A Contribution To Complementarity Theory

Aritra Dutta

Jadavpur University
188, Raja S.C. Mallick Road
Kolkata: 700 032
India

A Contribution To Complementarity Theory

Submitted to Jadavpur University in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

by

Aritra Dutta

JADAVPUR UNIVERSITY

Kolkata, India

March, 2023

DECLARATION

I hereby declare that the thesis entitled **A Contribution To Complementarity Theory** submitted by me, for the award of the degree of *Doctor of Philosophy* to Jadavpur University is a record of bonafide work carried out by me under the supervision of Dr. Arup Kumar Das, SQC & OR Unit, Indian Statistical Institute, Kolkata.

I further declare that the work reported in this thesis has not been submitted and will not be submitted, either in part or in full, for the award of any other degree or diploma in this institute or any other institute or university.

Place: Kolkata

Date: 28/03/2023

Aritra Dutta

Signature of the Candidate

CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled "A Contribution To Complementarity Theory" submitted by Ms. ARITRA DUTTA who got her name registered on June 6, 2018 for the award of Ph.D. (Science) degree of JADAVPUR UNIVERSITY, is absolutely based upon her own work under the supervision of DR. ARUP KUMAR DAS, SQC & OR Unit, Indian Statistical Institute, Kolkata, West Bengal, India and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

डॉ. अरुप कुमार दास / Dr. Arup Kumar Das
एसक्युसी एंड औआर यूनिट / SQC & OR Unit
भारतीय सांख्यिकीय संस्थान
INDIAN STATISTICAL INSTITUTE
203, बी.टी. रोड / 203, B. T. Road
कोलकाता-700 108 / Kolkata-700 108

Arup Das. 28/03/2023
....

(Dr. Arup Kumar Das)

(Signature of the Supervisor date with official seal)

To My Parents

Acknowledgements

More than a thesis, a PhD is a journey. As I look back to the years that led to the completion of this thesis, I realize how fortunate I am to find such kind people around me whose consistent support, supervision, and confidence in me made this journey possible.

I am indebted to Dr. Arup Kumar Das, my thesis supervisor, for introducing me to the field of complementarity theory and also for his excellent guidance and encouragement to carry out this work. His constructive suggestions and criticisms have always put me in the right track.

I express my deepest gratitude to Prof. Sagnik Sinha, Department of Mathematics, Jadavpur University for his constant encouragement and cooperation to carry out the dissertation. I express my sincere gratitude to the Head of the Department of Mathematics and Dean of Science of Jadavpur University for giving me an opportunity to complete the dissertation. I extend my sincere gratitude to all of my teachers for their valuable suggestions.

My sincere thanks are due to Dr. Rwitam Jana (a coauthor of my two papers) for his helpful comments about the work.

I wish to express my sincere thanks to Mr. Ajijul and Mrs. Swarnali and other official staffs for their efforts to help me out in the official matters smoothly.

I would like to take this opportunity to extend my sincere gratitude to my beloved parents Mr. Joydeep Dutta and Mrs. Anjali Dutta for being the most substantial support not only during my PhD career but also throughout my life. Both of them are my source of inspiration. My father's love for mathematics leads me to achieve the highest degree in mathematics. I am moved by his simplicity and deeply appreciate his belief in me. I express my deepest gratitude to Dr. Dhanadeep Dutta, Mr. Pratyush Dutta and other family members for all the supports I got from them.

I wish to express my earnest gratitude to Sri Sri Ravishankar Ji. His knowledge, wisdom helps me to overcome all the miseries and move forward in life. I express my sincere gratitude to Dr. Subinoy Das and Dr. Banani Chakraborty. Without their companionship, this journey would not have been completed smoothly.

I owe a debt of gratitude to Mr. Asta for his support and understanding that made the completion of the thesis possible. He was always around at times when I found it hard to continue and helped me to keep things in perspective. I wish to express my sincere thanks to Adrija Dutta, Abhinav Chakraborty and Abhiroop Ganguly. I would like to thank to all of my seniors and friends. Their support gives me strength and happiness during my research period. Finally, I would like to thank to everyone who directly and indirectly supported me throughout this journey.

Kolkata, March, 2023 *Aritra Dutta*

Preface

The complementarity problem is identified as a mathematical programming problem and provides a framework for several optimization problems. Optimization problems which arise in different branches of science, technology, economics and applied fields may be identified with finding a best solution of an objective function defined on a given domain. More specifically, it refers to the minimization (or maximization) problem of a given objective function subject to a set of constraints. Linear programming, an important class of optimization problems which is used to solve decision problems, became popular during Second World War. In nonlinear programming at least one of the objective function and the constraints is nonlinear. Among many facets of research in complementarity theory, the issue that has received wide attention is the existence of the solutions and development of efficient algorithms for finding solutions. In complementarity theory many of the available algorithms are developed based on a pivotal kind of technique that converges to a solution with a finite number of steps. The role of complementary slackness principal is an important consideration in complementarity theory. This principle holds not only for linear programming problems but also for more general programming problems. The complementary slackness principle for more general programming problems is based on the Karush-Kuhn-Tucker condition of optimality. For linear and quadratic programs, the Karush-Kuhn-Tucker optimality conditions finally reduce to the study of linear complementarity problems (LCP) and this observation was the early motivation for studying the linear complementarity problem. More specifically, the problem which can be posed as an LCP includes linear programming, linear fractional programming, convex quadratic programming and the bimatrix game problem. It is well studied in the literature on mathematical programming and a number of applications are reported in operations research, multiple objective programming problem, mathematical economics, geometry and engineering.

The linear complementarity problem (LCP) is the problem of finding a complementary pair of nonnegative vectors in a finite dimensional real vector space that satisfies the feasibility condition or to show that no such vector exists.

Many of the matrix classes encountered in the context of linear complementarity problems are commonly found in several applications. Matrix classes characterize properties of the linear complementarity problems and offer certain features from the view point of algorithms. Several algorithms have been designed for the solution of the linear complementarity problem. The algorithm presented by Lemke and Howson to compute an equilibrium pair of strategies to a bimatrix game, later extended by Lemke known as Lemke's algorithm, a pivotal kind of technique to solve the linear complementarity problem contributed significantly to the development of the linear complementarity theory. This algorithm does not solve every instance of the linear complementarity problem and in some instances, the problem may terminate inconclusively without either computing a solution to it or showing that no solution exists. This observation motivated me to pursue research in the area of complementarity theory.

Many of the results of linear complementarity problems can be stated in terms of the value of a matrix game. In this connection Kaplansky's result on matrix games is useful for deriving certain results. The principal pivot transform (PPT) is a fundamental concept for developing many theories and algorithms in complementarity theory and plays an important role in the study of matrix classes. Tucker introduced the concept of principal pivot transform and proved that if the diagonal entries for every principal pivot transform of a matrix are positive, then the matrix is a P-matrix. The notion of principal pivot transform is originally motivated by the well-known linear complementarity problem.

The idea of nonlinear complementarity problem (NCP) is based on the concept of linear complementarity problem. The concept of complementarity is synonymous with the notion of system equilibrium. A number of applications of nonlinear complementarity problems are reported in operations research, multiple objective programming problem, mathematical economics and engineering. A wide class of problems, which arise in complementarity theory, can be studied based on the nonlinear system of equations using various techniques. Finding a solution of a system of nonlinear equations has an important role to deal with problems in various fields such as chemical production processes, engineering design, economic equilibrium, transportation and applied physics. A number of methods are proposed to solve systems of equations. In this context Newton and quasi-Newton methods are well-known iterative methods to solve nonlinear systems of equations.

The fundamental idea of many iterative methods is to solve a problem by tracing a continuous path that leads to a solution of the problem. Defining an appropriate mapping that yields a finite continuation path plays an essential role in a homotopy continuation method. Homotopy methods are proposed for constructive proof of the existence of solutions to systems of nonlinear equations, nonlinear optimization problems, Brouwer fixed point problems, nonlinear programming, game problem and complementarity problems.

The results included in this dissertation are divided into eight chapters. The chapterwise summary is given below.

Chapter 1 includes the general introduction about the research work alongwith the required definitions and notations which will be used in the subsequent chapters. This section also includes a survey of the results in complementarity theory.

Chapter 2 considers the study of hidden Z-matrix in the context of linear complementarity problem. It is shown that the linear complementarity problem with hidden Z-matrix is processable by Lemke's algorithm as well as criss-cross method. To prove the results, the concept of principal pivot transform and game theoretic approach are applied. Certain matrix theoretic characterizations of hidden Z-matrix are provided to establish P_0 properties. Mangasarian showed that a linear complementarity problem with hidden Z-matrix can be solved with the help of a linear programming problem. We extend the result of Fiedler and Pták that a Z-matrix to be P-matrix is also true for hidden Z-matrix and propose a new formulation of linear complementarity problem as a linear programming problem. It is shown that for a non-degenerate feasible basis alongwith some additional assumptions, the linear complementarity problem with hidden Z-matrix has a unique non-degenerate solution.

Chapter 3 contains a study of column competent matrix and its matrix theoretic properties. The local w-uniqueness of the solution to the linear complementarity problem can be identified by the column competent matrices. Some new results on w-uniqueness as well as locally w-uniqueness properties in connection with column competent matrices are established. These results are significant in the context of matrix theory as well as algorithms in operations research. Finally, a connection between column competent matrices and column adequate matrices is established with the help of degree theory.

In chapter 4, K-type block matrices are introduced which include two new classes of block matrices namely block triangular K-matrices and hidden block triangular K-matrices. It is shown that the block triangular K-matrices satisfy least element property and the solution of linear complementarity problem with K-type block matrices can be obtained by solving a linear programming problem. It is also proved that the hidden block triangular K-matrices are Q_0 and processable by Lemke's algorithm. The purpose of this article is to study the properties of K-type block matrices in the context of the solution of linear complementarity problem.

Chapter 5 deals with solution approach of linear complementarity problem as an initial value problem. A new function alongwith interior point approach is proposed to trace a path for finding a solution. We parameterize the path with respect to arc length and obtain the solution of the proposed function by solving the initial value problem with predictor-corrector method. It is shown that the path approaching to the solution is smooth and bounded. To ensure continuous trajectory we introduce a new scheme of choosing step length with the help of predictor-corrector method. We show that under some conditions the solution of the proposed function can provide the solution of linear complementarity problem. We ensure that the solution of linear complementarity problem with P_0 matrix or nondegenerate matrix is obtained by the predictor-corrector method. Several examples are illustrated to show the effectiveness of the proposed algorithm.

Chapter 6 contains a solution method for finding the solution of two-person zero-sum discounted stochastic game with additive rewards and additive transitions (ARAT) structure as an application of LCP. Usually, two-person zero-sum discounted stochastic ARAT game is solved by pivoting algorithm namely, Lemke method and Cottole-Dantzig algorithm. Here we take an approach to solve this problem by an iterative method introducing a new function alongwith the complementarity condition. It is shown that the algorithm has higher order of convergence and the trajectory as obtained by the algorithm is bounded.

The results in chapter 7 are concerned with the solution approaches to nonlinear complementarity problem using homotopy approach. A new homotopy function is developed for finding the solution of nonlinear complementarity problem through a continuous path ensuring the boundedness property of the trajectory obtained from the homotopy function. It is proved that a path approaching to solution is smooth and bounded. We establish some conditions under which the continuation method gives a solution of nonlinear complementarity problem. Some numerical examples are considered to show the method approaching to the soution along a smooth and bounded homotopy path.

Chapter 8 contains a formulation of oligopolistic market equilibrium problem as an application of NCP. In this study the equivalence between nonlinear complementarity problem and the system of nonlinear equations is established and a homotopy method with vector parameter is proposed for finding the solution of oligopoly market equilibrium problem through system of nonlinear equations. It is shown that the trajectory to obtain the solution of the system of nonlinear equations with the help of the proposed method with vector parameter is smooth and bounded under some additional conditions. In this context it is shown that a newly introduced modified Newton method with higher order convergence can also be applied to obtain the solution of oligopoly market equilibrium problem.

Numbering

For internal referencing, Section j in Chapter i is denoted by $i.j$ and $i.j.k$ is used to refer Item k of Section j in Chapter i. For example, the triple 2.3.5 refers to Item 5 in Section 3 of Chapter 2. All items (e.g., Lemma, Theorem, Example, Remark etc.) are identified in this fashion. Equation $(i.j.k)$ is used to refer Equation k in Section j of Chapter i. We use brackets [ ] for a bibliographical reference.

List of Notations

The special notations pertaining to a particular chapter are provided in Section 2 of each chapter. The most frequently used notations are given below:

Spaces

$\mathbb{R}^n$	real n-dimensional space
$\mathbb{R}$	the real line
$\mathbb{R}_+$	the nonnegative orthant of $\mathbb{R}$
$\mathbb{R}^{n\times n}$	the space of $n \times n$ real matrices
$\mathbb{R}^n_+$	the nonnegative orthant of $\mathbb{R}^n$
$\mathbb{R}^n_{++}$	positive orthant of $\mathbb{R}^n$
$\mathbb{N}$	the set of natural numbers

Vectors

x^T	the transpose of a vector x
x^Ty	the standard inner product of vectors in $\mathbb{R}^n$
$x \geq y$	$x_i \geq y_i,\ i = 1, \ldots, n$
$x > y$	$x_i > y_i,\ i = 1, \ldots, n$
$y \in R^n$ is *unisigned*	if either $y \in \mathbb{R}^n_+$ or $-y \in \mathbb{R}^n_+$
e	the vector of all 1.

Sets

$\in$	element membership
$\notin$	not an element of
$\subseteq$	set inclusion
$\subset$	proper set inclusion
$\cup, \cap, \times$	union, intersection, cartesian product
$\emptyset$	the empty set
$\bar{\alpha}$	complement of an index set α
$\lvert\alpha\rvert$	cardinality of a finite set α

Matrices

$A = [a_{ij}]$	a matrix with real entries a_{ij}
$\det(A)$	the determinant of a square matrix A
A^{-1}	the inverse of a matrix A
A^T	the transpose of a matrix A
I	the identity matrix
$A_{\alpha\beta}$	submatrix formed by the rows and columns of A whose indices are in α and β, respectively
$A_{\alpha\cdot}$	submatrix formed by the rows of A whose indices are in α
$A_{\cdot\alpha}$	submatrix formed by the columns of A whose indices are in α
$A_{\alpha\alpha}$	the *principal submatrix* of A
$\det(A_{\alpha\alpha})$	the *principal minor* of A

Miscellaneous Symbols

$\mathrm{LCP}(q, A)$	the LCP with data (q, A)
$\mathrm{FEA}(q, A)$	the feasible region of $\mathrm{LCP}(q, A)$
$\mathrm{SOL}(q, A)$	the solution set of $\mathrm{LCP}(q, A)$

List Of Publications

Published:

1. R. Jana, A. Dutta and A. K. Das (2021). More On Hidden Z-Matrices And Linear Complementarity Problem, Linear and Multilinear Algebra,$69 : 6, 1151 - 1160$, DOI: 10.1080/03081087.2019.1623857.

2. R. Jana, A.K. Das and A. Dutta (2019). On Hidden Z-Matrix And Interior Point Algorithm. OPSEARCH $56, 1108$–1116, https://doi.org/10.1007/$s12597-019-00412-0$.

3. A. Dutta, R. Jana and A.K. Das (2022). On Column Competent Matrices And Linear Complementarity Problem. In: Giri, D., Raymond Choo, KK., Ponnusamy, S., Meng, W., Akleylek, S., Prasad Maity, S. (eds) Proceedings of the Seventh International Conference on Mathematics and Computing . Advances in Intelligent Systems and Computing, vol 1412. Springer, Singapore, https://doi.org/10.1007/$978 - 981 - 16 - 6890 - 6\_46$.

4. A. Dutta and A.K. Das (2022). On Some Properties Of K-type Block Matrices In The Context Of Complementarity Problem. In: Rushi Kumar, B., Ponnusamy, S., Giri, D., Thuraisingham, B., Clifton, C.W., Carminati, B. (eds) Mathematics and Computing. ICMC 2022. Springer Proceedings in Mathematics & Statistics, vol 415. Springer, Singapore, https://doi.org/10.1007/$978 - 981 - 19 - 9307 - 7\_12$.

Accepted:

1. A. Dutta and A.K. Das (2023). Homotopy Continuation Method For Discounted Zero-Sum Stochastic Game With ARAT Structure (Accepted for publication in the journal International Game Theory Review).

Communicated:

1. Bounded Homotopy Path Approach To The Solution Of Linear Complementarity Problems.

2. On Some Approaches To Compute A Nash Equilibrium Of An Oligopolistic Market By Means Of Nonlinear Complementarity Problem.

Contents

Chapter 1

General Introduction And Some Basic Concepts

1.1 Introduction

The complementarity problem is identified as a mathematical programming problem and provides a framework for several optimization problems. The role of complementary slackness principal is an important consideration in complementarity theory. This principle holds not only for linear programming problems but also for more general programming problems. The complementary slackness principle for more general programming problems is based on the Karush-Kuhn-Tucker condition of optimality. For linear and quadratic programs, the Karush-Kuhn-Tucker optimality conditions finally reduce to the study of linear complementarity problems (LCP) and this observation was the early motivation for studying the linear complementarity problem. It is well studied in the literature on mathematical programming. Many of the matrix classes encountered in the context of linear complementarity problems are commonly found in several applications. Matrix classes characterize properties of the linear complementarity problems and offer certain features from the view point of algorithms. Several algorithms have been designed for the solution of the linear complementarity problem. Some of the available algorithms are developed based on a pivotal kind of technique that converges to a solution with a finite number of steps.

The idea of nonlinear complementarity problem (NCP) is based on the concept of linear complementarity problem. The concept of complementarity is synonymous with the notion of system equilibrium. A number of applications of nonlinear complementarity problems are reported in operations research, multiple objective programming problem, mathematical economics and engineering. A wide class of problems in complementarity theory can be studied based on the nonlinear system of equations using various techniques. Finding a solution of a system of nonlinear equations has an important role to deal with problems in various fields such as chemical production processes, engineering design, economic equilibrium, transportation and applied physics. The idea of many iterative methods is to solve a problem by tracing a continuous path that leads to a solution of the problem.

The algorithm presented by Lemke and Howson to compute an equilibrium pair of strategies to a bimatrix game, later extended by Lemke known as Lemke's algorithm, a pivotal kind of technique to solve the linear complementarity problem contributed significantly to the development of the linear complementarity theory. This algorithm does not solve every instance of the linear complementarity problem and in some instances, the algorithm may terminate inconclusively without either computing a solution to it or showing that no solution exists. This observation motivated me to pursue research in the area of complementarity theory. The dissertation highlights a contribution to complementarity theory in terms of the matrix theoretic properties and the computational aspects. Attempts have been taken to establish w-uniqueness as well as least element properties and show that various matrix classes play a significant roles. A connection between two matrix classes considered for the study is being established with the help of degree theory. It is shown that the linear complementarity problem with the newly introduced matrix class is processable by Lemke's algorithm. A new type of function is considered to find the solution of the nonlinear complementarity problem through a continuous path ensuring the boundedness property of the trajectory. As applications, two-person zero-sum discounted stochastic game with additive rewards and additive transitions (ARAT) structure and the oligopolistic market equilibrium problem are considered to address their computational aspects. Some basic concepts, definitions, notations and results which will be used in the next chapters are discussed in the next sections. The details of the studies are given in the subsequent chapters.

1.2 Linear Complementarity Problem

The linear complementarity problem is considered as a problem of mathematical programming to unify several optimization problems. The problem may be stated as follows:
Given a square matrix A of order n with real entries and an n dimensional vector q, find n dimensional vectors w and z satisfying

$$w - Az = q, \ \ w \geq 0, \ z \geq 0 \tag{1.2.1}$$

$$w^T z = 0. \tag{1.2.2}$$

This problem is denoted as LCP(q, A). The condition 1.2.1 is called feasibility condition and the condition 1.2.2 is called complementarity condition. If a pair of vectors (w, z) satisfies (1.2.1), then the problem LCP(q, A) has a feasible solution. A pair (w, z) of vectors satisfying (1.2.1) and (1.2.2) is said to be a solution to the LCP(q, A). For details see [69]. The problem has undergone several name changes, from composite problem to complementary pivot problem. Cottle [113, p. 37] proposed the current name linear complementarity problem.

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the feasible set is defined by $\text{FEA}(q, A) = \{z \in \mathbb{R}^n : z \geq 0, q + Az \geq 0\}$ and the solution set of $\text{LCP}(q, A)$ is defined by $\text{SOL}(q, A) = \{z \in \text{FEA}(q, A) : z^T(q + Az) = 0\}$.

The role of complementary slackness principal is an important consideration in optimization theory. The problems which can be posed as an LCP include linear programming, linear fractional programming, convex quadratic programming and the bimatrix game problem. The problem of computing the value vector and optimal stationary strategies for structured stochastic games for discounted and undiscounded zero-sum games and quadratic multi-objective programming problem are formulated as linear complementary problems. For details see [201], [192], [198] and [199]. It is stated in the literature on mathematical programming and a number of applications are available in operations research [24], multiple objective programming problem [117], mathematical economics [61], geometry and engineering [25] and [206].

1.2.1 Some Preliminaries in Linear Complementarity Theory

We introduce required terminologies related to linear complementarity problem $\text{LCP}(q, A)$. The idea of using complementary cones to study LCP was considered by Samelson et. al. [133]. Later Murty studied LCP through complementary cones extensively and obtained some remarkable results. For details see [60].

Definition 1.2.1. *Given $A \in \mathbb{R}^{n\times n}$ and $\alpha \subseteq \{1, 2, \cdots, n\}$, $C_A(\alpha)$ is called a complementary matrix of A with respect to α where $C_A(\alpha)_{.j} = -A_{.j}$ if $j \in \alpha$ and $C_A(\alpha)_{.j} = I_{.j}$ if $j \notin \alpha$. The associated cone $C_A(\alpha)$ is called complementary cone relative to A with respect to α. If $\det(C_A(\alpha)) \neq 0$, then it is called complementary* *basis.*

DEFINITION 1.2.2. *The complementary cone with respect to* α *is said to be nondegenerate if* $\det(A_{\alpha\alpha}) \neq 0$. *Otherwise it is said to be degenerate. A degenerate pos* $C_A(\alpha)$ *is said to be strongly degenerate if there exists* $0 \neq x \geq 0,\ x \in \mathbb{R}^n$ *such that* $C_A(\alpha)x = 0$.

DEFINITION 1.2.3. *Given* $A \in \mathbb{R}^{n\times n}$ *and* $\alpha \subseteq \{1, 2, \cdots, n\}$, *the matrix* A *is said to be nondegenerate if* $\det(A_{\alpha\alpha}) \neq 0\ \forall\ \alpha \subseteq \{1, 2, \cdots, n\}$. *Any solution* (w, z) *of LCP*(q, A) *is said to be nondegenerate if* $w + z > 0$. *Otherwise it is called a degenerate solution. A vector* $q \in \mathbb{R}^n$ *is said to be nondegenerate with respect to* A *if every solution of LCP*(q, A) *is nondegenerate.*

Let $\mathcal{C}(A)$ be the union of the strongly degenerate complementary cones of A and let $\mathcal{K}(A)$ denote the union of all facets of all the complementary cones of A.

DEFINITION 1.2.4. *A set* $C \subseteq \mathbb{R}^n$ *is connected if there do not exist disjoint open sets* $U, V \subseteq \mathbb{R}^n$ *such that* $U \cap C \neq \emptyset$, $V \cap C \neq \emptyset$ *and* $C \subseteq U \cup V$. *A connected component of a set* S *containing a point* x *is defined as the union of all connected sets* C *such that* $x \in C \subseteq S$.

Tucker [10] introduced the concept of principal pivot transforms (PPTs) which is an important concept in the context of linear complementarity problem. The principal pivot transform (PPT) of A with respect to $\alpha \subseteq \{1, 2, \ldots, n\}$ is defined as the matrix M given by

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $\bar{\alpha} = \{1, 2, \ldots, n\} \setminus \alpha$, $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}}=-(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. The PPT of LCP$(q, A)$ with respect to α (obtained by pivoting on $A_{\alpha\alpha}$) is given by LCP(q', M) where $q'_{\alpha} = -A^{-1}_{\alpha\alpha}q_{\alpha}$ and $q'_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha}A^{-1}_{\alpha\alpha}q_{\alpha}$.

Note that PPT is only defined with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. When $\alpha = \emptyset$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For further details, see [113] and [90] in this connection. $A_{\alpha\alpha}$.

Let $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$, where $\alpha \subseteq \{1, 2, \ldots, n\}$, $\bar{\alpha} = \{1, 2, \ldots, n\} \setminus \alpha$. If $A_{\alpha\alpha}$ is invertible, then the schur complement of $A_{\alpha\alpha}$ of the matrix A is the matrix defined by $A/A_{\alpha\alpha} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$.

If $A_{\bar{\alpha}\bar{\alpha}}$ is invertible, then the schur complement of $A_{\bar{\alpha}\bar{\alpha}}$ of the matrix A is the matrix defined by $A/A_{\bar{\alpha}\bar{\alpha}} = A_{\alpha\alpha} - A_{\alpha\bar{\alpha}}(A_{\bar{\alpha}\bar{\alpha}})^{-1}A_{\bar{\alpha}\alpha}$.

1.2.2 Matrix Games

The linear complementarity problem and the matrix game have some important connections. We state the results of two person matrix games in linear complementarity problem due to von Neumann [66] and Kaplansky [51]. The results say that $\exists\ x^* \in \mathbb{R}^m, y^* \in \mathbb{R}^n$ and $v \in \mathbb{R}$ such that

$$\sum_{i=1}^{m} x_i^* a_{ij} \leq v, \ \forall\ j = 1, 2, \cdots, n,$$
$$\sum_{j=1}^{n} y_j^* a_{ij} \geq v, \ \forall\ i = 1, 2, \cdots, m.$$

The strategies (x^*, y^*) are said to be optimal strategies where m and n are pure strategies for player I and player II respectively and v is said to be minimax value of the game. The amount a_{ij} may be positive, negative or zero. The probability vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are the mixed strategies for player I and player II where $x_i \geq 0\ \forall i$, $\sum_{i=1}^{m} x_i = 1$ and $y_j \geq 0\ \forall j$, $\sum_{j=1}^{n} y_j = 1$.

The value of the game $v(A)$ is said to be positive(nonnegative) if there exists a $0 \neq x \geq 0$ such that $Ax > 0$ $(Ax \geq 0)$. Likewise, $v(A)$ is negative(nonpositive) if there exists a $0 \neq y \geq 0$ such that $y^T A < 0$ $(y^T A \leq 0.)$ For a payoff matrix

$A \in \mathbb{R}^{n\times n}$, $v(A)$ is preserved in its PPTs.

1.2.3 Lemke's Method

To solve (1.2.1) and (1.2.2) the complementary pivot method as a result of Lemke [22] which is identified as Lemke's algorithm, has become faster with an appreciable amount of investigation in the matrix class of what the algorithm is in a position to process LCP(q, A). If nondegeneracy is considered, the method determines either a ray termination or a solution to (1.2.1) and (1.2.2). Eaves [17] has pointed out some procedures to bypass cycling in the event that the degenerate almost complementary solutions are developed. For further details on Lemke's algorithm see [113].

An algorithm is appropriate for a given problem if the algorithm is able to work out a solution in case of its existence or confirm the existence of no solution. Assuming $A \in \ L(d)$ for which $d > 0$, the processability of Lemke method with $d > 0$ for LCP(q, A) with all matrices $A \in L_1(d)$ was proved by Todd [91]. Moreover Lemke method process LCP(q, A) when A is row sufficient. For more details see [115]. Ramamurthy [68] show that Lemke's algorithm for the linear complementarity problem can be used to check whether a given Z-matrix is a P_0 matrix and it can also be used to analyze the structure of finite Markov chains.

1.3 Some Relevant Definitions and Results

Various matrix classes arise in linear complementarity problem which are found in many applications. Some of the matrix classes characterize specific properties of the linear complementarity problem and provide interesting features from the view point of algorithms. Most of the algorithms depend on matrix classes. Hence the study of matrix classes is an important issue and form the basis for further discussions. In addition, some relevant results in connection with mapping are included in this section.

Let A be a given $n \times n$ real square matrix, not necessarily symmetric.

DEFINITION 1.3.5. *$A = [a_{ij}]$ is said to be nonnegative or $A \geq 0$ if $a_{ij} \geq 0 \; \forall \; i, j \in \{1, 2, \cdots, n\}$.*

DEFINITION 1.3.6. *$A = [a_{ij}]$ is said to be positive if $a_{ij} > 0 \; \forall \; i, j \in \{1, 2, \cdots, n\}$.*

Let A and B be two matrices with $A \geq B$, then $A - B \geq 0$.

DEFINITION 1.3.7. *$A = [a_{ij}]$ is said to be a Z-matrix if $a_{ij} \leq 0, \; \forall \, i \neq j$.*

The class of Z-matrices has been introduced by Fiedler and Pták [83].

DEFINITION 1.3.8. *A is said to be hidden Z-matrix if there exist Z-matrices X, $Y \in \mathbb{R}^{n \times n}$ and $r, s \in \mathbb{R}^n_+$ such that*

$$AX = Y, \tag{1.3.1}$$

$$r^T X + s^T Y > 0. \tag{1.3.2}$$

A hidden Z-matrix is said to be completely hidden Z-matrix if all its principal submatrices are hidden Z-matrix. For details on hidden Z-matrix, see [136], [134], [138].

DEFINITION 1.3.9. *A is said to be a positive semidefinite (PSD) if $z^T A z \geq 0 \; \forall \; z \in \mathbb{R}^n$ and A is positive definite (PD) if $z^T A z > 0 \; \forall \; 0 \neq z \in \mathbb{R}^n$.*

A is called PSD(PD) of order k, $0 \leq k \leq n$, if every principal submatrix of order k is PSD (PD).

DEFINITION 1.3.10. *A is said to be a $P \, (P_0)$-matrix if all its principal minors are positive (nonnegative).*

DEFINITION 1.3.11. *A is called a $N(N_0)$-matrix if all its principal minors are negative (nonpositive).*

An N-matrix is called an N-matrix of the first category if it contains atleast one positive entry otherwise it is called an N-matrix of the second category.

DEFINITION 1.3.12. *A is called copositive (C_0) (strictly copositive (C)) if $z^T Az \geq 0 \; \forall \; z \geq 0$ $(z^T Az > 0 \; \forall \; 0 \neq z \geq 0)$. A is said to be copositive-plus (C_0^+) if $A \in C_0$ and the following implication holds:*

$$[z^T Az = 0, z \geq 0] \Rightarrow (A + A^T)z = 0.$$

We say that $A \in \mathbb{R}^{n \times n}$ is copositive-star (C_0^*) if $A \in C_0$ and the following implication holds:

$$[z^T Az = 0, Az \geq 0, z \geq 0] \Rightarrow A^T z \leq 0.$$

DEFINITION 1.3.13. *A matrix A is called fully copositive (C_0^f) matrix if every legitimate PPT of A is C_0.*

DEFINITION 1.3.14. *A matrix A is called P_*-matrix if $\exists$ a constant $\tau > 0$ such that for any $x \in \mathbb{R}^n$,*

$$(1 + \tau) \sum_{i \in I_+(x)} x_i(Mx)_i + \sum_{i \in I_-(x)} x_i(Mx)_i \geq 0$$

where $I_+(x) = \{i \in N : x_i(Mx)_i > 0\}$ and $I_-(x) = \{i \in N : x_i(Mx)_i \leq 0\}$.

DEFINITION 1.3.15. *A is said to be column sufficient if for all $z \in \mathbb{R}^n$ the fol-*

lowing implication holds:

$$z_i(Az)_i \leq 0 \;\forall\, i \;\;\Rightarrow\;\; z_i(Az)_i = 0 \;\;\forall\, i.$$

A is said to be row sufficient if A^T is column sufficient.

A is sufficient if A and A^T are both column sufficient.

For details on sufficient matrices, see [112], [115] and [223].

DEFINITION 1.3.16. *A is said to be column competent if* $z_i(Az)_i = 0, \quad i = 1, 2, \cdots, n \implies Az = 0$.

A is said to be row competent if A^T is column competent.

A is competent if A and A^T are both column competent.

DEFINITION 1.3.17. *A is said to be column adequate if* $z_i(Az)_i \leq 0, \;\; i = 1, 2, \cdots, n \implies Az = 0$.

A is said to be row adequate if A^T is column adequate.

A is competent if A and A^T are both column adequate. For details on competent and adequate matrices, see [139].

DEFINITION 1.3.18. *$A \in \mathbb{R}^{n\times n}$ is called a Q-matrix (or a matrix satisfying Q-property) if for every $q \in \mathbb{R}^n$, LCP(q, A) has a solution.*

We say that A is a Q_0-matrix (or a matrix satisfying Q_0-property) if $F(q, A) \neq \emptyset$ implies $S(q, A) \neq \emptyset$.

A is said to be a completely Q (Q_0)-matrix if all its principal submatrices are Q (Q_0)-matrices.

DEFINITION 1.3.19. *$A \in \mathbb{R}^{n\times n}$ is said to be a semimonotone matrix (E_0-matrix) if for every $0 \neq z \geq 0$, $z \in \mathbb{R}^n$, $\exists$ an i such that $z_i > 0$ and $(Az)_i \geq 0$.*

DEFINITION 1.3.20. *$A \in \mathbb{R}^{n\times n}$ is said to be a strictly semimonotone matrix (E-matrix) if for every $0 \neq z \geq 0$, $z \in \mathbb{R}^n$, $\exists$ an i such that $z_i > 0$ and $(Az)_i > 0$.*

Definition 1.3.21. *A matrix A is called fully semimonotone (E_0^f) matrix if every legitimate PPT of A is E_0.*

Definition 1.3.22. *A is said to be an R-matrix (introduced by Karamardian) if for all $t \geq 0$, LCP(te, A) has only the trivial solution.*

Definition 1.3.23. *A is said to be an R_0-matrix if LCP$(0, A)$ has only the trivial solution.*

Definition 1.3.24. *A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an L_2-matrix if for every $0 \neq z \geq 0$, $z \in \mathbb{R}^n$, such that $Az \geq 0$, $z^T Az = 0$, $\exists$ two diagonal matrices $D_1 \geq 0$ and $D_2 \geq 0$ such that $D_2 z \neq 0$ and $(D_1 A + A^T D_2)z = 0$.*
$A \in \mathbb{R}^{n\times n}$ is said to be an L-matrix if $A \in E_0 \cap L_2$.

Definition 1.3.25. A real square matrix $A \in E(d)$, $d \in \mathbb{R}^n$ if (y, x), $x \neq 0$ is a solution for given LCP(d, A) indicates that $\exists$ nonzero $\bar{z} \geq 0$, $\bar{w} = -A^T \bar{z} \geq 0$, $\bar{z} \leq x$, $\bar{w} \leq y$.

Definition 1.3.26. A square matrix A with real entries belongs to $E^*(d)$ for $d \in \mathbb{R}^n$ if (y, x) is a solution of the given LCP$(d, A) \Rightarrow y = d$, $x = 0$.

According to [17] $E(d) = E^*(d)$ for each $d > 0$ or $d < 0$, $E(0) = L_2$ and $L(d) = E(d) \cap E(0)$. Hence, for $d > 0$, $A \in E(d)$ if LCP(d, A) has only zero solution $y = d$, $x = 0$. Todd [91] identifies a wider $E_1(d)$ and $L_1(d)$ enlarging $E(d)$ and $L(d)$ of Garcia [131] as follows:
Suppose (y, x) solves LCP(d, A) at least one $d \in \mathbb{R}^n$ where $A \in \mathbb{R}^{n\times n}$.

(a) For each β identified as $\{j \mid x_j > 0\} \subseteq \beta \subseteq \{j \mid y_j = 0\}$, determinant of principal submatrix of A considering β has positive value.

(b) For nonzero $z \geq 0$ with $w = -A^T z \geq 0$ and $z \leq x$, $w \leq y$.

Todd identifies $E_1(d) = \{A \mid \text{Condition (a) or (b) holds}\}$ and $L_1(d) = E_1(d) \cap E_1(0)$. Note that $L(d) \subseteq Q_0$ [131] and $L_1(d) \subseteq Q_0$ [91] if $d > 0$.

DEFINITION 1.3.27. *Let $S \subset \mathbb{R}^n$. If for any two vectors $x, y \in S$, their meet, which is defined by the vector $z = \min(x, y)$, also belongs to S, then S is called a meet semi-sublattice,*

THEOREM 1.3.1. *[113] Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix and $q \in \mathbb{R}^n$ be an arbitrary vector. Then the feasible region of LCP(q, A), which is denoted by FEA(q, A) is a meet semi-sublattice.*

DEFINITION 1.3.28. *Let $S \subset \mathbb{R}^n$. If there exists a vector $u \in \mathbb{R}^n$ such that $x \geq u \ \forall \ x \in S$, then S is called bounded below. If such a vector u belongs to S, then u is called a least element of the set S.*

Note that, if there exists a least element of a set, it must be unique.

THEOREM 1.3.2. *[113] Let S be a nonempty meet semi sub-lattice, which is closed and bounded below. Then S has a least element.*

THEOREM 1.3.3. *[113] Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix and $q \in \mathbb{R}^n$ be an arbitrary vector. If LCP(q, A) is feasible, then FEA(q, A) contains a least element u, which solves LCP(q, A).*

THEOREM 1.3.4. *[113] $A \in \mathbb{R}^{n\times n}$ is a Z-matrix if and only if for all vectors $q \in pos(I, -A)$, the feasible region of LCP(q, A) contains a least element, which is the solution of LCP.*

LEMMA 1.3.1. *[135] If z solves the linear program*

$$\min \;\; p^T x$$
$$\textit{subject to } Ax + q \geq 0,\; x \geq 0,$$
$$\textit{for an easily determined } p \in \mathbb{R}^n$$

and if a corresponding optimal dual variable y satisfies $(I - A^T)y + p > 0$, where I is the identity matrix, then z solves the linear complementarity problem LCP(q, A).

THEOREM 1.3.5. *[135] Consider the linear complementarity problem LCP*(q, A)*, where* A *is a* Z*-matrix. If FEA*$(q, A) \neq \emptyset$*, the least element solution* u *can be computed by the following linear program.*

$$\min \ p^T x$$
$$\textit{subject to } Ax + q \geq 0, \ x \geq 0,$$
$$\textit{where } p \geq 0.$$

COROLLARY 1.3.1. *Consider the linear complementarity problem LCP*(q, A)*, where* A *is a* Z*-matrix. If FEA*$(q, A) \neq \emptyset$*, then LCP*(q, A) *has a solution which can be obtained by solving the linear program*

$$\min \ e^T x$$
$$\textit{subject to } Ax + q \geq 0, \ x \geq 0,$$
$$\textit{where } e \textit{ is the vector of all } 1\textit{'s}.$$

THEOREM 1.3.6. *[135] Consider the linear complementarity problem LCP*(q, A)*, where* A *is a hidden* Z*-matrix with* $X,\ Y \in \mathbb{R}^{n\times n}$ *and* $r, s \in \mathbb{R}^n_+$ *such that* $AX = Y, \quad r^T X + s^T Y > 0$*. If FEA*$(q, A) \neq \emptyset$*, then the solution of linear complementarity problem can be computed by the following linear program.*

$$\min \ p^T x$$
$$\textit{subject to } Ax + q \geq 0, \ x \geq 0,$$
$$\textit{where } p = r + A^T s.$$

DEFINITION 1.3.29. *The function sgn* : $\mathbb{R} \to \mathbb{R}$ *is defined by*

$$sgn(x) = \begin{cases} 1 & \textit{if } x > 0 \\ -1 & \textit{if } x < 0 \end{cases}$$

LEMMA 1.3.2. *(Generalizations of Sard's Theorem[49]) Let* $U \subset \mathbb{R}^n$ *be an open set and* $f : \mathbb{R}^n \to \mathbb{R}^p$ *be smooth. We say* $y \in \mathbb{R}^p$ *is a regular value for* f *if*

Range $Df(x) = \mathbb{R}^p$ $\forall x \in f^{-1}(y)$, *where* $Df(x)$ *denotes the* $n \times p$ *matrix of partial derivatives of* $f(x)$.

LEMMA 1.3.3. *(Parameterized Sard Theorem [39]) Let* $V \subset \mathbb{R}^n, U \subset \mathbb{R}^m$ *be open sets, and let* $\phi : V \times U \to \mathbb{R}^k$ *be a* C^α *mapping, where* $\alpha > max\{0, m-k\}$. *If* $0 \in \mathbb{R}^k$ *is a regular value of* ϕ, *then for almost all* $a \in V, 0$ *is a regular value of* $\phi_a = \phi(a, .)$.

LEMMA 1.3.4. *(The inverse image theorem [39]) Let* $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^p$ *be* C^α *mapping, where* $\alpha > max\{0, n-p\}$. *Then* $\phi^{-1}(0)$ *consists of some* $(n-p)$ *dimensional* C^α *manifolds.*

LEMMA 1.3.5. *(Classification theorem of one-dimensional smooth manifold [40]) One-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.*

1.4 Nonlinear Complementarity Problem

The nonlinear complementarity problems is well studied in the literature on operations research [113], multiple objective programming problem [140], control theory, mathematical economics and engineering.

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$, and a vector $z \in \mathbb{R}^n$ such that $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$ and $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$. The complementarity problem is to find a vector $z \in \mathbb{R}^n$ such that

$$z^T f(z) = 0, \quad f(z) \geq 0, \quad z \geq 0. \tag{1.4.1}$$

When the function f is a nonlinear function, then it is called nonlinear complementarity problem. For details see [144].

In the literature, various techniques are developed to solve nonlinear complementarity problems. Several approaches such as fixed point, homotopy, projection and Newton method have appeared. For an extensive survey of the nonlinear complementarity problem see [67] and [70]. Josephy [71] presented a generalized Newton method for solving nonlinear complementarity problem. The Josephy–Newton method was shown to be convergent locally as well as quadratically. The basic idea is to linearize the nonlinear function $f(x)$ around the current iteration x^k and generate the next iteration x^{k+1} by solving the following problem:

$$f(x^k) + \nabla f(x^k)(x - x^k) \geq 0, \ x \geq 0,$$
$$[f(x^k) + \nabla f(x^k)(x - x^k)]^T x^k = 0.$$

However, this method does not converge globally. Therefore, the design of efficient global methods for solving the nonlinear complementarity problem becomes a challenging endeavor. Based on the above motivation, Pang [141] reformulated the nonlinear complementarity problem as a system of nonsmooth equations. They extend the classical Newton method for solving smooth equations generating from nonsmooth function. Harker and Xiao [72] also converted the nonlinear complementarity problem into a system of nonsmooth equations in a different way. They formulated the nonlinear complementarity problem as a system of B-differentiable equations through the use of the Minty-map. Pang and Gabriel [143] further combined the nonsmooth equations reformulation with sequential quadratic programming which is based on nonsmooth equations/successive quadratic programming (NE/SQP) based method. They established that the NE/SQP method converges both globally and locally under certain conditions. Mangasarian [186] reformulated the nonlinear complementarity problem as a system of smooth equations. Chen and Mangasarian [99] proposed smoothing methods where a class of smooth functions approximate certain nonsmooth functions arising in the reformulations of the nonlinear complementarity problem. The further study of those methods has been done by Chen and Harker [160]. Another approach is to reformulate the nonlinear complementarity problem as a smooth unconstrained minimization problem. Mangasarian and Solodov [152] introduced a smooth function in such a way that any global minimizer of the unconstrained minimization problem is a solution of the nonlinear complementarity problem. Yamashita and Fukushima [153] prove that any stationary point of the unconstrained minimization problem proposed by Mangasarian and Solodov is a solution of the nonlinear complementarity problem if the associated function f is continuously differentiable and strongly monotone in $\mathbb{R}^n$. This shows that any method for solving unconstrained minimization problem is applicable for the nonlinear complementarity problem as a special case. Kanzow [159] gave some approaches to characterize the nonlinear complementarity problem as unconstrained minimization problems. Reformulating the nonlinear complementarity problem as a smooth constrained minimization problem due to Fukushima's [168] and Auchmuty [169] is considered to be another direction. For further details of nonsmooth constrained minimization reformulations of the nonlinear complementarity problem, see [167]. Jiang and Qi [147] proposed a new nonsmooth equations-based method for the nonlinear complementarity problem. This work is more related to nonsmooth equations and smooth unconstrained minimization based methods. They transformed the nonlinear complementarity problem into a system of nonsmooth equations by employing a function introduced by Fischer [173]. Fischer [173] introduced the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $\phi(a,b) = \sqrt{a^2+b^2} - a - b$ to reformulate the Karush–Kuhn–Tucker (KKT) optimality conditions of nonlinear programming problems as systems of nonsmooth equations. Kanzow [159], [164] used this same function to reformulate nonlinear complementarity problems as smooth nonlinear programs or systems of smooth equations. In this connection Kostreva [204], [156] proposed a block-pivoting algorithm to exchange the basic and nonbasic variables. This algorithm extended Murty's scheme [155] known as Bard-type scheme for nonlinear complementarity problem. This method solves a system of nonlinear equations at each iteration. Kostreva established some convergence results for the case in which the nonlinear function f is a nondegenerate P-function. Kojima, Mizuno, Nome and Yoshise [187], [185], [80], [86] have developed interior-point algorithms for solving monotone linear and nonlinear complementarity problems. For the nonlinear case, the convergence theory exists for uniform P-functions, but only limited convergence results exist when the nonlinear function f is monotone.

