3.2 entails that $A \in s\mathcal{A}$ and we get $\mathcal{N}' \subsetneq s\mathcal{A}$. Finally the strictness of the inclusion follows from Corollary 6.4.14. □

Corollary 6.4.8. *For any increasing sequence of naturals (a_n), the* $\mathbf{N}$*-set $O_{(a_n)}(\mathbb{T})$ is contained in the s*$\mathbf{A}$*-set $t^s_{(a_n)}(\mathbb{T})$.*

Proof. The proof uses similar technics as in Theorem 6.4.7 and so is omitted. □

Corollary 6.4.9. $\mathcal{N}' \nsubseteq \mathcal{P}_\sigma$.

Proof. In 1985, S.V. Konyagin had shown that the set $O_{(n!)}(\mathbb{T})$ is not σ-porous (for a proof see [89]). Thus $\mathcal{N}' \nsubseteq \mathcal{P}_\sigma$. □

Corollary 6.4.10. $s\mathcal{A} \nsubseteq \mathcal{P}_\sigma$.

Proof. The proof follows directly from Theorem 6.4.7 and Corollary 6.4.9. For a general view consider the set $t^s_{(n!)}(\mathbb{T})$. Then Corollary 6.4.8 entails that

$$O_{(n!)}(\mathbb{T}) \subseteq t^s_{n!}(\mathbb{T}).$$

Since $\mathcal{P}_\sigma$ is an ideal, we conclude that $t^s_{n!}(\mathbb{T}) \notin \mathcal{P}_\sigma$. Consequently we have $s\mathcal{A} \nsubseteq \mathcal{P}_\sigma$. □

Corollary 6.4.11. $s\mathcal{A} \nsubseteq \mathcal{H}_\sigma$.

Proof. Since $\mathcal{H}_\sigma \subseteq \mathcal{P}_\sigma$ (see [89]), the result follows directly from Corollary 6.4.16. □

Corollary 6.4.12. $s\mathcal{A} \nsubseteq \mathcal{A}_\sigma$.

Proof. Since $\mathcal{A} \subseteq \mathcal{H}_\sigma$ (see [3]), the proof follows directly from Corollary 6.4.17. □

Recall that an uncountable set X is a Luzin set if every meager subset of X is countable.

Proposition 6.4.13. $s\mathcal{A} \subseteq w\mathcal{D}$.

Proof. Take a s-characterized subgroup $t^s_{(a_n)}(\mathbb{T})$ of $\mathbb{T}$. In [38] it has been shown that $t^s_{(a_n)}(\mathbb{T})$ is closed and therefore a complete subgroup of $\mathbb{T}$ and $\mu(t^s_{(a_n)}(\mathbb{T}) \setminus t_{(a_n)}(\mathbb{T})) = 0$. Therefore, $(e^{2\pi i a_n x})$ converges to 1 μ-almost every where on $t^s_{(a_n)}(\mathbb{T})$. Since $|e^{2\pi i a_n x}| \leq 1$, in view of Dominated Convergence Theorem we observe that

$$\lim_{n\to\infty} \int_{t^s_{(a_n)}(\mathbb{T})} |e^{2\pi i a_n x} - 1| d\mu = 0.$$

Consequently $t^s_{(a_n)}(\mathbb{T})$ is a $\mathbf{wD}$-set. Therefore, from Proposition 6.3.2 we conclude that $s\mathcal{A} \subseteq w\mathcal{D}$. □

Corollary 6.4.14. *Under CH (the continuum hypothesis),* $s\mathcal{A} \subsetneq w\mathcal{D}$.

Proof. Since every Luzin set which is non-meager while having strong measure zero is a $\mathbf{wD}$-set [22] (for example see [61]), Corollary 6.3.8 ensures that these sets do not belong to the family $s\mathcal{A}$. Thus we get $s\mathcal{A} \subsetneq w\mathcal{D}$. □

The following three corollaries give a clearer picture about the class $s\mathcal{A}$ as it is seen that though the classes $\mathcal{D}$ and $\mathcal{A}$ are contained in $s\mathcal{A}$ the same do not remain true when countable unions of $\mathbf{D}$-sets and $\mathbf{A}$-sets come into consideration. Further it is also noted that the class $\mathcal{N}_\sigma$ is definitely not contained in the class $s\mathcal{A}$.

Proposition 6.4.15. *[22, Corollary 8.13] There are two perfect* $\mathbf{D}$*-sets whose union is not a* $\mathbf{wD}$*-set. Consequently,* $\mathcal{D}_\sigma \nsubseteq w\mathcal{D}$.

Corollary 6.4.16. $\mathcal{D}_\sigma \nsubseteq s\mathcal{A}$.

Proof. Follows directly from Proposition 6.4.13 and Proposition 6.4.15. □

Our next two corollaries follow from Corollary 6.4.16 and from the fact that a $\mathbf{D}$-set is an $\mathbf{A}$-set as well as a $\mathbf{N}$-set i.e. $\mathcal{D}_\sigma \subseteq \mathcal{A}_\sigma \cap \mathcal{N}_\sigma$.

Corollary 6.4.17. $\mathcal{A}_\sigma \nsubseteq s\mathcal{A}$.

Corollary 6.4.18. $\mathcal{N}_\sigma \nsubseteq s\mathcal{A}$.

Proposition 6.4.19. *[60] Every* F_σ $\mathbf{wD}$*-set is an* $\mathbf{N}$*-set.*

Corollary 6.4.20. *Every* F_σ $\mathbf{sA}$*-set is an* $\mathbf{N}$*-set.*

6.5 Conclusion

In Section 6.2 of this chapter it is shown (in Theorem 6.2.3) that there are statistically characterized subgroups which can't be characterized by any sequence of integers establishing the "novelty" of the notion which was missing when the notion of statistically characterized subgroups was introduced in [38]. This naturally paves the way for a new class of sets generated by the class of statistically characterized subgroups as basis namely statistical Arbault sets which is introduced in Section 6.3 where some basic properties are established. Finally the last section is devoted to the comparison of this new class with the existing classes of trigonometric thin sets.

Chapter 7

OPEN PROBLEMS

The thesis consists significant contributions in the area of generalized characterized subgroups. Here we present some open questions and possible directions for future work.

7.1 Generalized characterized subgroups

In Theorem 5.1.3, we have shown that for any analytic P-ideal $\mathcal{I}$ the corresponding $\mathcal{I}$-characterized subgroup is a Borel subgroup of $\mathbb{T}$ (Observe that this is the most general version of a characterized subgroup which encompass all such notions exist in the literature of characterized subgroups). After that a characterization of these subgroups are given in Theorem 5.2.8 for an analytic translation invariant P-ideal. So one can easily think of the following problem:

Problem 7.1.1. *Give a characterization of topologically $\mathcal{I}$-torsion elements for any analytic P-ideal $\mathcal{I}$.*

Problem 7.1.2. *For any $\mathcal{I} \in \Im$ and for any increasing sequence of naturals (a_n) is it true that $|t^{\mathcal{I}}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$?*

Problem 7.1.3. *For any $\mathcal{I} \in \Im$ and for any increasing sequence of naturals (a_n) is it true that $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \supsetneq t_{(a_n)}(\mathbb{T})$?*

A notoriously non-arithmetic sequence is the Fibonacci sequence (f_n), defined by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. It is known that $t_{(f_n)}(\mathbb{T})$ is infinite cyclic [9, 15, 37, 70].

Problem 7.1.4. *For any $\mathcal{I} \in \Im$ give a characterization of $t^{\mathcal{I}}_{(f_n)}(\mathbb{T})$. Is it distinct from $t_{(f_n)}(\mathbb{T})$?*

The Fibonacci sequence is recursive. It obviously satisfies the condition $f_{n-1}|f_n - f_{n-2}$, so one can consider the most general sequences a_n with this property, which means that

$$a_n = b_{n-1}a_{n-1} + a_{n-2} \tag{7.1}$$

for some sequences (b_n) of naturals. One can consider even a more complicated recursion as

$$a_n = b^{(1)}_{n-1}a_{n-1} + b^{(2)}_{n-1}a_{n-2} + \ldots + b^{(k)}_{n-1}a_{n-k} \tag{7.2}$$

for some k-tuple of sequences $(b_n^{(j)})$ $(j = 1, 2, \ldots, k)$ of naturals (see [10] for topologically torsion related to such sequences). In particular, one may extend the above question to recursive sequences of integers satisfying (7.1) and (7.2):