Mangasarian[186] showed the equivalence of the nonlinear complementarity problem to a system of nonlinear equations. A number of methods are proposed to solve systems of equations. Newton and quasi-Newton methods are well-known iterative methods to solve nonlinear systems of equations. In recent years, researchers are interested to solve system of nonlinear equations both analytically and numerically. Several iterative methods have been developed using different techniques such as Taylor's series expansion, quadrature formulas, interpolation, decomposition and its various modification. For details, see [171], [172], [174], [157], [158], [177] and [178].

Chapter 2

Matrix Theoretic Properties And Solution Aspects Of Linear Complementarity Problem With Hidden Z- Matrix

2.1 Introduction

A generalization of Z-matrix was addressed by Mangasarian [135] to study the linear complementarity problems solvable as linear program. Pang [65] proposed this class as hidden Z-matrix. Though the class of hidden Z-matrix generalizes the class of Z-matrix, the completeness property of the class of Z-matrix is not carried over to the class of hidden Z matrix. That is for a hidden Z-matrix A, it is not guranteed that all proper principal submatrices of A are hidden Z. If all the principal submatrices of A are hidden Z then A is called completely hidden

*A part of this work has been published in the journal Linear and Multilinear Algebra [123].

†Another part of this work has been published in the journal Opsearch [124].

Z-matrix. Chu [138] studied the generalization of the class of hidden $Z \cap P$ matrix. Neogy [134] proved that hidden Z-matrix is a Q_0-matrix. They showed that the LCP(q, A) with hidden Z-matrix A is processable by Lemke's method. The linear complementarity problem alongwith a hidden Z-matrix received wide attention in the literature. The class of hidden Z-matrix is important in the context of mathematical programming and game theory.

The purpose of this chapter is to study some properties of hidden Z-matrix. Fiedler and Pták [83] studied Z-matrix in the context of linear complementarity problem. They showed that existence of a strictly positive vector x for a Z-matrix A such that $Ax \geq 0$ allows A to be P_0-matrix. In this chapter we extend this result in terms of hidden Z-matrix. The chapter is organized as follows. Section 2.2 presents some basic notations, required definitions and some relevant results used in this chapter. In section 2.3 main results are proved. We show a hidden Z-matrix under some additional conditions to be a P_0 matrix. We settle a result related to singular hidden Z-matrix. We illustrate our result by giving a suitable example of singular hidden Z-matrix. We show that linear complementarity problem with hidden Z-matrix has unique nondegenerate solution under some assumptions. Finally we show that a linear complementarity problem with hidden Z-matrix can be solved using linear programming problem. In this chapter we propose an interior point based iterative algorithm to solve LCP(q, A). We show that the proposed algorithm converges to a solution under certain feasibility conditions.

2.2 Preliminaries

Definition 2.2.1. *[113] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be almost P (P_0)-matrix if all its principal minors upto order $(n-1)$ are positive (nonnegative) and* $\det(A) < 0$.

Definition 2.2.2. *[113] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be S-matrix if there exists a vector $z > 0$ such that $Az > 0$ and $\bar{S}$-matrix if all its principal submatrices are S-matrix.*

Definition 2.2.3. *[113] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be $K\,(K_0)$-matrix if it is a Z-matrix as well as $P\,(P_0)$-matrix.*

Definition 2.2.4. *[77] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be type D if there exist some real numbers $\{\alpha_i\}_{i=1}^n$ with $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1$, such that*

$$a_{ij} = \begin{cases} \alpha_i \ \textit{if}\ i \le j; \\ \alpha_j \ \textit{if}\ i > j. \end{cases}$$

Now we give some theorems which will be required for discussion in the next section.

Theorem 2.2.1. *[113] Every Z-matrix is a hidden Z-matrix.*

Theorem 2.2.2. *[113] Let $A \in \mathbb{R}^{n\times n}$ be a K-matrix. Then the schur complement $A/A_{\alpha\alpha}$ is a K-matrix, where $\alpha \subset \{12, \cdots n\}$.*

Theorem 2.2.3. *[113] Let A be a Z matrix and $A_{\alpha\alpha}^{-1} \ge 0$. Then $A/A_{\alpha\alpha} \in Z$.*

Theorem 2.2.4. *[1] Let A be a K-matrix. Then $A^{-1} \ge 0$.*

Theorem 2.2.5. *[113] Let $A \in \mathbb{R}^{n\times n}$ be a hidden Z-matrix. Then for any two Z-matrices X and Y satisfying $AX = Y$ and $r^T X + s^T Y > 0$ for some $r, s \in \mathbb{R}^n_+$, the followings hold.*

(i) X is nonsingular and

(ii) there exists an index set $\alpha \subseteq \{1, 2, \cdots, n\}$ such that the matrix $\begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ is a K-matrix.

THEOREM 2.2.6. *[136] Let $A \in \mathbb{R}^{n\times n}$ be a hidden Z-matrix. Then A is a P-matrix if and only if A is an S-matrix.*

THEOREM 2.2.7. *[83] Let A be a Z-matrix. If $v(A)$ is strictly greater than zero then A is a P-matrix.*

THEOREM 2.2.8. *[83] Let A be a Z-matrix. If there exists a vector x strictly greater than zero such that $Ax \geq 0$, then A is a P_0-matrix.*

THEOREM 2.2.9. *[113] The classes E and $\bar{S}$ are identical i.e. $E = \bar{S}$.*

THEOREM 2.2.10. *[134] PPT of hidden Z-matrix is hidden Z.*

THEOREM 2.2.11. *[134] Let $A \in$ hidden Z. Then $A \in Q_0$.*

THEOREM 2.2.12. *[113] The classes Q_0 and Q are related through the following equation.*

$$Q = Q_0 \cap S.$$

2.3 Main Results On Hidden Z-Matrix

We first show that hidden Z-matrices are invariant under principal rearrangement.

THEOREM 2.3.1. *Suppose $A \in \mathbb{R}^{n\times n}$ is a hidden Z-matrix. Then PAP^T is a hidden Z-matrix for any permutation matrix P.*

Proof. Let A be a hidden Z-matrix. Then by the definition of hidden Z-matrix there exist two Z-matrices X, Y and two nonnegative vectors r, s such that $AX = Y$ and $r^TX + s^TY > 0$. Now for any permutation matrix P, $P^{-1}PAP^T(P^T)^{-1}X = Y$. Thus, $(PAP^T)(P^T)^{-1}X = PY$. Therefore $(PAP^T)(P^T)^{-1}XP^T = PYP^T$. Now letting $X_1 = (P^T)^{-1}XP^T$ and $Y_1 = PYP^T$, we get $(PAP^T)X_1 = Y_1$. It is easy to show that X_1 and Y_1 are Z-matrices. Now for $r_1, s_1 \in \mathbb{R}^n_+$, $r_1^TX_1 +$

$s_1^T Y_1 = r^T P^T ((P^T)^{-1} X P^T) + s^T P^{-1} (P Y P^T)$. Therefore $r_1^T X_1 + s_1^T Y_1 = r^T X P^T + s^T Y P^T = (r^T X + s^T Y) P^T > 0$. Thus PAP^T satisfies the definition of hidden Z-matrix. ■

Fiedler and Pták [83] proved that if there exists a vector $x > 0$ such that $Ax \geq 0$ for a Z-matrix A, then A is a P_0-matrix. We extend this result to hidden Z-matrix.

THEOREM 2.3.2. *Let A be a hidden Z-matrix with real entries. Suppose there exists a vector $x > 0$ such that $Ax \geq 0$. Then A is a P_0-matrix.*

Proof. We prove this result by induction method on n. The result is trivially true for $n = 1$. Consider that the result holds for all matrices of order less than n. Now A is a hidden Z-matrix with real entries. Then for some Z-matrices X and Y, $AX = Y$. Then there exists a vector $x > 0$ such that $Ax \geq 0$. This implies $YX^{-1}x \geq 0$ since X is nonsingular. Let $x_1 = X^{-1}x$ which implies $Xx_1 = x > 0$. Hence $Yx_1 \geq 0$. Then by the Theorem 2.2.5, there exists an index set $\alpha \subseteq \{1, 2, \cdots, n\}$ such that the matrix $W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ is a K-matrix and $Wx_1 \geq 0$. This gives $x_1 \geq 0$. Therefore X is a K-matrix and for $x > 0$, $X^{-1}x > 0$ since X is nonsingular. Therefore $x_1 > 0$ with $Yx_1 \geq 0$. Then by the Theorem 2.2.8 of [83], Y is a P_0-matrix. This implies Y is a K_0-matrix. Therefore $\det(A) \geq 0$. Now it is sufficient to prove that for any $\bar{\beta} \subset \{1, 2, \cdots, n\}$, the principal submatrix $A_{\bar{\beta}\bar{\beta}}$ of A is a hidden Z-matrix and there exists a $y > 0$ such that $A_{\bar{\beta}\bar{\beta}} y \geq 0$. Now $A_{\bar{\beta}\bar{\beta}}(X_{\bar{\beta}\bar{\beta}} - X_{\bar{\beta}\beta} X_{\beta\beta}^{-1} X_{\beta\bar{\beta}}) = Y_{\bar{\beta}\bar{\beta}} - Y_{\bar{\beta}\beta} X_{\beta\beta}^{-1} X_{\beta\bar{\beta}}$ which implies $A_{\bar{\beta}\bar{\beta}}(X/X_{\beta\beta}) = (M/X_{\beta\beta})$, where $M = \begin{bmatrix} X_{\beta\beta} & X_{\beta\bar{\beta}} \\ Y_{\bar{\beta}\beta} & Y_{\bar{\beta}\bar{\beta}} \end{bmatrix}$. Since $X \in K$, $X_{\beta\beta} \in K$. By Theorem 2.2.4, $X_{\beta\beta}^{-1} \geq 0$. Hence $(X/X_{\beta\beta})$ is a K-matrix and $(M/X_{\beta\beta})$ is a Z-matrix by Theorems 2.2.2 and 2.2.3. Therefore $A_{\bar{\beta}\bar{\beta}}$ is a hidden Z-matrix. Consider $x_1 = \begin{bmatrix} u_\beta \\ u_{\bar{\beta}} \end{bmatrix} > 0$ such that $Xx_1 > 0$ and $Yx_1 \geq 0$. Then

$$\begin{bmatrix} -X_{\bar\beta\beta}X_{\beta\beta}^{-1} & I \end{bmatrix}\begin{bmatrix} X_{\beta\beta} & X_{\beta\bar\beta} \\ X_{\bar\beta\beta} & X_{\bar\beta\bar\beta} \end{bmatrix}\begin{bmatrix} u_\beta \\ u_{\bar\beta} \end{bmatrix} = \begin{bmatrix} 0 & X/X_{\beta\beta} \end{bmatrix}\begin{bmatrix} u_\beta \\ u_{\bar\beta} \end{bmatrix}.$$

Now as $Xx_1 > 0$, $\begin{bmatrix} -X_{\bar\beta\beta}X_{\beta\beta}^{-1} & I \end{bmatrix}\begin{bmatrix} X_{\beta\beta} & X_{\beta\bar\beta} \\ X_{\bar\beta\beta} & X_{\bar\beta\bar\beta} \end{bmatrix}\begin{bmatrix} u_\beta \\ u_{\bar\beta} \end{bmatrix} > 0$. Then $(X/X_{\beta\beta})u_{\bar\beta} > 0$. Now consider $y_{\bar\beta} = (X/X_{\beta\beta})u_{\bar\beta} > 0$. Since $Xx_1 > 0$, $Yx_1 \geq 0$, then $Mx_1 \geq 0$. Therefore $\begin{bmatrix} -Y_{\bar\beta\beta}X_{\beta\beta}^{-1} & I \end{bmatrix}\begin{bmatrix} X_{\beta\beta} & X_{\beta\bar\beta} \\ Y_{\bar\beta\beta} & Y_{\bar\beta\bar\beta} \end{bmatrix}\begin{bmatrix} u_\beta \\ u_{\bar\beta} \end{bmatrix} = \begin{bmatrix} 0 & M/X_{\beta\beta} \end{bmatrix}\begin{bmatrix} u_\beta \\ u_{\bar\beta} \end{bmatrix} = (M/X_{\beta\beta})u_{\bar\beta} \geq 0$. So for $y_{\bar\beta} > 0$, $A_{\bar\beta\bar\beta}y_{\bar\beta} \geq 0$. This implies $\det(A_{\bar\beta\bar\beta}) \geq 0$. Therefore A is a P_0-matrix. ∎

REMARK 2.3.1. *The above result may not hold if the condition $x > 0$ is changed to $x \geq 0$. To illustrate our result we consider $A = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$. It is easy to show that there exists an $x \geq 0$ such that $Ax \geq 0$. Note that A is a hidden Z-matrix but not a P_0-matrix..*

COROLLARY 2.3.1. *Let A be a hidden Z-matrix with real entries and there exists a vector $x > 0$ such that $Ax \geq 0$. Then every Schur complement in A is a hidden Z-matrix as well as P_0-matrix.*

COROLLARY 2.3.2. *Let A be a hidden Z-matrix with real entries. If there exists a vector $x > 0$ such that $Ax \geq 0$, then the class of all the linear complementarity problems with the matrix A is NP-complete.*

Proof. Suppose A is a hidden Z-matrix with real entries and there exists a vector $x > 0$ such that $Ax \geq 0$. Then it follows from Theorem 2.3.2, A is a P_0-matrix. Now by using the result of [187] the class of LCP(q, A) is NP-complete. ∎

Consider a singular matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It is easy to show that $v(A) > 0$. We show the following result in the context of singular hidden Z-matrix.

THEOREM 2.3.3. *Let A be a singular hidden Z-matrix. Then $v(A) \not> 0$.*

Proof. We prove this result by contradiction. Let A be a singular hidden Z-matrix and $v(A) > 0$. We show that there exists an $\tilde{x} > 0$ such that $A\tilde{x} > 0$. By definition of value positivity there exists an $x \in \mathbb{R}^n_+$ such that $Ax > 0$. Let $\tilde{x} = x + \epsilon e > 0$, where $\epsilon > 0$. Then $A\tilde{x} = A(x + \epsilon e) = Ax + \epsilon Ae$. If $Ae \geq 0$, it is enough to choose any $\epsilon > 0$. If not. Let $a = \min_i (Ax)_i > 0$ and $b = \max_i |(Ae)_i|$. Now choose ϵ such that $a > \epsilon b$. This implies $\epsilon < a/b$. Now for $0 < \epsilon < a/b$, we can get $\tilde{x} = x + \epsilon e$ such that $\tilde{x} > 0$ and $A\tilde{x} > 0$. Now A is a hidden Z-matrix with $v(A) > 0$. We say that there exists an $\tilde{x} > 0$ such that $A\tilde{x} > 0$. Again as A is hidden Z-matrix then for some Z-matrices X and Y, $AX = Y$. Since X is nonsingular by the Theorem 2.2.5, then $YX^{-1}\tilde{x} > 0$. Let $\tilde{x}_1 = X^{-1}\tilde{x}$. Then $X\tilde{x}_1 > 0$ and $A\tilde{x} > 0$ implies $Y\tilde{x}_1 > 0$. Then by the Theorem 2.2.5, there exists an index set $\alpha \subseteq \{1, 2, \cdots, n\}$ such that the matrix $W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ is a K-matrix and $W\tilde{x}_1 > 0$. Let $\tilde{x}_2 = W\tilde{x}_1 > 0$. Then $\tilde{x}_1 = W^{-1}\tilde{x}_2 > 0$ since $W^{-1} \geq 0$. Hence for any $\tilde{x}_1 \geq 0$, $X\tilde{x}_1 > 0$ and $Y\tilde{x}_1 > 0$. Therefore $v(X) > 0$ and $v(Y) > 0$. Now as X and Y are Z-matrices then by the Theorem 2.2.7, X and Y are P-matrices. Thus we have $\det(Y) > 0$ and $\det(X^{-1}) > 0$. Therefore $\det(A) > 0$ which contradicts the fact that A is singular matrix. ∎

We consider a singular hidden Z-matrix A to show that $v(A) \not> 0$ with the help of the Theorem 2.3.3.

EXAMPLE 2.3.1. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Note that A is singular matrix. Now A is a hidden Z-matrix with* $X = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix}$ *and* $Y = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 3 \end{bmatrix}$.

Take $r = \begin{bmatrix} 1.6 \\ 4 \\ 2 \end{bmatrix}$ *and* $s = \begin{bmatrix} 4 \\ 0 \\ 0.1 \end{bmatrix}$, *then it is concluded that* $r^T X + s^T Y = \begin{bmatrix} 3.2 & 0.3 & 6.3 \end{bmatrix} > 0$. *Then by Theorem 2.3.4,* $v(A) \not> 0$.

Neogy et al. [134] show that if A is a hidden Z-matrix with $v(A) > 0$ and some additional assumptions, then A is a E_0-matrix. In this paper we show that a hidden Z-matrix with $v(A) > 0$ is a P-matrix.

THEOREM 2.3.4. *Let A be a hidden Z-matrix and $v(A) > 0$. Then A is a P-matrix.*

Proof. Let $A \in \mathbb{R}^{n \times n}$ with $v(A) > 0$. Then there exists an $x \in \mathbb{R}^n_+$ such that $Ax > 0$. In view of Theorem 2.3.3 $\exists\ \tilde{x} > 0$ such that $A\tilde{x} > 0$. Then by Theorem 2.2.6, A is a P-matrix. ■

REMARK 2.3.2. *For a hidden Z-matrix A with $v(A) > 0$, LCP(q, A) is processable by criss-cross method [5].*

Now we illustrate our result considering the following example.

EXAMPLE 2.3.2. *Let* $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. *For* $X = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$ *and* $Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $r = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$ *and* $s = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, *we obtain* $r^T X + s^T Y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} > 0$. *Hence A is a hidden Z-matrix. For* $x = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, $v(A) > 0$. *Therefore by Theorem 2.3.4, the system LCP(q, A) has a unique solution for each $q \in \mathbb{R}^n$.*

Here we propose a method to find whether a hidden Z-matrix A is P-matrix or P_0-matrix or not.

Algorithm:

Step I: Choose $\epsilon,\ \delta > 0$. Consider the following linear programming problem

$$\begin{aligned} \text{minimize} \quad & s \\ \text{subject to} \quad & Ax - se \geq 0 \\ & x \geq \delta e \\ & s \geq \epsilon \end{aligned} \tag{2.3.1}$$

If solution of the linear programming problem exists then by Theorem 2.3.4, A is a P-matrix, else go to Step II.

Step II: Choose $\epsilon = 0,\ \delta > 0$ and consider the linear programming problem (2.3.1). If the solution of the linear programming problem exists then by Theorem 2.3.2, A is a P_0-matrix, else decision is inconclusive.

Note that all 2×2 P-matrices are hidden Z but in general there are P-matrices which are not hidden Z [65]. Now we show the condition under which a P-matrix is a hidden Z-matrix. For this purpose we consider the D-matrix.

THEOREM 2.3.5. *Suppose A is positive type D-matrix. Then A is a hidden Z-matrix.*

Proof. Suppose A is positive type D-matrix. It is easy to show that positive type D-matrices are P-matrices. Then A is nonsingular and A^{-1} is Z-matrix as shown in [77] which in turn implies A^{-1} is hidden Z-matrix. Now A is a PPT of A^{-1}. Therefore, by Theorem 2.2.10 A is a hidden Z-matrix. ∎

It is known that inverse of an almost P-matrix is an N-matrix. To illustrate our result we consider $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. It is easy to show that A is an almost

P-matrix and A^{-1} is an N-matrix of first category. For further details see [42]. Now we prove the following theorem.

THEOREM 2.3.6. *Let A be a hidden Z-matrix with real entries. If A is an almost P-matrix then A^{-1} is an N-matrix of second category.*

Proof. Let A be a hidden Z-matrix with real entries. If A is an almost P-matrix then A^{-1} is an N-matrix. Suppose A^{-1} is an N-matrix of first category. Then by [203], A^{-1} is a Q-matrix. Therefore by Theorem 2.2.12 and Theorem 2.3.4 we arrive at a contradiction. This implies A^{-1} is an N-matrix of second category. ■

THEOREM 2.3.7. *Let A be a hidden Z-matrix with real entries. Assume that A is a E_0-matrix and every feasible basis of FEA(q, A) is nondegenerate. Let LCP(q, A) have a solution. Then the problem has a unique nondegenerate solution.*

Proof. Suppose A is a hidden Z-matrix with real entries then there exist two Z-matrices X, Y with two nonnegative vectors r, s such that

$$AX = Y,$$
$$r^T X + s^T Y > 0.$$

Consider $A(\epsilon) = A + \epsilon I$ for all $\epsilon \in (0, l)$, where

$$l = \frac{\min_i (r^T X + s^T Y)_i}{\max_i \mid (s^T X)_i \mid}.$$

As X is nonsingular by the Theorem 2.2.5, it is clear that $s^T X \neq 0$. Now $(A + \epsilon I)X = AX + \epsilon X = Y + \epsilon X$. Note that $Y + \epsilon X$ is a Z-matrix. Again $r^T X + s^T (Y + \epsilon X) > 0$ by the choice of ϵ. Hence $A(\epsilon)$ is a hidden Z-matrix. Note that A is a E_0-matrix. It is easy show that $(A + \epsilon I)$ is a E-matrix. Let $(-A_{\cdot k}, I_{\cdot \bar{k}})$, where $k \subseteq \{1, 2, \cdots, n\}$ and $\bar{k} \subseteq \{1, 2, \cdots n\} \setminus k$ denote a basis. By our assumption, $z_k = -(A_{kk})^{-1} q_k > 0$, $w_{\bar{k}} = q_{\bar{k}} - A_{\bar{k}k}(A_{kk})^{-1} q_k > 0$. For sufficiently small $\epsilon \in (0, l)$, $A(\epsilon)_{kk}$ is nonsingular and $z'_k = -(A(\epsilon)_{kk})^{-1}q_k > 0$ and $w'_{\bar{k}} = q_{\bar{k}} - A(\epsilon)_{\bar{k}k}(A(\epsilon)_{kk})^{-1}q_k > 0$. Therefore $z' = (z'_k, 0)$, $w' = (0, w'_{\bar{k}})$ is a nondegenerate solution to $LCP(q, A(\epsilon))$. Assume that $(-A_{\cdot p}, I_{\cdot \bar{p}})$ denotes another complementary feasible basis for LCP(q, A), where $k \neq p \subseteq \{1, 2, \cdots n\}$ and $\bar{p} \subseteq \{1, 2, \cdots n\} \setminus p$. Hence (w'', z'') is another nondegenerate solution to the LCP$(q, A(\epsilon))$. Note that by the Theorem 2.2.9, $A(\epsilon)$ is an $\bar{S}$-matrix. Then by the property 2 of [127], it contradicts that LCP$(q, A(\epsilon))$ has unique solution as $A(\epsilon)$ is a P-matrix. Therefore LCP(q, A) has a unique nondegenerate solution. ∎

Now we show some sufficient conditions under which a principal submatrix of a hidden Z-matrix will be hidden Z.

THEOREM 2.3.8. *Let A be a hidden Z-matrix with real entries such that $AX = Y$ and $r^T X + s^T Y > 0$ where X, Y are Z-matrices and $r, s \in \mathbb{R}^n_+$. If there exists an index set $\alpha \subset \{1, 2, \cdots, n\}$ such that $W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ and $\bar{W} = \begin{bmatrix} Y_{\alpha\alpha} & Y_{\alpha\bar{\alpha}} \\ X_{\bar{\alpha}\alpha} & X_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ are E-matrices. Then $A_{\alpha\alpha}$ and $A_{\bar{\alpha}\bar{\alpha}}$ are hidden Z-matrices.*

Proof. Note that A is a hidden Z-matrix with real entries then there exist two Z-matrices X, Y with two nonnegative vectors r, s such that

$$AX = Y,$$
$$r^T X + s^T Y > 0.$$

For an index set $\alpha \subset \{1, 2, \cdots n\}$, $W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ and $\bar{W} = \begin{bmatrix} Y_{\alpha\alpha} & Y_{\alpha\bar{\alpha}} \\ X_{\bar{\alpha}\alpha} & X_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ are Z-matrices as well as E-matrices. By Theorems 2.2.9, 2.2.1 and 2.2.6, W and $\bar{W}$ are K-matrices. Therefore, by Theorem 2.2.2 $W/X_{\alpha\alpha}$, $\bar{W}/X_{\bar{\alpha}\bar{\alpha}}$ are K-matrices. Also note that $X/X_{\alpha\alpha}$, $X/X_{\bar{\alpha}\bar{\alpha}}$ are Z-matrices. This implies that $A_{\alpha\alpha}$ is a hidden Z-matrix with nonsingular Z-matrix $X/X_{\bar{\alpha}\bar{\alpha}}$ and K–matrix $\bar{W}/X_{\bar{\alpha}\bar{\alpha}}$ such that $A_{\alpha\alpha}(X/X_{\bar{\alpha}\bar{\alpha}}) = \bar{W}/X_{\bar{\alpha}\bar{\alpha}}$. Similarly the principal submatrix $A_{\bar{\alpha}\bar{\alpha}}$ is a hidden Z-matrix with nonsingular Z-matrix $X/X_{\alpha\alpha}$ and K–matrix $W/X_{\alpha\alpha}$ such that $A_{\bar{\alpha}\bar{\alpha}}(X/X_{\alpha\alpha}) = W/X_{\alpha\alpha}$. ■

COROLLARY 2.3.3. *Let A be a hidden Z-matrix with real entries such that $AX = Y$ and $r^T X + s^T Y > 0$ where X, Y are Z-matrices and $r, s \in \mathbb{R}^n_+$. If there exists an index set $\alpha = \{1, 2, \cdots, n\}$ such that $W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ and $\bar{W} = \begin{bmatrix} Y_{\alpha\alpha} & Y_{\alpha\bar{\alpha}} \\ X_{\bar{\alpha}\alpha} & X_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ are E-matrices, then A is a completely hidden Z-matrix.*

Proof. Note that A is a hidden Z-matrix with real entries then there exist two Z-matrices X, Y with two nonnegative vectors r, s such that

$$AX = Y,$$
$$r^T X + s^T Y > 0.$$

For an index set $\alpha = \{1, 2, \cdots, n\}$, $W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} = X$ and $\bar{W} = \begin{bmatrix} Y_{\alpha\alpha} & Y_{\alpha\bar{\alpha}} \\ X_{\bar{\alpha}\alpha} & X_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} = Y$ are Z-matrices as well as E-matrices. By Theorems 2.2.9, 2.2.1 and 2.2.6, X and Y are K-matrices. This implies that for any $\beta \subset \{1, 2, \cdots, n\}$, by Theorem 2.2.2 $X/X_{\beta\beta}$ and $X/X_{\bar{\beta}\bar{\beta}}$ are K-matrices. Then the principal submatrix $A_{\beta\beta}$ of A is a hidden Z-matrix with K-matrix $X/X_{\bar{\beta}\bar{\beta}}$ and Z-matrix $\bar{M}/X_{\bar{\beta}\bar{\beta}}$ such that $A_{\beta\beta}(X/X_{\bar{\beta}\bar{\beta}}) = \bar{M}/X_{\bar{\beta}\bar{\beta}}$, where $\bar{M} = \begin{bmatrix} Y_{\beta\beta} & Y_{\beta\bar{\beta}} \\ X_{\bar{\beta}\beta} & X_{\bar{\beta}\bar{\beta}} \end{bmatrix}$. ■

COROLLARY 2.3.4. *Let A be a hidden Z-matrix with real entries such that $AX = Y$ and $r^T X + s^T Y > 0$ where X, Y are Z-matrices and $r, s \in \mathbb{R}^n_+$ and suppose there exists an empty index set α such that $W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ and $\bar{W} = \begin{bmatrix} Y_{\alpha\alpha} & Y_{\alpha\bar{\alpha}} \\ X_{\bar{\alpha}\alpha} & X_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ are E-matrices. Then A is a completely hidden Z-matrix.*

Now we introduce an alternative linear programming problem to solve linear complementarity problem with hidden Z-matrix.

THEOREM 2.3.9. *Let A be a hidden Z-matrix with real entries such that $AX = Y$ and $r^T X + s^T Y > 0$ where X, Y are Z-matrices and $r, s \in \mathbb{R}^n_+$. Then the linear complementarity problem denoted by LCP(q, A) can be written as*

$$\begin{aligned} \textit{minimize} \quad & (r + A^T s)^T z_1 + q^T z_2 \\ \textit{subject to} \quad & A^T s + r - A^T z_2 \geq 0, \\ & Az_1 + q \geq 0, \\ & z_1, z_2 \geq 0. \end{aligned}$$

Proof. To prove our result we consider LCP$(\tilde{q}, \mathcal{A})$ where $\mathcal{A} = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}$ and $\tilde{q} = \begin{bmatrix} p \\ q \end{bmatrix}$ with $p = r + A^T s$. By Lemma 1 of [135] and Lemma 3.3 of [134], LCP(q, A) and LCP$(\tilde{q}, \mathcal{A})$ are equivalent. Assume $\tilde{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ be the solution of LCP$(\tilde{q}, \mathcal{A})$. Note that $\mathcal{A}$ is a skew symmetric matrix. Now LCP$(\tilde{q}, \mathcal{A})$ can be written as

$$\begin{aligned} \text{minimize} \quad & \tilde{q}^T \tilde{z} + \tfrac{1}{2} \tilde{z}^T (\mathcal{A} + \mathcal{A}^T) \tilde{z} \\ \text{subject to} \quad & \tilde{q} + \mathcal{A} \tilde{z} \geq 0, \\ & \tilde{z} \geq 0. \end{aligned}$$

Again equivalent quadratic programming problem can be rewritten as

$$\begin{aligned} \text{minimize} \quad & (r + A^T s)^T z_1 + q^T z_2 \\ \text{subject to} \quad & A^T s + r - A^T z_2 \geq 0, \\ & Az_1 + q \geq 0, \\ & z_1, z_2 \geq 0. \end{aligned}$$

■

2.4 Hidden Z-matrix and Interior Point Algorithm

It is well-known that the linear complementarity problem can be solved by a linear program if A or its inverse is a Z-matrix. A number of authors have considered the special case of the linear complementarity problem under the restriction that A is a Z-matrix. Chandrasekharan [6] considered solving a sequence of linear inequalities. Lemke's algorithm is a well-known technique for solving linear complementarity problem [5]. In this chapter we propose an interior point based iterative algorithm to solve LCP(q, A). We show that the proposed algorithm converges to a solution under certain feasibility conditions.

Consider the potential function

$$\psi(u, v) = \kappa \log(u^T v) - \textstyle\sum_{i=1}^{n} \log\,(u_i v_i),$$

where $\kappa > n$. We start with a interior feasible point (u^0, v^0) such that $\psi(u^0, v^0) \leq O(\kappa L)$, where L is the size of input data A and q. The algorithm generates a sequence of interior feasible points $\{u^k, v^k : k \in \mathbb{N}\}$ so that $\psi(u^k, v^k) \leq -(\kappa - n)L$.

Algorithm:

Step 1: u^0 be a strictly feasible point of LCP(q, A) so that $v^0 = q + Au^0 > 0$ and $\beta \in (0, 1)$.

Step 2: Let (d_u^k, d_v^k) be the direction in the kth iteration. Now to find the search direction, consider the following problem

$$\text{minimize} \qquad (\nabla_u \psi_k)^T d_u + (\nabla_v \psi_k)^T d_v$$

$$\text{subject to} \qquad d_v = Ad_u, \qquad \|(U^k)^{-1} d_u\|^2 + \|(V^k)^{-1} d_v\|^2 \leq \beta^2.$$

Step 3: Then we have

$$\begin{bmatrix} (U^k)^{-1} d_u^k \\ (V^k)^{-1} d_v^k \end{bmatrix} = -\beta \frac{\alpha^k}{\|\alpha^k\|},$$

where

$$\alpha^k = \begin{bmatrix} \alpha_u^k \\ \alpha_v^k \end{bmatrix} = \begin{bmatrix} \frac{\kappa}{(u^T v)^k} U^k (v^k + A^T \pi^k) - e \\ \frac{\kappa}{(u^T v)^k} V^k (u^k - \pi^k) - e \end{bmatrix},$$

$\pi^k = ((V^k)^2 + A(U^k)^2 A^T)^{-1} (V^k - AU^k)(U^k v^k - \frac{(u^T v)^k}{\kappa} e)$ and e be the vector of all 1.

Step 4: Let $\beta = \min\{\frac{\|\alpha^k\|}{\kappa+2}, \frac{1}{\kappa+2}\} \leq 1/2$. We can write,

$$\psi(u^k + d_u^k, v^k + d_v^k) - \psi(u^k, v^k) \leq -\xi(\|\alpha^k\|^2),$$

where

$$\xi(\|\alpha^k\|^2) = \begin{cases} \frac{\|\alpha^k\|^2}{2(\kappa+2)}, & \text{if } \|\alpha^k\|^2 \leq \frac{(\kappa+2)^2}{4} \\ \frac{(\kappa+2)}{8}, & \text{otherwise.} \end{cases}$$

Step 5: Set

$$u^{k+1} = u^k + d_u^k,$$
$$v^{k+1} = v^k + d_v^k.$$

Step 6: If $(u^{k+1})^T v^{k+1} \leq \epsilon$, stop where ϵ is a very small positive quantity, else $k = k + 1$.

Now $\|\alpha^k\|^2 = h^T(u^k, v^k)H(u^k, v^k)h(u^k, v^k)$. We define the condition number $\zeta(q, A)$ for the LCP(q, A) as

$$\zeta(q, A) = inf\{\|h(u, v)\|_H^2 : u^T v \geq 2^{-L}, \psi(u, v) \leq O(\kappa L), u > 0, v = q + Au > 0\}.$$

The condition number $\zeta(q, A)$ represents the degree of difficulty for the proposed algorithm. If the condition number is bounded away from zero, the potential reduction algorithm will solve LCP(q, A) where A is a PSD matrix [129].

PROPOSITION 2.4.1. *[80] Let $\kappa \geq 2n + \sqrt{2n}$. Then, for A being a PSD matrix and any $q \in \mathbb{R}^n$, $\zeta(q, A) \geq 1$.*

Now we prove the following result.

THEOREM 2.4.1. *The proposed algorithm with $\kappa \geq 2n+\sqrt{2n}$ solves the LCP(q, A) in polynomial time where A is a hidden Z-matrix.*

Proof. To prove our result consider the LCP(q, A) where $A \in \mathbb{R}^{n\times n}$ is a hidden Z-matrix with $AX = Y$ and $r^T X + s^T Y > 0$ for some $r, s \geq 0$. Now construct LCP$(\tilde{q}, \mathcal{A})$ such that $\mathcal{A} = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}$ and $\tilde{q} = \begin{bmatrix} p \\ q \end{bmatrix}$, where $p = r + A^T s$, $r, s \geq 0$. If $\begin{bmatrix} x \\ y \end{bmatrix} \in F(\tilde{q}, \mathcal{A})$, then it can be derived from the Lemma 1.3.1 in line with Mangasarian [135] that $(I - A^T)y + p > 0$. Note that, $\mathcal{A}$ given in the LCP$(\tilde{q}, \mathcal{A})$

is a skew symmetric matrix. Therefore $\mathcal{A}$ is a PSD matrix. Now as $\kappa \geq 2n+\sqrt{2n}$ and $\mathcal{A}$ is a PSD matrix then $\zeta(\tilde{q},\mathcal{A}) \geq 1$ by the Proposition 2.4.1.
We consider the merit function $\psi(u,v) = \kappa \log(u^T v) - \sum_{i=1}^{n} \log\,(u_i v_i)$ for solving LCP$(\tilde{q},\mathcal{A})$, provided LCP$(\tilde{q},\mathcal{A})$ has a strictly feasible solution. Based on the concavity of log function and Lemma 3.1 of [129], we have

$$\psi(u^k + d_u^k, v^k + d_v^k) - \psi(u^k, v^k) \leq -\beta \left\|\alpha^k\right\| + \tfrac{\beta^2}{2}(\kappa + \tfrac{1}{1-\beta}).$$

Now by using Step 4 of the algorithm, we can write

$$\psi(u^k + d_u^k, v^k + d_v^k) - \psi(u^k, v^k) \leq -\xi(\left\|\alpha^k\right\|^2), \tag{2.4.1}$$

where

$$\xi(\left\|\alpha^k\right\|^2) = \begin{cases} \frac{\left\|\alpha^k\right\|^2}{2(\kappa+2)}, & \text{if } \left\|\alpha^k\right\|^2 \leq \frac{(\kappa+2)^2}{4} \\ \frac{\kappa+2}{8}, & \text{otherwise.} \end{cases} \tag{2.4.2}$$

Here $\left\|\alpha^k\right\|^2$ is used as the amount of reduction of the potential function at kth iteration. Now we find an interior feasible point for which each component is less than 2^L. The resulting point has a potential value less than $O(nL)$. Now from Equation (2.4.1) we say that the potential function is reduced by $O(\xi(\zeta(\tilde{q},\mathcal{A})))$ at every step of iteration. Hence in total of $O(\frac{nL}{\xi(\zeta(\tilde{q},\mathcal{A}))})$ iterations. Now we have $\psi(u^k, v^k) < -(\kappa - n)L$ and $(u^k)^T v^k < 2^{-L}$ [80]. It is easy to show from Lemma 1.3.1 that LCP(q,A) has a solution if and only if LCP$(\tilde{q},\mathcal{A})$ has a solution. Therefore the proposed algorithm with $\kappa \geq 2n + \sqrt{2n}$ solves the LCP(q,A) in polynomial time where A is a hidden Z-matrix. ■ ■

2.5 Numerical Illustration

Two numerical examples are considered to demonstrate the effectiveness and efficiency of the proposed algorithm.

EXAMPLE 2.5.3. *We consider the following example of LCP*(q, A)*, where*

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 6 & -2 \\ 1 & -1 & 3 \end{bmatrix} \text{ and } q = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}.$$

Note that A is a hidden Z-matrix with $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. *Therefore*

$$Y = AX = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 6 & -2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 6 & -2 \\ -2 & -1 & 3 \end{bmatrix}.$$

It is clear that Y is a Z-matrix.

Now consider $r,\ s \geq 0$ such that $r = \begin{bmatrix} 10 \\ 2 \\ 0 \end{bmatrix}$ *and* $s = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

Hence $r^T X + s^T Y = \begin{bmatrix} 10 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 6 & -2 \\ -2 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 10 & 2 & 0 \end{bmatrix} + \begin{bmatrix} -8 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 10 \end{bmatrix} > 0.$

Therefore A is a hidden Z-matrix. Consider a matrix $\mathcal{A}$ in the following form $\mathcal{A} = \begin{bmatrix} O & -A^T \\ A & O \end{bmatrix}$, *where* $A = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 6 & -2 \\ 1 & -1 & 3 \end{bmatrix}$. *Therefore* $\mathcal{A} =$

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 1 & 2 & -3 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ -2 & 6 & -2 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 \end{bmatrix}.$$

Now we have to find $p = r + A^T s$ *as shown in Theorem 1.3.6. Hence* $p = \begin{bmatrix} 10 \\ 2 \\ 0 \end{bmatrix}$

$$+ \begin{bmatrix} 1 & -2 & 1 \\ -1 & 6 & -1 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 10 \end{bmatrix}.$$

Now consider the LCP$(\tilde{q}, \mathcal{A})$, *where* $\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 1 & 2 & -3 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ -2 & 6 & -2 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 \end{bmatrix}$ *and*

$\tilde{q} = \begin{bmatrix} 12 \\ 4 \\ 10 \\ 5 \\ 0 \\ -2 \end{bmatrix}$. *Now consider* $u^0 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$. *It is easy to show that* $\mathcal{A}\ u^0 + \tilde{q} > 0$.

Therefore u^0 *is strictly feasible vector of the LCP*$(\tilde{q}, \mathcal{A})$. *Let us consider* $v^k = \mathcal{A}\ u^k + \tilde{q}$.

Iteration (k)	u^k	v^k	$\psi(u^k, v^k)$	d_u^k	d_v^k
1	$\begin{pmatrix} 2.8719 \\ 3.9097 \\ 2.9305 \\ 2.0487 \\ 1.0095 \\ 1.0458 \end{pmatrix}$	$\begin{pmatrix} 10.9244 \\ 1.0378 \\ 10.9302 \\ 1.0317 \\ 11.8535 \\ 5.7537 \end{pmatrix}$	22.6493	$\begin{pmatrix} -0.1281 \\ -0.0902 \\ -0.0694 \\ 0.0486 \\ 0.0095 \\ 0.0458 \end{pmatrix}$	$\begin{pmatrix} -0.0756 \\ 0.0378 \\ -0.0698 \\ 0.0317 \\ -0.1465 \\ -0.2463 \end{pmatrix}$
2	$\begin{pmatrix} 2.7492 \\ 3.8209 \\ 2.8640 \\ 2.0962 \\ 1.0189 \\ 1.0943 \end{pmatrix}$	$\begin{pmatrix} 10.8472 \\ 1.0772 \\ 10.8508 \\ 1.0642 \\ 11.6990 \\ 5.5203 \end{pmatrix}$	22.4402	$\begin{pmatrix} -0.1226 \\ -0.0888 \\ -0.0664 \\ 0.0474 \\ 0.0094 \\ 0.0485 \end{pmatrix}$	$\begin{pmatrix} -0.0772 \\ 0.0394 \\ -0.0793 \\ 0.0325 \\ -0.1545 \\ -0.2333 \end{pmatrix}$
3	$\begin{pmatrix} 2.6317 \\ 3.7335 \\ 2.8003 \\ 2.1422 \\ 1.0282 \\ 1.1458 \end{pmatrix}$	$\begin{pmatrix} 10.7684 \\ 1.1184 \\ 10.7611 \\ 1.0977 \\ 11.5373 \\ 5.2992 \end{pmatrix}$	22.2348	$\begin{pmatrix} -0.1174 \\ -0.0873 \\ -0.0636 \\ 0.0460 \\ 0.0093 \\ 0.0515 \end{pmatrix}$	$\begin{pmatrix} -0.0787 \\ 0.0412 \\ -0.0897 \\ 0.0335 \\ -0.1616 \\ -0.2211 \end{pmatrix}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	$\begin{pmatrix} 0.0581 \\ 0.3856 \\ 0.8266 \\ 0.1532 \\ 1.0813 \\ 3.8672 \end{pmatrix}$	$\begin{pmatrix} 10.1421 \\ 1.5326 \\ 0.7140 \\ 3.8458 \\ 0.5445 \\ 0.1523 \end{pmatrix}$	13.3456	$\begin{pmatrix} -0.0020 \\ -0.0053 \\ -0.0029 \\ -0.0055 \\ 0.0058 \\ 0.0093 \end{pmatrix}$	$\begin{pmatrix} 0.0078 \\ -0.0312 \\ -0.0219 \\ 0.0062 \\ -0.0218 \\ -0.0056 \end{pmatrix}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	$\begin{pmatrix} 0.0000001 \\ 0.25 \\ 0.75 \\ 0.0000004 \\ 1.3749 \\ 4.2499 \end{pmatrix}$	$\begin{pmatrix} 10.5 \\ 0.000006 \\ 0.000002 \\ 4 \\ 0.000001 \\ 0.0000003 \end{pmatrix}$	-12.21552	$\begin{pmatrix} -0 \\ -0 \\ -0 \\ -0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -0 \\ -0 \\ 0 \\ -0 \\ -0 \end{pmatrix}$

Table 2.1: Summary of computation for the proposed algorithm

Table 2.1 summarizes the computations for the first 3 iterations, 100th iteration and 500th iteration. At the 500th iteration, sequence $\{u^k\}$ *and* $\{v^k\}$ *produced by the proposed algorithm converges to the solution* u^* *and* v^* *of the* *given LCP(q, A) where* $z = u^* = \begin{pmatrix} 0 \\ 0.25 \\ 0.75 \end{pmatrix}$ *and* $w = v^* = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$.