Problem 7.1.5. *Let (a_n) be a recursive sequence as in (7.1) or (7.2). For an analytic P-ideal $\mathcal{I}$, compute $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$. When is it countable? Is it distinct from $t_{(a_n)}(\mathbb{T})$?*

Finally, one is left with the general problem to explore generalized characterized subgroups of (compact-like) topological abelian groups:

Problem 7.1.6. *For an analytic P-ideal $\mathcal{I}$, study the $\mathcal{I}$-characterized subgroups of the compact metrizable abelian groups, following the standard way already used for the characterized subgroups in [40, 42, 44, 45, 55].*

7.2 Statistical Arbault sets

Kunen [67] had proved that there are no Luzin sets under the assumption of MA (Martin's Axiom) and the negation of CH. As in Corollary 6.4.14 the strict inclusion could only be obtained using Luzin sets, so a natural question arises whether one can obtain the result without using Luzin sets, i.e. without explicitly using CH.

Problem 7.2.1. *Is $s\mathcal{A} \subsetneq w\mathcal{D}$ provable in ZFC?*

Problem 7.2.2. *Does there exists an $\mathbf{N}$-set which is not an $\mathbf{sA}$-set?*

We have just observed sets from the family $\mathcal{N}$ belonging to $s\mathcal{A}$ still it seems unlikely that $\mathcal{N} \subseteq s\mathcal{A}$. Though we are unable to provide a clear answer in this direction we conjecture this problem positively.

Conjecture 7.2.3. *The $\mathbf{N}$-set*

$$E = \{x \in [0, 1] : \sum_{n=1}^{\infty} \frac{1}{n \ln n} |\sin 2^n \pi x| < \infty\}$$

is not an $\mathbf{sA}$-set.

A set $F \subseteq \mathbb{T}$ is $s\mathcal{A}$-permitted if $X \cup F \in s\mathcal{A}$ for all $X \in s\mathcal{A}$.

Problem 7.2.4. *Is every countable set $s\mathcal{A}$-permitted?*

More generally we consider the following problem.

Problem 7.2.5. *Give a characterization of $s\mathcal{A}$-permitted Sets.*

In this direction the following problem will be crucial.

Problem 7.2.6. *Given two sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{Z}$ when can we conclude $t^s_{\mathbf{u}}(\mathbb{T}) \subseteq t^s_{\mathbf{v}}(\mathbb{T})$ or not?*

A subgroup K of $\mathbb{R}$ is called s-characterized if

$$K = \tau^s_{\mathbf{u}}(\mathbb{R}) := \{x \in \mathbb{R} \ : \ \|u_n x\| \to 0 \text{ statistically}\}$$

for some sequence $\mathbf{u}$ in $\mathbb{R}$. Then naturally next question arises.

Problem 7.2.7. *Given two sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}$ when can we conclude $\tau^s_{\mathbf{u}}(\mathbb{R}) \subseteq \tau^s_{\mathbf{v}}(\mathbb{R})$ or not?*

For two subgroups H, K of an infinite group G, say that H is almost contained in K if $[H : K \cap H]$ is finite. Similarly, say that H is weakly contained in K if $[H : K \cap H]$ is at most countable.

Problem 7.2.8. *What can you conclude when the inclusion in Problem 7.2.6, Problem 7.2.7 are replaced by almost inclusion and weak inclusion in the above sense?*

A subgroup K of $\mathbb{T}$ is s-factorizable if $K = t^s_{\mathbf{v}}(\mathbb{T}) + t^s_{\mathbf{w}}(\mathbb{T})$ for proper s-characterized subgroups $t^s_{\mathbf{v}}(\mathbb{T})$ and $t^s_{\mathbf{w}}(\mathbb{T})$ of K.

Problem 7.2.9. *When a given s-factorizable subgroup is s-characterized and when a given s-characterized subgroup is s-factorizable?*

Problem 7.2.10. *Does there exist a s-characterized subgroup of $\mathbb{T}$ which is F_σ?*

It is well known that the group generated by a Kronecker set cannot be characterized.

Problem 7.2.11. *Can a group generated by a Kronecker set be s-characterized?*

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