Example 2.5.4. *We consider another example of LCP*(q, A), *where*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \text{ and } q = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}.$$

It is easy to show that A is not a positive definite matrix. Note that A is a hidden Z-matrix with $X = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0.8 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & -0.2 & 0 \\ -1 & 1 & 0 \\ -0.5 & -1.6 & 1 \end{bmatrix}$ *and* $r,\ s \geq 0$ *such that* $r = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ *and* $s = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *respectively. Now consider a matrix $\mathcal{A}$ in the following form* $\mathcal{A} = \begin{bmatrix} O & -A^T \\ A & O \end{bmatrix}$. *Again* $p = r + A^T s = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$. *We consider the LCP*$(\tilde{q}, \mathcal{A})$, *where* $\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \end{bmatrix}$ *and* $\tilde{q} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ -2 \\ 1 \\ 4 \end{bmatrix}$. *Now* *consider* $u^0 = \begin{bmatrix} 0.5 \\ 2 \\ 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$. *It is easy to show that* $\mathcal{A}\ u^0 + \tilde{q} > 0$. *Therefore* u^0 *is strictly feasible vector of the* LCP$(\tilde{q}, \mathcal{A})$. *Let us consider* $v^k = \mathcal{A}\ u^k + \tilde{q}$.

Iteration (k)	u^k	v^k	$\psi(u^k, v^k)$	d_u^k	d_v^k
1	$\begin{pmatrix} 0.5067 \\ 1.9926 \\ 2.9724 \\ 1.0738 \\ 2.0847 \\ 0.9702 \end{pmatrix}$	$\begin{pmatrix} 4.0109 \\ 5.8666 \\ 1.0297 \\ 0.4993 \\ 0.4932 \\ 2.9871 \end{pmatrix}$	19.8622	$\begin{pmatrix} 0.0067 \\ -0.0073 \\ -0.0275 \\ 0.0738 \\ 0.0847 \\ 0.0297 \end{pmatrix}$	$\begin{pmatrix} 0.0109 \\ -0.1333 \\ 0.0297 \\ -0.0006 \\ -0.0067 \\ -0.0128 \end{pmatrix}$
2	$\begin{pmatrix} 0.5131 \\ 1.9851 \\ 2.9438 \\ 1.1545 \\ 2.1721 \\ 0.9421 \end{pmatrix}$	$\begin{pmatrix} 4.0175 \\ 5.7297 \\ 1.0578 \\ 0.4983 \\ 0.4868 \\ 2.9734 \end{pmatrix}$	19.6938	$\begin{pmatrix} 0.0064 \\ -0.0074 \\ -0.0285 \\ 0.0807 \\ 0.0873 \\ -0.0281 \end{pmatrix}$	$\begin{pmatrix} 0.0066 \\ -0.1369 \\ 0.0281 \\ -0.0009 \\ -0.0064 \\ -0.0136 \end{pmatrix}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
600	$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 5 \\ 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$	-21.4802	$\begin{pmatrix} 0 \\ -0 \\ -0 \\ -0 \\ -0 \\ -0 \end{pmatrix}$	$\begin{pmatrix} -0 \\ -0 \\ 0 \\ -0 \\ -0 \\ 0 \end{pmatrix}$

Table 2.2: Summary of computation for the proposed algorithm

Table 2.2 summarizes the computations for the first 2 iterations and 600th iteration. At the 600th iteration, sequence $\{u^k\}$ *and* $\{v^k\}$ *produced by the proposed algorithm converges to the solution* u^* *and* v^* *of the given* LCP(q, A) *where*

$$z = u^* = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \textit{ and } w = v^* = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

In this chapter, we study the class of hidden Z-matrix in the context of linear complementarity problem. We extend the results of Fiedler and Pták for the linear system in complementarity problem using game theoretic approach. We establish a result related to singular hidden Z-matrix. We show that for a nondegenerate feasible basis, linear complementarity problem with hidden Z-matrix has unique nondegenerate solution under some assumptions. To prove our result we apply the concept of principal pivot transform and game theoretic approach. We establish certain matrix theoretic characterization of hidden Z-matrix to show the P_0 properties. We show that linear complementarity problem with hidden Z-matrix is processable by Lemke's algorithm as well as criss-cross method. We propose an interior point method to compute solution of linear complementarity problem LCP(q, A) given that A is a real square hidden Z-matrix and q is a real vector. We observe that our proposed algorithm can process LCP(q, A) in polynomial time under some assumptions. Two numerical examples are illustrated to support our result.

Chapter 3

Column Competent Matrices And Linear Complementarity Problem

3.1 Introduction

The w-uniqueness property is important in the context of dynamical systems under smooth unilateral constraints. Xu [139] introduced the column competent matrices. On uniqueness, quite a large number of results are available in the literature of operations research. The study of uniqueness property of the solution is important in the context of the theory of the complementarity system as well as the method applied for finding the solution. For details see [133], [42], [120], [5]. Ingleton [132] studied the w-uniqueness solutions to linear complementarity problem in the context of adequate matrices. Pang [111] studied local z-uniqueness of solutions of a linear complementarity problem. The LCP(q, A) has unique z-solution for all $q \in \mathbb{R}^n$ iff A is a P-matrix [113]. The w-uniqueness

*The work of this chapter has been published in an international conference proceedings ICMC 2021 by Springer Publication [76].

property is identified by a condition on A related to the notion of sign-reversing. Motivated by the w-uniqueness results, we consider column competent matrices in the context of LCP(q, A). The sufficient matrices capture many properties of positive semi definite matrices. The aim of this chapter is to study some matrix theoretic properties of the class of column competent matrix and establish some new results which are useful to the solution of the LCP(q, A).

The chapter is organised as follows. In section 3.2, we include few related notations and results. Section 3.3 presents some new results related to column competent matrices. We develop several matrix theoretic results of column competent matrices which are related to the solution of linear complementarity problem.

3.2 Preliminaries

We write $z = z^+ - z^-$ where $z_i^+ = \max(0, z_i)$ and $z_i^- = \max(0, -z_i)$ for any index i. A z-solution, $\tilde{z}$ is called locally unique if $\exists$ a neighborhood of $\tilde{z}$ within which $\tilde{z}$ is the only z-solution. A w-solution, $\tilde{w} = Az + q$, is called locally unique if $\exists$ a neighborhood of $\tilde{w}$ in which $\tilde{w}$ is the only w-solution. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ and the kernel of the function ψ is defined by $\ker\psi = \{z \in \mathbb{R}^n : \psi(z) = 0\}$. The kernel of a matrix $A \in \mathbb{R}^{n\times n}$ is defined by $\ker A = \{z \in \mathbb{R}^n : Az = 0\}$.

Column competent matrices can be singular or nonsingular matrices. Note that all singular matrices need not be column competent matrices. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ which is a singular matrix. For any $z = \begin{bmatrix} 0 \\ k \end{bmatrix}$, $k \in R,\ z_i(Az)_i = 0\ \forall\ i$ implies that $Az = 0$. Consider another $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to show that $z_i(Az)_i = 0\ \forall\ i$ does not imply $Az = 0$. Hence A is not a column competent matrix. Let $A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ 3 & 2 & 0 \end{bmatrix}$. For $z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $z_i(Az)_i = 0 \ \forall \ i$. But $Az = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \neq 0$. Here A is a nonsingular matrix but not a column competent matrix.

Now we define $\psi : \mathbb{R}^n \to \mathbb{R}^n$ where $\psi(z) = z * (Az)$. $z * (Az)$ is the Hadamard product defined by $(z * (Az))_i = z_i * (Az)_i, \ \forall \ i$. Note that the product is associative, distributive and commutative.

DEFINITION 3.2.1. *[139] In view of Hadamard product, a matrix A is said to be column competent if $ker\psi = kerA$.*

Column adequate matrices are related to column competent matrices. We start with definition of column adequate matrix.

DEFINITION 3.2.2. *[113] The matrix A is said to be column adequate if $z_i(Az)_i \leq 0, \ \forall \ i \implies Az = 0$.*

We state the following lemma and theorems which are useful for the subsequent sections.

LEMMA 3.2.1. *[139] The matrix A is said to be nondegenerate if and only if $ker\psi = \{0\}$.*

THEOREM 3.2.1. *[139] The following statements are equivalent.*

(i) A is column competent.

(ii) For all vector q, the LCP(q, A) has a finite number (possibly zero) of w-solutions.

(iii) For all vector q, any w-solution of the LCP(q, A), if it exists, must be locally w-unique.

THEOREM 3.2.2. *[139] The following statements are equivalent.*

(i) (a) A is column competent.

(b) A is a P_0-matrix.

(ii) A is column adequate.

THEOREM 3.2.3. *[113] Let $A \in \mathbb{R}^{n\times n}$ be a E_0-matrix. Then the following statements are equivalent.*

(i) $A \in R_0$.

(ii) $A \in R$.

DEFINITION 3.2.3. *[113] $A \in \mathbb{R}^{n\times n}$ is a principally nondegenerate matrix if it has no principal submatrix which has determinant zero.*

We establish a connection between competent matrices and adequate matrices using degree theoretic approach. We provide a brief details about degree theory in the subsequent section.

3.2.1 Degree theory

Let $f_A : \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise linear mapping for a given matrix $A \in \mathbb{R}^{n\times n}$ defined as $f_A(e_i) = e_i$ and $f_A(-e_i) = -Ae_i \ \forall\ i$. We write for any $z \in \mathbb{R}^n$,

$$f_A(z) = z^+ - Az^-.$$

For details see [120]. It is clear that LCP(q, A) is equivalent to find a vector $z \in \mathbb{R}^n$ such that $f_A(z) = q$. If z belongs to the interior of some orthants of $\mathbb{R}^n$

and $\det A_{\alpha\alpha} \neq 0$ where $\alpha = \{i : z_i < 0\}$, then the index of $f_A(z)$ at z is well defined and can be written as

$$\text{ind} f_A(q, z) = \text{sgn}(\det A_{\alpha\alpha}).$$

Note that the cardinality of $f_A^{-1}(q)$ denotes the number of solutions of LCP(q, A). Particularly, if q is nondegenerate with respect to A, each index of f_A is well defined and we can define local degree of A at q. It can be denoted as $\deg_A(q)$. For details see [113]. We state the following theorem from [120], which will be required to prove one of our result.

Theorem 3.2.4. *[113] Let $A \in \mathbb{R}^{n\times n}$ and $K(A)$ denotes the union of all the facets of the complementary cones of $(I, -A)$. Consider $q \in \mathbb{R}^n \setminus K(A)$ where q is nondegenerate with respect to A. Let $\beta \subseteq \{1, 2, \cdots, n\}$ be such that* $\det A_{\beta\beta} \neq 0$. *Suppose A' is a PPT of A with respect to β. Then $deg_{A'}(q') = sgn(\det A_{\beta\beta}) deg_A(q)$.*

3.3 Results on Column Competent Matrices

Hadamard product is important to characterize the complementary condition. Here we show that the property of column competent matrix is related to Hadamard product.

Theorem 3.3.1. *Suppose A is a column competent matrix and the function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\psi(z) = z * (Az)$ where $z * (Az)$ is the Hadamard product. Then $ker\psi = kerA$.*

Proof. Let A be a column competent matrix. Then for a vector $z \in \mathbb{R}^n, z_i(Az)_i = 0,\ i = 1, 2, \cdots, n \implies Az = 0$. Hence $z \in \ker\psi$ implies $z \in \ker A$. So we write $\ker\psi \subseteq \ker A$. Again by definition $\ker A \subseteq \ker\psi$. Therefore $\ker\psi = \ker A$. ■

The following result provides a characterization of nondegenerate column competent matrices.

THEOREM 3.3.2. *Let $A \in \mathbb{R}^{n\times n}$ be a nondegenerate column competent matrix. Then $A \in R_0$.*

Proof. Let A be a nondegenerate column competent matrix. By Lemma 3.2.1, $\ker\psi = \{0\}$ where $\psi(z) = z * Az$. By Theorem 3.3.1, we can write $\ker\psi = \ker A = \{0\}$. Let z be the solution of LCP$(0, A)$. Then $z_i(Az)_i = 0 \ \forall\ i$. This implies that $Az = 0$. Hence $z = 0$. Therefore, LCP$(0, A)$ has only one solution which is zero. Hence A is an R_0- matrix. ■

Note that column competent matrix need not be a P_0- matrix in general. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$. We show that A is a column competent matrix but not a P_0-matrix. Now we establish the following result.

THEOREM 3.3.3. *Suppose A is a column competent matrix with $A \in P_0$. Then for $0 \neq z \geq 0$, $(z, 0)$ is the solution of LCP$(0, A)$.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a column competent matrix with $A \in P_0$. Then for each $0 \neq z \in \mathbb{R}^n$, $\max_i z_i(Az)_i \geq 0, z_i \neq 0$. If $z_i(Az)_i = 0 \ \forall\ i$ implies that $Az = 0$, then $(z, 0), z \geq 0$ is the solution of LCP$(0, A)$. ■

Now we consider the matrix $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ is column competent as well as P_0 and $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$ is a solution of LCP$(0, A)$. Note that this can be explained using the Theorem 3.3.3.

THEOREM 3.3.4. *Let A be a column competent matrix. Suppose $z \geq 0$ and $z_i(Az)_i = 0 \ \forall\ i$. Then LCP$(0, A)$ has the solution $(z, 0)$.*

Proof. Since A is a column competent matrix, then for $z \geq 0$ and $z_i(Az)_i = 0\ \forall\ i$. This implies that $Az = 0$. Therefore $(z, 0)$ is the solution of LCP$(0, A)$. ■

Xu [139] showed that if A is a column competent matrix then DAD^T is a column competent matrix where D is a diagonal matrix. In the next theorem, we prove that column competent matrices with some additional assumptions are invariant under principal rearrangement.

THEOREM 3.3.5. *Suppose A is a column competent matrix. If for any $z \in \mathbb{R}^n$, either $z_i(Az)_i \geq 0$ or $z_i(Az)_i \leq 0$ for all i, then PAP^T is also column competent where P is a permutation matrix.*

Proof. Let for any $z \in \mathbb{R}^n$, $y = Pz$. Consider $y_i(PAP^Ty)_i = 0$ for all i. This implies that $(Pz)_i(PAP^TPz)_i = 0$ for all i. We know that

$$z^TP^TPAP^TPz = \sum_{i=1}^n (Pz)_i(PAP^TPz)_i = 0.$$

Hence $z^TAz = 0$ as $P^TP = I$. We write $\sum_{i=1}^n z_i(Az)_i = 0$. Considering the additional assumption that either $z_i(Az)_i \geq 0$ or $z_i(Az)_i \leq 0$ for all i, we obtain $z_i(Az)_i = 0\ \forall\ i$. As A is a column competent matrix, $Az = 0$. Therefore $AP^TPz = 0$. Hence $(PAP^T)(Pz) = 0$. Hence PAP^T is column competent. ■

Consider $A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 4 & 1 \end{bmatrix}$. Note that A is an R_0-matrix. Now for $z = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $z_i(Az)_i = 0\ \forall\ i$ but $Az \neq 0$. Hence A is not a column competent matrix. The class of nondegenerate matrices plays an important role to characterize certain uniqueness properties of the solutions of LCP(q, A). We prove the following theorem to establish the relation between principally nondegenerate matrices and column competent matrices.

THEOREM 3.3.6. *Let A be a principally nondegenerate matrix. Then A is column competent.*

Proof. Let A be a principally nondegenerate matrix. Assume that A is not a column competent matrix. Hence $\exists$ a $0 \neq z \in \mathbb{R}^n$ such that $z_i(Az)_i = 0,\ i = 1, 2, \cdots, n$ but $Az \neq 0$. Without loss of generality, consider $z = \begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix} \neq 0$ where $z_\alpha \neq 0, z_\beta = 0$ and $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix}$. Then we consider the following cases.

Case 1: Let $\alpha = \{1, 2, \cdots, n\}$ and $\beta = \emptyset$. Then $z = z_\alpha$ and $z_i(Az)_i = 0,\ i \in \alpha$. This implies $(Az)_i = 0\ \forall\ i$. As $z \neq 0$, $Az = 0$ implies that A is singular. This contradicts that the matrix A is a principally nondegenerate matrix.

Case 2: Let $\alpha \subset \{1, 2, \cdots, n\}$ and $\beta = \{1, 2, \cdots, n\} \backslash \alpha$. Consider $(z_\alpha)_i(A_{\alpha\alpha}z_\alpha)_i = 0$, for $i \in \alpha$. This implies $A_{\alpha\alpha}z_\alpha = 0$. As $z_\alpha \neq 0$, $A_{\alpha\alpha}$ is a singular matrix. This contradicts that the matrix A is a principally nondegenerate matrix.

Therefore A is a column competent matrix. ∎

Here we consider $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 1 & 1 \\ -3 & 2 & 0 \end{bmatrix}$. For $z = \begin{bmatrix} 2k \\ 3k \\ k \end{bmatrix}, k \in R,\ z_i(Az)_i = 0\ \forall\ i$ implies that $Az = 0$. Hence A is a column competent matrix. However A is neither an adequate matrix nor a sufficient matrix. For details of sufficient matrix see [44], [45], [46].

Now we prove the following sufficient condition related to the PPT of column competent matrices.

THEOREM 3.3.7. *Let $A_{\alpha\alpha}$ and the Schur complement $A/A_{\alpha\alpha}$ of the square matrix $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix}$ be nonsingular, where $\alpha \cup \beta = \{1, 2, \cdots, n\}$ and $\alpha \cap \beta = \emptyset$. If*

A is column competent, then $A' = \mathcal{P}_\alpha(A)$ is column competent, where the matrix A' is the principal pivot transform of the matrix A.

Proof. Let $w = A'z$ and $z * w = 0$ where $*$ is the Hadamard product. Thus we write

$$\begin{bmatrix} w_\alpha \\ w_\beta \end{bmatrix} = \begin{bmatrix} A'_{\alpha\alpha} & A'_{\alpha\beta} \\ A'_{\beta\alpha} & A'_{\beta\beta} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix}. \tag{3.3.1}$$

Now $z * w = 0$ can be written as $\begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix} * \begin{bmatrix} w_\alpha \\ w_\beta \end{bmatrix} = \begin{bmatrix} w_\alpha * z_\alpha \\ w_\beta * z_\beta \end{bmatrix} = 0$. Hence $\begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix}_i \left(\begin{bmatrix} A'_{\alpha\alpha} & A'_{\alpha\beta} \\ A'_{\beta\alpha} & A'_{\beta\beta} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix} \right)_i = 0 \ \forall\ i$. Since $A' = \mathcal{P}_\alpha(A)$, we have

$$\begin{bmatrix} z_\alpha \\ w_\beta \end{bmatrix} = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_\beta \end{bmatrix}. \tag{3.3.2}$$

For $z * w = 0$, $\begin{bmatrix} w_\alpha \\ z_\beta \end{bmatrix}_i \left(\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_\beta \end{bmatrix} \right)_i = 0 \ \forall\ i.$
The matrix A is column competent. This implies that $\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_\beta \end{bmatrix} = 0$. This follows that $\begin{bmatrix} z_\alpha \\ w_\beta \end{bmatrix} = 0$. From (3.3.1), we obtain $A'_{\beta\alpha} z_\alpha + A'_{\beta\beta} z_\beta = 0$. Then $A'_{\beta\beta} z_\beta = 0$ implies that $z_\beta = 0$ as $A'_{\beta\beta} = A/A_{\alpha\alpha}$ is nonsingular. Clearly, $w_\alpha = 0$. Hence $\begin{bmatrix} w_\alpha \\ w_\beta \end{bmatrix} = \begin{bmatrix} A'_{\alpha\alpha} & A'_{\alpha\beta} \\ A'_{\beta\alpha} & A'_{\beta\beta} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix} = 0$. Therefore A' is column competent. ∎

THEOREM 3.3.8. *Let A be a column competent matrix where $A_{\alpha\alpha}$ and the Schur complement $A/A_{\alpha\alpha}$ of the square matrix $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix}$ be nonsingular. If $A \in E_0 \cap R_0$ and A' is an R-matrix, where A' is the PPT of A, then A is column* *adequate.*

Proof. Suppose A is not a column adequate matrix but is column competent. By Theorem 3.2.2, A is not a P_0- matrix. Then there exists $\beta \subseteq \{1, 2, \cdots, n\}$ such that $\det(A_{\beta\beta}) < 0$. Let $A \in E_0 \cap R_0$. It follows from the Theorem 3.2.3 that $A \in R$. Then $\deg_A(q) = 1$ for any q. Let A' be a PPT of A and $A' \in R$. Hence $\deg_{A'}(q') = 1$. By Theorem 3.2.4, $\deg_{A'}(q') = \deg_A(q).sgn(\det(A_{\beta\beta}))$. It implies that $\deg_{A'}(q') = -1$. This contradicts that A is not a P_0-matrix. Therefore A is column adequate matrix. ∎

3.3.1 Solution of Linear Complementarity Problem with Column Competent Matrices

We begin with some examples of w-uniqueness of the solution. Consider the column competent matrix $A = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}, q = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This LCP$(q, A)$ has solution $z = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. In the neighbourhood of z there is another solution $z' = \begin{bmatrix} 4.0099 \\ 1.0033 \end{bmatrix}$ and $w' = w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We consider another matrix $A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \\ 1 & -1 & -1 \end{bmatrix}, q = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. For $z = \begin{bmatrix} 2k \\ k \\ k \end{bmatrix}$, $k \in R, z_i(Az)_i = 0 \ \forall \ i$ implies that $Az = 0$. So A is a column competent matrix. This LCP(q, A) has solution $z = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. In the neighbourhood of z there is another solution $z' = \begin{bmatrix} 4.02 \\ 4.01 \\ 1.01 \end{bmatrix}$ and $w' = w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now we prove the following two results in connection with locally w-uniqueness property of the column competent matrices. The following two results state the necessary and sufficient condition that A is a column competent matrix in the system of linear complementarity problem.

THEOREM 3.3.9. *Suppose (w^*, z^*) is the solution of LCP(q, A) such that $w^* = q + Az^*$. Let $\alpha = \{i : w_i{}^* > 0\}$, $\beta = \{i : w_i{}^* = 0\}$ be the index sets. Further consider that the submatrix $A_{\alpha\alpha}$ is nonsingular. If $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix}$ is a column competent matrix, then $(w_\alpha, z_\beta) = (0, 0)$ is the only solution of the system:*

$$\begin{aligned} z_\alpha = A'_{\alpha\alpha} w_\alpha + A'_{\alpha\beta} z_\beta = 0 \\ w_\beta = A'_{\beta\alpha} w_\alpha + A'_{\beta\beta} z_\beta = 0 \\ w_\alpha > 0 \\ z_\beta > 0, \end{aligned} \tag{3.3.3}$$

where $A'_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $A'_{\alpha\beta} = -(A_{\alpha\alpha})^{-1} A_{\alpha\beta}$, $A'_{\beta\alpha} = A_{\beta\alpha}(A_{\alpha\alpha})^{-1}$ and $A'_{\beta\beta} = A_{\beta\beta} - A_{\beta\alpha}(A_{\alpha\alpha})^{-1} A_{\alpha\beta}$.

Proof. Let A be a column competent matrix. Then by Theorem 3.2.1, it is locally w-unique. Suppose w^* is locally unique solution of LCP(q, A) such that $w^* = q + Az^*$ and the system (3.3.3) has a nonzero solution $(\bar{w}_\alpha, \bar{z}_\beta)$.

Now $\begin{bmatrix} \bar{z}_\alpha \\ \bar{w}_\beta \end{bmatrix} = \begin{bmatrix} A'_{\alpha\alpha} & A'_{\alpha\beta} \\ A'_{\beta\alpha} & A'_{\beta\beta} \end{bmatrix} \begin{bmatrix} \bar{w}_\alpha \\ \bar{z}_\beta \end{bmatrix} = 0$ implies that $\begin{bmatrix} \bar{w}_\alpha \\ \bar{w}_\beta \end{bmatrix} = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix} \begin{bmatrix} \bar{z}_\alpha \\ \bar{z}_\beta \end{bmatrix}$. Clearly, $\bar{w} = A\bar{z}$ and $(w^*)^T \bar{z} = 0$, $(\bar{w})^T z^* = 0$. Hence

$(w^* + \lambda\bar{w}, z^* + \lambda\bar{z})$ solves LCP(q, A) for all $\lambda \geq 0$. This contradicts the local uniqueness of w^*. Therefore, $(w_\alpha, z_\beta) = (0, 0)$ is the only solution of the system (3.3.3). ■

THEOREM 3.3.10. *Suppose* (w^*, z^*) *is the solution of LCP*(q, A) *such that* $w^* = q + Az^*$ *where* $\alpha = \{i : w_i{}^* > 0\}$ *and* $\beta = \{i : w_i{}^* = 0\}$. *Further suppose* $\begin{bmatrix} z_\alpha \\ w_\beta \end{bmatrix} = \begin{bmatrix} A'_{\alpha\alpha} & A'_{\alpha\beta} \\ A'_{\beta\alpha} & A'_{\beta\beta} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_\beta \end{bmatrix} = 0,\ w_\alpha > 0, z_\beta > 0$. *If* $(z_\alpha, z_\beta) = (0, 0)$ *is the only solution of* $w_\beta = A_{\beta\alpha}z_\alpha + A_{\beta\beta}z_\beta = 0$ *then* $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix}$ *is column competent.*

Proof. Suppose the matrix A is not column competent. So w^* is not locally unique. Now $\begin{bmatrix} z_\alpha \\ w_\beta \end{bmatrix} = \begin{bmatrix} A'_{\alpha\alpha} & A'_{\alpha\beta} \\ A'_{\beta\alpha} & A'_{\beta\beta} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_\beta \end{bmatrix} = 0$ implies that $\begin{bmatrix} w_\alpha \\ w_\beta \end{bmatrix} = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_\beta \end{bmatrix}$ and $(w^*)^T z = 0$, $(w)^T z^* = 0$. Hence $(w^* + \lambda w, z^* + \lambda z)$ solves LCP(q, A) for all $\lambda \geq 0$. Hence $\exists$ a sequence of vectors $\{\bar{w}^k\}$ such that each $(\bar{w}^k, \bar{z}^k) = (w^* + \lambda^k w, z^* + \lambda^k z)$ is a solution of LCP(q, A) with $\bar{w}^k = q + A\bar{z}^k$ and $\bar{w}^k_\alpha > 0, \bar{z}^k_\beta > 0$. By complementarity $\bar{z}^k_\alpha = 0, \bar{w}^k_\beta = 0$. Consider $v^k = \bar{w}^k - w^*$ and $u^k = \bar{z}^k - z^*$. The normalized sequence $\{v^k/\|v^k\|\}$ is bounded and converges to v^* as $k \to \infty$. Similarly, the normalized sequence $\{u^k/\|u^k\|\}$ is bounded and converges to u^* as $k \to \infty$. Now for all large k, we have $\bar{w}^k_\beta - w^*_\beta = \lambda^k w_\beta = 0 = A_{\beta\alpha}u^k_\alpha + A_{\beta\beta}u^k_\beta$. Thus dividing by $\|u^k\|$ and $k \to \infty$, we have $A_{\beta\alpha}u_\alpha{}^* + A_{\beta\beta}u_\beta{}^* = 0$. Therefore, $u^* = \begin{bmatrix} u_\alpha{}^* \\ u_\beta{}^* \end{bmatrix} \neq 0$ is the nonzero solution of system $w_\beta = A_{\beta\alpha}z_\alpha + A_{\beta\beta}z_\beta = 0$. It contradicts that $(z_\alpha, z_\beta) = (0, 0)$ is the only solution of the system $w_\beta = A_{\beta\alpha}z_\alpha + A_{\beta\beta}z_\beta = 0$. Hence A is column competent.
■

The complementary condition is an important issue in operations research.

The concept of matrix theoretic approach helps to develop many theories of linear complementary problem. In this study we consider column competent matrix in the context of local w-uniqueness property which is important both for the theory as well as solution method of complementarity problrm. We study some matrix theoretic properties of this class. The local w-uniqueness of the solutions to the linear complementarity problem can be identified by the column competent matrices. We establish some new results on w-uniqueness properties in connection with column competent matrices. These results are significant in the context of matrix theory as well as algorithms in operations research. We prove some results in connection with locally w-uniqueness property of column competent matrices. Finally we establish a connection between column competent matrices and column adequate matrices with the help of degree theory.

Chapter 4

Properties Of K- Type Block Matrices In The Context Of Complementarity Problem

4.1 Introduction

It is well-known that the linear complementarity problem can be solved by a linear program if A or its inverse is a Z-matrix [135]. A number of authors have considered the special case of the linear complementarity problem under the restriction that A is a Z-matrix. Chandrasekharan [6] considered Z-matrix solving a sequence of linear inequalities. Lemke's algorithm is a well-known technique for solving linear complementarity problem [113]. Mangasarian [135] proved that least element of the polyhedral set $\{u : q + Au \geq 0, u \geq 0\}$ in the sense of Cottle-Veinott can be obtained by a single linear program. It is

*Results of this chapter have been published in an international conference proceedings by Springer [225].

well-known that the quadratic programming problem

$$\begin{array}{ll} \text{minimize} & q^T u + \frac{1}{2} u^T A u \\ \text{subject to} & u \geq 0 \end{array}$$

can be formulated as a linear complementarity problem when A is symmetric positive semidefinite. Mangasarian showed that this problem can be solved using single linear program if A is a Z-matrix. For details see the Theorems 1.3.5, 1.3.6 given in chapter 1. In this chapter we introduce block triangular K-matrix and hidden block triangular K-matrix. We call these two classes collectively as K-type block matrix. We discuss the class of K-type block matrices in solution aspects for linear complementarity problem.

This chapter is organized as follows. Section 4.2 presents some basic notations, definitions and results. In section 4.3, we establish some results of these two matrix classes. We show that a linear complementarity problem with K-type block matrix can be solved using linear programming problem.

4.2 Preliminaries

Now we give some definitions, lemmas, theorems which will be required for discussions in the next section.

Lemma 4.2.1. *[113] If A is a P-matrix, then A^T is also a P-matrix.*

Lemma 4.2.2. *[113] Let A be a P-matrix. Then $v(A) > 0$.*

Definition 4.2.1. *[83] The spectral radius $\sigma(A)$ of A is defined as the maximum of the moduli $|\lambda|$ of all proper values λ of A.*

Lemma 4.2.3. *[83] Let A be a nonnegative matrix. Then there exists a proper value $p(A)$ of A, the Perron root of A, such that $p(A) \geq 0$ and $|\lambda| \leq p(A)$*

for every proper value λ of A. If $0 \leq A \leq B$ then $p(A) \leq p(B)$. Moreover, if A is irreducible, the Perron-Frobenius root $p(A)$ is positive, simple and the corresponding proper value may be chosen positive. According to the Perron-Frobenius theorem, we have $\sigma(A) = p(A)$ for nonnegative matrices.

DEFINITION 4.2.2. *[83] A matrix A is said to have dominant principal diagonal if $|a_{ii}| > \sum_{k \neq i} |a_{ik}|$ for each i.*

LEMMA 4.2.4. *[83] If A is a matrix with dominant principal diagonal, then $\sigma(I - H^{-1}A) < 1$, where H is the diagonal of A.*

THEOREM 4.2.1. *[83] The following four properties of a matrix are equivalent:*
(i) All principal minors of A are positive.
(ii) To every vector $x \neq 0$ there exists an index k such that $x_k y_k > 0$ where $y = Ax$.
(iii)To every vector $x \neq 0$, there exists a diagonal matrix D_x with positive diagonal elements such that the inner product $(Ax, D_x x) > 0$.
(iv)To every vector $x \neq 0$ there exists a diagonal matrix $H_x \geq 0$ such that the inner product $\langle Ax, H_x x \rangle > 0$.
(v)Every real proper value of A as well as of each principal minor of A is positive.

4.3 Main Results

In this chapter we introduce block triangular K-matrix and hidden block triangular K-matrix, which are defined as follows:
A matrix $A \in \mathbb{R}^{mn \times mn}$ is said to be a block triangular K-matrix if it is formed with block of K-matrices $A_{ij} \in \mathbb{R}^{m \times m}$, either in upper triangular forms or in lower triangular forms. Here i and j vary from 1 to n. For block upper triangular form of A, the blocks $A_{ij} = 0$ for $i < j$ and for block lower triangular form of A,

the blocks $A_{ij} = 0$ for $i > j$.

$$\text{Consider } A = \left[\begin{array}{cc|cc|cc} 1 & -1 & 0 & 0 & 0 & 0 \\ -1.5 & 2 & 0 & 0 & 0 & 0 \\ \hline 3 & -1 & 1 & -1 & 0 & 0 \\ -1 & 4 & -0.75 & 1 & 0 & 0 \\ \hline 1 & -1 & 1 & -0.5 & 5 & -1 \\ -0.5 & 1 & -0.5 & 1 & -10 & 6 \end{array}\right],$$

which is a block triangular K-matrix.

The matrix $A \in \mathbb{R}^{mn \times mn}$ is said to be hidden block triangular K-matrix if there exist two block triangular K-matrices X and Y such that $AX = Y$. As X and Y are block triangular K-matrices, there exist vectors $r, s \geq 0$ such that $r^T X + s^T Y > 0$. A is formed with block matrices either in upper triangular forms or in lower triangular forms. For block upper triangular form of A, the blocks $A_{ij} = 0$ for $i < j$ and X, Y are formed with K matrices in upper triangular form. Similarly for block lower triangular form of A, the blocks $A_{ij} = 0$ for $i > j$ and X, Y are formed with K matrices in lower triangular form.

$$\text{Consider } A = \left[\begin{array}{cc|cc} -1 & -1 & 0 & 0 \\ 5 & 4 & 0 & 0 \\ \hline -4.5 & -3 & 1 & 0.5 \\ 6.5 & 3.875 & -0.25 & 0.3125 \end{array}\right],$$

$$X = \left[\begin{array}{cc|cc} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ \hline 3 & 0 & 4 & -1 \\ -2 & 1 & 0 & 2 \end{array}\right] \text{ and } Y = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ \hline 2 & -1 & 4 & 0 \\ 0 & 1.5625 & -1 & 0.875 \end{array}\right],$$

such that $AX = Y$. Consider $r = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and $s = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$. Note that $r^T X + s^T Y > 0$. Then A is a hidden block triangular K-matrix.

THEOREM 4.3.1. *Let A be a block triangular K-matrix. Then LCP(q, A) is processable by Lemke's algorithm.*

Proof. Let A be a block triangular K-matrix. Then $\exists\ z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^{mn}$, where $z_i \in \mathbb{R}^m$ is a block column vector such that $z_i(Az)_i \leq 0\ \forall i \implies (z_1)_i(A_{11}z_1)_i \leq 0\ \forall i \implies z_1 = 0$, as $A_{11} \in K$; $(z_2)_i(A_{21}z_1 + A_{22}z_2)_i \leq 0\ \forall i \implies (z_2)_i(A_{22}z_2)_i \leq 0\ \forall i \implies z_2 = 0$, as $A_{22} \in K$. In similar way $(z_n)_i(A_{n1}z_1 + A_{n2}z_2 + \cdots + A_{nn}z_n)_i \leq 0\ \forall i \implies (z_n)_i(A_{nn}z_n)_i \leq 0\ \forall i \implies z_n = 0$, as $A_{nn} \in K$ and $z_1 = z_2 = \cdots = z_{n-1} = 0$. Hence A is a P-matrix. Therefore LCP(q, A) is processable by Lemke's algorithm. ∎

REMARK 4.3.1. *Let A be a block triangular K-matrix. Then LCP(q, A) is solvable by criss-cross method. For details see [5].*

THEOREM 4.3.2. *If A is a block triangular K-matrix and q is an arbitrary vector, then the feasible region of LCP(q, A) is a meet semi-sublattice.*

Proof. Let $F =$FEA(q, A). Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \in F$

are two feasible vectors, where $x, y \in \mathbb{R}^{mn}$ and $x_i, y_i \in \mathbb{R}^m\ \forall\ i$. So $x \geq 0, y \geq 0, Ax + q \geq 0, Ay + q \geq 0$.

Let $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} = \min(x, y)$. Then

$$Ax + q = \begin{bmatrix} A_{11}x_1 + q_1 \\ A_{21}x_1 + A_{22}x_2 + q_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + q_3 \\ \vdots \\ A_{n1}x_1 + A_{n2}x_2 + A_{n3}x_3 + \cdots + A_{nn}x_n + q_n \end{bmatrix} \geq 0.$$

$\implies x_1 \in \text{FEA}(q_1, A_{11},), x_2 \in \text{FEA}(A_{21}x_1 + q_1, A_{22}), \cdots, x_n \in \text{FEA}(A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{n(n-1)}x_{n-1} + q_n, A_{nn})$. In similar way $Ay + q \geq 0 \implies y_1 \in \text{FEA}(q_1, A_{11}),\ y_2 \in \text{FEA}(A_{21}x_1 + q_1, A_{22}), \cdots, y_n \in \text{FEA}(A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{n(n-1)}x_{n-1} + q_n, A_{nn})$. Suppose $z = \min(x, y) \implies z_1 = \min(x_1, y_1), z_2 = \min(x_2, y_2), \cdots, z_n = \min(x_n, y_n)$. $A_{ij} \in K \implies z_1 \in \text{FEA}(q_1, A_{11}) \implies A_{11}z_1 + q_1 \geq 0, z_2 \in \text{FEA}(A_{21}z_1 + q_2, A_{22}) \implies A_{22}z_2 + A_{21}z_1 + q_2 \geq 0, \cdots, z_n \in \text{FEA}(A_{n1}z_1 + A_{n2}z_2 + \cdots + A_{n(n-1)}z_{n-1} + q_n, A_{nn}) \implies A_{n1}z_1 + A_{n2}z_2 + \cdots + A_{n(n-1)}z_{n-1} + A_{nn}z_n + q_n \geq 0$. So $z = \min(x, y) \in \text{FEA}(q, A)$. Hence the feasible region is a meet semi-sublattice. ∎

Cottle et al. [113] showed that if F is a nonempty meet semi-sublattice, i.e. closed and bounded below, then F has a least element by Lemma 1.3.2. Now we show that if $\text{LCP}(q, A)$ is feasible, where A is a block triangular K-matrix, then $\text{FEA}(q, A)$ contains a least element l.

THEOREM 4.3.3. *Let A be a block triangular K-matrix and q be an arbitrary vector. If the LCP(q, A) is feasible, then FEA(q, A) contains a least element $l \in \mathbb{R}^{mn}$, where l solves the LCP(q, A).*

Proof. Let F =FEA(q, A). By Theorem 4.3.2, F is a meet semi-sublattice. Let LCP(q, A) be feasible. Then the set F is obviously nonempty and bounded below by zero. Then the existence of the least element $l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_n \end{bmatrix} \in \mathbb{R}^{mn}$, $l_i \in \mathbb{R}^m \ \forall \ i$ follows from Lemma 1.3.2. That is $l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_n \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = x \ \forall \ x \in F$ and $l \in F$.

Let F_i = FEA$(A_{i(i-1)}x_{i-1} + A_{i(i-2)}x_{i-2} + \cdots + A_{i2}x_2 + A_{i1}x_1 + q_i, A_{ii})$. Now it is clear that $x_1, l_1 \in F_1, x_2, l_2 \in F_2, \cdots, x_n, l_n \in F_n$, where $x, l \in F$. As A_{ii} are Z- matrices, l_i is the least element of $F_i \ \forall \ i \in \{1, 2, \cdots, n\}$ and l_i solves LCP$(A_{i(i-1)}l_{i-1} + A_{i(i-2)}l_{i-2} + \cdots + A_{i2}l_2 + A_{i1}l_1 + q_i, A_{ii})$. So $l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_n \end{bmatrix}$ solves LCP(q, A). ■

Mangasarian [135] showed that if z solves the linear program to minimize $p^T x$ subject to $Ax + q \geq 0, x \geq 0$ and the corresponding optimal dual variable y

satisfies $(I - A^T)y + p > 0$, then x solves the linear complementarity problem LCP(q, A) by Lemma 1.3.1. Here we show that if LCP(q, A) where A belongs to a block triangular K-matrix, has a solution which can be obtained by solving the linear program to minimize $p^T x$ subject to $Ax + q \geq 0$, $x \geq 0$.

THEOREM 4.3.4. *The linear complementarity problem LCP(q, A), where A is a block triangular K-matrix, has a solution which can be obtained by solving the linear program to minimize $p^T x$ subject to $Ax + q \geq 0$, $x \geq 0$, where $p = r \geq 0$ and Z_1 is a block triangular K-matrix with $r^T Z_1 > 0$.*

Proof. Let A be a block triangular K-matrix. The linear program

$$\min\ p^T x$$
$$\text{subject to } Ax + q \geq 0,\ \ x \geq 0$$

and the dual linear program,

$$\max -q^T y$$
$$\text{subject to } -A^T y + p \geq 0,\ \ y \geq 0$$

have solutions x and y respectively. The matrix A can be written as $D - U$, where

$$D = \begin{bmatrix} D_{11} & 0 & 0 & \cdots & 0 \\ D_{21} & D_{22} & 0 & \cdots & 0 \\ D_{31} & D_{32} & D_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{n1} & D_{n2} & D_{n3} & \cdots & D_{nn} \end{bmatrix},$$

D_{ij}'s are diagonal matrices with positive entries and

$$U = \begin{bmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ U_{21} & U_{22} & 0 & \cdots & 0 \\ U_{31} & U_{32} & U_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_{n1} & U_{n2} & U_{n3} & \cdots & U_{nn} \end{bmatrix},$$

U_{ij}'s are matrices with nonnegative entries. Consider $Z_1 = D - V$, a block triangular K-matrix and the matrix product $AZ_1 = D - W$, where

$$V = \begin{bmatrix} V_{11} & 0 & 0 & \cdots & 0 \\ V_{21} & V_{22} & 0 & \cdots & 0 \\ V_{31} & V_{32} & V_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n1} & V_{n2} & V_{n3} & \cdots & V_{nn} \end{bmatrix},$$

V_{ij}'s are matrices with nonnegative entries and

$$W = \begin{bmatrix} W_{11} & 0 & 0 & \cdots & 0 \\ W_{21} & W_{22} & 0 & \cdots & 0 \\ W_{31} & W_{32} & W_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_{n1} & W_{n2} & W_{n3} & \cdots & W_{nn} \end{bmatrix},$$

W_{ij}'s are matrices with nonnegtive entries. Since Z_1 is a block triangular K-matrix, Z_1 is a P-matrix. Hence $v(Z_1) > 0$ and by Lemma 4.2.1 $v({Z_1}^T) > 0$. Let $r \geq 0$ be the value of Z_1^T, then $r^T Z_1 > 0$. Now $0 < r^T Z_1 = p^T Z_1 = p^T Z_1 + y^T(-AZ_1 + D - W) = (p^T - y^T A)Z_1 + y^T(D - W) = (p^T - y^T A)(D - V) + y^T(D - W) \leq (p^T - y^T A + y^T)D$ as $p^T - y^T A \geq 0, y \geq 0, U \geq 0, V \geq 0$. This implies $(I - A^T)y + p > 0$, since D_{ij}'s are positive diagonal matrices. So by Lemma 1.3.1, x solves LCP(q, A), which is a solution of the problem to minimize $p^t x$ subject to $Ax + q \geq 0$, $x \geq 0$. ■

Corollary 4.3.1. *The solution of linear complementarity problem $LCP(q, A)$ with $A \in$ block triangular K-matrix can be obtained by solving the linear program* *to minimize $e^T x$ subject to $Ax + q \geq 0$, $x \geq 0$.*

THEOREM 4.3.5. *Let A be a block triangular K-matrix. Then A^{-1} exists and $A^{-1} \geq 0$.*

Proof. Assume that $Q = I - tA \geq 0, t > 0$. Let $p(Q)$ be the Perron-root of Q. Then we have $\det((1-p(Q))I - tA) = \det(Q - p(Q)I) = 0$. By Theorem 4.2.1, $0 < p(Q) < 1$. Thus the series $I + Q + Q^2 + \cdots$ converges to the matrix $(I-Q)^{-1} = (tA)^{-1} \geq 0$, since $Q^k \geq 0$ for $k = 1, 2, \cdots$. Therefore A^{-1} exists and $A^{-1} \geq 0$. ∎

THEOREM 4.3.6. *Let A be a block triangular K-matrix and M be a Z-matrix such that $M \leq A$. Then both M^{-1} and A^{-1} exist and $M^{-1} \geq A^{-1} \geq 0$.*

Proof. Let A be a block triangular K-matrix and M ba a Z-matrix such that $M \leq A$. Assume that $R = I - \alpha A \geq 0, \alpha > 0$. Then $S = I - \alpha M \geq R \geq 0$. Let $p(R)$ be a Perron root of R. Then we have $\det((1-p(R))I - \alpha A) = \det(R - p(R)I) = 0$. By Theorem 4.2.1, $0 < p(R) < 1$. Thus the series $I + R + R^2 + \cdots$ converges to the matrix $(I-R)^{-1} = (\alpha A)^{-1}$. Since $S^k \geq R^k \geq 0$, for $k = 1, 2, \cdots$, and the series $I + S + S^2 + \cdots$ converges to the matrix $(I-S)^{-1} = (\alpha M)^{-1}$. Therefore M^{-1} and A^{-1} exist and $M^{-1} \geq A^{-1} \geq 0$.

∎

COROLLARY 4.3.2. *Assume that A, B are block triangular K-matrices such that $A \leq B$. Then both A^{-1} and B^{-1} exist and $A^{-1} \geq B^{-1} \geq 0$.*

THEOREM 4.3.7. *Let A be a hidden block triangular K- matrix. Then every diagonal block of A is a hidden Z- matrix.*

Proof. Let A be a hidden block triangular K- matrix with $AX = Y$, where X and Y are block triangular K-matrices. Let

$$A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix},$$

$$X = \begin{bmatrix} X_{11} & 0 & 0 & \cdots & 0 \\ X_{21} & X_{22} & 0 & \cdots & 0 \\ X_{31} & X_{32} & X_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{n1} & X_{n2} & X_{n3} & \cdots & X_{nn} \end{bmatrix} \text{ and}$$

$$Y = \begin{bmatrix} Y_{11} & 0 & 0 & \cdots & 0 \\ Y_{21} & Y_{22} & 0 & \cdots & 0 \\ Y_{31} & Y_{32} & Y_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{n1} & Y_{n2} & Y_{n3} & \cdots & Y_{nn} \end{bmatrix}.$$

The block diagonal of AX are $A_{ii}X_{ii}$ for $i \in \{1, 2, \cdots n\}$. So $A_{ii}X_{ii} = Y_{ii}$ for $i \in \{1, 2, \cdots n\}$. X_{ii}, Y_{ii} are K-matrices. Then X_{ii}^T, Y_{ii}^T are also K-matrices. So $v(X_{ii}^T) > 0$, $v(Y_{ii}^T) > 0$. Let $r_i, s_i \in \mathbb{R}^m{}_+$ such that $X_{ii}^T r_i + Y_{ii}^T s_i > 0 \implies r_i^T X_{ii} + s_i^T Y_{ii} > 0$. Hence the block diagonals of A are hidden Z-matrices. ∎

THEOREM 4.3.8. *Let A be a hidden block triangular K-matrix. Then every determinant of block diagonal matrices of A are positive.*

Proof. Let A be a hidden block triangular K-matrix with $AX = Y$, where X and Y are block triangular K-matrices. Let

$$A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix},$$

$$X = \begin{bmatrix} X_{11} & 0 & 0 & \cdots & 0 \\ X_{21} & X_{22} & 0 & \cdots & 0 \\ X_{31} & X_{32} & X_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{n1} & X_{n2} & X_{n3} & \cdots & X_{nn} \end{bmatrix} \text{ and } Y = \begin{bmatrix} Y_{11} & 0 & 0 & \cdots & 0 \\ Y_{21} & Y_{22} & 0 & \cdots & 0 \\ Y_{31} & Y_{32} & Y_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{n1} & Y_{n2} & Y_{n3} & \cdots & Y_{nn} \end{bmatrix}.$$

The block diagonal of AX are $A_{ii}X_{ii}$ for $i \in \{1, 2, \cdots n\}$. So $A_{ii}X_{ii} = Y_{ii}$ for $i \in \{1, 2, \cdots n\}$. X_{ii}, Y_{ii} are K-matrices. Then $\det(X_{ii}), \det(Y_{ii}) > 0 \ \forall\ i$. Hence $\det(A_{ii}) > 0 \ \forall\ i$. ■

Corollary 4.3.3. *Every block triangular K-matrix is a hidden block triangular K-matrix.*

Proof. Let A be a block triangular K-matrix. Taking $X = I$, the identity matrix, it is clear that A is a hidden block triangular K-matrix. ■

Theorem 4.3.9. *The linear complementarity problem LCP(q, A), where A is a hidden block triangular K-matrix with $AX = Y$, X and Y are block triangular K-matrices, has a solution which can be obtained by solving the linear program to minimize $p^T x$ subject to $Ax + q \geq 0$, $x \geq 0$, where $p = r + A^T s \geq 0$ and $r, s \geq 0$ such that $X^T r > 0$ and $Y^T s > 0$.*

Proof. Let A be a hidden block triangular K- matrix with $AX = Y$, where X and Y are block triangular K-matrices. The linear program to minimize $p^T x$ subject to $Ax + q \geq 0$, $x \geq 0$ and the dual linear program to maximize $-q^T y$ subject to $-A^T y + p \geq 0$, $y \geq 0$ have solutions x and y respectively. X can be written as $D-U$, where

$$D=\begin{bmatrix} D_{11} & 0 & 0 & \cdots & 0 \\ D_{21} & D_{22} & 0 & \cdots & 0 \\ D_{31} & D_{32} & D_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{n1} & D_{n2} & D_{n3} & \cdots & D_{nn} \end{bmatrix},$$

D_{ij}'s are diagonal matrices with positive entries and

$$U=\begin{bmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ U_{21} & U_{22} & 0 & \cdots & 0 \\ U_{31} & U_{32} & U_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_{n1} & U_{n2} & U_{n3} & \cdots & U_{nn} \end{bmatrix},$$

U_{ij}'s are matrices with nonnegative entries.

Y can be written as $D-V$. Then the matrix product AX can be written as $D-V$, where

$$V=\begin{bmatrix} V_{11} & 0 & 0 & \cdots & 0 \\ V_{21} & V_{22} & 0 & \cdots & 0 \\ V_{31} & V_{32} & V_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n1} & V_{n2} & V_{n3} & \cdots & V_{nn} \end{bmatrix},$$

V_{ij}'s are matrices with nonnegative entries.

As X, Y are block triangular K-matrices, so they are P-matrices. So $v(X) > 0, v(Y) > 0$. Let $r \geq 0$ be the value of X^T and $s \geq 0$ be the value of Y^T. Then

$0 < r^TX + s^TY = (r^T + s^TA)X = p^TX = p^T(D-U)$

$= p^T(D-U) + y^T(-AD + AU + D - V), \quad \text{since } A(D-U) = D-V$

$= (p^T - y^TA)(D-U) + y^T(D-V)$

$\leq (y^T(I-A) + p^T)D, \quad \text{since } -y^TA + p^T \geq 0, U \geq 0, V \geq 0, y \geq 0.$

Now D_{ij}'s are diagonal matrices with positive entries and D is formed with the block matrices D_{ij}'s. Hence $y^T(I-A)+p^T > 0$. By Lemma 1.3.1, x solves the LCP(q,A), which is a solution of the problem to minimize p^Tx subject to $Ax+q \geq 0$, $x \geq 0$, where $p = r + A^Ts \geq 0$ and $r,s \geq 0$ such that $X^Tr > 0$ and $Y^Ts > 0$. ∎

LEMMA 4.3.1. *Let A be a hidden block triangular K-matrix. Consider the LCP($\mathcal{A}$,$\bar{q}$), where* $\mathcal{A} = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}$, $\tilde{q} = \begin{bmatrix} r + A^Ts \\ q \end{bmatrix}$ *and r,s as mentioned in Theorem 4.3.9. If* $\begin{bmatrix} x \\ y \end{bmatrix} \in$ *FEA($\tilde{q}$, $\mathcal{A}$), then $(I-A^T)y+p > 0$, where $p = r + A^Ts$.*

Proof. Suppose $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{FEA}(\tilde{q}, \mathcal{A})$. Since A is a hidden block triangular K-matrix, there exist two block triangular K-matrices X and Y such that $AX = Y$ and $r,s \geq 0$, $r^TX + s^TY > 0$. Let $X = D - U$ and $Y = D - V$, where U and V are two square matrices with all nonnegative entries and D is a block triangular diagonal matrix with positive entries as mentioned in Theorem 4.3.9. Then $0 < r^TX + s^TY = r^TX + s^TAX = p^T(D-U) = p^T(D-U) + y^T(Y-AX) = p^T(D-U) + y^T(D-V-A(D-U)) = (-y^TA+p^T)(D-U) + y^T(D-V) \leq (y^T(I-A)+p^T)D$ since $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{FEA}(\tilde{q}, \mathcal{A})$, $U \geq 0, V \geq 0$. Since D is a positive block triangular diagonal matrix, $(I-A^T)y+p > 0$. ∎

THEOREM 4.3.10. *LCP($\tilde{q}$, $\mathcal{A}$) has a solution iff LCP(q,A) has a solution.*

Proof. Suppose LCP($\tilde{q}$, $\mathcal{A}$) has a solution. Let $\bar{z} = \begin{bmatrix} x \\ y \end{bmatrix} \in \text{SOL}(\tilde{q}, \mathcal{A})$. From the complementarity condition it follows that $x^T(p-A^Ty) + y^T(Ax+q) = 0$. Since $p - A^Ty, Ax+q, x, y \geq 0$, and $x^T(p-A^Ty) = 0, y^T(Ax+q) = 0$. By Lemma 4.3.1, it follows that $y + (p - A^Ty) > 0$. This implies for all i either $(p-A^Ty)_i > 0$

or $y_i > 0$. Now if $(p - A^T y)_i > 0$, then $x_i = 0$. If $y_i > 0$ then $(q + Ax)_i = 0$. This implies $x_i(q + Ax)_i = 0\ \forall\ i$. Therefore x solves LCP(q, A).

Conversely, x solves LCP(q, A). Let $y = s$, where s as mentioned in Theorem 4.3.9. Here $p - A^T y = r + A^T s - A^T y = r + A^T s - A^T s = r \geq 0$. So $\bar{z} = \begin{bmatrix} x \\ s \end{bmatrix} \in$ FEA($\tilde{q}$, $\mathcal{A}$). Further $\mathcal{A}$ is PSD-matrix, which implies that $\mathcal{A} \in Q_0$. Therefore $\bar{z}$ solves the LCP($\tilde{q}$, $\mathcal{A}$). ∎

Theorem 4.3.11. *All hidden block triangular K-matrices are Q_0.*

Proof. Let A be a hidden block triangular K-matrix. It is clear that the feasibility of LCP(q, A) implies the feasibility of LCP($\tilde{q}$, $\mathcal{A}$). Note that $\mathcal{A} \in Q_0$. This implies that the feasible point of LCP($\tilde{q}$, $\mathcal{A}$) is also a solution of LCP($\tilde{q}$, $\mathcal{A}$). Hence by Theorem 4.3.10, feasibility of LCP(q, A) ensures the solvability of LCP(q, A). Therefore A is a Q_0-matrix. ∎

Remark 4.3.2. *Let* $A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix}$, *where $A_{ij} \in \mathbb{R}^{m \times m}$ are K-matrices.*

$$\text{Let } z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} \text{ and } q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix}, \text{ where } z_i, q_i \in \mathbb{R}^m.$$

$$\textit{Then } Az+q = \begin{bmatrix} A_{11}z_1+q_1 \\ A_{21}z_1+A_{22}z_2+q_2 \\ A_{31}z_1+A_{32}z_2+A_{33}z_3+q_3 \\ \vdots \\ A_{n1}z_1+A_{n2}z_2+A_{n3}z_3+\cdots+A_{nn}z_n+q_n \end{bmatrix}.$$

First we solve LCP(q_1, A_{11}) *and obtain the solution* $w_1 = A_{11}z_1 + q_1, {w_1}^T z_1 = 0$. *Then we solve LCP*$(A_{21}z_1 + q_2, A_{22})$ *and obtain the solution* $w_2 = A_{22}z_2 + A_{21}z_1 + q_2, {w_2}^T z_2 = 0$. *Finally we solve LCP*$(A_{n1}z_1 + A_{n2}z_2 + A_{n3}z_3 + \cdots + A_{n(n-1)}z_{n-1} + q_n, A_{nn})$ *and obtain the solution* $w_n = A_{nn}z_n + A_{n1}z_1 + A_{n2}z_2 + A_{n3}z_3 + \cdots + A_{n(n-1)}z_{n-1} + q_n, {w_n}^T z_n = 0$.

So $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^{mn}$ *and* $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^{mn}$ *solve LCP*(q, A), *where* $w_i, z_i \in \mathbb{R}^m$.

In this chapter, we introduce K-type block matrices which include two new classes of block matrices namely block triangular K-matrices and hidden block triangular K-matrices in the context of solution of linear complementarity problem. We show that the linear complementarity problem with K-type block matrix is solvable by linear program. The linear complementarity problem with block triangular K-matrix is also processable by Lemke's algorithm as well as criss-cross method. We show that the hidden block triangular K-matrix is a Q_0-matrix. The purpose of this article is to study the properties of newly introduced K-type block matrices in the context of the solution of linear complementarity problem.

Chapter 5

Solution Method Of Linear Complementarity Problem Using Predictor-Corrector Approach

5.1 Introduction

Among the many facets of research in linear complementarity problems, the area that has received thorough attention in recent years is the development of robust and efficient algorithms for solving linear complementarity problems. Lemke's algorithm is a well-known technique to solve linear complementarity problem. But this algorithm does not solve every instance of the linear complementarity problem and in some situations the algorithm may terminate inconclusively without either computing a solution of it or showing that no solution exists. Later a path following method to solve linear complementarity problem was developed by Kojima et al. [187] based on Karmarkar's polynomial time algorithm [98] for linear programming. This polynomial time-bound method is widely used to

*Results of this chapter have been communicated.

†The work has been arxived [179].

solve LCP(q, A), but for some matrix classes the LCP(q, A) is not processable by this method. The linear complementarity problem arising from a free boundary problem can be reformulated as a fixed-point equation. Zhang [36] presented a modified modulus-based multigrid method to solve this fixed-point equation. The concept of complementarity is synonymous with the notion of system equilibrium. Modulus based algorithm is one of the proposed iterative method to solve linear complementarity problem. It was proved by van Bokhoven that the modulus algorithm works when the matrix involved is a symmetric P-matrix. Kappel et al.[188] extended van Bokhoven's results by showing that the modulus algorithm can be applied to a class of non-symmetric P-matrices. Schafer [189] showed the convergence of the modulus algorithm for three subclasses of P -matrices. Hadjidimos et al. [190], [191] proposed a new method, the scaled extrapolated block modulus algorithm as well as an improved version of recently introduced modulus-based matrix splitting modified accelerated overrelaxation (AOR) iterative method to find the solution of the linear complementarity problem with H_+-matrix. For the large sparse linear complementarity problem, Zheng et al. [210], [211], [212] established a relaxation modulus-based matrix splitting iteration method, a class of accelerated modulus-based matrix splitting iteration methods by reformulating it as a general implicit fixed-point equation which covered the known modulus-based matrix splitting iteration methods and presented the convergence conditions when the matrix involved is either a positive definite matrix or an H_+-matrix. Dai et al. [213] proposed a preconditioned two-step modulus-based matrix splitting iteration method for linear complementarity problems associated with an M-matrix. For further details see [215], [216] and [217]. Eaves and Saigal [182] formed an important class of globally convergent methods for solving systems of non-linear equations. Such methods have been used to constructively prove the existence of solutions to many economic and engineering problems. A continuation method was proposed based on such methods to solve LCP(q, A) with some restricted matrix classes. For details see [37], [38] and [41].

Now the purpose of this chapter is to solve linear complementarity problem with various matrix classes through a initial value problem. Milne's method [180] is the classic predictor-corrector method for solving ordinary differential equation with initial condition. Based on this predictor corrector approach along with interior point we find the solution of LCP(q, A) through a continuous trajectory. The chapter is organized as follows. Section 5.2 presents some basic notations and results. In section 5.3, we propose a new function to find the solution of LCP(q, A). We construct a smooth and bounded path approaching to the solution. To ensure a continuos trajectory approaching to the solution a new scheme of choosing step length is introduced. We show that under some instances the proposed function can give the solution of linear complementarity problem. Finally in section 5.4, we consider various classes of numerical examples to present the effectiveness of the algorithm.

5.2 Preliminaries

An ordinary differential equation with initial values is known as initial value problem. Consider the initial value problem

$$y' = f(x, y), y(x_0) = y_0 \tag{5.2.1}$$

We can obtain the solution of the problem (5.2.1) by using predictor-corrector approach. A simple predictor-corrector method can be constructed from the explicit method, known as Euler method and the implicit method known as trapezoidal method.

5.2.1 Predictor-Corrector Approach

Predictor–corrector method is designed to solve ordinary differential equations with initial values. Considering the initial value problem, a predictor–corrector method typically uses an explicit method for the predictor step and an implicit method for the corrector step. Such algorithm is processed in two steps. The first step known as prediction step, starts from a function fitted to the function-values and derivative-values at a preceding set of points to extrapolate this function's value at a subsequent point. The next step known as corrector step refines the initial approximation by using the predicted value of the function and interpolate the function's value at the same subsequent point.

5.3 Main Results

We define $\mathcal{F} = \{x \in \mathbb{R}^n : x > 0, Ax + q > 0\}$, $\bar{\mathcal{F}} = \{x \in \mathbb{R}^n : x \geq 0, Ax + q \geq 0\}, \mathcal{F}_1 = \mathcal{F} \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$ and $\bar{\mathcal{F}}_1 = \bar{\mathcal{F}} \times \mathbb{R}^n_{+} \times \mathbb{R}^n_{+}$. $\partial\mathcal{F}_1$ denotes the boundary of $\mathcal{F}_1$. We propose a new function to solve wider classes of LCP(q, A)

$$\mathcal{G}(y, y^{(0)}, \lambda) = \begin{bmatrix} (1-\lambda)[(A + A^T)x + q - z_1 - A^T z_2] + \lambda(x - x^{(0)}) \\ Z_1 x - \lambda Z_1^{(0)} x^{(0)} \\ Z_2(Ax + q) - \lambda Z_2^{(0)}(Ax^{(0)} + q) \end{bmatrix} = 0 \tag{5.3.1}$$

where $Z_1 = \text{diag}(z_1), Z_2 = \text{diag}(z_2), Z_1^{(0)} = \text{diag}(z_1^{(0)}), Z_2^{(0)} = \text{diag}(z_2^{(0)}), y = (x, z_1, z_2) \in \bar{\mathcal{F}}_1, y^{(0)} = (x^{(0)}, {z_1}^{(0)}, {z_2}^{(0)}) \in \mathcal{F}_1$ and $\lambda \in (0, 1]$. We denote $\Gamma_y^{(0)} = \{(y, \lambda) \in \mathbb{R}^{3n} \times (0, 1] : \mathcal{G}(y, y^{(0)}, \lambda) = 0\} \subset \mathcal{F}_1 \times (0, 1]$. To find the solution of LCP(q, A), we go along the path $\Gamma_y^{(0)}$ for the scalar λ goes from 1 to 0.

First we show that the smooth curve for our proposed function exists.

Theorem 5.3.1. *For almost all initial points $y^{(0)} \in \mathcal{F}_1$, 0 is a regular value of* *the proposed function* $\mathcal{G} : \mathbb{R}^{3n} \times (0,1] \to \mathbb{R}^{3n}$ *and the zero point set* $\mathcal{G}^{-1}_{y^{(0)}}(0) = \{(y,\lambda) \in \mathcal{F}_1 : \mathcal{G}_{y^{(0)}}(y,\lambda) = 0\}$ *contains a smooth curve* $\Gamma^{(0)}_y$ *starting from* $(y^{(0)}, 1)$.

Proof. The Jacobian matrix of the proposed function $\mathcal{G}(y, y^{(0)}, \lambda)$ is denoted by $D\mathcal{G}(y, y^{(0)}, \lambda))$ and we have $D\mathcal{G}(y, y^{(0)}, \lambda)) = \begin{bmatrix} \frac{\partial \mathcal{G}(y,y^{(0)},\lambda)}{\partial y} & \frac{\partial \mathcal{G}(y,y^{(0)},\lambda)}{\partial y^{(0)}} & \frac{\partial \mathcal{G}(y,y^{(0)},\lambda)}{\partial \lambda} \end{bmatrix}$. For all $y^{(0)} \in \mathcal{F}_1$ and $\lambda \in (0,1]$, we have $\frac{\partial \mathcal{G}(y,y^{(0)},\lambda)}{\partial y^{(0)}} = \begin{bmatrix} -\lambda I & 0 & 0 \\ -\lambda Z_1^{(0)} & -\lambda X^{(0)} & 0 \\ -\lambda Z_2^{(0)} A & 0 & -\lambda Y^{(0)} \end{bmatrix}$, where $Y^{(0)} = \mathrm{diag}(Ax^{(0)} + q), X^{(0)} = \mathrm{diag}(x^{(0)})$ and $\det(\frac{\partial \mathcal{G}}{\partial y^{(0)}}) = (-1)^{3n}\lambda^{3n}\prod_{i=1}^{n} x_i^{(0)} y_i^{(0)} \neq 0$ for $\lambda \in (0,1]$. Thus $D\mathcal{G}(y, y^{(0)}, \lambda)$ is of full row rank. Therefore, 0 is a regular value of $\mathcal{G}(y, y^{(0)}, \lambda)$ by the Lemma 1.3.2. By Lemma 1.3.3 and Lemma 1.3.4, for almost all $y^{(0)} \in \mathcal{F}_1$, 0 is a regular value of $\mathcal{G}_{y^{(0)}}(y,\lambda)$ and $\mathcal{G}^{-1}_{y^{(0)}}(0)$ consists of some smooth curves and $\mathcal{G}_{y^{(0)}}(y^{(0)}, 1) = 0$. Hence there must be a smooth curve $\Gamma^{(0)}_y$ starting from $(y^{(0)}, 1)$. ■

Now we show that the smooth curve $\Gamma^{(0)}_y$ for the proposed function (5.3.1) is bounded and convergent.

THEOREM 5.3.2. *Let* $\mathcal{F}$ *be a non-empty set and* A *be a nonsingular matrix and assume that there exists a sequence of points* $\{(x^k, z_1^k, z_2^k, \lambda^k)\} \subset \Gamma^{(0)}_y \subset \mathcal{F}_1 \times (0,1]$ *such that* $\|x^k\| < \infty$ *as* $k \to \infty$. *Further suppose for* $\lambda^k \to 1$, $\|z_2^k\| < \infty$ *as* $k \to \infty$. *Suppose that for a given* $y^{(0)} \in \mathcal{F}_1$, 0 *is a regular value of* $\mathcal{G}(y, y^{(0)}, \lambda)$. *Then* $\Gamma^{(0)}_y$ *is a bounded curve in* $\mathcal{F}_1 \times (0,1]$.

Proof. Note that 0 is a regular value of $\mathcal{G}(y, y^{(0)}, \lambda)$ by Theorem 5.3.1. Now we assume that $\Gamma^{(0)}_y \subset \mathcal{F}_1 \times (0,1]$ is an unbounded curve. Then there exists a sequence of points $\{(y^k, \lambda_k)\} \subset \Gamma^{(0)}_y$ such that $\|(y^k, \lambda_k)\| \to \infty$. As $(0,1]$ is a bounded set and x component of $\Gamma^{(0)}_y$ is bounded, there exists a subsequence of points $\{(y^k, \lambda_k)\}$ such that

$x^k \to \bar{x}, \lambda_k \to \bar{\lambda} \in [0,1]$ and $\|z^k\| \to \infty$ as $k \to \infty$, where $z^k = \begin{bmatrix} z_1^k \\ z_2^k \end{bmatrix}$. Since $\Gamma_y^{(0)} \subset \mathcal{G}^{-1}_{y^{(0)}}(0)$, we have

$$(1-\lambda_k)[(A+A^T)x^k + q - z_1^k - A^T z_2^k] + \lambda_k(x^k - x^{(0)}) = 0 \tag{5.3.2}$$

$$Z_1^k x^k - \lambda_k Z_1^{(0)} x^{(0)} = 0 \tag{5.3.3}$$

$$Z_2^k(Ax^k + q) - \lambda_k Z_2^{(0)}(Ax^{(0)} + q) = 0 \tag{5.3.4}$$

where $Z_1^k = \text{diag}(z_1^k)$ and $Z_2^k = \text{diag}(z_2^k)$. Now we consider following three cases:

Case 1: $\bar{\lambda} \in [0,1]$, $\|z_1^k\| = \infty$ and $\|z_2^k\| < \infty$.
Let $\|z_1^k\| = \infty$. Then $\exists\ i \in \{1,2,\cdots n\}$ such that $z_{1i}^k \to \infty$ as $k \to \infty$. Let $I_{1z} = \{i \in \{1,2,\cdots n\}; \lim\limits_{k\to\infty} z_{1i}^k = \infty\}$. When $\bar{\lambda} \in [0,1)$ for $i \in I_{1z}$, we can obtain from Equation (5.3.2)
$(1-\lambda_k)[((A+A^T)x^k)_i + q_i - z_{1i}^k - (A^T z_2^k)_i] + \lambda_k(x_i^k - x_i^{(0)}) = 0 \implies (1-\lambda_k)z_{1i}^k = (1-\lambda_k)[((A+A^T)x^k)_i + q_i - (A^T z_2^k)_i] + \lambda_k(x_i^k - x_i^{(0)}) \implies z_{1i}^k = [((A+A^T)x^k)_i + q_i - (A^T z_2^k)_i] + \frac{\lambda_k}{(1-\lambda_k)}(x_i^k - x_i^{(0)})$. As $k \to \infty$ right hand side is bounded, but left hand side is unbounded. It contradicts that $\|z_1^k\| = \infty$. When $\bar{\lambda} = 1$, then from Equation (5.3.3) we obtain $x_i^k = \frac{\lambda_k z_{1i}^{(0)} x_i^{(0)}}{z_{1i}^k}$ for $i \in I_{1z}$. As $k \to \infty, x_i^k \to 0$. Again from Equation (5.3.2), we obtain $x_i^{(0)} = \frac{(1-\lambda_k)}{\lambda_k}[((A+A^T)x^k)_i + q_i - z_{1i}^k - (A^T z_2^k)_i] + x_i^k$ for $i \in I_{1z}$. As $k \to \infty$, we have $x_i^{(0)} = -\lim\limits_{k\to\infty} \frac{(1-\lambda_k)}{\lambda_k} z_{1i}^k \le 0$. It contradicts that $\|z_1^k\| = \infty$.

Case 2: $\bar{\lambda} \in [0,1]$, $\|z_1^k\| < \infty$ and $\|z_2^k\| = \infty$.
Let $\|z_2^k\| = \infty$. Then $\exists\ j \in \{1,2,\cdots n\}$ such that $z_{2j}^k \to \infty$ as $k \to \infty$. Let $I_{2z} = \{j \in \{1,2,\cdots n\}; \lim\limits_{k\to\infty} z_{2j}^k = \infty\}$. When $\bar{\lambda} \in [0,1)$, for $j \in I_{2z}$ we can obtain from Equation (5.3.2) $z_{2j}^k = (A^{-t}(A+A^T)x^k)_j + (A^{-t}q)_j - (A^{-t}z_1^k)_j + \frac{\lambda_k}{1-\lambda_k}(x_j^k - x_j^{(0)})$. As $k \to \infty$, right hand side is bounded, but left hand side is not. This also contradicts that $\|z_2^k\| = \infty$. By our assumption $\|z_2^k\| < \infty$ as $k \to \infty$ for $\bar{\lambda} = 1$.

Case 3: $\bar{\lambda} \in [0,1]$, $\|z_1^k\| = \infty$ and $\|z_2^k\| = \infty$.

Let $\|z_1^k\| = \infty, \|z_2^k\| = \infty$. Then either $\exists\ i \in \{1, 2, \cdots n\}$ such that $z_{1i}^k \to \infty$ $z_{2i}^k \to \infty$ as $k \to \infty$ or $\exists\ i, j \in \{1, 2, \cdots n\}, i \neq j$ such that $z_{1i}^k \to \infty$ and $z_{2j}^k \to \infty$ as $k \to \infty$. When $z_{1i}^k \to \infty$ $z_{2i}^k \to \infty$ as $k \to \infty$ and $\bar{\lambda} \in [0,1)$, we have, $z_{1i}^k + (A^T z_2^k)_i = ((A + A^T)x^k)_i + q_i + \frac{\lambda_k}{(1-\lambda_k)}(x_i^k - x_i^{(0)})$. Now as $k \to \infty$, right hand side is bounded, but left hand side is not, which is impossible. When $\bar{\lambda} = 1$, then our assumption $\|z_2^k\| < \infty$ as $k \to \infty$ and the argument of case 1 contradicts that $z_{1i}^k \to \infty$, $z_{2i}^k \to \infty$ as $k \to \infty$. As $k \to \infty$, when $z_{1i}^k \to \infty$, $z_{2j}^k \to \infty$ for $i \neq j$ as $k \to \infty$ then considering the ith and jth component and using same argument similar to case 1 and case 2, we will obtain a contradiction.

Thus $\Gamma_y^{(0)}$ is a bounded curve in $\mathcal{F}_1 \times (0,1]$. ■

Theorem 5.3.3. *Let $\mathcal{F}$ be a non-empty set and A be a nonsingular matrix and assume that there exists a sequence of points $\{(x^k, z_1^k, z_2^k, \lambda^k)\} \subset \Gamma_y^{(0)} \subset \mathcal{F}_1 \times (0,1]$ such that $\|x^k\| < \infty$ as $k \to \infty$. Further suppose for $\lambda^k \to 1$, $\|z_2^k\| < \infty$ as $k \to \infty$. Suppose that for a given $y^{(0)} \in \mathcal{F}_1$, 0 is a regular value of $\mathcal{G}(y, y^{(0)}, \lambda)$. For $y^{(0)} = (x^{(0)}, z_1^{(0)}, z_2^{(0)}) \in \mathcal{F}_1$, the proposed function finds a bounded smooth curve $\Gamma_y^{(0)} \subset \mathcal{F}_1 \times (0,1]$ which starts from $(y^{(0)}, 1)$ and approaches the hyperplane at $\lambda = 0$. As $\lambda \to 0$, the limit set $L \times \{0\} \subset \bar{\mathcal{F}}_1 \times \{0\}$ of $\Gamma_y^{(0)}$ is nonempty and every point in L is a solution of the following system:*

$$\begin{aligned} (A + A^T)x + q - z_1 - A^T z_2 = 0, \\ Z_1 x = 0, \\ Z_2(Ax + q) = 0. \end{aligned} \tag{5.3.5}$$

Proof. Note that $\Gamma_y^{(0)}$ is diffeomorphic to a unit circle or a unit interval $(0,1]$ in view of Lemma 1.3.5. As $\frac{\partial \mathcal{G}(y, y^{(0)}, 1)}{\partial y^{(0)}}$ is nonsingular, $\Gamma_y^{(0)}$ is diffeomorphic to a unit interval $(0, 1]$. Again $\Gamma_y^{(0)}$ is a bounded smooth curve by the Theorem 5.3.2. Let $(\bar{y}, \bar{\lambda})$ be a limit point of $\Gamma_y^{(0)}$. We consider four cases:

Case 1: $(\bar{y}, \bar{\lambda}) \in \mathcal{F}_1 \times \{1\}$.
Case 2: $(\bar{y}, \bar{\lambda}) \in \partial\mathcal{F}_1 \times \{1\}$.
Case 3: $(\bar{y}, \bar{\lambda}) \in \partial\mathcal{F}_1 \times (0, 1)$.
Case 4: $(\bar{y}, \bar{\lambda}) \in \bar{\mathcal{F}}_1 \times \{0\}$.

As $\mathcal{G}_{y^{(0)}}(y, 1) = 0$ has only one solution $y^{(0)} \in \mathcal{F}_1$, the case 1 is impossible. In case 2 and 3, there exists a subsequence of $(y^k, \lambda_k) \in \Gamma_y^{(0)}$ such that $x_i^k \to 0$ or $(Ax^k + q)_i \to 0$ for $i \subseteq \{1, 2, \cdots n\}$. From the last two equations of the proposed function (5.3.1), we have $z_1^k \to \infty$ or $z_2^k \to \infty$. Hence it contradicts the boundedness of the path obtained from the proposed function by the Theorem 5.3.2. Therefore case 4 is the only possible option. Hence $\bar{y} = (\bar{x}, \bar{z}_1, \bar{z}_2)$ is a solution of the system $(A + A^T)x + q - z_1 - A^T z_2 = 0,\ Z_1 x = 0,\ Z_2(Ax + q) = 0$.
■

REMARK 5.3.1. *From the proposed function (5.3.1), we obtain* $\bar{z}_{1i}\bar{x}_i = 0$ *and* $\bar{z}_{2i}(A\bar{x} + q)_i = 0\ \forall i$. *Now* $\bar{z}_1$ *and* $\bar{z}_2$ *can be decomposed as* $\bar{z}_1 = \bar{w} - \Delta\bar{w} \geq 0$ *and* $\bar{z}_2 = \bar{x} - \Delta\bar{x} \geq 0$ *where* $\bar{w} = A\bar{x} + q$. *It is clear that* $\bar{w}_i\bar{x}_i = \Delta\bar{w}_i\bar{x}_i = \Delta\bar{x}_i\bar{w}_i\ \forall i$.

We demonstrate the condition under which the proposed function will give the solution of LCP(q, A).

THEOREM 5.3.4. *The component* $\bar{x}$ *of* $(\bar{x}, \bar{z}_1, \bar{z}_2, 0) \in L \times \{0\}$ *gives the solution of LCP*(q, A) *if and only if* $\Delta\bar{x}_i\Delta\bar{w}_i = 0$ *or* $\bar{z}_{1i} + \bar{z}_{2i} > 0\ \forall i$.

Proof. Suppose $\bar{x} \geq 0$ and $\bar{w} = A\bar{x} + q \geq 0$ give the solution of LCP(q, A). Then $\bar{x}_i\bar{w}_i = 0\ \ \forall i$. This implies that $\bar{x}_i = 0$ or $\bar{w}_i = 0\ \ \forall i$. We consider the following three cases.

Case 1: For atleast one $i \in \{1, 2, \cdots n\}$, let $\bar{w}_i > 0, \bar{x}_i = 0$. In view of Remark 5.3.1, this implies that $\Delta\bar{x}_i = 0 \implies \Delta\bar{x}_i\Delta\bar{w}_i = 0$.

Case 2: For atleast one $i \in \{1, 2, \cdots n\}$, let $\bar{x}_i > 0, \bar{w}_i = 0$. In view of 5.3.1, this implies that $\Delta\bar{w}_i = 0 \implies \Delta\bar{x}_i\Delta\bar{w}_i = 0$.

Case 3: For atleast one $i \in \{1, 2, \cdots n\}$, let $\bar{w}_i = 0, \bar{x}_i = 0$. This implies that either $\Delta\bar{w}_i\Delta\bar{x}_i = 0$ or $\bar{z}_{1i} + \bar{z}_{2i} > 0$.

For the converse part, consider $\Delta\bar{x}_i\Delta\bar{w}_i = 0$ or $\bar{z}_{1i} + \bar{z}_{2i} > 0\ \forall i$. Let for each $i \in \{1, 2, \cdots n\}$, $\Delta\bar{x}_i\Delta\bar{w}_i = 0$ implies either $\Delta\bar{x}_i = 0$ or $\Delta\bar{w}_i = 0$. This implies that $\bar{w}_i\bar{x}_i = 0\ \forall i$. Therefore $\bar{w}$ and $\bar{x}$ are the solution of given LCP(q, A). Consider $\bar{z}_{1i} + \bar{z}_{2i} > 0\ \forall\ i$. Then following three cases will arise.

Case 1: Let $\bar{z}_{1i} > 0, \bar{z}_{2i} = 0$ for atleast one $i \in \{1, 2, \cdots n\}$. This implies that $\bar{x}_i = 0$ and $\bar{w}_i \geq 0$.

Case 2: Let $\bar{z}_{1i} = 0, \bar{z}_{2i} > 0$ for atleast one $i \in \{1, 2, \cdots n\}$. This implies that $\bar{x}_i \geq 0$ and $\bar{w}_i = 0$.

Case 3: Let $\bar{z}_{1i} > 0, \bar{z}_{2i} > 0$ for atleast one $i \in \{1, 2, \cdots n\}$. This implies that $\bar{x}_i = 0$ and $\bar{w}_i = 0$. Considering the above three cases $\bar{x}, \bar{w}$ solve the LCP(q, A).
■

Corollary 5.3.1. *If A is a P_0-matrix, then the component $\bar{x}$ of $(\bar{x}, \bar{z}_1, \bar{z}_2, 0) \in L \times \{0\}$ gives the solution of LCP(q, A).*

Proof. Let A be a P_0-matrix. Assume that the component $\bar{x}$ of $(\bar{x}, \bar{z}_1, \bar{z}_2, 0) \in L \times \{0\}$ does not give the solution of LCP(q, A). Hence $\Delta\bar{x}_i\Delta\bar{w}_i \neq 0$ and $\bar{z}_{1i} + \bar{z}_{2i} = 0$ for atleast one i. Then $\Delta\bar{x}_i \neq 0, \Delta\bar{w}_i \neq 0, \bar{z}_{1i} = 0, \bar{z}_{2i} = 0$. Now $\bar{z}_{1i} = \bar{w}_i - \Delta\bar{w}_i = 0$ and $\Delta\bar{x}_i\Delta\bar{w}_i \neq 0 \implies \bar{w}_i = \Delta\bar{w}_i > 0$. In similar way $\bar{z}_{2i} = \bar{x}_i - \Delta\bar{x}_i = 0$ and $\Delta\bar{x}_i\Delta\bar{w}_i \neq 0 \implies \bar{x}_i = \Delta\bar{x}_i > 0$. From Equation (5.3.5), $\Delta\bar{w}_i + (A^T\Delta\bar{x})_i = 0$. This implies that $(A^T\Delta\bar{x})_i < 0$ and also $(\bar{x})_i(A^T\Delta\bar{x})_i < 0$. This contradicts that A is a P_0-matrix. Therefore the component $\bar{x}$ of $(\bar{x}, \bar{z}_1, \bar{z}_2, 0) \in L \times \{0\}$ gives the solution of LCP(q, A). ■

THEOREM 5.3.5. *Suppose the matrix* $(\bar{W} + \bar{X}A^T)$ *is nonsingular where* $\bar{W} =$ *diag*$(\bar{w})$ *and* $\bar{X} =$ *diag*$(\bar{x})$*. Then* $\bar{x}$ *solves the LCP*(q, A)*.*

Proof. Let the matrix $(\bar{W} + \bar{X}A^T)$ be nonsingular. We obtain from the solution of the system of equation (5.3.5), $\Delta\bar{w} + A^T\Delta\bar{x} = 0$ and $\bar{X}\Delta\bar{w} = \bar{W}\Delta\bar{x}$, where $\bar{W}$ =diag$(\bar{w})$ =diag$(A\bar{x} + q)$. Now $\bar{X}\Delta\bar{w} + \bar{X}A^T\Delta\bar{x} = 0$ implies that $\bar{W}\Delta\bar{x} + \bar{X}A^T\Delta\bar{x} = 0 \implies (\bar{W} + \bar{X}A^T)\Delta\bar{x} = 0$. Since the matrix $(\bar{W} + \bar{X}A^T)$ is nonsingular, this implies that $\Delta\bar{x} = 0$. Then $\bar{x}$ solves the LCP(q, A). ■

5.3.1 Tracing Path Using Predictor-Corrector Approach

We trace the path $\Gamma_y^{(0)} \subset \mathcal{F}_1 \times (0, 1]$ from the initial point $(y^{(0)}, 1)$ as $\lambda \to 0$. To find the solution of the given LCP(q, A) we consider continuous path. Let s denote the arc length of $\Gamma_y^{(0)}$. We parameterize the path $\Gamma_y^{(0)}$ with respect to s in the following form

$$\mathcal{G}_{y^{(0)}}(y(s), \lambda(s)) = 0,\ y(0) = y^{(0)},\ \lambda(0) = 1. \tag{5.3.6}$$

Differentiating (5.3.6) with respect to s, we obtain the following system of ordinary differential equations with given initial values

$$\mathcal{G}'_{y^{(0)}}(y(s), \lambda(s)) \begin{bmatrix} \frac{dy}{ds} \\ \frac{d\lambda}{ds} \end{bmatrix} = 0,\ \|(\frac{dy}{ds}, \frac{d\lambda}{ds})\| = 1,\ y(0) = y^{(0)},\ \lambda(0) = 1, \tag{5.3.7}$$

Hence the system (5.3.7) reduces to the following initial value problem

$$(\tfrac{d\lambda}{ds})^{-1}\tfrac{dy}{ds} = p(y, \lambda),\ y(0) = y^{(0)},\ \lambda(0) = 1$$

where

$$p(y, \lambda) = -\tfrac{\partial}{\partial y}\mathcal{G}(y, \lambda)^{-1}\tfrac{\partial}{\partial \lambda}\mathcal{G}(y, \lambda)$$

This problem will be solved by predictor-corrector method and the y-component of $(y(s), \lambda(s))$ gives the solution of LCP(q, A).

Note that the parameter λ is updated from Moore-Penrose inverse of the Jacobian matrix for tracing the path. However this approach does not ensure that the updated value of the parameter λ is in $(0, 1]$. Value of λ beyond $(0, 1]$ leads to a nonconsiderable path. To eliminate deviation, we propose a modification by introducing a method ensuring feasibility by changing step length. In this method it is necessary to check whether $0 < (\tilde{\lambda}_i - \hat{\lambda}_i) < 1$ and $(\tilde{y}^{(i)} - \hat{y}^{(i)}) \in \bar{\mathcal{F}}_1$ holds or not. If any of the above mentioned criteria fails, then the step length will be changed appropriately using geometric series to trace the path $\Gamma_y^{(0)}$. This guarantees a continuous trajectory leading to the solution of the proposed function (5.3.1).

Algorithm

Step 0: Initialize $(y^{(0)}, \lambda_0)$. Set $l_0 \in (0, 1)$. Choose $\epsilon_3 >> \epsilon_1 > 0$ which are small positive quantities.

Step 1: $\tau^{(0)} = \xi^{(0)} = (\frac{1}{n}) \begin{bmatrix} s \\ -1 \end{bmatrix}$ for $i = 0$, where $n = \| \begin{bmatrix} s \\ -1 \end{bmatrix} \|$ and $s = (\frac{\partial \mathcal{G}}{\partial y}(y^{(0)}, \lambda_0))^{-1}(\frac{\partial \mathcal{G}}{\partial \lambda}(y^{(0)}, \lambda_0))$. If $\det(\frac{\partial \mathcal{G}}{\partial y}(y^{(i)}, \lambda_i)) > 0$, $\tau^{(i)} = \xi^{(i)}$ else $\tau^{(i)} = -\xi^{(i)}$, $i \geq 1$. Set $l = 0$.

Step 2: (Predictor point calculation) $(\tilde{y}^{(i)}, \tilde{\lambda}_i) = (y^{(i)}, \lambda_i) + a\tau^{(i)}$, where $a = {l_0}^l$. Compute $(\hat{y}^{(i)}, \hat{\lambda}_i) = \mathcal{G}'_{y^{(0)}}(\tilde{y}^{(i)}, \tilde{\lambda}_i)^{+}\mathcal{G}(\tilde{y}^{(i)}, \tilde{\lambda}_i)$. If $0 < (\tilde{\lambda}_i - \hat{\lambda}_i) < 1$, go to Step 3. Otherwise if $m = \min(a, \|(\tilde{y}^{(i)}, \tilde{\lambda}_i) - (\hat{y}^{(i)}, \hat{\lambda}_i) - (y^{(i)}, \lambda_i)\|) > a_0$, update l by $l + 1$, and recompute $(\tilde{\lambda}_i, \hat{\lambda}_i)$ else go to Step 3.

(Corrector point calculation) $(y^{(i+1)}, \lambda_{i+1}) = (\tilde{y}^{(i)}, \tilde{\lambda}_i) - (\hat{y}^{(i)}, \hat{\lambda}_i)$. Determine the norm, $r = \|\mathcal{G}(y^{(i+1)}, \lambda_{i+1})\|$. If $r \leq 1$ and $y^{(i+1)} > 0$ go to Step 3, otherwise if

$a > \epsilon_3$, update l by $l+1$ and compute $(\tilde{y}^{(i)}, \tilde{\lambda}_i)$ else go to Step 3.

Step 3: If $|\lambda_{i+1}| \leq \epsilon_1$, then stop with solution $(y^{(i+1)}, \lambda_{i+1})$, else $i = i+1$ and go to Step 1.

Note that in Step 2, $\mathcal{G}'_{y^{(0)}}(y,\lambda)^+ = \mathcal{G}'_{y^{(0)}}(y,\lambda)^T(\mathcal{G}'_{y^{(0)}}(y,\lambda)\mathcal{G}'_{y^{(0)}}(y,\lambda)^T)^{-1}$ is the Moore-Penrose inverse of $\mathcal{G}'_{y^{(0)}}(y,\lambda)$. We prove the following result to obtain the positive direction of the proposed algorithm.

THEOREM 5.3.6. *If the curve $\Gamma_y^{(0)}$ is smooth, then the positive predictor direction $\tau^{(0)}$ at the initial point $y^{(0)}$ satisfies* $\det\left(\begin{bmatrix} \frac{\partial \mathcal{G}}{\partial y \partial \lambda}(y^{(0)}, 1) \\ \tau^{(0)^T} \end{bmatrix}\right) < 0$.

Proof. From the Equation (5.3.1), we consider the following function

$$\mathcal{G}(y, y^{(0)}, \lambda) = \begin{bmatrix} (1-\lambda)[(A+A^T)x + q - z_1 - A^T z_2] + \lambda(x - x^{(0)}) \\ Z_1 x - \lambda Z_1^{(0)} x^{(0)} \\ Z_2(Ax+q) - \lambda Z_2^{(0)}(Ax^{(0)}+q) \end{bmatrix} = 0.$$

Now,

$$\frac{\partial \mathcal{G}}{\partial y \partial \lambda}(y,\lambda) = \begin{bmatrix} (1-\lambda)(A+A^T) + \lambda I & -(1-\lambda)I & -(1-\lambda)A^T & P \\ Z_1 & X & 0 & -Z_1^{(0)} x^{(0)} \\ Z_2 A & 0 & Y & -Z_2^{(0)}(Ax^{(0)}+q) \end{bmatrix},$$

where $P = (x - x^{(0)}) - [(A+A^T)x + q - z_1 - A^T z_2]$ and $Y = \mathrm{diag}(Ax+q)$. At the initial point $(y^{(0)}, 1)$

$$\frac{\partial \mathcal{G}}{\partial y \partial \lambda}(y^{(0)}, 1) = \begin{bmatrix} I & 0 & 0 & -[(A+A^T)x^{(0)} + q - z_1^{(0)} - A^T z_2^{(0)}] \\ Z_1^{(0)} & X^{(0)} & 0 & -Z_1^{(0)} x^{(0)} \\ Z_2^{(0)} A & 0 & Y^{(0)} & -Z_2^{(0)}(Ax^{(0)}+q) \end{bmatrix}.$$

Let positive predictor direction be $\tau^{(0)} = \begin{bmatrix} \kappa \\ -1 \end{bmatrix} = \begin{bmatrix} (R_1^{(0)})^{(-1)} R_2^{(0)} \\ -1 \end{bmatrix}$, where

$$R_1^{(0)} = \begin{bmatrix} I & 0 & 0 \\ Z_1^{(0)} & X^{(0)} & 0 \\ Z_2^{(0)} A & 0 & Y^{(0)} \end{bmatrix},$$

$R_2^{(0)} = \begin{bmatrix} -[(A+A^T)x^{(0)} + q - z_1^{(0)} - A^T z_2^{(0)}] \\ -Z_1^{(0)} x^{(0)} \\ -Z_2^{(0)}(Ax^{(0)} + q) \end{bmatrix}$ and κ is a column vector.

Hence, $\det(\begin{bmatrix} \frac{\partial \mathcal{G}}{\partial y \partial \lambda}(y^{(0)}, 1) \\ \tau^{(0)^T} \end{bmatrix})$

$= \det(\begin{bmatrix} R_1^{(0)} & R_2^{(0)} \\ (R_2^{(0)})^T (R_1^{(0)})^{(-T)} & -1 \end{bmatrix})$

$= \det(\begin{bmatrix} R_1^{(0)} & R_2^{(0)} \\ 0 & -1 - (R_2^{(0)})^T (R_1^{(0)})^{(-T)} (R_1^{(0)})^{(-1)} R_2^{(0)} \end{bmatrix})$

$= \det(R_1^{(0)}) \det(-1 - (R_2^{(0)})^T (R_1^{(0)})^{(-T)} (R_1^{(0)})^{(-1)} R_2^{(0)})$

$= -\det(R_1^{(0)}) \det(1 + (R_2^{(0)})^T (R_1^{(0)})^{(-T)} (R_1^{(0)})^{(-1)} R_2^{(0)})$

$= -\prod_{i=1}^n x_i^{(0)} y_i^{(0)} \det(1 + (R_2^{(0)})^T (R_1^{(0)})^{(-T)} (R_1^{(0)})^{(-1)} R_2^{(0)}) < 0.$

■

REMARK 5.3.2. *We conclude from the Theorem 5.3.6 that the positive tangent direction τ of the path $\Gamma_y^{(0)}$ at any point (y, λ) be negative and it depends on*

$$\det(R_1),\ where\ R_1 = \begin{bmatrix} (1-\lambda)(A+A^T)+\lambda I & -(1-\lambda)I & -(1-\lambda)A^T \\ Z_1 & X & 0 \\ Z_2A & 0 & Y \end{bmatrix}.$$

5.4 Numerical Examples

In this section we consider some examples of LCP(q, A) to demonstrate the effectiveness of the proposed algorithm. Note that many of the examples given below are not processable by Lemke's algorithm. We show that the proposed algorithm can process these examples to find the solution. Consider $\epsilon_1 = 10^{-9},\ \epsilon_3 = 10^{-5},\ a_0 = 10^{-12},\ l_0 = \frac{1}{2}$.

EXAMPLE 5.4.1. *Consider* $A = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$ *and* $q = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$. *Note that A is an N-matrix. Now choose the initial point* $x^{(0)} = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}$, ${z_1}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* ${z_2}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Using the proposed algorithm we obtain the optimal solution of the function (5.3.1) after 20 iterations and the solution is given by* $(\bar{y}, \bar{\lambda}) = (1, 0, 0, 2.5, 1, 0, 0)$. *Therefore* $\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *solves LCP(q, A). The path shown in Figure 5.1 illustrates the convergence with respect to the solution vector x and λ.*

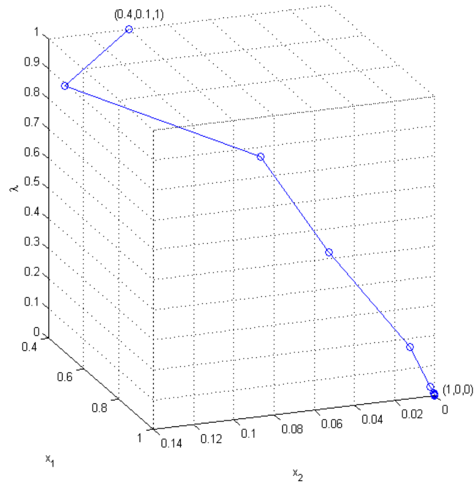

Figure 5.1:

EXAMPLE 5.4.2. *Let* $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -0.50 & -0.25 \\ -0.50 & -1 & -1 \end{bmatrix}$ *and* $q = \begin{bmatrix} -0.25 \\ -0.10 \\ 3 \end{bmatrix}$.

Now choose the initial point $x^{(0)} = \begin{bmatrix} 2.3 \\ 1 \\ 0.7 \end{bmatrix}$, $z_1{}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $z_2{}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. *Using the proposed algorithm we obtain* $(\bar{y}, \bar{\lambda}) = (1.8333, 0, 2.0833, 0, 1, 2125, 0, 1.8333, 0, 2.0833, 0)$ *after* 17 *iterations. Note that* $\bar{x} = \begin{bmatrix} 1.8333 \\ 0 \\ 2.0833 \end{bmatrix}$ *is the solution of LCP*(q, A)*. The convergence of the path is shown in the Figure 5.2. The first, second and third components of* x *are represented by data1, data2 and data3 respectively.*

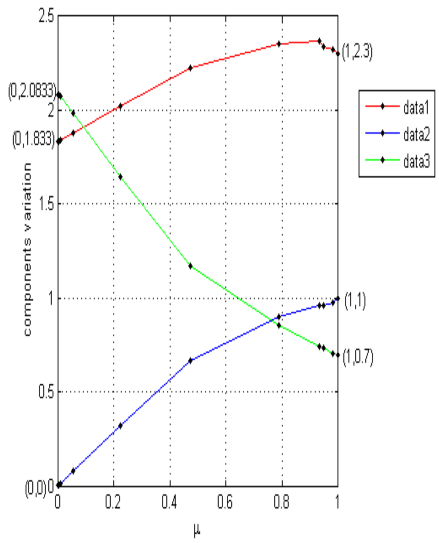

Figure 5.2:

EXAMPLE 5.4.3. *Let* $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix}$ *and* $q = \begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix}$. *It is easy to show that* A *is an almost* C_0*-matrix. Now choose the initial point* $x^{(0)} = \begin{bmatrix} 3 \\ 0.5 \\ 0.5 \end{bmatrix}$, ${z_1}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* ${z_2}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. *Using the proposed algorithm we obtain* $(\bar{y}, \bar{\lambda}) = (1, 0, 0, 0, 1, 5, 1, 0, 0, 0)$ *after* 24 *iterations. Note that* $\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *solves LCP*(q, A)*, which is a degenerate solution. The convergence of the homotopy function is shown in the Figure 5.3. The first, second and third components of* x *are represented by data1, data2 and data3 respectively.*

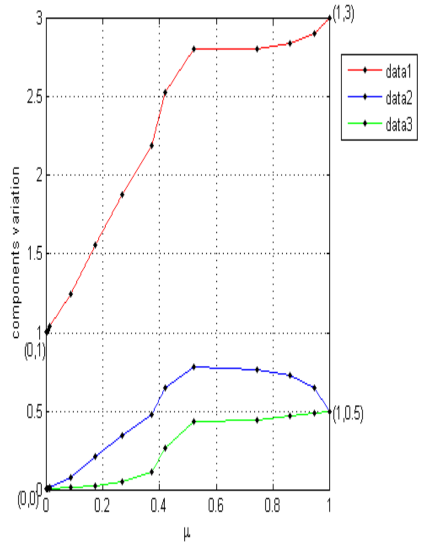

Figure 5.3:

EXAMPLE 5.4.4. *Let* $A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$ *and* $q = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. *A is a Q-matrix. See [42]. Now choose the initial point* $x^{(0)} = \begin{bmatrix} 4 \\ 4 \\ 1 \\ 1 \end{bmatrix}$, ${z_1}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ *and* ${z_2}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. *We apply our proposed algorithm to this LCP*(q, A) *and after 17 iterations we obtain the approximate optimal solution of the function*

(5.3.1), which is $(\bar{y}, \bar{\lambda}) = (1,0,2,0,0,2,0,0,1,0,2,0,0)$. *Note that* $\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ *solves LCP*(q, A), *which gives a degenerate solution. The convergence of the function is shown in the Figure 5.4. Data1, data2, data3 and data4 represent the first, second, third and fourth components of* x *respectively.*

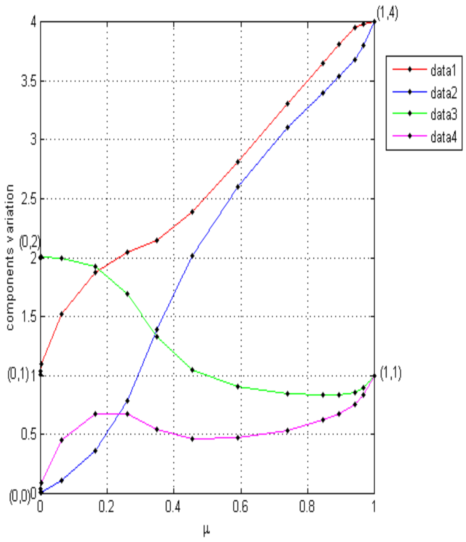

Figure 5.4:

EXAMPLE 5.4.5. *Consider* $A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}$ *and* $q = \begin{bmatrix} -2 \\ -1 \\ 7 \\ 2 \\ -1 \end{bmatrix}$.

A *is an* N_0*-matrix of exact order* 2. *Now choose the initial point* $x^{(0)} =$

$\begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$, $z_1{}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $z_2{}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. *Using the proposed algorithm, we obtain the optimal solution of the function (5.3.1),* $(\bar{y}, \bar{\lambda}) = (0.5, 0, 0, 0, 1, 0, 1, 8, 2.5, 0, 0.5, 0, 0, 0, 1, 0)$ *after 27 iterations. Note that* $\bar{x} = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ *solves LCP*(q, A)*. The convergence of the function is shown in Figure 5.5. Data1, data2, data3, data4 and data5 represent the first, second, third, fourth and fifth components of* x *respectively.*

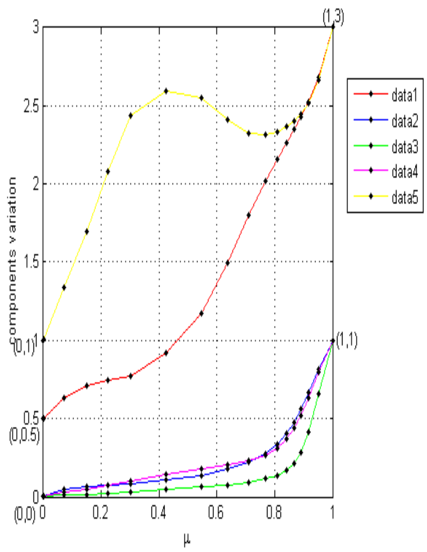

Figure 5.5:

In this study, we propose a new function to solve linear complementarity problem. The key idea to solve LCP(q, A) by the method is to solve an initial value problem using predictor-corrector approach. The value of λ will start from 1 and goes to 0. In this way one can find the solution of LCP(q, A) tracing a continuous path. We prove that the smooth curve for the proposed function is bounded and also converges. To ensure a continuous trajectory approaching to the solution we introduce a new scheme of choosing step length. Several numerical examples are provided to demonstrate the processability of larger classes of LCP(q, A).

Chapter 6

Solution Approaches Of Discounted Zero-Sum Stochastic Game With ARAT Structure

.

.

6.1 Introduction

In this chapter, we consider two-person zero-sum discounted stochastic game with additive reward and additive transition (ARAT) structure. Shapley [218] introduced stochastic game and showed that there exist an optimal value and optimal stationary strategies for a stochastic game with discounted payoff, which depends only on the current state and not on the history. There are many applications of stochastic games like search problems, military applications, advertising

---

[*]Results of this chapter have been accepted in the journal International Game Theory Review

[†]Results of this chapter have been arxived [181]

problems, the traveling inspector model and various economic applications. For details see [63]. There are significant research on theoretical as well as computational aspects of stochastic games. For details see [227], [209], [226],[219] and [228].

Raghaban et al. [209] studied ARAT games and showed that for a β-discounted zero-sum ARAT game, the value exists and both players have stationary optimal strategies, which may also be taken as pure strategies. A stochastic game is said to be an Additive Reward & Additive Transition game (ARAT game) if the reward and and the transition probabilities satisfy
(i) $r(s,i,j) = R_i^1(s) + R_j^2(s)$ for $i \in A_s, j \in B_s, s \in S$.
(ii) $p_{ij}(s,s') = p_i^1(s,s') + p_j^2(s,s')$ for $i \in A_s, j \in B_s, (s,s') \in S \times S$.

We denote the matrix $((p_i^1(s,s'), s, s' \in S, i \in A_s)))$ as $P_1(s)$ where S is the set of states. This is a $m_1(s) \times d$ matrix where $m_1(s)$ is the cardinality of A_s and d is the cardinality of s. Similarly the matrix $P_2(s)$ of order $m_2(s) \times d$ is defined where $m_2(s)$ denotes the cardinality of the set B_s. The Shapley equations for state $s, s' \in S$ can be stated as

$$\text{Val}[r(s,i,j) + \beta \textstyle\sum_{s'} p_{ij}(s,s')v_\beta(s')] = v_\beta(s).$$

This implies the following inequalities.
For player I: For any fixed j

$$r(s,i,j) + \beta \sum_{s'} p_{ij}(s,s')v_\beta(s') \le v_\beta(s) \quad \forall\ i. \tag{6.1.1}$$

For playeer II: For any fixed i

$$r(s,i,j) + \beta \sum_{s'} p_{ij}(s,s')v_\beta(s') \ge v_\beta(s) \quad \forall\ j \tag{6.1.2}$$

Both the discounted and limiting average criterion of evaluation of strategies have been considered. For details see [209],[208] and [63]. A finite step method to compute a pair of pure stationary optimal strategies and the value of the game is suggested in [209]. This approach involve solving a series of Markov decision problems. In recent days various approaches have been proposed for solving different classes of stochastic games. One such approach is to formulate the ARAT game as complementarity problem. A pair of pure stationary optimal strategies and the corresponding value for a zero-sum discounted ARAT game with some additional assumptions can be computed by solving a single vertical linear complementarity problem. The well known Lemke's algorithm solves LCPs when the underlying matrix class belongs to a particular class. Cottle and Dantzig [229] extended Lemke's algorithm to vetical linear complementarity problems (VLCPs). There are some pivotal kind techniques to solve VLCP based on Lemke's algorithm. The processability of Lemke's algorithm and Cottle-Dantzig's algorithm is restricted on some classes of matrices. For details see [17] and [230]. One sufficient condition for the processibility of Lemke's algorithm and Cottle-Dantzig algorithm is that the underlying matrix should be both E_0 and R_0 matrix [231],[69],[70] and [73].

In earlier the methods which are proposed to solve discounted zero-sum stochastic game with ARAT structure are pivotal kind techniques. But in this chapter we introduce an iterative method to obtain the solution of discounted zero-sum stochastic game with ARAT structure. The chapter is organized as follows. In section 6.2 we define the vertical linear complementarity problem and supply relevant results which will be used in the next section. In section 6.3, we propose a new iterative method to find the solution of discounted zero-sum stochastic ARAT game. We show that the proposed iterative method possesses a smooth and bounded path to find the solution. To find the solution of the proposed function we modify the steps of the iterative method to increase the order of convergency of the algorithm. We also find the sign of the tangential direction of the path. Finally, in section 6.4, we illustrate two numerical examples of ARAT stochastic games to present the effectiveness of the proposed iterative method.

6.2 Preliminaries

6.2.1 Discounted Stochastic Game with the Structure of Additive Reward and Additive Transition

Consider a state space $S = \{1, 2, \cdots, N\}$. For each $s \in S$, consider the finite action sets $A_s = \{1, 2, ..., m_s\}$ for Player I and $B_s = \{1, 2, ..., n_s\}$ for Player II. For state $s \in S$ a reward law $R(s) = [r(s, i, j)]$ is an $m_s \times n_s$ matrix whose (i, j)th entry is the payoff from Player II to Player I when Player I chooses an action $i \in A_s$ and player II chooses an action $j \in B_s$, while the game is being played in state s and the payoff from player I to player II is $-r(s, i, j)$. Let $p_{ij}(s, s')$ denotes the probability of a transition from state s to state s', given that Player I and Player II choose actions $i \in A_s, j \in B_s$ respectively. Then transition law is defined by

$$p = (p_{ij}(s, s') : (s, s') \in S \times S, i \in A_s, j \in B_s).$$

Let the game be played in stages $t = 0, 1, 2, \cdots$. At some stage t, the players find themselves in a state $s \in S$ and independently choose actions $i \in A_s, j \in B_s$. Player II pays Player I an amount $r(s, i, j)$ and at stage $(t + 1)$, the new state is s' with probability $p_{ij}(s, s')$. Play continues at this new state. The players guide the game via strategies and in general, strategies can depend on complete histories of the game until the current stage. We are however concerned with the simpler class of stationary strategies which depend only on the current state s and not on stages. So for Player I, a stationary strategy $k \in K_s = \{k_i(s)|s \in S, i \in A_s, k_i(s) \geq 0, \sum_{i\in A_s} k_i(s) = 1\}$ indicates that the action $i \in A_s$ should be chosen by Player I with probability $k_i(s)$ when the game is in state s.

Similarly for Player II, a stationary strategy $l \in L_s = \{l_j(s)|s \in S, j \in B_s, l_j(s) \geq 0, \sum_{j\in B_s} l_j(s) = 1\}$ indicates that the action $j \in B_s$ should be chosen with probability $l_j(s)$ when the game is in state s. Here K_s and L_s denote the set of all stationary strategies for Player I and Player II respectively. Let $k(s)$ and $l(s)$ be the corresponding m_s and n_s dimensional vectors respectively. Fixed stationary strategies k and l induce a Markov chain on S with transition matrix $P(k,l)$ whose (s,s')th entry is given by

$$P_{ss'}(k,l) = \sum_{i\in A_s} \sum_{j\in B_s} p_{ij}(s,s')k_i(s)l_j(s)$$

and the expected current reward vector has entries defined by

$$R_s(k,l) = \sum_{i\in A_s} \sum_{j\in B_s} r(s,i,j)k_i(s)l_j(s) = k^T(s)R(s)l(s) \ .$$

With fixed general strategies k, l and an initial state s, the stream of expected payoff to Player I at stage t, denoted by $v_s^T(k,l), t = 0,1,2,\cdots$ is well defined and the resulting discounted payoff is $\phi_s^\beta(k,l) = \sum_0^\infty \beta^T v_s^T(k,l)$ for a $\beta \in (0,1)$, where β is the discount factor. Due to this additive property assumed on the transition and reward functions, the game is called β-discounted zero-sum ARAT(Additive Reward Additive Transition) game. For futther details see [232], [209] and [233].

Vertical Linear Complementarity Problem

Cottle and Dantzig [234] extended the linear complementarity problem to vertical linear complementarity problem. Consider a vertical block matrix $A \in$

$\mathbb{R}^{m\times k}(m \geq k)$, $A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_k \end{bmatrix}$ such that $A_j \in \mathbb{R}^{m_j \times k}, 1 \leq j \leq k, \sum_{j=1}^{k} m_j = m.$

This matrix is called vertical block matrix of type $(m_1, m_2, \cdots m_k)$ and consider $q \in \mathbb{R}^m$ where $m = \sum_{j=1}^{k} m_j$, the generalized linear complementarity problem is to find $w \in \mathbb{R}^m$ and $x \in \mathbb{R}^k$ such that

$$w - Ax = q, w \geq 0, x \geq 0, \tag{6.2.1}$$

$$x_j \prod_{i}^{m_j} w^i{}_j, j = 1, 2, \cdots k. \tag{6.2.2}$$

This generalization is known as vertical linear complementarity problem and denoted by VLCP(q, A). For further details see [234]. The vertical block matrix arises naturally in the literature of stochastic games where the states are represented by the columns and actions in each state are represented by rows in a particular block. For details see [235], [236] and [200].

An equivalent square matrix M can be constructed from a vertical block matrix A of type $(m_1, ..., m_k)$ by copying $A_{\cdot j}, m_j$ times for $j = 1, 2, \cdots, k$. Therefore $M_{\cdot p} = A_{\cdot j}$ $\forall p \in J_j$. LCP(q, M) is called as equivalent LCP of VLCP(q, A). For more details see [236] and [237]. Mohan et al. [236] proposed techniques to convert a VLCP to an LCP and also showed that processibility conditions as well. Mohan et al. [200] formulated zero-sum discounted Additive Reward Additive Transition (ARAT) games as a VLCP.

Definition 6.2.1. *[200] A is said to be a vertical block $E(d)$-matrix for some $d > 0$ if VLCP(d, A) has a unique solution $w = d, z = 0$.*

Definition 6.2.2. *[200] A is said to be a vertical block R_0-matrix if VLCP$(0, A)$*

has a unique solution $w = 0, z = 0$.

We denote the class of vertical block $E(d)$ matrices as VBE(d) and the class of vertical block R_0 matrices by VBR_0.

THEOREM 6.2.1. *[63] For ARAT stochastic games*
(i) Both players possess β discounted optimal stationary strategies that are pure.
(ii) These strategies are optimal for the average reward criterion as well.
(iii) The ordered field property holds for the discounted as well as the average reward criterion.

Now we observe the following property of the additive components P_1 and P_2 of the transition probability matrix P. For details see [200].

LEMMA 6.2.1. *If* $p_j^2(s, s') = 0$ *for all* $s' \in S$ *and for some* $j \in B(s)$*, then* $P_2(s) = 0$.

THEOREM 6.2.2. *[200]Consider the vertical block matrix A arising from the zero-sum ARAT game. Then* $A \in VBE(e)$ *where e is the vector each of whose entries is* 1.

THEOREM 6.2.3. *[200] Consider the vertical block matrix A arising from zero-sum ARAT game. Then* $A \in VBR_0$ *if either the condition* (a) *or the set of conditions* (b) *stated below is satisfied.*
(a) *For each s and each* $j \in B_s$, $p_j^2(s, s) > 0$.
(b) (i) *For each s, the matrix* $P_1(s)$ *does not contain any zero column and*
(ii)*the matrix* $P_2(s)$ *is not a null matrix.*

LEMMA 6.2.2. *[88] Let* $f : R^n \to R^n$ *be a sufficiently differentiable function in a neighborhood D of* α*, that is a solution of the system* $f(x) = 0$*, whose Jacobian matrix is continuous and nonsingular in D. Consider the iterative method* $z^k = \phi(x^k, y^k)$, $w^k = z^k - f'(y^k)^{-1} f(z^k)$*, where* $y^k = x^k - f'(x^k)^{-1} f(x^k)$ *and* $z^k =$

$\phi(x^k, y^k)$ is the iteration function of a method of order p. Then for an initial approximation sufficiently close to α, this method has order of convergence $p+2$.

LEMMA 6.2.3. *[88] Consider the function $f : R^n \to R^n$ and the iterative method $y^k = x^k - f'(x^k)^{-1}f(x^k)$, $z^k = x^k - 2(f'(y^k) + f'(x^k))^{-1}f(x^k)$, $w^k = z^k - f'(y^k)^{-1}f(z^k)$ has $5th$ order convergence.*

6.3 Main Results

In this chapter we consider the followings:

$$\mathcal{R} = \{x \in \mathbb{R}^n : x > 0, Ax + q > 0\}$$
$$\bar{\mathcal{R}} = \{x \in \mathbb{R}^n : x \geq 0, Ax + q \geq 0\}$$
$$\mathcal{R}_1 = \mathcal{R} \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$$
$$\bar{\mathcal{R}}_1 = \bar{\mathcal{R}} \times \mathbb{R}^n_{+} \times \mathbb{R}^n_{+}.$$
$$\partial\mathcal{R}_1 \text{ denotes the boundary of } \bar{\mathcal{R}}_1.$$

In this section, we consider the two-person zero-sum discounted stochstic ARAT game and introduce an iterative method to find the solution of the discounted zero-sum ARAT game. We state that a pair of strategies (k^*, l^*) is optimal for Player I and Player II in the discounted game if for all $s \in S$ $\phi_s(k, l^*) \leq \phi_s(k^*, l^*) = v^*_s \leq \phi_s(k^*, l)$ for any strategies k and l of Player I and Player II. The number v^*_s is called the value of the game starting in state s and $v^* = (v^*_1, v^*_2, \cdots v^*_N)$ is called the value vector. To find the optimal strategy of player I and player II of the two-person zero-sum discounted ARAT stochastic game we propose a new function based on the concept of iterative process.

In this section, we consider linear complementarity problem LCP(q, A) with the matrix A from various matrix classes and also consider the two-person zero-sum discounted stochstic ARAT game. Now we introduce a function to find the solution of LCP(q, A) and the discounted zero-sum ARAT game.

$$H(u,t) = \begin{bmatrix} (1-t)[(A+A^T)x + q - y_1 - A^T y_2] + t(x - x^{(0)}) \\ Y_1 x - tY_1^{(0)} x^{(0)} + (1-t)X(Ax+q) \\ Y_2(Ax+q) - tY_2^{(0)}(Ax^{(0)}+q) \end{bmatrix} = 0 \quad (6.3.1)$$

where $Y_1 = \text{diag}(y_1)$, $X = \text{diag}(x)$, $Y_2 = \text{diag}(y_2)$, $Y_1^{(0)} = \text{diag}(y_1^{(0)})$, $Y_2^{(0)} = \text{diag}(y_2^{(0)})$, $u = (x, y_1, y_2) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$, $u^{(0)} = (x^{(0)}, {y_1}^{(0)}, {y_2}^{(0)}) \in \mathcal{R}_1$, and $\lambda \in (0, 1]$.

Now we establish the conditions under which the solution exists for the proposed function (6.3.1). We prove the following result to show that the smooth curve $\Gamma_u^{(0)}$ exists for the proposed function (6.3.1).

Theorem 6.3.1. *Let initial point $u^{(0)} \in \mathcal{R}_1$. Then 0 is a regular value of the function $H : \mathbb{R}^{3n} \times (0, 1] \to \mathbb{R}^{3n}$ and the zero point set $H^{-1}(0) = \{(u, t) \in \mathcal{R}_1 : H(u, t) = 0\}$ contains a smooth curve $\Gamma_u^{(0)}$ starting from $(u^{(0)}, 1)$.*

Proof. The Jacobian matrix of the above function $H(u, u^{(0)}, t)$ is $DH(u, u^{(0)}, t) = \begin{bmatrix} \frac{\partial H(u,t)}{\partial u} & \frac{\partial H(u,t)}{\partial u^{(0)}} & \frac{\partial H(u,t)}{\partial t} \end{bmatrix}$. For all $u^{(0)} \in \mathcal{R}_1$ and $t \in (0, 1]$,

$$\frac{\partial H(u,t)}{\partial u^{(0)}} = \begin{bmatrix} -\lambda I & 0 & 0 \\ -tY_1^{(0)} & -tX^{(0)} & 0 \\ -tY_2^{(0)} A & 0 & -tY^{(0)} \end{bmatrix},$$

where $Y^{(0)} = \text{diag}(Ax^{(0)} + q), X^{(0)} = \text{diag}(x^{(0)})$, $y^{(0)} = Ax^{(0)} + q$.

Now $\det(\frac{\partial H}{\partial u^{(0)}}) = (-1)^{3n} t^{3n} \prod_{i=1}^{n} x_i^{(0)} y_i^{(0)} \neq 0$ for $t \in (0, 1]$. Therefore, 0 is a regular value of $H(u, u^{(0)}, t)$ by the Lemma 1.3.2. By Lemma 1.3.3 and Lemma 1.3.4, for almost all $u^{(0)} \in \mathcal{R}_1$, 0 is a regular value of $H(u, t)$ and $H^{-1}(0)$ consists of some smooth curves and $H(u^{(0)}, 1) = 0$. Hence there must be a smooth curve $\Gamma_u^{(0)}$ starting from $(u^{(0)}, 1)$. ∎

Hence by Implicit Function Theorem for every t sufficiently close to 1, the function (6.3.1) has a unique solution $(u^{(0)}, 1)$, which is smooth in the parameter t in a neighbourhood of $(u^{(0)}, 1)$. We prove the following result to show that the smooth curve $\Gamma_u^{(0)}$ for the proposed function (6.3.1) is bounded and convergent.

THEOREM 6.3.2. *Let $\mathcal{R}$ be a nonempty set and $A \in \mathbb{R}^{n\times n}$ a matrix and assume that there exists a sequence of points $\{u^k\} \subset \Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0, 1]$, where $u^k = (x^k, y_1^k, y_2^k, t^k)$ such that $\|x^k\| < \infty$ as $k \to \infty$ and $\|y_2^k\| < \infty$ as $k \to \infty$ and for a given $u^{(0)} \in \mathcal{R}_1$, 0 is a regular value of $H(u, u^{(0)}, t)$, then $\Gamma_u^{(0)}$ is a bounded curve in $\mathcal{R}_1 \times (0, 1]$.*

Proof. Note that 0 is a regular value of $H(u, u^{(0)}, t)$ by Theorem 6.3.1. By contradiction we assume that $\Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0, 1]$ is an unbounded curve. Then there exists a sequence of points $\{v^k\}$ where $v^k = (u^k, t^k) \subset \Gamma_u^{(0)}$ such that $\|(u^k, t^k)\| \to \infty$. As $(0, 1]$ is a bounded set and x component and y_2 component of $\Gamma_u^{(0)}$ are bounded, there exists a subsequence of points $\{v^k\} = \{(u^k, t^k)\} = \{x^k, y_1^k, y_2^k, t^k\}$ such that $x^k \to \bar{x}$, $y_2^k \to \bar{y_2}$, $t^k \to \bar{t} \in [0, 1]$ and $\|y^k\| \to \infty$ as $k \to \infty$, where $y^k = \begin{bmatrix} y_1^k \\ y_2^k \end{bmatrix}$. Since $\Gamma_u^{(0)} \subset H^{-1}(0)$, we have

$$(1 - t^k)[(A + A^T)x^k + q - y_1^k - A^T y_2^k] + t^k(x^k - x^{(0)}) = 0 \tag{6.3.2}$$

$$Y_1^k x^k - t^k Y_1^{(0)} x^{(0)} + (1 - t^k)X^k(Ax^k + q) = 0 \tag{6.3.3}$$

$$Y_2^k(Ax^k + q) - t^k Y_2^{(0)}(Ax^{(0)} + q) = 0 \tag{6.3.4}$$

where $Y_1^k = \text{diag}(y_1^k)$, $X^k = \text{diag}(x^k)$ and $Y_2^k = \text{diag}(y_2^k)$.

Let $\bar{t} \in [0, 1]$, $\|y_1^k\| = \infty$ and $\|y_2^k\| < \infty$ as $k \to \infty$. Then $\exists\ i \in \{1, 2, \cdots, n\}$ such that $y_{1i}^k \to \infty$ as $k \to \infty$. Let $I_{1y} = \{i \in \{1, 2, \cdots n\} : \lim\limits_{k\to\infty} y_{1i}^k = \infty\}$. When $\bar{t} \in [0, 1)$, for $i \in I_{1y}$ we write from Equation (6.3.2),

$(1-t^k)[((A+A^T)x^k)_i + q_i - y_{1i}^k - (A^T y_2^k)_i] + t^k(x_i^k - x_i^{(0)}) = 0$
$\implies (1-t^k)y_{1i}^k = (1-t^k)[((A+A^T)x^k)_i + q_i - (A^T y_2^k)_i] + t^k(x_i^k - x_i^{(0)})$
$\implies y_{1i}^k = [((A+A^T)x^k)_i + q_i - (A^T y_2^k)_i] + \frac{t^k}{(1-t^k)}(x_i^k - x_i^{(0)})$.
As $k \to \infty$, right hand side is bounded, but left hand side is unbounded. It contradicts that $\|y_1^k\| = \infty$.
When $\bar{t} = 1$, from Equation (6.3.3), we obtain, $x_i^k = \frac{t^k y_{1i}^{(0)} x_i^{(0)}}{y_{1i}^k}$ for $i \in I_{1y}$. As $k \to \infty, x_i^k \to 0$.
Again from Equation (6.3.2), we obtain $x_i^{(0)} = \frac{(1-t^k)}{t^k}[((A+A^T)x^k)_i + q_i - y_{1i}^k - (A^T y_2^k)_i] + x_i^k$ for $i \in I_{1y}$. As $k \to \infty$, we have $x_i^{(0)} = -\lim\limits_{k\to\infty} \frac{(1-t^k)}{t^k} y_{1i}^k \leq 0$. It contradicts that $\|y_1^k\| = \infty$.
So $\Gamma_u^{(0)}$ is a bounded curve in $\mathcal{R}_1 \times (0,1]$. ■

Therefore the boundedness of the sequences $\{x^k\}$ and $\{y_2^k\}$ gurantee the boundedness of the sequence $\{y_1^k\}$, i.e. the boundedness of the sequence $\{v^k\}$.

THEOREM 6.3.3. *Suppose the solution set $\Gamma_u^{(0)}$ of the function $H(u, u^{(0)}, t) = 0$ is unbounded for $t \in [0,1)$. If there exists $(\xi, \eta, \zeta) \in \mathbb{R}_+^{3n}$ such that $e^T\xi = 1$, then $\xi^T A \xi \leq 0$.*

Proof. Assume that the solution set $\Gamma_u^{(0)}$ is unbounded for $t \in [0,1)$. Then there exists a sequence of points $\{v^k\} \subset \Gamma_u^{(0)} \subset \mathcal{R}_1 \times [0,1)$ where $v^k = (u^k, t^k) = (x^k, y_1^k, y_2^k, t^k)$ such that $\lim_{k\to\infty} t^k = \bar{t} \in [0,1)$. Now we consider following two cases.
Case 1: $\|y_2^k\| < \infty$ as $k \to \infty$. Since the solution set $\Gamma_u^{(0)}$ is unbounded we consider the following two subcases.
Subcase (i) $\lim_{k\to\infty} e^T x^k = \infty$:
Let $\lim_{k\to\infty} \frac{x^k}{e^T x^k} = \xi \geq 0$ and $\lim_{k\to\infty} \frac{y_1^k}{e^T x^k} = \eta \geq 0$. So it is clear that $e^T\xi = 1$. Then dividing by $e^T x^k$ and taking $k \to \infty$ from Equations (6.3.2), and dividing by $(e^T x^k)^2$ and taking $k \to \infty$ from Equations (6.3.3) and (6.3.4), we obtain

$$(1-\bar{t})[(A+A^T)\xi-\eta]+\bar{t}\xi=0 \tag{6.3.5}$$

$$\xi_i\eta_i+\xi_i(A\xi)_i=0\ \forall\ i \tag{6.3.6}$$

From Equations (6.3.5) and (6.3.6) we write $\eta=(A+A^T)\xi+\frac{\bar{t}}{(1-\bar{t})}\xi$ and $-\xi^TA\xi=\xi^T\eta$. These two imply that $\xi^T[(A+A^T)\xi+\frac{\bar{t}}{(1-\bar{t})}\xi]=\xi^T\eta=-\xi^TA\xi$ for $\bar{t}\in[0,1)$. This implies that $2\xi^TA\xi+\xi^TA^T\xi=-\frac{\bar{t}}{(1-\bar{t})}\xi^T\xi\leq 0$ i.e. $\xi^TA\xi\leq 0$ for $\bar{t}\in[0,1)$. Specifically for $\bar{t}=0$, $\xi^TA\xi=0$ and for $\bar{t}\in(0,1)$, $\xi^TA\xi<0$.

Subcase (ii) $\lim_{k\to\infty}(1-t^k)e^Tx^k=\infty$:

Let $\lim_{k\to\infty}\frac{(1-t^k)x^k}{(1-t^k)e^Tx^k}=\xi'\geq 0$. Then $e^T\xi'=1$. Let $\lim_{k\to\infty}\frac{y_1^k}{(1-t^k)e^Tx^k}=\eta'\geq 0$. Then multiplying the Equation (6.3.2) with $(1-t^k)$ and dividing by $(1-t^k)e^Tx^k$, multiplying the Equation (6.3.3) with $(1-t^k)$ and dividing by $((1-t^k)e^Tx^k)^2$ and multiplying the Equation (6.3.4) with $(1-t^k)$ and dividing by $((1-t^k)e^Tx^k)^2$ and taking $k\to\infty$, we obtain

$$(1-\bar{t})[(A+A^T)\xi'-(1-\bar{t})\eta']+\bar{t}\xi'=0 \tag{6.3.7}$$

$$\xi_i'\eta_i'+\xi_i'(A\xi')_i=0\ \forall\ i \tag{6.3.8}$$

Multiplying $(\xi')^T$ in both sides of Equation (6.3.7), we have $(\xi')^T(A+A^T)\xi'-(1-\bar{t})(\xi')^T\eta'=-\frac{\bar{t}}{(1-\bar{t})}(\xi')^T\xi'$. Now using Equation (6.3.8), we write $(\xi')^T(A+A^T)\xi'+(1-\bar{t})(\xi')^TA\xi'=-\frac{\bar{t}}{(1-\bar{t})}(\xi')^T\xi' \implies (\xi')^TA^T\xi'+(2-\bar{t})(\xi')^TA\xi'=-\frac{\bar{t}}{(1-\bar{t})}(\xi')^T\xi'$ for $\bar{t}\in[0,1)$. Hence $(3-\bar{t})(\xi')^TA\xi'=-\frac{\bar{t}}{(1-\bar{t})}(\xi')^T\xi' \implies (\xi')^TA\xi'=-\frac{\bar{t}}{(1-\bar{t})(3-\bar{t})}(\xi')^T\xi'\leq 0$. So we have $(\xi')^TA\xi'\leq 0$ for $\bar{t}\in[0,1)$. Specifically for $\bar{t}=0$, $(\xi')^TA\xi'=0$ and for $\bar{t}\in(0,1)$, $(\xi')^TA\xi'<0$.

Case 2: $\lim_{k\to\infty}e^Ty_2^k=\infty$. Since the solution set of $\Gamma_u^{(0)}$ is unbounded we consider following two subcases.

Subcase (i) $\lim_{k\to\infty}e^Tx^k=\infty$:

Let $\lim_{k\to\infty} \frac{x^k}{e^T x^k} = \xi \geq 0$, $\lim_{k\to\infty} \frac{y_1^k}{e^T x^k} = \eta \geq 0$ and $\lim_{k\to\infty} \frac{y_2^k}{e^T x^k} = \zeta \geq 0$. It is clear that $e^T\xi = 1$. Then dividing by $e^T x^k$ and taking $k \to \infty$ from Equation (6.3.2), dividing by $(e^T x^k)^2$ and taking $k \to \infty$ from Equations (6.3.3), (6.3.4), we obtain

$$(1-\bar{t})[(A+A^T)\xi - \eta - A^T\zeta] + \bar{t}\xi = 0 \tag{6.3.9}$$

$$\xi_i\eta_i + \xi_i(A\xi)_i = 0 \ \forall\ i \tag{6.3.10}$$

$$\zeta_i(A\xi)_i = 0 \ \forall\ i \tag{6.3.11}$$

From Equation (6.3.9), we have $\eta + A^T\zeta = (A+A^T)\xi + \frac{\bar{t}}{1-\bar{t}}\xi$ for $\bar{t} \in [0,1)$. Now multiplying ξ^T in both sides we obtain $\xi^T(A+A^T)\xi + \frac{\bar{t}}{1-\bar{t}}\xi^T\xi = \xi^T\eta + \xi^T A^T\zeta$. From Equations (6.3.10) and (6.3.11), we write $\xi^T(A+A^T)\xi + \frac{\bar{t}}{1-\bar{t}}\xi^T\xi = -\xi^T A\xi$. Hence $\xi^T A\xi + \xi^T(A+A^T)\xi = -\frac{\bar{t}}{1-\bar{t}}\xi^T\xi \leq 0$ for $\bar{t} \in [0,1)$. Therefore $\xi^T A\xi \leq 0$ for $\bar{t} \in [0,1)$. Specifically for $\bar{t} = 0$, $\xi^T A\xi = 0$ and for $\bar{t} \in (0,1)$, $\xi^T A\xi < 0$.

Subcase(ii) $\lim_{k\to\infty}(1-t^k)e^T x^k = \infty$:

Let $\lim_{k\to\infty} \frac{(1-t^k)x^k}{(1-t^k)e^T x^k} = \xi' \geq 0$. Then $e^T\xi' = 1$. Let $\lim_{k\to\infty} \frac{y_1^k}{(1-t^k)e^T x^k} = \eta' \geq 0$ and $\lim_{k\to\infty} \frac{y_2^k}{(1-t^k)e^T x^k} = \zeta' \geq 0$ Then multiplying the Equation (6.3.2) with $(1-t^k)$ and dividing by $(1-t^k)e^T x^k$, multiplying the Equation (6.3.3) with $(1-t^k)$ and dividing by $((1-t^k)e^T x^k)^2$ and multiplying the Equation (6.3.4) with $(1-t^k)$ and dividing by $((1-t^k)e^T x^k)^2$ and taking $k \to \infty$, we obtain

$$(1-\bar{t})(A+A^T)\xi' - (1-\bar{t})^2\eta' - (1-\bar{t})^2 A^T\zeta' + \bar{t}\xi' = 0 \tag{6.3.12}$$

$$\xi_i'\eta_i' + \xi_i'(A\xi')_i = 0 \ \forall\ i \tag{6.3.13}$$

$$\zeta_i'(A\xi')_i = 0 \ \forall\ i \tag{6.3.14}$$

Multiplying $(\xi')^T$ in both side of Equation (6.3.12), we have $(\xi')^T(A+A^T)\xi' - (1-\bar{t})(\xi')^T\eta' - (1-\bar{t})(\xi')^T A^T\zeta' = -\frac{\bar{t}}{(1-\bar{t})}(\xi')^T\xi'$. Now from Equations (6.3.13) and (6.3.14), we write $(\xi')^T(A+A^T)\xi' + (1-\bar{t})(\xi')^T A\xi' = (\xi')^T A^T\xi' + (2-\bar{t})(\xi')^T A\xi' =$

$(3-\bar{t})(\xi')^T A\xi' = -\frac{\bar{t}}{(1-\bar{t})}(\xi')^T\xi' \implies (\xi')^T A\xi' = -\frac{\bar{t}}{(1-\bar{t})(3-\bar{t})}(\xi')^T\xi' \leq 0$ for $\bar{t} \in [0,1)$. Specifically for $\bar{t}=0$, $(\xi')^T A\xi' = 0$ and for $\bar{t} \in (0,1)$, $(\xi')^T A\xi' < 0$.
Hence considering all the cases, it is proved that the unboundedness of the solution set $\Gamma_u^{(0)}$ of the homotopy function $H(u, u^{(0)}, t) = 0$ and the existence of $(\xi, \eta, \zeta) \in R_+^{3n}$ such that $e^T\xi = 1$, imply that $\xi^T A\xi \leq 0$ for $\bar{t} \in [0,1)$. ■

Corollary 6.3.1. *For $\bar{t}=1$, the curve is bounded.*

Proof. Consider that the curve is unbounded in the neighbourhood of $\bar{t}=1$. Then there exists a sequence of points $\{v^k\} \subset \Gamma_u^{(0)} \subset \mathcal{R}_1 \times [0,1)$, where $v^k = (u^k, t^k) = (x^k, y_1^k, y_2^k, t^k)$ such that $\lim_{k\to\infty} t^k = \bar{t} = 1$. Now we consider following two cases.
Case 1. Let $\lim_{k\to\infty} e^T x^k = \infty$ and $\lim_{k\to\infty} \frac{x^k}{e^T x^k} = \xi \geq 0$. Hence $e^T\xi = 1$. If $\|y_2^k\| < \infty$ as $k \to \infty$, then from Equation (6.3.5), we obtain $\xi = 0$ for $\bar{t}=1$. If $\lim_{k\to\infty} e^T y_2^k = \infty$, then from Equation (6.3.9), we obtain $\xi = 0$ for $\bar{t}=1$. This contradicts that $e^T\xi = 1$.
Case 2. Let $\lim_{k\to\infty}(1-t^k)e^T x^k = \infty$ and $\lim_{k\to\infty} \frac{(1-t^k)x^k}{(1-t^k)e^T x^k} = \xi' \geq 0$. Hence $e^T\xi' = 1$. If $\|y_2^k\| < \infty$ as $k \to \infty$, then from Equation (6.3.7), we obtain $\xi' = 0$ for $\bar{t}=1$. If $\lim_{k\to\infty} e^T y_2^k = \infty$, then from Equation (6.3.12), we obtain $\xi' = 0$ for $\bar{t}=1$. This contradicts that $e^T\xi' = 1$.
Therefore the curve is bounded for $\bar{t}=1$. ■

Theorem 6.3.4. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix. If the set $\mathcal{R}_1$ be nonempty and 0 is a regular value of $H(u, u^{(0)}, t)$, then the path $\Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0,1]$ is bounded.*

Proof. Suppose $A \in \mathbb{R}^{n\times n}$ is a matrix and there exists a sequence of points $\{v^k\} \subset \Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0,1]$, where $v^k = (x^k, y_1^k, y_2^k, t^k)$. Hence by the definition of $\mathcal{R}_1$ $x^k, y_1^k, y_2^k, Ax^k + q > 0$. From Corollary 6.3.1 the curve is bounded for $\bar{t}=1$. Assume that the curve $\Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0,1)$ is unbounded. Then from Theorem 6.3.3 $(\xi)^T A\xi < 0$ for $\bar{t} \in (0,1)$. For $\lim_{k\to\infty} e^T x^k = \infty$, $\xi = \lim_{k\to\infty} \frac{x^k}{e^T x^k} \geq 0$. Again for

$\lim_{k\to\infty}(1-t^k)e^Tx^k = \infty$, $\xi = \lim_{k\to\infty}\frac{(1-t^k)x^k}{(1-t^k)e^Tx^k} \geq 0$. $Ax^k + q > 0$ implies that $A\xi \geq 0$. Hence $\xi, A\xi \geq 0$ imply that $\xi^TA\xi \geq 0$ for $\bar{t} \in (0,1)$, which contradicts that the path is unbounded for $\bar{t} \in (0,1)$. Hence the curve $\Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0,1]$ is bounded. ■

Therefore the curve $\Gamma_u^{(0)}$ is bounded for the parameter t starting from 1 to 0 if the set $\mathcal{R}_1$ be nonempty and 0 is a regular value of the function (6.3.1). For an initial point $u^{(0)} \in \mathcal{R}_1$ we obtain a smooth bounded path which leads to the solution of function (6.3.1) as the parameter $t \to 0$.

THEOREM 6.3.5. *For $u^{(0)} = (x^{(0)}, y_1^{(0)}, y_2^{(0)}) \in \mathcal{R}_1$, the equation finds a bounded smooth curve $\Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0,1]$ which starts from $(u^{(0)}, 1)$ and approaches the hyperplane at $t = 0$. As $t \to 0$, the limit set $L \times \{0\} \subset \bar{\mathcal{R}}_1 \times \{0\}$ of $\Gamma_u^{(0)}$ is nonempty and every point in L is a solution of the following system:*

$$\begin{aligned} (A + A^T)x + q - y_1 - A^Ty_2 &= 0 \\ Y_1x + X(Ax + q) &= 0 \\ Y_2(Ax + q) &= 0. \end{aligned} \tag{6.3.15}$$

Proof. Note that $\Gamma_u^{(0)}$ is diffeomorphic to a unit circle or a unit interval $(0,1]$ in view of Lemma 1.3.5. As $\frac{\partial H(u,u^{(0)},1)}{\partial u^{(0)}}$ is nonsingular, $\Gamma_u^{(0)}$ is diffeomorphic to a unit interval $(0,1]$. Again $\Gamma_u^{(0)}$ is a bounded smooth curve by the Theorem 6.3.2. Let $(\bar{u}, \bar{t})$ be a limit point of $\Gamma_u^{(0)}$. Now consider the followings:

$(i)(\bar{u}, \bar{t}) \in \mathcal{R}_1 \times \{1\}$: As the equation $H(u,1) = 0$ has only one solution $u^{(0)} \in \mathcal{R}_1$, this case is impossible.

$(ii)(\bar{u}, \bar{t}) \in \partial\mathcal{R}_1 \times \{1\}$: there exists a subsequence of $(u^k, t^k) \in \Gamma_u^{(0)}$ such that $x_i^k \to 0$ or $(Ax^k + q)_i \to 0$ for $i \subseteq \{1, 2, \cdots n\}$. From the last two equations of the function (6.3.1), we have $y_1^k \to \infty$ or $y_2^k \to \infty$. Hence it contradicts the boundedness of the path by the Theorem 6.3.2.

$(iii)(\bar{u},\bar{t}) \in \partial\mathcal{R}_1 \times (0,1)$: Also impossible followed by the case (ii).

$(iv)(\bar{u},\bar{t}) \in \bar{\mathcal{R}}_1 \times \{0\}$: The only possible case.

Hence $\bar{u} = (\bar{x}, \bar{y_1}, \bar{y_2})$ is a solution of the system (6.3.15)

$$(A + A^T)x + q - y_1 - A^T y_2 = 0$$
$$Y_1 x + X(Ax + q) = 0$$
$$Y_2(Ax + q) = 0.$$

■

6.3.1 Computing Solution of ARAT Stochastic Game based on Iterative Process

We state that a pair of strategies $(k^\star, l^\star)$ is optimal for Player I and Player II in the discounted game if for all $s \in S$ $\phi_s(k, l^\star) \leq \phi_s(k^\star, l^\star) = v_s^\star \leq \phi_s(k^\star, l)$ for any strategies k and l of Player I and Player II. The number $v_s^\star$ is called the value of the game starting in state s and $v^\star = (v_1^\star, v_2^\star, \cdots, v_N^\star)$ is called the value vector. To find the optimal strategy of player I and player II of the two-person zero-sum discounted ARAT stochastic game we propose a new function based on the concept of iterative method.

Now we show that the solution of the proposed function will give the solution of discounted ARAT stochastic game.

THEOREM 6.3.6. *Suppose* $\Gamma_u^{(0)} = \{(u,t) \in \mathbb{R}^{3n} \times (0,1] : H(u, u^{(0)}, t) = 0\} \subset \mathcal{R}_1 \times (0,1]\},$ *and* $\mathcal{A} = \begin{bmatrix} -\beta P_1 & E - \beta P_1 \\ -E + \beta P_2 & \beta P_2 \end{bmatrix}$ *and* $q = \begin{bmatrix} -R_i^1(s) \\ R_j^2(s) \end{bmatrix}$, *where* *the matrix* $P_1 = P_1(s), P_2 = P_2(s)$ *and* $E = \begin{bmatrix} e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e_d \end{bmatrix}$ *is a vertical block identity matrix where* $e_j, 1 \leq j \leq d$, *is a column vector of all* $1's$. *Then the function (6.3.1) solves discounted zero-sum stochastic ARAT game.*

Proof. Suppose for a zero-sum discounted ARAT game the optimal pure strategy in state s is i_0 for Player I and j_0 for Player II.
Then the inequality (6.1.1) and the inequality (6.1.2) reduces to

$$R_i^1(s) + R_{j_0}^2(s) + \beta \sum_{s'} p_i^1(s,s')v_\beta(s') + \beta \sum_{s'} p_{j_0}^2(s,s')v_\beta(s') \leq v_\beta(s) \ \forall\ i. \quad (6.3.16)$$

$$R_{i_0}^1(s) + R_j^2(s) + \beta \sum_{s'} p_{i_0}^1(s,s')v_\beta(s') + \beta \sum_{s'} p_j^2(s,s')v_\beta(s') \geq v_\beta(s) \ \forall\ j. \quad (6.3.17)$$

Thus the inequalities are

$$R_i^1(s) + \beta \sum_{s'} p_i^1(s,s')\xi_\beta(s') - \xi_\beta(s) + \beta \sum_{s'} p_i^1(s,s')\eta_\beta(s') \leq 0 \ \forall i \in A_s, s \in S \quad (6.3.18)$$

and similarly the inequalities for Player II are

$$R_j^2(s) + \beta \sum_{s'} p_j^2(s,s')\eta_\beta(s') - \eta_\beta(s) + \beta \sum_{s'} p_j^2(s,s')\xi_\beta(s') \geq 0 \ \forall j \in B_s, s \in S. \quad (6.3.19)$$

Also for each s, in (6.3.18) there is an $i(s)$ such that equality holds. Similarly, for each s in (6.3.19) there is a $j(s)$ such that equality holds. Let for $i \in A_s$,

$$w_i^1(s) = -R_i^1(s) - \beta \sum_{s'} p_i^1(s,s')\eta_\beta(s') + \xi_\beta(s) - \beta \sum_{s'} p_i^1(s,s')\xi_\beta(s') \geq 0, \quad (6.3.20)$$

and for $j \in B_s$,

$$w_j^2(s) = R_j^2(s) - \eta_\beta(s) + \beta \sum_{s'} p_j^2(s,s')\eta_\beta(s') + \beta \sum_{s'} p_j^2(s,s')\xi_\beta(s') \geq 0. \quad (6.3.21)$$

We may assume without loss of generality that $\eta_\beta(s), \xi_\beta(s)$ are strictly positive. Since there is atleast one inequality in (6.3.20) for each $s \in S$ that holds as an equality and one inequality in (6.3.21) for each $s \in S$ that holds as an equality, the following complementarity conditions will hold.

$$\eta_\beta(s) \prod_{i \in A_s} w_i^1(s) = 0 \; for \; 1 \leq s \leq d \quad (6.3.22)$$

and

$$\xi_\beta(s) \prod_{j \in B_s} w_j^2(s) = 0 \; for \; 1 \leq s \leq d. \quad (6.3.23)$$

The inequality (6.3.20) and inequality (6.3.21) along with the complementarity conditions (6.3.22) and (6.3.23) lead to the VLCP$(q, \mathcal{A})$ where the matrix $\mathcal{A}$ is of the form

$$\mathcal{A} = \begin{bmatrix} -\beta P_1 & E - \beta P_1 \\ -E + \beta P_2 & \beta P_2 \end{bmatrix} \text{ and } q = \begin{bmatrix} -R_i^1(s) \\ R_j^2(s) \end{bmatrix},$$

where the matrix $P_1 = P_1(s), P_2 = P_2(s)$ and

$E = \begin{bmatrix} e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e_d \end{bmatrix}$ is a vertical block identity matrix where $e_j, 1 \leq j \leq d$, is a column vector of all $1'$s. Now an equivalent square matrix A can be constructed from the vertical block matrix $\mathcal{A}$ of type $(m_1, ..., m_c)$ by copying $\mathcal{A}_{\cdot j}$, m_j times for $j = 1, 2, \cdots, c$. Therefore $A_{\cdot p} = \mathcal{A}_{\cdot j}$ $\forall p \in J_j$ and the LCP(q, A) is the equivalent LCP of VLCP$(q, \mathcal{A})$. We consider the proposed function (6.3.1)

$$H(u,t) = \begin{bmatrix} (1-t)[(A+A^T)x + q - y_1 - A^T y_2] + t(x - x^{(0)}) \\ Y_1 x - tY_1^{(0)} x^{(0)} + (1-t)X(Ax+q) \\ Y_2(Ax+q) - tY_2^{(0)}(Ax^{(0)}+q) \end{bmatrix} = 0 \quad (6.3.24)$$

where $Y_1 = \text{diag}(y_1)$, $X = \text{diag}(x)$, $Y_2 = \text{diag}(y_2)$, $Y_1^{(0)} = \text{diag}(y_1^{(0)})$, $Y_2^{(0)} = \text{diag}(y_2^{(0)})$, $u = (x, y_1, y_2) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$, $u^{(0)} = (x^{(0)}, {y_1}^{(0)}, {y_2}^{(0)}) \in \mathcal{R}_1$, and $\lambda \in (0,1]$. We denote $\Gamma_u^{(0)} = \{(u,t) \in \mathbb{R}^{3n} \times (0,1] : H(u, u^{(0)}, t) = 0\} \subset \mathcal{R}_1 \times (0,1]\}$. For the proposed function t varies from 1 to 0. Starting from $t = 1$ to $t \to 0$ if we have a smooth bounded curve, then we obtain a finite solution of the equation (6.3.1) at $t \to 0$. As $t \to 1$, the equation (6.3.1) gives the solution $(u^{(0)}, 1)$, and as $t \to 0$, the equation (6.3.1) gives the solution of the system of following equations:

$$(A + A^T)x + q - y_1 - A^T y_2 = 0$$
$$Y_1 x + X(Ax + q) = 0$$
$$Y_2(Ax + q) = 0$$

where $Y_1 = \text{diag}(y_1)$ and $Y_2 = \text{diag}(y_2)$. Hence the solution of the function (6.3.1) gives the solution of discounted zero-sum ARAT game. ∎

Therefore if the function (6.3.1) converges to its solution as the parameter $t \to 0$, we obtain the solution of discounted ARAT stochastic game.
Note that Theorem 6.3.5 establishes the solution of the proposed function which validates the Theorem 6.3.6. This in turn leads to the solution of discounted ARAT stochastic game.

In this approach the initial point $u^{(0)} = (x^{(0)}, y_1^{(0)}, y_2^{(0)}) \in \mathcal{R}_1$ has to be a feasible point. Hence choose the initial point such that $x^{(0)} > 0$, $Ax^{(0)} + q > 0$. Here $(\bar{u}, 0)$ is the solution of the function (6.3.1). Therefore $\bar{u} \in \bar{R}_1$ is the solution of the system of equations (6.3.15). Hence $\bar{Y}_1 \bar{x} = 0$ and $\bar{X}(A\bar{x} + q) = 0$, where

$\bar{Y}_1$ =diag($\bar{y}_1$) and $\bar{X}$ =diag($\bar{x}$). It is clear that the component $\bar{x}$ of $\bar{u} = (\bar{x}, \bar{y}_1, \bar{y}_2)$ provides the solution of discounted ARAT stochastic game.

6.3.2 Tracing Path by iterative process

We trace the path $\Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0, 1]$ from the initial point $(u^{(0)}, 1)$. To find the solution of the discounted ARAT stochastic game we consider path alongwith other assumptions. Let s denote the arc length of $\Gamma_u^{(0)}$. We parameterize the path $\Gamma_u^{(0)}$ with respect to s in the following form

$$H(u,t) = 0,\ u(0) = u^{(0)},\ t(0) = 1. \tag{6.3.25}$$

The solution of the Equation (6.3.25) satisfies the following problem

$$\dot{u} = -\frac{\partial}{\partial u} H(u,t)^{-1} \frac{\partial}{\partial t} H(u,t),\ u(0) = u^{(0)},\ t(0) = 1 \tag{6.3.26}$$

From equation (6.3.1) the choice of H is $H(u,t) = (1-t)f(u) + tg(u) = 0$, where $f(u) = \begin{bmatrix} (A + A^T)x + q - y_1 - A^T y_2 \\ Y_1 x + X(Ax + q) \\ Y_2(Ax + q) \end{bmatrix}$ and $g(u) = \begin{bmatrix} x - x^{(0)} \\ Y_1 x - Y_1^{(0)} x^{(0)} \\ Y_2(Ax + q) - Y_2^{(0)}(Ax^{(0)} + q) \end{bmatrix}$. Hence the system (6.3.26) becomes

$$\dot{u} = p(u,t),\ u(0) = u^{(0)} \text{ where } p(u,t) = -\tilde{J}^{-1}\tilde{f}$$

This problem will be solved by iterative process

$$u_{(i+1)} = P(u_i, t_i, h_i), \text{ where } h_i = t_{i+1} - t_i.$$

Here u_i is an approximation of u. $P(u_i, t_i, h_i)$ is given by

$$P(u,t,h) = I_m(u,t,h), \text{ where } I_0(u,t,h) = u$$
$$\text{and } K_j = \tfrac{\partial}{\partial u} H(I_j, t+h)^{+} H(I_j, t+h)$$
$$L_j = I_j - K_j$$
$$KK_j = (\tfrac{\partial}{\partial u} H(L_j, t+h) + \tfrac{\partial}{\partial u} H(I_j, t+h))^{+} H(I_j, t+h)$$
$$LL_j = I_j - 2 * KK_j$$

The next iteration

$$I_{j+1} = LL_j - \tfrac{\partial}{\partial u} H(L_j, t+h)^{+} H(L_j, t+h), \text{ for } j = 0, 1, 2, \cdots, m-1.$$

Algorithm:

Step 0: Initialize $(u^{(0)}, t_0)$. Set $l_0 \in (0,1)$. Choose $\epsilon_3 >> \epsilon_1 > 0$ which are small positive quantity.

Step 1: $\tau^{(0)} = \xi^{(0)} = (\frac{1}{n_0}) \begin{bmatrix} s^{(0)} \\ -1 \end{bmatrix}$ for $i = 0$, where $n_0 = \| \begin{bmatrix} s^{(0)} \\ -1 \end{bmatrix} \|$ and $s^{(0)} = (\frac{\partial H}{\partial u}(u^{(0)}, t_0))^{-1}(\frac{\partial H}{\partial t}(u^{(0)}, t_0))$.
For $i > 0$, $s^{(i)} = (\frac{\partial H}{\partial u}(u^{(i)}, t_i))^{-1}(\frac{\partial H}{\partial t}(u^{(i)}, t_i))$, $n_i = \| \begin{bmatrix} s^{(i)} \\ -1 \end{bmatrix} \|$, $\xi^{(i)} = (\frac{1}{n_i}) \begin{bmatrix} s^{(i)} \\ -1 \end{bmatrix}$.
If $\det(\frac{\partial H}{\partial u}(u^{(i)}, t_i)) > 0$, $\tau^{(i)} = \xi^{(i)}$ else $\tau^{(i)} = -\xi^{(i)}$, $i \geq 1$.
Set $l = 0$.

Step 2: (Predictor and corrector point calculation) $(\tilde{u}^{(i)}, \tilde{t}_i) = (u^{(i)}, t_i) + a\tau^{(i)}$, where $a = {l_0}^l$. Compute $(\hat{u}^{(i)}, \hat{t}_i) = H'_{u^{(0)}}(\tilde{u}^{(i)}, \tilde{t}_i)^+ H(\tilde{u}^{(i)}, \tilde{t}_i)$ and $(\bar{u}^{(i)}, \bar{t}_i) = (\tilde{u}^{(i)}, \tilde{t}_i) - (\hat{u}^{(i)}, \hat{t}_i)$. Now compute $(\hat{uu}^{(i)}, \hat{tt}_i) = (H'_{u^{(0)}}(\tilde{u}^{(i)}, \tilde{t}_i) + H'_{u^{(0)}}(\bar{u}^{(i)}, \bar{t}_i))^+ H(\tilde{u}^{(i)}, \tilde{t}_i)$ and $(\bar{uu}^{(i)}, \bar{tt}_i) = (\tilde{u}^{(i)}, \tilde{t}_i) - 2(\hat{uu}^{(i)}, \hat{tt}_i)$.
Compute $(u^{(i+1)}, t_{i+1}) = (\bar{uu}^{(i)}, \bar{tt}_i) - H'_{u^{(0)}}(\bar{u}^{(i)}, \bar{t}_i)^+ H(\bar{uu}^{(i)}, \bar{tt}_i)$.
If $0 < \|t_{i+1} - t_i\| < 1$, go to step 3. Otherwise if $m' = \min(a, \|(u^{(i+1)}, t_{i+1}) - (u^{(i)}, t_i)\|) > a_0$, update l by $l+1$, and recompute $(\tilde{u}_i, \tilde{t}_i)$ else go to step 3.

Step 3: Determine the norm $r = \|H(u^{(i+1)}, t_{i+1})\|$. If $r \leq 1$ and $u^{(i+1)} > 0$ go to step 5, otherwise if $a > \epsilon_3$, update l by $l+1$ and go to step 2 else go to step 4.

Step 4: If $|t_{i+1}| \leq \epsilon_1$, then stop with solution $(u^{(i+1)}, t_{i+1})$, else $i = i+1$ and go to step 1.

Note that in step 2, $H'_{u^{(0)}}(u,t)^+ = H'_{u^{(0)}}(u,t)^T (H'_{u^{(0)}}(u,t) H'_{u^{(0)}}(u,t)^T)^{-1}$ is the Moore-Penrose inverse of $H'_{u^{(0)}}(u,t)$.

We prove the following theorem to obtain the positive direction of the proposed algorithm.

THEOREM 6.3.7. *If the curve $\Gamma_u^{(0)}$ is smooth, then the positive predictor direction $\tau^{(0)}$ at the initial point $u^{(0)}$ satisfies* $\det(\begin{bmatrix} \frac{\partial H}{\partial u \partial t}(u^{(0)}, 1) \\ \tau^{(0)^T} \end{bmatrix}) < 0$.

Proof. From the Equation (6.3.1), we consider the following function

$$H(u,t) = \begin{bmatrix} (1-t)[(A+A^T)x+q-y_1-A^Ty_2]+t(x-x^{(0)}) \\ Y_1x - tY_1^{(0)}x^{(0)} + (1-t)X(Ax+q) \\ Y_2(Ax+q) - tY_2^{(0)}(Ax^{(0)}+q) \end{bmatrix} = 0.$$

Now $\frac{\partial H}{\partial u \partial t}(u,t) =$

$$\begin{bmatrix} (1-t)(A+A^T)+tI & -(1-t)I & -(1-t)A^T & Q \\ Y_1 + (1-t)(Y+XA) & X & 0 & -Y_1^{(0)}x^{(0)} - X(Ax+q) \\ Y_2A & 0 & Y & -Y_2^{(0)}(Ax^{(0)}+q) \\ & & & \end{bmatrix},$$

where $Q = (x-x^{(0)}) - [(A+A^T)x+q-y_1-A^Ty_2]$ and $Y = \mathrm{diag}(Ax+q)$.

At the initial point $(u^{(0)},1)$

$$\frac{\partial H}{\partial u \partial t}(u^{(0)},1) = \begin{bmatrix} I & 0 & 0 & -[(A+A^T)x^{(0)}+q-y_1^{(0)}-A^Ty_2^{(0)}] \\ Y_1^{(0)} & X^{(0)} & 0 & -Y_1^{(0)}x^{(0)} - X^{(0)}(Ax^{(0)}+q) \\ Y_2^{(0)}A & 0 & Y^{(0)} & -Y_2^{(0)}(Ax^{(0)}+q) \end{bmatrix}.$$

Let positive predictor direction be $\tau^{(0)} = \begin{bmatrix} \kappa \\ -1 \end{bmatrix} = \begin{bmatrix} (Q_1^{(0)})^{(-1)}Q_2^{(0)} \\ -1 \end{bmatrix}$,

where

$$Q_1^{(0)} = \begin{bmatrix} I & 0 & 0 \\ Y_1^{(0)} & X^{(0)} & 0 \\ Y_2^{(0)}A & 0 & Y^{(0)} \end{bmatrix},$$

$$Q_2^{(0)} = \begin{bmatrix} -[(A+A^T)x^{(0)}+q-y_1^{(0)}-A^Ty_2^{(0)}] \\ -Y_1^{(0)}x^{(0)} - X^{(0)}(Ax^{(0)}+q) \\ -Y_2^{(0)}(Ax^{(0)}+q) \end{bmatrix}$$ and κ is an $n \times 1$ column vector.

Hence, $\det(\begin{bmatrix} \frac{\partial H}{\partial u \partial \lambda}(u^{(0)},1) \\ \tau^{(0)T} \end{bmatrix})$

$$= \det(\begin{bmatrix} Q_1^{(0)} & Q_2^{(0)} \\ (Q_2^{(0)})^T(Q_1^{(0)})^{(-T)} & -1 \end{bmatrix})$$

$$= \det\left(\begin{bmatrix} Q_1^{(0)} & Q_2^{(0)} \\ 0 & -1-(Q_2^{(0)})^T(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)} \end{bmatrix}\right)$$

$$= \det(Q_1^{(0)})\det(-1-(Q_2^{(0)})^T(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)})$$

$$= -\det(Q_1^{(0)})\det(1+(Q_2^{(0)})^T(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)})$$

$$= -\prod_{i=1}^{n} x_i^{(0)} y_i^{(0)} \det(1+(Q_2^{(0)})^T(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)}) < 0.$$

So the positive predictor direction $\tau^{(0)}$ at the initial point $u^{(0)}$ satisfies
$\det\left(\begin{bmatrix} \frac{\partial H}{\partial u \partial \lambda}(u^{(0)},1) \\ \tau^{(0)^T} \end{bmatrix}\right) < 0.$ ■

Remark 6.3.1. *We conclude from the Theorem 6.3.7 that the positive tangent direction τ of the path $\Gamma_u^{(0)}$ at any point (u,t) be negative and it depends on* $\det(Q_1)$, *where* $Q_1 = \begin{bmatrix} (1-t)(A+A^T)+tI & -(1-t)I & -(1-t)A^T \\ Y_1+(1-t)(Y+XA) & X & 0 \\ Y_2A & 0 & Y \end{bmatrix}$.

Based on the earlier work to solve the initial value problem (6.3.26) was formulated with the iterative process as

$$I_{j+1} = I_j - \frac{\partial}{\partial u}H(I_j, t+h)^{+}H(I_j, t+h), \text{ for } j = 0,1,2,\cdots,m-1.$$

For details see [109]. However the proposed iterative method solves the function by solving the problem (6.3.26) with the following iterative process

$$K_j = \frac{\partial}{\partial u}H(I_j, t+h)^{+}H(I_j, t+h)$$
$$L_j = I_j - K_j$$

$$KK_j = (\tfrac{\partial}{\partial u}H(L_j, t+h) + \tfrac{\partial}{\partial u}H(I_j, t+h))^{+}H(I_j, t+h)$$
$$LL_j = I_j - 2 * KK_j$$
$$I_{j+1} = LL_j - \tfrac{\partial}{\partial u}H(L_j, t+h)^{+}H(L_j, t+h), \text{ for } j = 0, 1, 2, \cdots, m-1.$$

By this iterative process the proposed function achieves the order of convergence as $5^m - 1$.

THEOREM 6.3.8. *Suppose that the function has derivative which is Lipschitz continuous in a convex neighbourhood $\mathcal{N}$ of c, where c is the solution of the function $H(u, t) = 0$, whose Jacobian matrix is continuous and nonsingular and bounded on $\mathcal{N}$. Then the iterative method has order $5^m - 1$.*

Proof. By the Implicit Function Theorem ensures the existence of a unique continuous solution $z(h) \in \mathcal{N}$ of $\dot{z}(h) = -\tilde{J}^{-1}\tilde{f}$, $z(0) = u$ and $h \in (-\delta, \delta)$, for some $\delta > 0$. Define $\beta_j = \|z(h) - I_j(u, h)\|$. From Lemma 6.2.3 $\beta_j = O(h^{5^j})$. Then $\beta_{j+1} = \|z(h) - I_{j+1}\| \leq K{\beta_j}^5$. Hence $\beta_{j+1} = O(h^{5^{j+1}})$. By induction the proposed iterative method has convergency of order $5^m - 1$ ∎

6.3.3 Solving Discounted Zero-Sum Stochastic Game with ARAT Structure

EXAMPLE 6.3.1. *Consider a two player zero-sum discounted ARAT game with $s = 2$ states. In each state each player has 2 actions. The transition probabilities are given by*

$p_1^1(1,1) = \frac{1}{2},\ p_1^1(1,2) = 0,$

$p_2^1(1,1) = \frac{1}{2},\ p_2^1(1,2) = 0,$

$p_1^1(2,1) = 0,\ p_1^1(2,2) = \frac{1}{2},$

$p_2^1(2,1) = 0,\ p_2^1(2,2) = \frac{1}{2},$

$p_1^2(1,1) = \frac{1}{2},\ p_1^2(1,2) = 0,$

$p_2^2(1,1) = 0,\ p_2^2(1,2) = \frac{1}{2},$

$p_1^2(2,1) = 0,\ p_1^2(2,2) = \frac{1}{2},$

$p_2^2(2,1) = \frac{1}{2},\ p_2^2(2,2) = 0.$

Note that $p_{ij}(s,s') = p_i^1(s,s') + p_j^2(s,s')$.

$P_1 = P_1(s) = ((p_i^1(s,s'), s, s' \in S, i \in A_s))$ *and*

$P_2 = P_2(s) = ((p_j^2(s,s'), s, s' \in S, j \in B_s))$.

Let the discount factor $\beta = \frac{1}{2}$.

The reward structure:

$R_1^1(1) = 4,\quad R_1^1(2) = 5,$

$R_2^1(1) = 3,\quad R_2^1(2) = 4,$

$R_1^2(1) = 3,\quad R_1^2(2) = 6,$

$R_2^2(1) = 6,\quad R_2^2(2) = 2.$

Note that $r(s,i,j) = R_i^1(s) + R_j^2(s)$.

Now we solve discounted ARAT game using the proposed function. The initial point is $u^{(0)} = (4,5,3,4,8,8,6,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)^T$. *Here* $l_0 = \frac{1}{2}, \epsilon_1 = 10^{-9}, \epsilon_3 = 10^{-5}$. *After* 47 *iterations we obtain the solution* $u = (0,7,0,6,9,0,0,7.33,1,0,1,0,0,2.33,3.33,0,0,7,0,6,9,0,0,7.33,0)$ *as* $t \to 0$.

Hence the solution of discounted ARAT is $x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2.33 \\ 3.33 \\ 0 \end{bmatrix}$.

Example 6.3.2. *Consider another two player zero-sum discounted ARAT game with* $s = 2$ *states. In each state each player has* 2 *actions. The transition probabilities are given by*

$p_1^1(1,1) = \frac{1}{4},\ p_1^1(1,2) = 0,$

$p_2^1(1,1) = \frac{1}{4},\ p_2^1(1,2) = 0,$

$p_1^1(2,1) = 0,\ p_1^1(2,2) = \frac{1}{2},$

$p_2^1(2,1) = 0,\ p_2^1(2,2) = \frac{1}{2},$

$p_1^2(1,1) = \frac{3}{4},\ p_1^2(1,2) = 0,$

$p_2^2(1,1) = 0,\ p_2^2(1,2) = \frac{3}{4},$

$p_1^2(2,1) = 0,\ p_1^2(2,2) = \frac{1}{2},$

$p_2^2(2,1) = \frac{1}{2},\ \ p_2^2(2,2) = 0.$

Note that $p_{ij}(s,s') = p_i^1(s,s') + p_j^2(s,s')$.

$P_1 = P_1(s) = ((p_i^1(s,s'), s, s' \in S, i \in A_s))$ *and*

$P_2 = P_2(s) = ((p_j^2(s,s'), s, s' \in S, j \in B_s))$.

Let the discount factor $\beta = \frac{1}{2}$.

The reward structure:

$R_1^1(1) = 4, \quad R_1^1(2) = 5,$

$R_2^1(1) = 3, \quad R_2^1(2) = 4,$

$R_1^2(1) = 3, \quad R_1^2(2) = 6,$

$R_2^2(1) = 6, \quad R_2^2(2) = 2.$

Note that $r(s,i,j) = R_i^1(s) + R_j^2(s)$.

Now we solve discounted ARAT game using the proposed function. The initial point is $u^{(0)} = (1,1,1,1,20,20,10,10,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)^T$. *Here* $l_0 = \frac{1}{2}, \epsilon_1 = 10^{-9}, \epsilon_3 = 10^{-5}$. *After* 31 *iterations we obtain the solution* $u = (0,9,0,6,7,0,0,7.33,1,0,1,0,0,2,3.33,0,0,9,0,6,7,0,0,7.33,0)$ *as* $t \to 0$.

Hence the solution of discounted ARAT is $x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 2 \\ 3.33 \\ 0 \\ 0 \end{bmatrix}.$

In this chapter, we introduce a iterative process to find the solution of discounted ARAT stochastic game. Mathematically, we obtain the positive tangent direction of the homotopy path. We prove that the smooth curve of the proposed homotopy function is bounded and convergent.

Chapter 7

Tracing Homotopy Path For The Solution Of Nonlinear Complementarity Problem

7.1 Introduction

In this chapter, we consider nonlinear complementarity problem. The concept of complementarity is synonymous with the notion of system equilibrium. The nonlinear complementarity problem is identified as an important mathematical programming problem. The idea of nonlinear complementarity problem is based on the concept of linear complementarity problem. In the literature so many techniques are developed to solve nonlinear complementarity problems. For details see chapter 1. In this chapter we solve nonlinear complementarity problem by using homotopy approach ensuring the boundedness property of the trajectory obtained from the proposed homotopy function. The basic idea of homotopy method is to construct a homotopy continuation path from the auxiliary mapping

*Work of this chapter has been communicated.

†Results of this chapter have been arxived [137].

to the object mapping.

The chapter is organized as follows. Section 7.2 is about continuation method with homotopy function. In section 7.3, we propose a new homotopy function to find the solution of nonlinear complementarity problem. We construct a smooth and bounded homotopy path to find the solution of the nonlinear complementarity problem as the homotopy parameter $\mu \in \mathbb{R}$ starts from 1 and tends to 0. To find the solution of homotopy function we modify homotopy continuation method to increase the order of convergency of the algorithm. We also find the sign of the positive tangent direction of the homotopy path. Finally, in section 7.4 we numerically solve some examples of nonlinear complementarity problem using the introduced homotopy function to demonstrate the effectiveness of our proposed approach.

7.2 Continuation Method with Homotopy Function

The fundamental idea of the homotopy continuation method is to solve a problem by tracing a certain continuous path that leads to a solution to the problem. Thus, defining a homotopy mapping that yields a finite continuation path plays an essential role in a homotopy continuation method. The homotopy method [183] is itself an important class of globally convergent methods. Many homotopy methods are proposed for constructive proof of the existence of solutions to systems of nonlinear equations, nonlinear optimization problems, Brouwer fixed point problems, nonlinear programming problems, game problems and complementarity problems [184]. Eaves and Saigal [182] attempted to use similar approaches for solving system of non-linear equations. Such methods have been used to constructively prove the existence of solutions to many economic and engineering problems.

Let P_1, P_2 be two topological spaces and $\mathrm{Map}(P_1, P_2)$ be the set of all continuous maps from P_1 to P_2. Homotopy is an equivalence relation on $\mathrm{Map}(P_1, P_2)$. Let $h_1, h_2 : P_1 \to P_2$ be continuous maps. A homotopy from h_1 to h_2 is a continuous function $H : P_1 \times [0, 1] \to P_2$ satisfying $H(y, 0) = h_1(y)$, $H(y, 1) = h_2(y) \; \forall \, y \in P_1$. If such a homotopy exists, then h_1 is homotopic to h_2 and it is denoted by $h_1 \simeq h_2$.

The basic idea of homotopy method is to construct a homotopy continuation path from the auxiliary mapping g to the object mapping f. Suppose the given problem is to find a root of the nonlinear equation $f(x) = 0$ and $g(x) = 0$ is an auxiliary equation with $g(x_0) = 0$. Then the homotopy function $H : \mathbb{R}^{n+1} \to \mathbb{R}^n$ can be defined as $H(x, \mu) = (1 - \mu)f(x) + \mu g(x)$, $0 \leq \mu \leq 1$. Based on this concept, we consider the homotopy function $H(x, \mu) = 0$, where $(x_0, 1)$ is a known solution of the homotopy function. Our aim is to find the solution of the equation $f(x) = 0$ from the known solution of $g(x) = 0$ by solving the homotopy function $H(x, \mu) = 0$ varying the values of μ from 1 to 0.

7.3 Continuation Method for Nonlinear Complementarity Problem

Some homotopy methods are developed to solve nonlinear complementarity problem under some assumptions such as the nonlinear function $f(z)$ associated with the nonlinear complementarity problem is a P function. Ding et al.[85] proposed a homotopy method. However this method can not ensure the existence of the bounded solution in many cases. For details see [146], [87], [85], [89].

Now we solve nonlinear complementarity problem by a new approach of continuation method. We consider two positive numbers m, l such that m is large positive number and l is positive number with $l << m$. First we define

$\mathcal{R}_{(m)} = \{(z, y, w_1, w_2, v_1, v_2) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} : m - (\sum_{i=1}^{n}(z + w_1)_i + v_2) > l, m - (\sum_{i=1}^{n}(y + w_2)_i + v_1) > l\}$,

$\bar{\mathcal{R}}_{(m)} = \{(z, y, w_1, w_2, v_1, v_2) \in \mathbb{R}^n_{+} \times \mathbb{R}^n_{+} \times \mathbb{R}^n_{+} \times \mathbb{R}^n_{+} \times \mathbb{R}_{+} \times \mathbb{R}_{+} : m - (\sum_{i=1}^{n}(z + w_1)_i + v_2) \geq l, m - (\sum_{i=1}^{n}(y + w_2)_i + v_1) \geq l\}$.

We choose the initial point

$x^{(0)} = (z^{(0)}, y^{(0)}, {w_1}^{(0)}, {w_2}^{(0)}, {v_1}^{(0)}, {v_2}^{(0)}) \in \mathcal{R}_{(m)}$ such that

$A^{(0)} = B^{(0)} \neq 0,\ {v_2}^{(0)} = {v_1}^{(0)},\ A^{(0)} \neq 2{v_1}^{(0)}$,

$l(B^{(0)}{v_2}^{(0)} - A^{(0)}{v_1}^{(0)}) + lB^{(0)}(l - A^{(0)}) + A^{(0)}{v_1}^{(0)}(A^{(0)} - {v_2}^{(0)}) \neq 0$,

$l(A^{(0)}{v_1}^{(0)} - B^{(0)}{v_2}^{(0)}) + lA^{(0)}(l - B^{(0)}) + B^{(0)}{v_2}^{(0)}(B^{(0)} - {v_1}^{(0)}) \neq 0$,

$l(B^{(0)} - l) \neq (A^{(0)} - {v_2}^{(0)}){v_1}^{(0)},\ l(A^{(0)} - l) \neq (B^{(0)} - {v_1}^{(0)}){v_2}^{(0)}$,

where $A^{(0)} = (m - (\sum_{i=1}^{n}(z^{(0)} + w_1^{(0)})_i)),\ B^{(0)} = m - (\sum_{i=1}^{n}(y^{(0)} + w_2^{(0)})_i)$. These are the criteria to be an initial point.

Now we define the feasible region

$\mathcal{F}_{(m)} = \{(z, y, w_1, w_2, v_1, v_2) \in \mathcal{R}_{(m)} : v_1 \neq \frac{\mu(A^{(0)} - {v_2}^{(0)}){v_1}^{(0)}}{l}; v_2 \neq \frac{\mu(B^{(0)} - {v_1}^{(0)}){v_2}^{(0)}}{l}\ \forall\ \mu \in (0, 1)\}$,

$\bar{\mathcal{F}}_{(m)} = \{(z, y, w_1, w_2, v_1, v_2) \in \bar{\mathcal{R}}_{(m)} : v_1 \neq \frac{\mu(A^{(0)} - {v_2}^{(0)}){v_1}^{(0)}}{l}; v_2 \neq \frac{\mu(B^{(0)} - {v_1}^{(0)}){v_2}^{(0)}}{l}\ \forall\ \mu \in (0, 1)\}$.

$\partial\mathcal{F}_{(m)} = \{(z, y, w_1, w_2, v_1, v_2) \in \partial\mathcal{R}_{(m)} : v_1 \neq \frac{\mu(A^{(0)} - {v_2}^{(0)}){v_1}^{(0)}}{l}; v_2 \neq \frac{\mu(B^{(0)} - {v_1}^{(0)}){v_2}^{(0)}}{l}\ \forall\ \mu \in (0, 1)\}$, where $\partial\mathcal{R}_{(m)}$ is the boundary of $\bar{\mathcal{R}}_{(m)}$.

We construct a suitable homotopy function

$$H(x,x^{(0)},\mu)=\begin{bmatrix}(1-\mu)(y-w_1+v_1e+J_f^T(z-w_2+v_2e))+\mu(z-z^{(0)})\\ W_1z-\mu W_1^{(0)}z^{(0)}\\ W_2y-\mu W_2^{(0)}y^{(0)}\\ y-(1-\mu)f(z)-\mu(y^{(0)})\\ (m-\sum_{i=1}^{n}(z+w_1)_i-v_2)v_1-\mu((m-\sum_{i=1}^{n}(z^{(0)}+w_1^{(0)})_i-v_2^{(0)})v_1^{(0)})\\ (m-\sum_{i=1}^{n}(y+w_2)_i-v_1)v_2-\mu((m-\sum_{i=1}^{n}(y^{(0)}+w_2^{(0)})_i-v_1^{(0)})v_2^{(0)})\end{bmatrix}=0 \tag{7.3.1}$$

where $e=[1,1,\cdots,1]^T,\ Z=\text{diag}(z);\ W_1=\text{diag}(w_1);\ W_2=\text{diag}(w_2);\ W_1^{(0)}=\text{diag}(w_1^{(0)});\ W_2^{(0)}=\text{diag}(w_2^{(0)});\ x=(z,y,w_1,w_2,v_1,v_2)\in\bar{\mathcal{F}}_{(m)};$
$x^{(0)}=(z^{(0)},y^{(0)},{w_1}^{(0)},{w_2}^{(0)},{v_1}^{(0)},{v_2}^{(0)})\in\mathcal{F}_{(m)};\ \mu\in(0,1]$ and J_f is the Jacobian of the nonlinear function $f(z)$.

7.3.1 Properties of the Trajectory for Single Parameter

First we prove the smoothness property of the trajectory obtained by the proposed homotopy function (7.3.1).

THEOREM 7.3.1. *For almost all initial points $x^{(0)}\in\mathcal{R}_{(m)}$ satisfying the criteria to be an initial point, 0 is a regular value of the homotopy function $H:\mathbb{R}^{4n+2}\times(0,1]\to\mathbb{R}^{4n+2}$ and the zero point set $H^{-1}_{x^{(0)}}(0)=\{(x,\mu)\in\mathcal{F}_{(m)}\times(0,1]:H_{x^{(0)}}(x,\mu)=0\}$ contains a smooth curve $\Gamma_x^{(0)}$ starting from $(x^{(0)},1)$.*

Proof. The Jacobian matrix of the above homotopy function $H(x,x^{(0)},\mu)$ is denoted by $DH(x,x^{(0)},\mu))$ and we have $DH(x,x^{(0)},\mu))=\begin{bmatrix}\frac{\partial H(x,x^{(0)},\mu)}{\partial x} & \frac{\partial H(x,x^{(0)},\mu)}{\partial x^{(0)}} & \frac{\partial H(x,x^{(0)},\mu)}{\partial \mu}\end{bmatrix}$. For all $x^{(0)}\in\mathcal{F}_{(m)}$ and $\mu\in(0,1]$, we have $\frac{\partial H(x,x^{(0)},\mu)}{\partial x^{(0)}}=\begin{bmatrix}K_1 & K_2\\ K_3 & K_4\end{bmatrix}$, where $K_1 = \begin{bmatrix} -\mu I & 0 & 0 & 0 \\ -\mu W_1^{(0)} & 0 & -\mu Z^{(0)} & 0 \\ 0 & -\mu W_2^{(0)} & 0 & -\mu Y^{(0)} \\ 0 & -\mu I & 0 & 0 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$$K_3 = \begin{bmatrix} \mu v_1^{(0)} e^T & 0 & \mu v_1^{(0)} e^T & 0 \\ 0 & \mu v_2^{(0)} e^T & 0 & \mu v_2^{(0)} e^T \end{bmatrix},$$

$$K_4 = \begin{bmatrix} -\mu(m - \sum\limits_{i=1}^{n}(z^{(0)} + w_1^{(0)})_i - v_2^{(0)}) & \mu v_1^{(0)} \\ \mu v_2^{(0)} & -\mu(m - \sum\limits_{i=1}^{n}(y^{(0)} + w_2^{(0)})_i - v_1^{(0)}) \end{bmatrix},$$

$Y^{(0)} = \mathrm{diag}(y^{(0)}), Z^{(0)} = \mathrm{diag}(z^{(0)})$, $W_1^{(0)} = \mathrm{diag}(w_1^{(0)})$, $W_2^{(0)} = \mathrm{diag}(w_2^{(0)})$.

Now $\det(\frac{\partial H}{\partial x^{(0)}}) = \det(K_4)\det(K_1 - K_2 {K_4}^{-1} K_3) = \det(K_4)\det(K_1) = \mu^{4n+2}((m - \sum\limits_{i=1}^{n}(z^{(0)} + w_1^{(0)})_i - v_2^{(0)})(m - \sum\limits_{i=1}^{n}(y^{(0)} + w_2^{(0)})_i - v_1^{(0)}) - v_1^{(0)} v_2^{(0)}) \prod_{i=1}^{n} z_i^{(0)} y_i^{(0)} \neq 0$ for $\mu \in (0, 1]$. Thus $DH(x, x^{(0)}, \mu))$ is of full row rank. Therefore, 0 is a regular value of $H(x, x^{(0)}, \mu))$. By Lemma 1.3.3 and Lemma 1.3.4, for almost all $x^{(0)} \in \mathcal{F}_{(m)}$, 0 is a regular value of $H_{x^{(0)}}(x, \mu)$ and $H_{x^{(0)}}^{-1}(0)$ consists of some smooth curves and $H_{x^{(0)}}(x^{(0)}, 1) = 0$. Hence there must be a smooth curve $\Gamma_x^{(0)}$ starting from $(x^{(0)}, 1)$. ■

Now we show that a smooth and bounded curve exists by the homotopy function (7.3.1).

THEOREM 7.3.2. *Let $\mathcal{F}_{(m)}$ be a nonempty set. For a given $x^{(0)} \in \mathcal{R}_{(m)}$ satisfying the criteria to be an initial point, if* 0 *is a regular value of $H(x, x^{(0)}, \mu)$, then $\Gamma_x^{(0)}$ is a bounded curve in $\bar{\mathcal{F}}_{(m)} \times (0, 1]$.*

Proof. We have that 0 is a regular value of $H(x, x^{(0)}, \mu)$ by Theorem 7.3.1 and $\mathcal{F}_{(m)}$ is a nonempty set. It is clear that the sets $\mathcal{F}_{(m)}$ and $(0, 1]$ are bounded. Hence there exists a sequence of points $\{z^k, y^k, w_1^k, w_2^k, v_1^k, v_2^k, \mu^k\} \subset \Gamma_x^{(0)} \times (0, 1]$, such that $\lim\limits_{k\to\infty} z^k = \bar{z}$, $\lim\limits_{k\to\infty} y^k = \bar{y}$, $\lim\limits_{k\to\infty} w_1^k = \bar{w}_1$, $\lim\limits_{k\to\infty} w_2^k = \bar{w}_2$, $\lim\limits_{k\to\infty} v_1^k =$

$\bar{v}_1, \lim_{k\to\infty} v_2^k = \bar{v}_2, \lim_{k\to\infty} \mu^k = \bar{\mu}$. Hence $\Gamma_x^{(0)}$ is a bounded curve in $\bar{\mathcal{F}}_{(m)} \times (0,1]$. ■

We show the convergence of the homotopy function (7.3.1).

THEOREM 7.3.3. *For* $x^{(0)} = (z^{(0)}, y^{(0)}, w_1^{(0)}, w_2^{(0)}, v_1^{(0)}, v_2^{(0)}) \in \mathcal{R}_{(m)}$ *such that*
$A^{(0)} = B^{(0)} \neq 0,\ {v_2}^{(0)} = {v_1}^{(0)},\ A^{(0)} \neq 2{v_1}^{(0)}$,
$l(B^{(0)}{v_2}^{(0)} - A^{(0)}{v_1}^{(0)}) + lB^{(0)}(l - A^{(0)}) + A^{(0)}{v_1}^{(0)}(A^{(0)} - {v_2}^{(0)}) \neq 0$,
$l(A^{(0)}{v_1}^{(0)} - B^{(0)}{v_2}^{(0)}) + lA^{(0)}(l - B^{(0)}) + B^{(0)}{v_2}^{(0)}(B^{(0)} - {v_1}^{(0)}) \neq 0$,
$l(B^{(0)} - l) \neq (A^{(0)} - {v_2}^{(0)}){v_1}^{(0)}$,
$l(A^{(0)} - l) \neq (B^{(0)} - {v_1}^{(0)}){v_2}^{(0)}$,
the homotopy function finds a bounded smooth curve $\Gamma_x^{(0)} \subset \mathcal{F}_{(m)} \times (0,1]$ *which starts from* $(x^{(0)}, 1)$ *and approaches the hyperplane at* $\mu \to 0$. *As* $\mu \to 0$, *the limit set* $\mathcal{L} \times \bar{\mu} \subset \bar{\mathcal{F}}_{(m)} \times \{0\}$ *of* $\Gamma_x^{(0)}$ *is nonempty and every point in* $\mathcal{L}$ *is a solution of the following system of equations:*

$$\begin{aligned}
(y - w_1 + v_1 e + J_f^T(z - w_2 + v_2 e)) &= 0\\
W_1 z &= 0\\
W_2 y &= 0\\
y - f(z) &= 0\\
(m - \sum_{i=1}^{n}(z + w_1)_i - v_2)v_1 &= 0\\
(m - \sum_{i=1}^{n}(y + w_2)_i - v_1)v_2 &= 0
\end{aligned} \tag{7.3.2}$$

Proof. Note that $\Gamma_x^{(0)}$ is diffeomorphic to a unit circle or a unit interval $(0,1]$ in view of Lemma 1.3.5. As $\frac{\partial H(x, x^{(0)}, 1)}{\partial x^{(0)}}$ is nonsingular, $\Gamma_x^{(0)}$ is diffeomorphic to a unit interval $(0,1]$. Again $\Gamma_x^{(0)}$ is a bounded smooth curve by the Theorem 7.3.2. Let $(\bar{x}, \bar{\mu})$ be a limit point of $\Gamma_x^{(0)}$. We consider four cases:

(i) $(\bar{x}, \bar{\mu}) \in \mathcal{F}_{(m)} \times \{1\}$.

(ii) $(\bar{x}, \bar{\mu}) \in \partial\mathcal{F}_{(m)} \times \{1\}$.

(iii) $(\bar{x}, \bar{\mu}) \in \partial\mathcal{F}_{(m)} \times (0, 1)$.

(iv) $(\bar{x}, \bar{\mu}) \in \bar{\mathcal{F}}_{(m)} \times \{0\}$.

Suppose for case (i) the homotopy function (7.3.1) has solution $(\bar{x}, 1)$, other than the initial solution $x^{(0)}$. As $\mu \to 1$, $\bar{z} = z^{(0)}, \bar{y} = y^{(0)}, \bar{w}_1 = z_1^{(0)}, \bar{w}_2 = z_2^{(0)}, \bar{v}_1 \neq 0, \bar{v}_2 \neq 0$. Hence for $\mu \to 1$, $(A - v_2) \to (A^{(0)} - \bar{v}_2), (B - v_1) \to (B^{(0)} - \bar{v}_1)$. Hence from homotopy function (7.3.1)

$$(A^{(0)} - \bar{v}_2)\bar{v}_1 = (A^0 - {v_2}^{(0)}){v_1}^{(0)} \tag{7.3.3}$$

$$(B^{(0)} - \bar{v}_1)\bar{v}_2 = (B^0 - {v_1}^{(0)}){v_2}^{(0)} \tag{7.3.4}$$

From (7.3.4) and (7.3.4), we obtain
$A^{(0)}(\bar{v}_1 - {v_1}^{(0)}) = (\bar{v}_1\bar{v}_2 - {v_1}^{(0)}{v_2}^{(0)})$
$B^{(0)}(\bar{v}_2 - {v_2}^{(0)}) = (\bar{v}_1\bar{v}_2 - {v_1}^{(0)}{v_2}^{(0)})$
This implies $A^{(0)}(\bar{v}_1 - {v_1}^{(0)}) = B^{(0)}(\bar{v}_2 - {v_2}^{(0)})$. As $A^{(0)} = B^{(0)}$ and ${v_1}^{(0)} = {v_2}^{(0)}$, this implies $\bar{v}_1 = {v_1}^{(0)}$, $\bar{v}_2 = {v_2}^{(0)}$. Hence the equation $H_{x^{(0)}}(x, 1) = 0$ has only one solution $x^{(0)} \in \mathcal{R}_{(m)}$. Hence the case (i) is impossible.
In case (ii) the homotopy function (7.3.1) implies that $\bar{z} = z^{(0)}, \bar{y} = y^{(0)}, \bar{w}_1 = w_1^0, \bar{w}_2 = w_2^0, \bar{v}_1 \neq 0, \bar{v}_2 \neq 0$. Hence $(A - v_2) \to (A^{(0)} - \bar{v}_2)$ and $(B - v_1) \to (B^{(0)} - \bar{v}_1)$ as $\mu \to 1$. From last two components of homotopy function (7.3.1), we have

$$(A^{(0)} - \bar{v}_2)\bar{v}_1 = (A^{(0)} - {v_2}^{(0)}){v_1}^{(0)}, (B^{(0)} - \bar{v}_1)\bar{v}_2 = (B^{(0)} - {v_1}^{(0)}){v_2}^{(0)}. \tag{7.3.5}$$

Three cases may arise.

Case 1: Let $A^{(0)} - \bar{v}_2 = l$. From Equation (7.3.5) and last two components of

the homotopy function (7.3.1), we obtain
$\bar{v}_1 = \frac{(A^{(0)}-v_2{}^{(0)})v_1{}^{(0)}}{l}$, $(B^{(0)} - \frac{(A^{(0)}-v_2{}^{(0)})v_1{}^{(0)}}{l})\bar{v}_2 = (B^{(0)} - v_1{}^{(0)})v_2{}^{(0)}$
$\implies \bar{v}_2 = \frac{l(B^{(0)}-v_1{}^{(0)})v_2{}^{(0)}}{lB^{(0)}-A^{(0)}v_1{}^{(0)}+v_1{}^{(0)}v_2{}^{(0)}} = A^{(0)} - l$
$\implies l(B^{(0)}v_2{}^{(0)}-A^{(0)}v_1{}^{(0)})+lB^{(0)}(l-A^{(0)})+A^{(0)}v_1{}^{(0)}(A^{(0)}-v_2{}^{(0)}) = 0$, contradicts the choosing of initial point.

Case 2: Let $B^{(0)} - \bar{v}_1 = l$. From Equation (7.3.5) and last two components of the homotopy function (7.3.1), we obtain
$\bar{v}_2 = \frac{(B^{(0)}-v_1{}^{(0)})v_2{}^{(0)}}{l}$, $(A^{(0)} - \frac{(B^{(0)}-v_1{}^{(0)})v_2{}^{(0)}}{l})\bar{v}_1 = (A^{(0)} - v_2{}^{(0)})v_1{}^{(0)}$
$\implies \bar{v}_1 = \frac{l(A^{(0)}-v_2{}^{(0)})v_1{}^{(0)}}{lA^{(0)}-B^{(0)}v_2{}^{(0)}+v_1{}^{(0)}v_2{}^{(0)}} = B^{(0)} - l$
$\implies l(A^{(0)}v_1{}^{(0)}-B^{(0)}v_2{}^{(0)})+lA^{(0)}(l-B^{(0)})+B^{(0)}v_2{}^{(0)}(B^{(0)}-v_1{}^{(0)}) = 0$, contradicts the choosing of initial point.

Case 3: Let $B^{(0)} - \bar{v}_1 = l,\ A^{(0)} - \bar{v}_2 = l$. From Equation (7.3.5) and last two components of the homotopy function (7.3.1), we obtain
$l\bar{v}_1 = (A^{(0)} - v_2{}^{(0)})v_1{}^{(0)}, l\bar{v}_2 = (B^{(0)} - v_1{}^{(0)})v_2{}^{(0)}$.
$\implies l(B^{(0)} - l) = (A^{(0)} - v_2{}^{(0)})v_1{}^{(0)}, l(A^{(0)} - l) = (B^{(0)} - v_1{}^{(0)})v_2{}^{(0)}$, contradicts the choosing of initial point.

In case (iii) from homotopy function (7.3.1), we have $\bar{z} > 0, \bar{y} > 0, \bar{w}_1 > 0, \bar{w}_2 > 0$. Three cases may arise.

Case1: Let $A-v_2 \to \bar{A}-\bar{v}_2 = l$, where $\bar{A} = (m-\sum\limits_{i=1}^{n}(\bar{z}+\bar{w}_1)_i$. Then from Equation (7.3.5) we have $\bar{v}_1 = \frac{\bar{\mu}(A^{(0)}-v_2{}^{(0)})v_1{}^{(0)}}{l}$, which contradicts that $\bar{v}_1 \in \partial\mathcal{F}_{(m)}$.

Case2: Let $B - v_1 \to \bar{B} - \bar{v}_1 = l$, where $\bar{B} = (m - \sum\limits_{i=1}^{n}(\bar{y} + \bar{w}_2)_i$. Then from Equation (7.3.5) we have $\bar{v}_2 = \frac{\bar{\mu}(B^{(0)}-v_1{}^{(0)})v_2{}^{(0)}}{l}$, which contradicts that $\bar{v}_2 \in \partial\mathcal{F}_{(m)}$.

Case3: Let $A - v_2 \to \bar{A} - \bar{v}_2 = l, B - v_1 \to \bar{B} - \bar{v}_1 = l$. Then from Equation (7.3.5) we have $\bar{v}_1 = \frac{\bar{\mu}(A^{(0)}-v_2{}^{(0)})v_1{}^{(0)}}{l}$ and $\bar{v}_2 = \frac{\bar{\mu}(B^{(0)}-v_1{}^{(0)})v_2{}^{(0)}}{l}$, which contradicts that $\bar{v}_1, \bar{v}_2 \in \partial\mathcal{F}_{(m)}$.

Therefore (iv) is the only possible case and $\bar{x} = (\bar{z}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{v}_1, \bar{v}_2)$ is a solution of the system of equations (7.3.2). Hence the homotopy function (7.3.1) leads to its solution $\bar{x}$ as $\mu \to 0$. ■

REMARK 7.3.1. *From the homotopy function (7.3.1) as* $\mu \to 0$ *we obtain* $\bar{y} - \bar{w}_1 + \bar{J}_f^T(\bar{z} - \bar{w}_2) = 0$, $\bar{y} = f(\bar{z})$, $\bar{w}_{1i}\bar{z}_i = 0$, $\bar{w}_{2i}\bar{y}_i = 0$, $\bar{v}_1 = 0$, $\bar{v}_2 = 0 \; \forall \; i \in \{1, 2, \cdots n\}$, *where* $\bar{J}_f$ *is the Jacobian of* $f(z)$ *at the point* $\bar{z}$. *Now* $\bar{w}_1$ *and* $\bar{w}_2$ *can be decomposed as* $\bar{w}_1 = \bar{y} - \Delta\bar{y} \geq 0$ *and* $\bar{w}_2 = \bar{z} - \Delta\bar{z} \geq 0$. *Now it is clear that* $\bar{y}_i\bar{z}_i = \Delta\bar{y}_i\bar{z}_i = \Delta\bar{z}_i\bar{y}_i \; \forall \; i$ *and* $\bar{J}_f^T\Delta\bar{z} + \Delta\bar{y} = 0$. *This implies that* $(\bar{Z}\bar{J}_f^T + \bar{Y})\Delta\bar{z} = 0$, *where* $\bar{Y} =$*diag*$(\bar{y})$ *and* $\bar{Z} =$*diag*$(\bar{z})$.

Here we find the condition under which the homotopy solution gives the solution of the complementarity problem (1.4.1).

THEOREM 7.3.4. *The component* $\bar{z}$ *of* $(\bar{z}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{\mu}) \in \mathcal{L} \times \{0\}$ *is the solution of the complementarity problem (1.4.1) if and only if* $\Delta\bar{z}_i\Delta\bar{y}_i = 0$ *or* $\bar{w}_{1i} + \bar{w}_{2i} > 0 \; \forall \; i$.

Proof. Suppose $\bar{z} \geq 0$ and $\bar{y} = f(\bar{z}) \geq 0$ give the solution of the complementarity problem (1.4.1). Then $\bar{z}_i\bar{y}_i = 0 \;\; \forall \; i$. This implies that $\bar{z}_i = 0$ or $\bar{y}_i = 0 \;\; \forall \; i$. Now we consider the following cases.

Case 1: For atleast one $i \in \{1, 2, \cdots n\}$, let $\bar{z}_i > 0, \bar{y}_i = 0$. In view of Remark 7.3.1, this implies that $\Delta\bar{y}_i = 0 \implies \Delta\bar{z}_i\Delta\bar{y}_i = 0$.

Case 2: For atleast one $i \in \{1, 2, \cdots n\}$, let $\bar{y}_i > 0, \bar{z}_i = 0$. In view of Remark 7.3.1, this implies that $\Delta\bar{z}_i = 0 \implies \Delta\bar{z}_i\Delta\bar{y}_i = 0$.

Case 3: For atleast one $i \in \{1, 2, \cdots n\}$, let $\bar{y}_i = 0, \bar{z}_i = 0$. This implies that either $\Delta\bar{y}_i\Delta\bar{z}_i = 0$ or $\bar{w}_{1i} + \bar{w}_{2i} > 0$.

Conversely, we consider $\Delta\bar{z}_i\Delta\bar{y}_i = 0$ or $\bar{w}_{1i} + \bar{w}_{2i} > 0 \; \forall \; i$. Let $\forall \; i, \; \Delta\bar{z}_i\Delta\bar{y}_i = 0$ implies either $\Delta\bar{z}_i = 0$ or $\Delta\bar{y}_i = 0$. This implies that $\bar{y}_i\bar{z}_i = 0 \; \forall \; i$. Therefore $\bar{y}$ and $\bar{z}$ give the solution of given complementarity problem (1.4.1). Let $\bar{w}_1 + \bar{w}_2 > 0$. Then three cases will arise.

Case 1: Let $\bar{w}_{1i} > 0, \bar{w}_{2i} = 0$ for atleast one $i \in \{1, 2, \cdots n\}$. This implies that $\bar{z}_i = 0$ and $\bar{y}_i \geq 0$.

Case 2: Let $\bar{w}_{1i} = 0, \bar{w}_{2i} > 0$ for atleast one $i \in \{1, 2, \cdots n\}$. This implies that $\bar{z}_i \geq 0$ and $\bar{y}_i = 0$.

Case 3: Let $\bar{w}_{1i} > 0, \bar{w}_{2i} > 0$ for atleast one $i \in \{1, 2, \cdots n\}$. This implies that $\bar{z}_i = 0$ and $\bar{y}_i = 0$.

Considering the above three cases $\bar{z}$ and $\bar{y}$ solves the compplementarity problem (1.4.1).

■

THEOREM 7.3.5. *If the nonlinear function $f(z)$ is a P_0 function, then the component $\bar{z}$ of $(\bar{z}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{v}_1, \bar{v}_2, \bar{\mu}) \in \mathcal{L} \times \{0\}$ gives the solution of the nonlinear complementarity problem (1.4.1).*

Proof. Let $f(z)$ be a P_0 function. Then the Jacobian matrix of the nonlinear function at a point z, $\bar{J}_f$ is a P_0 matrix. Assume that the component $\bar{z}$ of $(\bar{z}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{v}_1, \bar{v}_2, \bar{\mu}) \in \mathcal{L} \times \{0\}$ does not give the solution of the nonlinear complementarity problem (1.4.1). Hence $\Delta\bar{z}_i \Delta\bar{y}_i \neq 0$ and $\bar{w}_{1i} + \bar{w}_{2i} = 0$ for atleast one i. Then $\Delta\bar{z}_i \neq 0, \Delta\bar{y}_i \neq 0, \bar{w}_{1i} = 0, \bar{w}_{2i} = 0$. Now $\bar{w}_{1i} = \bar{y}_i - \Delta\bar{y}_i = 0$ and $\Delta\bar{z}_i\Delta\bar{y}_i \neq 0 \implies \bar{y}_i = \Delta\bar{y}_i > 0$. In similar way $\bar{w}_{2i} = \bar{z}_i - \Delta\bar{z}_i = 0$ and $\Delta\bar{z}_i\Delta\bar{y}_i \neq 0 \implies \bar{z}_i = \Delta\bar{z}_i > 0$. Since $(\bar{z}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{v}_1, \bar{v}_2, \bar{\mu}) \in \bar{\mathcal{F}}_{(m)} \times \{0\}$, we obtain that $\bar{v}_1 = 0, \bar{v}_2 = 0$. From Equation (7.3.2) and Remark 7.3.1, $\Delta\bar{y}_i + (\bar{J}_f^T \Delta\bar{z})_i = 0$. This implies that $(\bar{J}_f^T \Delta\bar{z})_i < 0$ and $(\Delta\bar{z})_i(\bar{J}_f^T \Delta\bar{z})_i < 0$. This contradicts that $\bar{J}_f$ is a P_0-matrix. Therefore the component $\bar{z}$ of $(\bar{z}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{v}_1, \bar{v}_2, \bar{\mu}) \in \mathcal{L} \times \{0\}$ is the solution of the nonlinear complementarity problem (1.4.1). ■

THEOREM 7.3.6. *Suppose the matrix $(\bar{Z}\bar{J}_f^T + \bar{Y})$ is nonsingular, where $\bar{Y}$ =diag($\bar{y}$) and $\bar{Z}$ =diag($\bar{z}$). Then $\bar{z}$ solves the complementarity problem (1.4.1).*

Proof. Let $(\bar{Z}\bar{J}_f^T + \bar{Y})$ be nonsingular matrix. Now from Remark 7.3.1 it is clear that $\Delta z = 0$. This implies that $\bar{z}$ solves the complementarity problem (1.4.1). ■

REMARK 7.3.2. *We trace the homotopy path $\Gamma_x^{(0)} \subset \mathcal{F}_{(m)} \times (0, 1]$ from the initial point $(x^{(0)}, 1)$ until $\mu \to 0$ and find the solution of the given complementarity* *problem (1.4.1). Let s denote the arc length of* $\Gamma_x^{(0)}$. *We can parameterize the homotopy path* $\Gamma_x^{(0)}$ *with respect to s in the following form*

$$H_{x^{(0)}}(x(s),\mu(s)) = 0,\ x(0) = x^{(0)},\ \mu(0) = 1. \tag{7.3.6}$$

Differentiating (7.3.6) with respect to s we obtain the following system of ordinary differential equations with initial values.

$$H'_{x^{(0)}}(x(s),\mu(s))\begin{bmatrix}\frac{dx}{ds}\\ \frac{d\mu}{ds}\end{bmatrix} = 0,\ \|(\frac{dx}{ds},\frac{d\mu}{ds})\| = 1,\ x(0) = x^{(0)},\ \mu(0) = 1,\ \frac{d\mu}{ds}(0) < 0. \tag{7.3.7}$$

and the x-component of $(x(s),\mu(s))$ *provides the solution of the complementarity problem for* $\mu(s) = 0$.

The Equation (7.3.7) implies that $[\frac{\partial H}{\partial x}\ \frac{\partial H}{\partial \mu}]\nu = 0$, $\nu^T\nu = 1$, $\frac{d\zeta}{ds} = \nu$, $\zeta = \begin{bmatrix}x\\ \mu\end{bmatrix}$, $\zeta(0) = \begin{bmatrix}x^{(0)}\\ 1\end{bmatrix}$. *Now it is obtained that*

$$\frac{d\zeta}{ds} = \begin{bmatrix}-(\frac{\partial H}{\partial x})^{-1}\frac{\partial H}{\partial \mu}\\ 1\end{bmatrix}\nu_2,\ \zeta(0) = \begin{bmatrix}x^{(0)}\\ 1\end{bmatrix},\ \textit{where}\ \nu = \begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix}. \tag{7.3.8}$$

We use the homotopy continuation method with some modifications in choosing step length and updating iterations to trace the homotopy path $\Gamma_x^{(0)}$ numerically.

7.3.2 Algorithm: Continuation Method with Single Parameter

Step 0: Parameter i counts the number of iterations and parameter i_s counts the number of shifting of the initial point. Set $i = i_s = 0$. Give an initial point $(x^{(0)},\mu_0)$ satisfying the criteria to be an initial point with $\mu_0 = 1$. η_1 and η_2 are small positive numbers where η_1 denotes the lower boundary of the norm of the direction vector and η_2 denotes the lower boundary of the step length. $\kappa_1 \in (1,2]$ and κ_2 is a positive number such that $\kappa_1^k \leq \kappa_2$, where κ_1^k is the step length. ϵ_1 is a small positive numbers, which is used as a threshold for the parameter μ.

Step 1: Set $\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} x^{(0)} \\ 1 \end{bmatrix}$, where $\mu = \mu_0 = 1$. Calculate $d^{(0)} = \det(\frac{\partial H}{\partial x}(x^{(0)}, \mu_0))$ and go to Step 2.

Step 2: Set $c_1 = c_2 = 0$. Calculate $d = \det(\frac{\partial H}{\partial x}(x,\mu))$. Then go to Step 3.

Step 3: Determine the unit predictor direction $\tau^{(n)}$ by the following method: If $\text{sign}(d) = -\text{sign}(d_0)$, then $t_d = 1 - \mu$, else $t_d = -\mu$.

Calculate $x_d = -t_d(\frac{\partial H}{\partial x}(x,\mu))^{-1}(\frac{\partial H}{\partial \mu}(x,\mu))$, $\tau^{(n)} = \begin{bmatrix} x_n \\ t_n \end{bmatrix} = \frac{1}{\|v_d\|} v_d$, where $v_d = \begin{bmatrix} x_d \\ t_d \end{bmatrix}$, $\tau = \dfrac{|t_d|}{\|v_d\|}$, where $\|v_d\| = \sqrt{x_d^2 + t_d^2}$, $x_d^2 = \sum_i x_{di}^2$, x_{di} is the ith component of x_d. If $\tau \leq \eta_1$, then set $c1 = c1 + 1$ else $c1 = 0$. If $t_d \leq \epsilon_1$, then stop with a solution else go to Step 4.

Step 4: Choosing step length: Set $i = i+1, k = 0, \gamma = [\nabla \lambda(x)]^T x_n$, $\lambda : \mathbb{R}^n \to \mathbb{R}$ and $\lambda(x) = [H_0(x)]^T [H_0(x)]$, where

$$H_0(x) = \begin{bmatrix} (y - w_1 + v_1 e + J_f^T(z - w_2 + v_2 e)) \\ W_1 z \\ W_2 y \\ y - f(z) \\ (m - \sum_{i=1}^{n}(z + w_1)_i - v_2)v_1 \\ (m - \sum_{i=1}^{n}(y + w_2)_i - v_1)v_2 \end{bmatrix}.$$

If $\gamma \geq 0$, $x + \kappa_1^{k+1} x_n \in \mathcal{F}_{(m)}, 0 < t + \kappa_1^{k+1} t_n < 1$, then set $k = k+1$ and go to Step 5, else if $\gamma < 0, \mu(x + \kappa_1^{k+1} x_n) < \mu(x + \kappa_1^k x_n), x + \kappa_1^{k+1} x_n \in \mathcal{F}_(m), 0 < t + \kappa_1^{k+1} t_n < 1$, then set $k = k+1$, and go to Step 5, else $c_2 = 0$, and go to Step 6.

Step 5: If $\kappa_1^k > \kappa_2$, then set $k = k - 1$, $c_2 = c_2 + 1$ and go to Step 6, else go to

Step 4.

Step 6: If $t_n \leq \epsilon_1$, then stop with solution else go to Step 7.

Step 7: Compute $\begin{bmatrix} x_p \\ t_p \end{bmatrix} = \begin{bmatrix} x \\ t \end{bmatrix} + \kappa_1^k \begin{bmatrix} x_n \\ t_n \end{bmatrix}$,

$$\begin{bmatrix} \bar{x}_p \\ \bar{t}_p \end{bmatrix} = \begin{bmatrix} x_p \\ t_p \end{bmatrix} - [J_H(x_p, t_p)^+ H(x_p, t_p)],$$

$$\begin{bmatrix} \tilde{x}_p \\ \tilde{t}_p \end{bmatrix} = \begin{bmatrix} x_p \\ t_p \end{bmatrix} - 2[(J_H(x_p, t_p) + J_H(\bar{x}_p, \bar{t}_p))^+ H(x_p, t_p)],$$

$$\begin{bmatrix} x_{cc} \\ t_{cc} \end{bmatrix} = \begin{bmatrix} \tilde{x}_p \\ \tilde{t}_p \end{bmatrix} - [J_H(\bar{x}_p, \bar{t}_p)]^+ H(\tilde{x}_p, \tilde{t}_p),$$

$$\begin{bmatrix} x_b \\ t_b \end{bmatrix} = \begin{bmatrix} x_{cc} \\ t_{cc} \end{bmatrix} - J_H(\tilde{x}_p, \tilde{t}_p)^+ H(x_{cc}, t_{cc}).$$

Then the next iteration is $\begin{bmatrix} x_c \\ t_c \end{bmatrix} = \begin{bmatrix} x_b \\ t_b \end{bmatrix}$. Let $r = \|H(x_c, t_c)\|$. If $r \leq 1, 0 < t_c < 1$, and $x_c \in \mathcal{F}_{(m)}$, then go to Step 10 else set $k = k - 1$ and go to Step 8.

Step 8: Calculate $a = \min(\kappa_1^k, \|x - x_c\|)$. If $a \leq \eta_2$, then go to Step 9 else go to Step 5.

Step 9: If $t_c \leq \epsilon_1$, then stop with a solution else $i = i - 1, i_s = i_s + 1$ and after changing the initial point as $x^{(0)} = x_c$ go to Step 1,

Step 10: Set $\begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} x_c \\ t_c \end{bmatrix}$. If $t_c \leq \epsilon_1$, then stop with homotopy solution else set $i = i + 1$ and go to Step 2.

Note that $J_H(x, t)^+$ is the Moore-Penrose inverse of the Jacobian matrix

$J_H(x,t)$, which is defined by $J_H(x,t)^+ = J_H(x,t)^T(J_H(x,t)J_H(x,t)^T)^{-1}$.

We show that the positive tangent direction at the initial point is negative.

THEOREM 7.3.7. *If the homotopy curve $\Gamma_x^{(0)}$ is smooth, then the positive tangent direction $\tau^{(0)}$ at the initial point $x^{(0)}$ satisfies sign(*$\det(\begin{bmatrix} D_{(x,\mu)}H(x^{(0)},1) \\ \tau^{(0)T} \end{bmatrix})) < 0$, *where* $D_{(x,\mu)}H(x^{(0)},1) = \begin{bmatrix} \frac{\partial H}{\partial x}(x^{(0)},1) & \frac{\partial H}{\partial \mu}(x^{(0)},1) \end{bmatrix}$.

Proof. From Equation (7.3.1) we have $H(x,x^{(0)},\mu) =$

$$\begin{bmatrix} (1-\mu)(y-w_1+v_1e+J_f^T(z-w_2+v_2e))+\mu(z-z^{(0)}) \\ W_1z-\mu W_1^{(0)}z^{(0)} \\ W_2y-\mu W_2^{(0)}y^{(0)} \\ y-(1-\mu)f(z)-\mu(y^{(0)}) \\ (m-\sum_{i=1}^{n}(z+w_1)_i-v_2)v_1-\mu((m-\sum_{i=1}^{n}(z^{(0)}+w_1^{(0)})_i-v_2^{(0)})v_1^{(0)}) \\ (m-\sum_{i=1}^{n}(y+w_2)_i-v_1)v_2-\mu((m-\sum_{i=1}^{n}(y^{(0)}+w_2^{(0)})_i-v_1^{(0)})v_2^{(0)}) \end{bmatrix} = 0.$$

Now at the point $(x=x^{(0)},\mu=1)$, the value of the partial derivative is $D_{(x,\mu)}H(x,\mu) = \begin{bmatrix} \frac{\partial H}{\partial x}(x^{(0)},1) & \frac{\partial H}{\partial \mu}(x^{(0)},1) \end{bmatrix} = \begin{bmatrix} K_5 & K_6 \end{bmatrix}$, where

$$K_5 = \begin{bmatrix} M' & N' \end{bmatrix},\ M' = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix},\ M_1' = \begin{bmatrix} I & 0 & 0 & 0 \\ W_1^{(0)} & 0 & Z^{(0)} & 0 \\ 0 & W_2^{(0)} & 0 & Y^{(0)} \\ 0 & I & 0 & 0 \end{bmatrix}_{4n\times 4n},$$

$$M_2' = \begin{bmatrix} -v_1^{(0)}e^T & 0 & -v_1^{(0)}e^T & 0 \\ 0 & -v_2^{(0)}e^T & 0 & -v_2^{(0)}e^T \end{bmatrix}_{2\times 4n},$$

$$N' = \begin{bmatrix} N_1' \\ N_2' \end{bmatrix},\ N_1' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4n\times 2},$$

$$N_2' = \begin{bmatrix} (m - \sum\limits_{i=1}^{n}(z^{(0)} + w_1^{(0)})_i - v_2^{(0)}) & -v_1^{(0)} \\ -v_2^{(0)} & (m - \sum\limits_{i=1}^{n}(y^{(0)} + w_2^{(0)})_i - v_1^{(0)}) \end{bmatrix}_{2\times 2},$$

$$K_6 = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix},$$

where $Y^{(0)} = \mathrm{diag}(y^{(0)}), Z^{(0)} = \mathrm{diag}(z^{(0)})$,
$W_1^{(0)} = \mathrm{diag}(w_1^{(0)})$, $W_2^{(0)} = \mathrm{diag}(w_2^{(0)})$,
$A = -[y^{(0)} - w_1^{(0)} + v_1^{(0)}e + J^T_{(f^{(0)})}(z^{(0)} - w_2^{(0)} + v_2^{(0)}e)], B = -W_1^{(0)}z^{(0)}$,
$C = -W_2^{(0)}y^{(0)}, D = f(z^{(0)}) - y^{(0)}, E = -(m - \sum\limits_{i=1}^{n}(z^{(0)} + w_1^{(0)})_i - v_2^{(0)})v_1^{(0)}$,
$F = -(m - \sum\limits_{i=1}^{n}(y^{(0)} + w_2^{(0)})_i - v_1^{(0)})v_2^{(0)}$.

Let positive tangent direction be $\tau^{(0)} = \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} (R_1^{(0)})^{(-1)}R_2^{(0)} \\ -1 \end{bmatrix}$,

$$R_1^{(0)} = K_5 = \begin{bmatrix} M_1' & N_1' \\ M_2' & N_2' \end{bmatrix} \text{ and } R_2^{(0)} = K_6 = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix}.$$

Here $\det(R_1^{(0)}) = \det(N_2')\det(M_1' - N_1'(N_2')^{-1}M_2') = \det(N_2')\det(M_1') = ((m - \sum\limits_{i=1}^{n}(z^{(0)} + w_1^{(0)})_i - v_2^{(0)})(m - \sum\limits_{i=1}^{n}(y^{(0)} + w_2^{(0)})_i - v_1^{(0)}) - v_1^{(0)}v_2^{(0)}) \prod_{i=1}^{n} z_i^{(0)}y_i^{(0)} \neq 0$.

Therefore $\det(\begin{bmatrix} D_{(x,\mu)}H(x^{(0)},1) \\ \tau^{(0)^T} \end{bmatrix}) = \det(\begin{bmatrix} R_1^{(0)} & R_2^{(0)} \\ (R_2^{(0)})^T(R_1^{(0)})^{(-T)} & -1 \end{bmatrix})$

$$= \det\left(\begin{bmatrix} R_1^{(0)} & R_2^{(0)} \\ 0 & -1-(R_2^{(0)})^T(R_1^{(0)})^{(-T)}(R_1^{(0)})^{(-1)}R_2^{(0)} \end{bmatrix}\right)$$
$$= \det(R_1^{(0)})\det(-1-(R_2^{(0)})^T(R_1^{(0)})^{(-T)}(R_1^{(0)})^{(-1)}R_2^{(0)})$$
$$= -\det(R_1^{(0)})\det(1+(R_2^{(0)})^T(R_1^{(0)})^{(-T)}(R_1^{(0)})^{(-1)}R_2^{(0)})$$
$$= -((m-\sum_{i=1}^{n}(z^{(0)}+w_1^{(0)})_i - v_2^{(0)})(m-\sum_{i=1}^{n}(y^{(0)}+w_2^{(0)})_i - v_1^{(0)}) - v_1^{(0)}v_2^{(0)})\prod_{i=1}^{n} z_i^{(0)}y_i^{(0)}\det(1+(R_2^{(0)})^T(R_1^{(0)})^{(-T)}(R_1^{(0)})^{(-1)}R_2^{(0)}) < 0. \qquad \blacksquare$$

7.3.3 Order of Convergence

We trace the homotopy path $\Gamma_x^{(0)} \subset \mathcal{F}_{(m)} \times (0,1]$ from the initial point $(x^{(0)},1)$ as $\mu \to 0$. Let s denote the arc length of $\Gamma_x^{(0)}$. We parameterize the homotopy path $\Gamma_x^{(0)}$ with respect to s in the following form

$$H_{x^{(0)}}(x(s),\mu(s)) = 0,\ x(0) = x^{(0)},\ \mu(0) = 1. \tag{7.3.9}$$

From Equation (7.3.1) the choice of H is $H(x,\mu) = (1-\mu)g_1(x) + \mu g_2(x) = 0$, where

$$g_1(x) = \begin{bmatrix} (y - w_1 + v_1 e + J_f^T(z - w_2 + v_2 e)) \\ W_1 z \\ W_2 y \\ y - f(z) \\ (m - \sum_{i=1}^{n}(z+w_1)_i - v_2)v_1 \\ (m - \sum_{i=1}^{n}(y+w_2)_i - v_1)v_2 \end{bmatrix} \text{ and}$$

$$g_2(x) = \begin{bmatrix} z - z^{(0)} \\ W_1 z - W_1^{(0)} z^{(0)} \\ W_2 y - W_2^{(0)} y^{(0)} \\ y - y^{(0)} \\ (m - \sum_{i=1}^{n}(z + w_1)_i - v_2)v_1 - ((m - \sum_{i=1}^{n}(z^{(0)} + w_1^{(0)})_i - v_2^{(0)})v_1^{(0)}) \\ (m - \sum_{i=1}^{n}(y + w_2)_i - v_1)v_2 - ((m - \sum_{i=1}^{n}(y^{(0)} + w_2^{(0)})_i - v_1^{(0)})v_2^{(0)}) \end{bmatrix}.$$

Hence the system (7.3.8) reduces to the following problem

$$\tfrac{d\zeta}{ds} = \begin{bmatrix} -((1-\mu)J_{g_1} + \mu J_{g_2})^{-1}(g_2(x) - g_1(x)) \\ 1 \end{bmatrix} \nu_2 = \begin{bmatrix} -\tilde{J}^{-1}\tilde{g} \\ 1 \end{bmatrix} \nu_2,$$

$$\zeta(0) = \zeta^{(0)} = \begin{bmatrix} x^{(0)} \\ 1 \end{bmatrix}.$$

where J_{g_1}, J_{g_2} are Jacobian matrices of the functions g_1 and g_2 and $\tilde{J} = (1-\mu)J_{g_1} + \mu J_{g_2}$ and $\tilde{g} = g_2(x) - g_1(x)$.

This problem reduces to

$$\dot{\zeta} = q(x,\mu),\ \zeta(0) = \zeta^{(0)} \text{ where } q(x,\mu) = -\tilde{J}^{-1}\tilde{g}$$

This problem will be solved by iterative process

$$\zeta_{(i+1)} = Q(x_i, \mu_i, h_i), \text{ for } i = 0, 1 \cdots,$$

Here ζ_i is an approximation of $\zeta(s)$ and $h_i = \mu(s_{i+1}) - \mu(s_i)$. $Q(x_i, \mu_i, h_i)$ is given by

$$Q(x,\mu,h) = I_m(x,\mu,h), \text{ where } I_0(x,\mu,h) = \zeta$$
$$\text{and } K_j = \tfrac{\partial}{\partial x} H(I_j, \mu + h)^{+} H(I_j, \mu + h)$$

$$L_j = I_j - K_j,$$
$$KK_j = (\tfrac{\partial}{\partial x} H(L_j, \mu + h) + \tfrac{\partial}{\partial x} H(I_j, \mu + h))^{+} H(I_j, \mu + h),$$
$$LL_j = I_j - 2 * KK_j,$$
$$S_j = \tfrac{\partial}{\partial x} H(L_j, \mu + h)^{+} H(LL_j, \mu + h),$$
$$SS_j = LL_j - S_j,$$
$$T_j = \tfrac{\partial}{\partial x} H(LL_j, \mu + h)^{+} H(SS_j, \mu + h),$$
$$TT_j = SS_j - T_j,$$

The next iteration is given by

$$I_{j+1} = TT_j \text{ for } j = 0, 1, 2, \cdots, m_0 - 1.$$

By this iterative process the proposed homotopy function achieves the order of convergence $7^{m_0} - 1$.

LEMMA 7.3.1. *[88] Consider the function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and the iterative method* $y^k = x^k - f'(x^k)^{-1} f(x^k)$, $z^k = x^k - 2(f'(y^k) + f'(x^k))^{-1} f(x^k)$, $w^k = z^k - f'(y^k)^{-1} f(z^k)$ *has fifth order convergence.*

LEMMA 7.3.2. *[88] Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a sufficiently differentiable function in a neighborhood* D *of* α, *that is a solution of the system* $F(x) = 0$, *whose Jacobian matrix is continuous and nonsingular in* D. *Then, for an initial approximation sufficiently close to* α, *the method defined by* $z^k = \phi(x^k, y^k)$, $w^k = z^k - f'(y^k)^{-1} f(z^k)$ *has order of convergence* $p + 2$, *where* $z^k = \phi(x^k, y^k)$ *is the iteration function of a method of order* p *and* $y^k = x^k - f'(x^k)^{-1} f(x^k)$.

Now we show that the modified method which is defined in the following Lemma has seventh order of convergence.

LEMMA 7.3.3. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a sufficiently differentiable function in a neighborhood* N *of* c^*, *which is a solution of the system* $f(x) = 0$, *whose Jacobian matrix is continuous and nonsingular in* N. *Then, for an initial approximation* *sufficiently close to* c^*, *the method defined by* $y^k = x^k - f'(x^k)^{-1}f(x^k)$, $z^k = x^k - 2(f'(y^k)+f'(x^k))^{-1}f(x^k)$, $w^k = z^k - f'(y^k)^{-1}f(z^k)$, $v^k = w^k - f'(z^k)^{-1}f(w^k)$ *has seventh order of convergence.*

Proof. By Lemma 7.3.1 and Lemma 7.3.2 it is clear that the following iterative method

$y^k = x^k - f'(x^k)^{-1}f(x^k)$,

$z^k = x^k - 2(f'(y^k) + f'(x^k))^{-1}f(x^k)$,

$w^k = z^k - f'(y^k)^{-1}f(z^k)$,

$v^k = w^k - f'(z^k)^{-1}f(w^k)$

has seventh order of convergence. ■

THEOREM 7.3.8. *Suppose the homotopy function has derivative, which is lipschitz continuous in a convex neighbourhood* $\mathcal{N}$ *of* c, *where* c *is the solution of the homotopy function* $H(x,\mu) = 0$, *whose Jacobian matrix is continuous and nonsingular and bounded on* $\mathcal{N}$. *Then the homotopy continuation method has order* $7^{m_0} - 1$.

Proof. Implicit Function Theorem ensures the existence of a unique continuous solution $\zeta(h) \in \mathcal{N}$ of $\dot{\zeta} = -\tilde{J}^{-1}\tilde{g}$, $\zeta(0) = \zeta^{(0)}$ and $h \in (-\delta,\delta)$, for some $\delta > 0$. Define $\alpha_j = \|\zeta(h) - I_j(\zeta,h)\|$. Hence $\alpha_0 = \|\zeta(h) - \zeta)\| = O(h)$. From the Lemma 7.3.3, $\alpha_j = O(h^{7^j})$. Then $\alpha_{j+1} = \|\zeta(h) - I_{j+1}\| \leq K{\alpha_j}^7$, where K is a constant. Hence $\alpha_{j+1} = O(h^{7^{j+1}})$. By induction method the modified homotopy continuation method has convergency of order $7^{m_0} - 1$. ■

7.4 Numerical Example

We consider some examples of nonlinear complementarity problems and determine the solutions with homotopy method. To illustrate the effectiveness of the Algorithm 7.3.2 we consider the initial point $x^{(0)}$ such that $z^{(0)} = e$,

$y^{(0)} = e$, ${w_1}^{(0)} = e$, ${w_2}^{(0)} = e$, ${v_1}^{(0)} = 0.001$, ${v_2}^{(0)} = 0.001$ and $\mu_0 = 1$. Set $\eta_1 = 10^{-12}, \eta_2 = 10^{-18}$, $\kappa_1 = \sqrt{2}, \kappa_2 = 9000$, $\epsilon_1 = 10^{-11}$.

EXAMPLE 7.4.1. $f(z) = \begin{bmatrix} x_1 + \frac{x_2x_3x_4x_5}{50} \\ x_2 + \frac{x_1x_3x_4x_5}{50} - 3 \\ x_3 + \frac{x_1x_2x_4x_5}{50} - 1 \\ x_4 + \frac{x_1x_2x_3x_5}{50} + \frac{1}{2} \\ x_5 + \frac{x_1x_2x_3x_4}{50} \end{bmatrix}$

After 15 *iterations the parameter* μ *converges to* 0 *and we obtain the solution of the homotopy function (7.3.1)* $(\bar{x}, \bar{\mu}) = (0, 3, 1, 0, 0, 0, 0, 0, 0.5, 0, 0, 0, 0, 0.5, 0, 0, 3, 1, 0, 0, 0, 0, 0)$. *The* $\bar{z}$ *components* $(0, 3, 1, 0, 0)$ *of* $(\bar{x}, \bar{\mu})$ *is the solution of the nonlinear complementarity problem.*

EXAMPLE 7.4.2. $f(z) = \begin{bmatrix} 3{x_1}^2 + 2x_1x_2 + 2{x_2}^2 + x_3 + 3x_4 - 6 \\ 2{x_1}^2 + x_1 + {x_2}^2 + 10x_3 + 2x_4 - 2 \\ 3{x_1}^2 + x_1x_2 + 2{x_2}^2 + 2x_3 + 3x_4 - 9 \\ {x_1}^2 + 3{x_2}^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}$

After 20 *iterations the parameter* μ *converges to* 0 *and we obtain the solution of the homotopy function (7.3.1)* $(\bar{x}, \bar{\mu}) = (1, 0, 3, 0, 0, 31, 0, 4, 0, 31, 0, 4, 1, 0, 3, 0, 0, 0, 0, 0)$. *The* $\bar{z}$ *components* $(1, 0, 3, 0)$ *of* $(\bar{x}, \bar{\mu})$ *is the solution of the nonlinear complementarity problem.*

EXAMPLE 7.4.3. $f(z) = \begin{bmatrix} -x_2 + x_3 + x_4 \\ x_1 - \frac{4.5x_3 + 2.7x_4}{1 + x_2} \\ 5 - x_1 - \frac{0.5x_3 + 0.3x_4}{1 + x_3} \\ 3 - x_1 \end{bmatrix}$

After 26 *iterations and changing the initial point* 3 *times the parameter* μ *converges to* 0 *and we obtain the solution of the homotopy function (7.3.1)* $(\bar{x}, \bar{\mu}) = (1.10, 0, 0, 0, 0, 1.10, 3.89, 1.89, 0, 1.21, 3.78, 1.78, 1.21, 0, 0, 0, 0, 0, 0)$. *The* $\bar{z}$ *components* $(1.10, 0, 0, 0)$ *of* $(\bar{x}, \bar{\mu})$ *is the solution of the nonlinear complementarity problem.*

EXAMPLE 7.4.4. $f(z) = \begin{bmatrix} {x_1}^2 - \sin(x_1) \\ {x_2}^3 + x_1x_3 \\ {x_3}^2 + x_1x_2 - 200 \end{bmatrix}$

After 28 iterations the parameter μ converges to 0 and we obtain the solution of the homotopy function (7.3.1) $(\bar{x}, \bar{\mu}) = (0.88, 0, 14.14, 0, 12.39, 0, 0, 12.39, 0, 0.88, 0, 14.14, 0, 0, 0)$. *The $\bar{z}$ components* $(0.88, 0, 14.14)$ *of* $(\bar{x}, \bar{\mu})$ *is the solution of the nonlinear complementarity problem.*

In this study, we consider homotopy path to solve nonlinear complementarity problem based on newly introduced homotopy function by modified homotopy continuation method. The homotopy function is developed based on KKT conditions and ensuring the boundedness property of the homotopy trajectory. We find the positive tangent direction of the homotopy path. We prove that the smooth curve for the proposed homotopy function is bounded and convergent under some conditions related to initial points. Some examples of nonlinear complementarity problem are numerically solved by the proposed modified homotopy continuation method to demonstrate the effectiveness of the method.

Chapter 8

Oligopolistic Market Equilibrium Problem In The Context Of Nonlinear Complementarity Problem

8.1 Introduction

Oligopoly is a fundamental economic market structure found in industrialized nations. The oligopoly problem consists of a finite number of firms involved in the production of homogenous commodities in a noncooperative manner. Cournot [161] studied the oligopoly problem which is one of the classical problems in economics. Cournot studied noncompetitive behaviour of competition between two producers, known as duopoly problem. The decisions taken by the producers are said to be in equilibrium if no one can increase his income by unilateral action assuming that the other producer does not alter his decision.

*Results of this chapter have been communicated.

†Work of this chapter has been arxived [58] and [57].

The oligopoly problem is one of the classical problems in economics, dating to Cournot [161], who first studied this problem. The oligopoly problem consists of a finite number of firms involved in the production of homogenous commodities in a noncooperative manner. In particular, Cournot investigated competition between two producers, which is known as duopoly problem and is credited with being the first to study noncompetitive behaviour. In his treatise, the decisions made by the producers are said to be in equilibrium if no one can increase his income by unilateral action, given that the other producer does not alter his decision. Subsequently to fundamental Cournot contributions, the problem has been studied extensively. Nash [163], [162] in turn generalized Cournot's concept of equilibrium for a behavioural model consisting of n players, each acting in its own self-interest, which is called a noncooperative game. Specially, consider m players, each player i having at his disposal a strategy vector $x_i = \{x_{i1}, x_{i2}, ..., x_{in}\}$ selected from a closed, convex set $X_i \subset \mathbb{R}^n$, with a utility function $v_i : X \to R$, where $X = X_1 \times X_2 \times \cdots \times X_m \subset \mathbb{R}^{mn}$. The rationality postulate is that each player i selects a strategy vector $x_i \in X_i$ that maximizes his utility level $v_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_m)$ given the decisions $(x_j)_{j\neq i}$ of the other players. we consider the oligopoly problems operating under the Nash equilibrium concept of noncooperative behaviour, as a problem of the game theory. In particular, we consider the firms as players and their commodity production output as their strategies. Oligopolies are a fundamental economic market structure found in industrialized nations. Recently, there have been a number of studies dealing with the numerical computation of various oligopoiistic market equilibrium problems. For details see Gabay and Moulin [165], Salant [166], Murphy et. al. [59], Harker [193], [194]. Murphy et al. [59] attempted to formulate oligopolistic market equilibrium problem. Later Harker [193] applied relaxation algorithm to find its solution.

8.2 Formulation of Oligopolistic Market Equilibrium

In this chapter we consider an oligopolistic market structure in which n firms supply a homogeneous product in a noncooperative fashion which is followed by Murphy et. al. [59]. Let $P(\tilde{Q})$, price at which consumers will purchase a quantity $\tilde{Q} \geq 0$, denotes the inverse demand function. With generally accepted economic behaviour, this is assumed that $P(\tilde{Q})$ is strictly decreasing and the industry revenue curve $\tilde{Q}P(\tilde{Q})$ is a concave function of $\tilde{Q}$ for $\tilde{Q} \geq 0$. It is further assumed that $P(\tilde{Q})$ is continuously differentiable. Note that $\tilde{Q} = \sum_{i=1}^{n} Q_i$, where $Q_i \geq 0$ denotes the ith firm's supply. Let $c_i(Q_i)$ be the total cost of supplying Q_i units. Now the Nash equilibrium solution is a set of nonnegative production levels ${Q_1}^*, {Q_2}^*, \cdots, {Q_n}^*$ for the n firms at which the market will be in a state of equilibrium, i.e. ${Q_i}^*$ is a Nash equilibrium if

$$({Q_i}^* P({Q_i}^* + \tilde{Q}_i^*) - c_i({Q_i}^*)) \geq (Q_i P(Q_i + \tilde{Q}_i^*) - c_i(Q_i)) \text{ for all } Q_i \in S_i$$

where S_i is the set of all possible strategies for ith firm and $\tilde{Q}_i^* = \sum_{j \neq i} {Q_j}^*$. Then ${Q_i}^*$ maximizes the profit of the ith firm given that the other firms produce quantities ${Q_j}^*$ for $j \neq i$. We restate the problem as ${Q_i}^*$ is an optimal solution to the following problem for $i \in \{1, 2 \cdots, n\}$

$$\max_{Q_i \geq 0} \; Q_i P(Q_i + \tilde{Q}_i^*) - c_i(Q_i) \;\; \text{where} \;\; \tilde{Q}_i^* = \sum_{j \neq i} {Q_j}^* \tag{8.2.1}$$

Murphy et al.[59] showed that if $c_i(Q_i)$ is convex and continuously differentiable $\forall \; i$ and the inverse demand function $P(\tilde{Q})$ is strictly decreasing and continuously differentiable and the industry revenue curve $\tilde{Q}P(\tilde{Q})$ is concave, then $(Q_1{}^*, Q_2{}^*, \cdots, Q_n{}^*)$ is a Nash equilibrium solution if and only if

$$[P(\tilde{Q}^*) + Q_i{}^* P'(\tilde{Q}^*) - c_i{}'(Q_i{}^*)]Q_i{}^* = 0 \tag{8.2.2}$$

$$c_i{}'(Q_i{}^*) - P(\tilde{Q}^*) - Q_i{}^* P'(\tilde{Q}^*) \geq 0 \tag{8.2.3}$$

$$Q_i{}^* \geq 0 \quad \forall\, i \tag{8.2.4}$$

where $\tilde{Q}^* = \sum_{i=1}^{n} Q_i{}^*$, which is a nonlinear complementarity problem with $f_i(z) = c_i{}'(Q_i{}^*) - P(\tilde{Q}^*) - Q_i{}^* P'(\tilde{Q}^*)$, and $z_i = Q_i{}^*$.
Note that here the functions $c_i(Q_i)$ and $-\tilde{Q}P(\tilde{Q})$ are convex. Hence the first order derivative of these two functions are increasing function. Hence the function $f_i(z) = c_i{}'(Q_i{}^*) - P(\tilde{Q}^*) - Q_i{}^* P'(\tilde{Q}^*)$ is an increasing function.

Murphy et al. [59] attempted to formulate oligopolistic market equilibrium problem. Later Harker [193] applied relaxation algorithm to find its solution.

In this study we propose some new methods to find the Nash equilibrium of oligopolistic market through nonlinear complementarity problem. In this context we introduce an approach to transform nonlinear complementarity problem to a system of nonlinear equations and consider a continuation method to find the oligopolistic market equilibrium. Mangasarian [186] gave an idea to connect nonlinear complementarity problem as a system of nonlinear equations. In addition we consider Newton method to obtain the solution of system of linear equations for finding Nash equilibrium and study Jacobian of the system of nonlinear equations to deal with its singularity. The chapter is organised as follows. In section 8.3 we discuss about system of nonlinear equations and establish the equivalence between nonlinear complementarity problem and system of nonlinear equations. In section 8.4 we introduce a continuation method with multiple parameters to solve system of nonlinear equations. We show that the trajectory obtained from that method is bounded under some conditions. In section 8.5 we propose modified Newton method to solve nonlinear system of equations. We show that this method has seventh order of convergence. Finally, we consider an example of oligopolistic market equilibrium problem [59] to demonstrate the effectiveness of our proposed algorithms.

8.3 Formulation of Nonlinear Complementarity Problem as System of Nonlinear Equations

The nonlinear complementarity problem is identified as an important mathematical programming problem which is defined in the first chapter (1.4.1). A wide class of problems, which arise in complementarity theory, can be studied via the system of nonlinear equations. Finding solution of system of nonlinear equations has an important role to deal with problems in various fields such as chemical production processes, engineering design, economic equilibrium, transportation and applied physics. A number of methods are proposed to solve system of equations. Newton and quasi-Newton methods are well known iterative methods to solve system of nonlinear equations. In recent years, researchers are interested to solve system of nonlinear equations both analytically and numerically. Several iterative methods have been developed using different techniques such as Taylor's series expansion, quadrature formulas, homotopy method, interpolation, decomposition and its various modification. For details, see [171], [172], [174] and [175].

Now we show the equivalence between nonlinear complementarity problem and system of nonlinear equations.

THEOREM 8.3.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function such that*

$\phi(0) = 0$. *Then z solves the complementarity problem (1.4.1) if and only if*

$$\phi((f_i(z) - z_i)^2) - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|) = 0 \ \forall \ i \tag{8.3.1}$$

Proof. Necessary. Suppose z is the solution of the nonlinear complementarity problem (1.4.1). Now we show that z satisfies the system of nonlinear equations (8.3.1).

For each $i = 1, 2, \ldots, n$ either $z_i = 0, f_i(z) \geq 0$ or $f_i(z) = 0, z_i \geq 0$.

If $z_i = 0, f_i(z) \geq 0$ then

$\phi((f_i(z) - z_i)^2) - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|)$

$= \phi((f_i(z))^2) - \phi(f_i(z)|f_i(z)|) = \phi((f_i(z))^2) - \phi((f_i(z))^2) = 0.$

If $f_i(z) = 0, z_i \geq 0$ then

$\phi((f_i(z) - z_i)^2) - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|)$

$= \phi((z_i)^2) - \phi(z_i|z_i|) = \phi((z_i)^2) - \phi((z_i)^2) = 0.$

Hence the solution of (1.4.1) satisfies the system of equations (8.3.1).

Sufficient. Suppose z satisfies the system of nonlinear equations (8.3.1). We show that (a)$f(z) \geq 0$, (b)$z \geq 0$ and (c)$z^T f(z) = 0$.

(a) To show $f(z) \geq 0$ assume that $f_i(z) < 0$ for atleast one $i \in \{1, 2, \ldots, n\}$. Since ϕ is a strictly increasing function with $\phi(0) = 0$, we obtain

$$\begin{aligned} 0 \leq \phi((f_i(z) - z_i)^2) = \phi(f_i(z)|f_i(z)|) + \phi(z_i|z_i|) &= \phi(-f_i(z)^2) + \phi(z_i|z_i|) \\ &< \phi(z_i|z_i|) \end{aligned}$$

This implies that $\phi(z_i|z_i|) > 0 \implies z_i|z_i| > 0 \implies z_i > 0$ and

$\phi((f_i(z) - z_i)^2) < \phi(z_i|z_i|) \implies (f_i(z) - z_i)^2 < z_i|z_i| \implies (f_i(z) - z_i)^2 < z_i^2.$

Now $z_i > 0$ and $f_i(z) < 0$ imply that $(f_i(z) - z_i) < 0$ and $|f_i(z) - z_i| > z_i$. Hence $(f_i(z) - z_i)^2 > z_i^2$. This contradicts that $(f_i(z) - z_i)^2 < z_i^2$. Hence $f_i(z) \geq 0 \ \forall \ i$.

(b) To show that $z \geq 0$ assume that $z_i < 0$ for atleast one $i \in \{1, 2, \ldots, n\}$. Since ϕ is a strictly increasing function with $\phi(0) = 0$, we obtain

$$0 \leq \phi((f_i(z) - z_i)^2) = \phi(f_i(z)|f_i(z)|) + \phi(z_i|z_i|) = \phi(f_i(z)|f_i(z)|) + \phi(-z_i^2)$$
$$< \phi(f_i(z)|f_i(z)|)$$

This implies that $\phi(f_i(z)|f_i(z)|) > 0 \implies f_i(z)|f_i(z)| > 0 \implies f_i(z) > 0$ and $\phi((f_i(z) - z_i)^2) < \phi(f_i(z)|f_i(z)|) \implies (f_i(z) - z_i)^2 < f_i(z)|f_i(z)| \implies (f_i(z) - z_i)^2 < (f_i(z))^2$.

Now $f_i(z) > 0$ and $z_i < 0$ imply that $(f_i(z) - z_i) > 0$ and $(f_i(z) - z_i) > f_i(z)$. Hence $(f_i(z) - z_i)^2 > (f_i(z))^2$. This contradicts that $(f_i(z) - z_i)^2 < (f_i(z))^2$. Hence $z_i \geq 0 \ \forall \ i$.

(c)From (a) and (b) we have $z \geq 0$ and $f(z) \geq 0$. To show $z^T f(z) = 0$ assume that $z_i > 0$ and $f_i(z) > 0$ for atleast one $i \in \{1, 2, \ldots, n\}$. Then $\phi((f_i(z) - z_i)^2) < \phi((f_i(z))^2) + \phi((z_i)^2) = \phi(f_i(z)|f_i(z)|) + \phi(z_i|z_i|)$. This contradicts that $\phi((f_i(z) - z_i)^2) = \phi(f_i(z)|f_i(z)|) + \phi(z_i|z_i|)$. Hence $z^T f(z) = 0$.

Therefore the solution of system of nonlinear equations (8.3.1) provides the solution of nonlinear complementarity problem (1.4.1). ■

REMARK 8.3.1. *Hence it is shown that the complementarity problem of finding a* $z \in \mathbb{R}^n$ *satisfying* $z^T f(z) = 0, \ f(z) \geq 0, \ z \geq 0$ *where* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a nonlinear function, is equivalent to the following problem of solving system of* n *nonlinear equations in* n *variables*

$$\psi_i(z) = \phi((f_i(z) - z_i)^2) - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|) = 0, \quad i \in \{1, 2, \cdots n\} \quad (8.3.2)$$

where ϕ *is a strictly increasing function and* $\phi(0) = 0$. *In this paper the function* $\phi : \mathbb{R} \to \mathbb{R}$ *is defined by* $\phi(x) = x^5$

Now we show that the Jacobian of the system of nonlinear equations (8.3.2) is nonsingular at the solution of nonlinear complementarity problem under some conditions. To show this we consider the followings.

$\frac{\partial \psi_i}{\partial z_j} = \phi'((f_i(z) - z_i)^2) 2(f_i(z) - z_i)(\frac{\partial f_i}{\partial z_j} - \delta_{ij}) \ -\phi'(f_i(z)|f_i(z)|) 2 f_i(z) sgn(f_i(z)) \frac{\partial f_i}{\partial z_j}$

$-\phi'(z_i|z_i|)2z_i sgn(z_i)\delta_{ij}$, where the function $sgn : \mathbb{R} \to \mathbb{R}$ is defined in the first chapter.

We show that the Jacobian of the system of nonlinear equations (8.3.2) is nonsingular at the solution of nonlinear complementarity problem under some conditions.

THEOREM 8.3.2. *Let z be a nondegenerate solution to the nonlinear complementarity problem (1.4.1) satisfying $z + f(z) > 0$. Let $\mathcal{J}\ (f(z))$, the Jacobian of the function f at z has nonsingular principal minors and let $\phi : \mathbb{R} \to \mathbb{R}$ be a differentiable strictly increasing function such that $\phi'(y) > 0$ for all $y > 0$ and $\phi(0) = 0$. Then z solves (8.3.2) and the Jacobian of the function ψ at z, $\mathcal{J}\ (\psi(z))$ is nonsingular.*

Proof. Let z be a nondegenerate solution of the nonlinear complementarity problem (1.4.1). The Jacobian matrix of the function ψ is defined by

$$\mathcal{J}(\psi) = \begin{bmatrix} \frac{\partial\psi_1}{\partial z_1} & \frac{\partial\psi_1}{\partial z_2} & \cdots & \frac{\partial\psi_1}{\partial z_n} \\ \frac{\partial\psi_2}{\partial z_1} & \frac{\partial\psi_2}{\partial z_2} & \cdots & \frac{\partial\psi_2}{\partial z_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\psi_n}{\partial z_1} & \frac{\partial\psi_n}{\partial z_2} & \cdots & \frac{\partial\psi_n}{\partial z_n} \end{bmatrix},$$

where $\frac{\partial\psi_i}{\partial z_i} = \phi'((f_i(z)-z_i)^2)2(f_i(z)-z_i)(\frac{\partial f_i}{\partial z_i} - 1) - \phi'(f_i(z)|f_i(z)|)2f_i(z)sgn(f_i(z))\frac{\partial f_i}{\partial z_i} - \phi'(z_i|z_i|)2z_i sgn(z_i)$ and $\frac{\partial\psi_i}{\partial z_j} = \phi'((f_i(z)-z_i)^2)2(f_i(z)-z_i)(\frac{\partial f_i}{\partial z_j}) - \phi'(f_i(z)|f_i(z)|)2f_i(z)sgn(f_i(z))\frac{\partial f_i}{\partial z_j}$ for $i \neq j$.

Assume that $f_i(z) = 0$ for $i = 1, 2, \cdots n_1,\ n_1 \leq n$ and $f_i(z) > 0$ for $i = n_1+1, n_1+2, \cdots n$. Hence by nondegeneracy of z, $z_i > 0$ for $i = 1, 2, \cdots n_1, n_1 \leq n$, $z_i = 0$ for $i = n_1 + 1, n_1 + 2, \cdots n$.

Now the Jacobian of the function ψ at z is given by,

$$\mathcal{J}\ (\psi(z)) = P'\ \mathcal{J}\ (f(z)) + Q', \text{ where } P'_{ij} = \begin{cases} -\phi'((z_i)^2)2z_i & \text{if } i = j,\ 1 \leq i \leq n_1 \\ 0 & \text{if } i = j,\ n_1 < i \leq n, \\ 0 & \text{if } i \neq j \end{cases}$$

$$Q'_{ij} = \begin{cases} 0 & \text{if } i = j,\ 1 \le i \le n_1 \\ -\phi'((f_i(z))^2)2f_i(z) & \text{if } i = j,\ n_1 < i \le n \\ 0 & \text{if } i \ne j \end{cases}$$

Since $\mathcal{J}\ (f(z))$ has nonsingular principal minors and $\phi'(y) > 0$ for $y > 0$, then the Jacobian of the function ψ at z, $\mathcal{J}\ (\psi(z))$ is nonsingular. ■

8.4 Continuation Method with Multiple Parameters

The basic idea of continuation method with vector parameter $\tilde{\lambda} \in \mathbb{R}^n$ is to construct a multi dimentional path to find the solution of the object fuction $\tilde{p}(\tilde{z}) = 0$ varying each component of $\tilde{\lambda}$ from 1 to 0. We consider the function $\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda}) = \tilde{p}(\tilde{z}) - \tilde{\lambda}\tilde{p}(\tilde{z}^{(0)})$, where each component of $\tilde{p}(\tilde{z}^{(0)})$, $p_i(\tilde{z}^{(0)}) \ne 0\ \forall\ i$ and the product term $\tilde{\lambda}\tilde{p}(\tilde{z}^{(0)})$ is a componentwise product i.e. $\tilde{\lambda}\tilde{p}(\tilde{z}^{(0)}) = \begin{bmatrix} \lambda_1 p_1(\tilde{z}^{(0)}) \\ \lambda_2 p_2(\tilde{z}^{(0)}) \\ \vdots \\ \lambda_n p_n(\tilde{z}^{(0)}) \end{bmatrix}$, where $\tilde{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$ and $\tilde{p}(\tilde{z}^{(0)}) = \begin{bmatrix} p_1(\tilde{z}^{(0)}) \\ p_2(\tilde{z}^{(0)}) \\ \vdots \\ p_n(\tilde{z}^{(0)}) \end{bmatrix}$.
In this method our main aim is to vary each component $\lambda_i = \frac{p_i(\tilde{z})}{p_i(\tilde{z}^{(0)})}$ of $\tilde{\lambda}$ from 1 to 0.

We solve the system of nonlinear equations (8.3.2) using the function $H(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda})$ with multiple parameters in vector form $\tilde{\lambda} = [\lambda_1, \lambda_2, \cdots, \lambda_n]^T \in \mathbb{R}^n$. Consider the system of nonlinear equations $\psi_i(z) = \phi((f_i(z) - z_i)^2) - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|) = 0\ \forall\ i$, where $\phi : \mathbb{R} \to \mathbb{R}$ is identified by $\phi(x) = x^5$. Consider the set $F_{\mathcal{H}} = \{\tilde{z} : \tilde{z} \in \mathbb{R}^n\}$ and $\tilde{F}_{\mathcal{H}} = \{(\tilde{z}, \tilde{\lambda}) : (\tilde{z}, \tilde{\lambda}) \in \mathbb{R}^n \times (0, 1]^n\}$.

The function can be given as,

$$\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda}) = \psi(\tilde{z}) - \tilde{\lambda}\psi(\tilde{z}^{(0)}) = \begin{bmatrix} \psi_1(\tilde{z}) - \lambda_1\psi_1(\tilde{z}^{(0)}) \\ \psi_2(\tilde{z}) - \lambda_2\psi_2(\tilde{z}^{(0)}) \\ \vdots \\ \psi_n(\tilde{z}) - \lambda_n\psi_n(\tilde{z}^{(0)}) \end{bmatrix} \tag{8.4.1}$$

where, $\tilde{z}^{(0)}$ is the initial value such that $\psi_i(\tilde{z}^{(0)}) > 0\ \forall\ i$, the Jacobian matrix $\frac{\partial\psi(\tilde{z}^{(0)})}{\partial\tilde{z}^{(0)}}$ at initial point $\tilde{z}^{(0)}$ is nonsingular and $\tilde{\lambda} \in \mathbb{R}^n$.

8.4.1 Properties of the Trajectory for Multiple Parameters

First we show that the continuation path with multiple parameters is smooth.

Theorem 8.4.1. *If the Jacobian matrix $\frac{\partial\psi(\tilde{z}^{(0)})}{\partial\tilde{z}^{(0)}}$ at initial point $\tilde{z}^{(0)}$ is nonsingular, then for almost all initial points $\tilde{z}^{(0)} \in F_{\mathcal{H}}$, 0 is a regular value of the function $\mathcal{H} : \mathbb{R}^n \times (0,1]^n \to \mathbb{R}^n$ and the zero point set $\mathcal{H}^{-1}_{\tilde{z}^{(0)}}(0) = \{(\tilde{z}, \tilde{\lambda}) \in \tilde{F}_{\mathcal{H}} : \mathcal{H}_{\tilde{z}^{(0)}}(\tilde{z}, \tilde{\lambda}) = 0\}$ contains a smooth curve $\Gamma_{\tilde{z}^{(0)}}$ starting from $(\tilde{z}^{(0)}, e)$.*

Proof. Jacobian matrix of the above function (8.4.1) $\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda})$ is denoted by $D\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda}))$ and we have $D\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda})) = \begin{bmatrix} \frac{\partial\mathcal{H}(\tilde{z},\tilde{z}^{(0)},\tilde{\lambda})}{\partial\tilde{z}} & \frac{\partial\mathcal{H}(\tilde{z},\tilde{z}^{(0)},\tilde{\lambda})}{\partial\tilde{z}^{(0)}} & \frac{\partial\mathcal{H}(\tilde{z},\tilde{z}^{(0)},\tilde{\lambda})}{\partial\tilde{\lambda}} \end{bmatrix}$. For all $\tilde{z}^{(0)} \in F_H$ such that $\psi(\tilde{z}^{(0)}) \neq 0$, the Jacobian matrix $\frac{\partial\psi(\tilde{z}^{(0)})}{\partial\tilde{z}^{(0)}}$ at initial point $\tilde{z}^{(0)}$ is nonsingular and $\tilde{\lambda} \in (0,1]^n$, we have

$$\frac{\partial\mathcal{H}(\tilde{z},\tilde{z}^{(0)},\tilde{\lambda})}{\partial\tilde{z}^{(0)}} = \begin{bmatrix} -\lambda_1\frac{\partial\psi_1(\tilde{z}^{(0)})}{\partial\tilde{z}_1^{(0)}} & -\lambda_1\frac{\partial\psi_1(\tilde{z}^{(0)})}{\partial\tilde{z}_2^{(0)}} & \cdots & -\lambda_1\frac{\partial\psi_1(\tilde{z}^{(0)})}{\partial\tilde{z}_n^{(0)}} \\ -\lambda_2\frac{\partial\psi_2(\tilde{z}^{(0)})}{\partial\tilde{z}_1^{(0)}} & -\lambda_2\frac{\partial\psi_2(\tilde{z}^{(0)})}{\partial\tilde{z}_2^{(0)}} & \cdots & -\lambda_2\frac{\partial\psi_2(\tilde{z}^{(0)})}{\partial\tilde{z}_n^{(0)}} \\ \vdots & \vdots & \vdots & \vdots \\ -\lambda_n\frac{\partial\psi_n(\tilde{z}^{(0)})}{\partial\tilde{z}_1^{(0)}} & -\lambda_n\frac{\partial\psi_n(\tilde{z}^{(0)})}{\partial\tilde{z}_2^{(0)}} & \cdots & -\lambda_n\frac{\partial\psi_n(\tilde{z}^{(0)})}{\partial\tilde{z}_n^{(0)}} \end{bmatrix}.$$

Now $\det(\frac{\partial\mathcal{H}(\tilde{z},\tilde{z}^{(0)},\tilde{\lambda})}{\partial\tilde{z}^{(0)}}) = (-1)^n \det(J_{\mathcal{H}}(\tilde{z}^{(0)})) \prod_{i=1}^{i=n} \lambda_i \neq 0$ for $\tilde{\lambda} \in (0,1]^n$, where

$J_{\mathcal{H}}(\tilde{z}^{(0)}) = \frac{\partial\psi(\tilde{z}^{(0)})}{\partial\tilde{z}^{(0)}}$. Thus $D\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda})$ is of full row rank. Therefore, 0 is a regular value of $\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda})$ and $\mathcal{H}^{-1}_{\tilde{z}^{(0)}}(0)$ consists of some smooth curves such that $\mathcal{H}_{\tilde{z}^{(0)}}(\tilde{z}^{(0)}, e) = 0$. Hence there must be a smooth curve $\Gamma_{\tilde{z}^{(0)}}$ starting from $(\tilde{z}^{(0)}, e)$. ■

Now we show the boundedness of the curve.

THEOREM 8.4.2. *Let the function $f(\tilde{z})$ be an increasing function and $\psi_i(\tilde{z}^{(0)}) > 0\ \forall\ i$. Then the smooth curve $\Gamma_{\tilde{z}^{(0)}}$ is bounded for $\lambda_i \in (0, 1]\ \forall\ i$.*

Proof. From the function (8.4.1) we obtain $\psi_i(\tilde{z}) = \lambda_i\psi_i(\tilde{z}^{(0)})$ for $i \in \{1, 2, \cdots, n\}$,where $\psi_i(\tilde{z}) = \phi((f_i(\tilde{z}) - \tilde{z}_i)^2) - \phi(f_i(\tilde{z})|f_i(\tilde{z})|) - \phi(\tilde{z}_i|\tilde{z}_i|)$. It is clear that $\psi_i(\tilde{z})$ is finite as $\lambda_i \in [0, 1]$. Hence $\|\psi(\tilde{z})\| < \infty$.
That is $\psi_i(\tilde{z}) = \phi((f_i(\tilde{z}) - \tilde{z}_i)^2) - \phi(f_i(\tilde{z})|f_i(\tilde{z})|) - \phi(\tilde{z}_i|\tilde{z}_i|) = \lambda_i\psi_i(\tilde{z}^{(0)})\ \forall\ i$. Let $\lambda_i\psi_i(\tilde{z}^{(0)}) = k_i$. For $\lambda_i \in (0, 1], k_i > 0$.
Hence $\phi((f_i(\tilde{z}) - \tilde{z}_i)^2) = \phi(f_i(\tilde{z})|f_i(\tilde{z})|) + \phi(\tilde{z}_i|\tilde{z}_i|) + k_i$.
Again $\forall\ i$ we have $\psi_i(\tilde{z}) = ((f_i(\tilde{z}) - \tilde{z}_i)^{10}) - (f_i(\tilde{z})|f_i(\tilde{z})|)^5 - (\tilde{z}_i|\tilde{z}_i|)^5 = (f_i(\tilde{z}))^{10} - 10(f_i(\tilde{z}))^9\tilde{z}_i + 45(f_i(\tilde{z}))^8(\tilde{z}_i)^2 - 120(f_i(\tilde{z}))^7(\tilde{z}_i)^3 + 210(f_i(\tilde{z}))^6(\tilde{z}_i)^4 - 252(f_i(\tilde{z}))^5(\tilde{z}_i)^5 + 210(f_i(\tilde{z}))^4(\tilde{z}_i)^6 - 120(f_i(\tilde{z}))^3(\tilde{z}_i)^7 + 45(f_i(\tilde{z}))^2(\tilde{z}_i)^8 - 10(f_i(\tilde{z}))(\tilde{z}_i)^9 + (\tilde{z}_i)^{10} - (f_i(\tilde{z}))^9|f_i(\tilde{z})| - (\tilde{z}_i)^9|(\tilde{z}_i)|$. Consider the path $\Gamma_{\tilde{z}^{(0)}}$ is unbounded. Then there exists a sequence of points $(\tilde{z}^{(k)}, \tilde{\lambda}^{(k)}) \subset \Gamma_{\tilde{z}^{(0)}}$ such that $\|\tilde{z}^{(k)}\| \to \infty$. Then there are two possibilities.

Case 1: Let $\|\tilde{z}^{(k)}\| \to \infty$. Then $\exists\ i \in \{1, 2, \cdots, n\}$ such that $\tilde{z}_i^{(k)} \to -\infty$ as $k \to \infty$. Let set $I_{1\tilde{z}} = \{i \in \{1, 2, \cdots, n\} : \lim\limits_{k\to\infty} \tilde{z}_i^{(k)} \to -\infty\}$.
Now consider $\|f(\tilde{z}^{(k)})\| < \infty$. Then $\forall\ i \in I_{1\tilde{z}}$, $\frac{\psi_i(\tilde{z}^{(k)})}{(\tilde{z}_i^{(k)})^{10}} \to 2$ as $k \to \infty$, which contradicts that $\psi(\tilde{z})$ is bounded.
Again consider that the nonlinear function $f(\tilde{z})$ is unbounded. It is noted that $f(\tilde{z})$ is an increasing function. Therefore there exists a nonempty set

$L_{1f(\tilde{z})}$ such that $L_{1f(\tilde{z})} = \{l \in \{1, 2, \cdots, n\} : \lim_{k\to\infty} f_l(\tilde{z}^{(k)}) \to -\infty\}$ and consider $\lim_{k\to\infty} \frac{f_i(\tilde{z}^{(k)})}{(\tilde{z}_i^{(k)})} = p \; \forall \; i \in I_{1\tilde{z}} \cap L_{1f(\tilde{z})}$. Hence $\forall \; i \in I_{1\tilde{z}} \cap L_{1f(\tilde{z})}$, $\frac{\psi_i(\tilde{z}^{(k)})}{(\tilde{z}_i^{(k)})^{10}} \to 2p^{10} - 10p^9 + 45p^8 - 120p^7 + 210p^6 - 252p^5 + 210p^4 - 120p^3 + 45p^2 - 10p + 2$ as $k \to \infty$. From the boundedness of $\psi(\tilde{z}^{(k)})$, it is clear that $2p^{10} - 10p^9 + 45p^8 - 120p^7 + 210p^6 - 252p^5 + 210p^4 - 120p^3 + 45p^2 - 10p + 2 = 0$, which has no real solution, which contradicts that $f(\tilde{z})$ is unbounded. Again for all $l \in L_{1f(\tilde{z})}, l \notin I_{1\tilde{z}}$, $\lim_{k\to\infty} \frac{\psi_l(\tilde{z}^{(k)})}{f_l(\tilde{z}^{(k)})^{10}} \to 2$, contradicts the boundedness of the function $\psi(\tilde{z})$. Therefore there exists no $i \in \{1, 2, \cdots, n\}$ such that $\tilde{z}_i^{(k)} \to -\infty$ as $k \to \infty$.

Case 2: Let $\|\tilde{z}^{(k)}\| \to \infty$. Then $\exists \; j \in \{1, 2, \cdots, n\}$ such that $\tilde{z}_j^{(k)} \to \infty$ as $k \to \infty$. Let set $I_{2\tilde{z}} = \{j \in \{1, 2, \cdots, n\} : \lim_{k\to\infty} \tilde{z}_j^{(k)} \to \infty\}$.
Assume that there exists atleast one $j \in \{1, 2, \cdots, n\}$ such that $\tilde{z}_j > 0$ and $f_j(\tilde{z}) \geq 0$. For $\lambda_j \in (0, 1]$, $\phi((f_j(\tilde{z}) - \tilde{z}_j)^2) = \phi(f_j(\tilde{z})|f_j(\tilde{z})|) + \phi(\tilde{z}_j|\tilde{z}_j|) + k_j = \phi((f_j(\tilde{z})^2) + \phi((\tilde{z}_j)^2) + k_j$, where $k_j > 0$. Now $0 \leq \phi((f_j(\tilde{z}) - \tilde{z}_j)^2) \leq \phi((f_j(\tilde{z})^2) + \phi((\tilde{z}_j)^2)$. This contradicts that $\phi((f_j(\tilde{z}) - \tilde{z}_j)^2) = \phi((f_j(\tilde{z})^2) + \phi((\tilde{z}_j)^2) + k_j$, where $k_j > 0$. Hence $\lambda_j > 0$, $\tilde{z}_j > 0$ imply $f_j(\tilde{z}) < 0$. Again f is an increasing function. Hence $f(\tilde{z})$ is bounded. Now considering that $\|f(\tilde{z}^{(k)})\| < \infty$. Then $\forall j \in I_{2\tilde{z}}$, as $k \to \infty$, $\frac{\psi_j(\tilde{z}^{(k)})}{(\tilde{z}_j^{(k)})^9} \to -10(f_j(\tilde{z}^{(k)})) \nrightarrow 0$, which contradicts that $\psi(\tilde{z})$ is bounded. Therefore there exists no $j \in \{1, 2, \cdots, n\}$ such that $\tilde{z}_j^{(k)} \to \infty$ as $k \to \infty$.
Therefore considering both the cases we conclude that the path $\Gamma_{\tilde{z}^{(0)}}$ is bounded for $\lambda_i \in (0, 1] \; \forall \; i$. ∎

REMARK 8.4.2. *Now we trace the path* $\Gamma_{\tilde{z}^{(0)}} \subset \mathcal{H}_{\tilde{z}^{(0)}}^{-1}(0) \subset \tilde{F}_{\mathcal{H}}$ *from the initial point* $(\tilde{z}^{(0)}, e)$ *until* $\tilde{\lambda} \to 0$ *and find the solution of the system of nonlinear equations (8.3.2). If the path is bounded and* $\tilde{\lambda}$ *goes to* 0 *starting from* e*, then* $\tilde{z}$ *is the solution of (8.3.2). Note that here we use the vector* $\tilde{\lambda}$*, such that* $\lambda_i = \exp(-t_i)$*, where* exp *is the exponential function. Hence for* $\lambda_i \in (0, 1]$*,* $t_i \in [0, \infty)$*.*

LEMMA 8.4.1. *The path $\Gamma_{\tilde{z}^{(0)}}$ is determined by the following problem*

$$\mathcal{H}'_{\tilde{z}^{(0)}}(\tilde{z}(s), \tilde{\lambda}(s)) \begin{bmatrix} \frac{d\tilde{z}}{ds} \\ \frac{d\tilde{\lambda}}{ds} \end{bmatrix} = 0, \ \|(\frac{d\tilde{z}}{ds}, \frac{d\tilde{\lambda}}{ds})\|_2 = 1, \ \tilde{z}(0) = \tilde{z}^{(0)}, \ \tilde{\lambda}(0) = e. \quad (8.4.2)$$

.

Proof. Let s denote the arc length of the path $\Gamma_{\tilde{z}^{(0)}}$. Now differentiating the function

$$\mathcal{H}_{\tilde{z}^{(0)}}(\tilde{z}(s), \tilde{\lambda}(s)) = 0, \ \tilde{z}(0) = \tilde{z}^{(0)}, \ \tilde{\lambda}(0) = e \quad (8.4.3)$$

we obtain $\frac{\partial \mathcal{H}}{\partial \tilde{z}} \frac{d\tilde{z}}{ds} + \frac{\partial \mathcal{H}}{\partial \tilde{\lambda}} \frac{d\tilde{\lambda}}{ds} = 0$. Let $\nu = \begin{bmatrix} \tilde{z} \\ \tilde{\lambda} \end{bmatrix}$. As s is the arc length of $\Gamma_{\tilde{z}^{(0)}}$, then $\|\nu'\| = \|(\frac{d\tilde{z}}{ds}, \frac{d\tilde{\lambda}}{ds})\|_2 = 1$. Then we obtain the following system of equation.

$$\begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \tilde{z}} & \frac{\partial \mathcal{H}}{\partial \tilde{\lambda}} \end{bmatrix} \mu = 0, \mu^T \mu = 1, \frac{d\nu}{ds} = \mu, \nu(0) = \begin{bmatrix} \tilde{z}^{(0)} \\ e \end{bmatrix} \quad (8.4.4)$$

Hence from the system (8.4.4) the first two equations are solvable on μ. Solving the following cauchy problem, the curve $\Gamma_{\tilde{z}^{(0)}}$ can be derived.

$$\frac{d\nu}{ds} = \mu, \nu(0) = \begin{bmatrix} \tilde{z}^{(0)} \\ e \end{bmatrix}. \quad (8.4.5)$$

■

8.4.2 Algorithm: Continuation Method with Multiple Parameters

Step 0: Let i be the count of iteration and i_c be the count of shifting of the initial point. Set $i = 0, i_c = 0$. Give an initial point $(\tilde{z}^{(0)}, \tilde{\lambda}_0) \in \tilde{F}_{\mathcal{H}} \times \{1\}^n$. η_1 is a small positive number and c_0 is a counter. ϵ_1 is a small positive number.

Consider $\kappa_1 \in (1, 2], \kappa_2$ such that the step length is determined by $\kappa_1^k, k \in Z$ and $\kappa_1^k \leq \kappa_2$. Consider two counters c_1 and c_2.

Set $\tilde{u}_1 = e$, $\tilde{\lambda}(t) = \exp(-t)$, where $(\exp(-t))_i = \exp(-t_i)$, $t = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$,

$\tilde{p}_1 = \tilde{\lambda}^{-1}(e)$, $e =$vector of all $1'$s$= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

Step 1: Set $\begin{bmatrix} \tilde{z} \\ t \end{bmatrix} = \begin{bmatrix} \tilde{z}^{(0)} \\ \tilde{p}_1 \end{bmatrix}$. Calculate $d^{(0)} = \det(\frac{\partial \mathcal{H}}{\partial \tilde{z}}(\tilde{z}^{(0)}, \tilde{\lambda}(\tilde{p}_1)))$ and $s^{(0)} = \frac{d\tilde{\lambda}}{dt}(t = \tilde{p}_1)$. Then go to Step 2.

Step 2: Set $c_1 = c_2 = 0$. Calculate $d = \det(\frac{\partial \mathcal{H}}{\partial \tilde{z}}(\tilde{z}, \tilde{\lambda}(t))), N = \frac{d\tilde{\lambda}(t)}{dt}$, and $s = Ne$. Then go to Step 3.

Step 3: If $\text{sgn}(d) = -\text{sgn}(d_0)$, then $\tilde{t}_d = N^{-1}(e - \tilde{\lambda}(t))$, else $\tilde{t}_d = -N^{-1}\tilde{\lambda}(t)$. Calculate $\tilde{z}_d = -(\frac{\partial \mathcal{H}}{\partial \tilde{z}}(\tilde{z}, \tilde{\lambda}(t))^{-1}(\frac{\partial \mathcal{H}}{\partial \tilde{\lambda}}(\tilde{z}, \tilde{\lambda}(t))\tilde{t}_d$, $\tau^{(n)} = \begin{bmatrix} \tilde{z}_n \\ \tilde{t}_n \end{bmatrix} = \frac{1}{\|\tilde{u}_d\|} \begin{bmatrix} \tilde{z}_d \\ \tilde{t}_d \end{bmatrix}$, $\tilde{\tau} = \dfrac{\|\tilde{t}_d\|}{\|\tilde{u}_d\|}$, where $\tilde{u}_d = \begin{bmatrix} \tilde{z}_d \\ \tilde{t}_d \end{bmatrix}$, $\|\tilde{u}_d\| = \sqrt{\tilde{z}_d^2 + \tilde{t}_d^2}$, $\tilde{z}_d^2 = \sum_i \tilde{z}_{di}^2$, $\tilde{z}_{di}$ is the ith component of $\tilde{z}_d$. If $\tilde{\tau} \leq \eta_1$, then $c1 = c1 + 1$ else $c1 = 0$. Set $\tilde{u}_1 = \tilde{\lambda}^{-1}(\tilde{t}_n)$. If $\tilde{t}_n \leq \epsilon_1$ then stop with a solution else go to Step 4.

Step 4: Set $i = i + 1, k = 0, \gamma = [\nabla \mu(\tilde{z})]^T \tilde{z}_n$, where the function $\mu : \mathbb{R}^n \to R$ is defined as $\mu(\tilde{z}) = [\mathcal{H}_0(\tilde{z})]^T [\mathcal{H}_0(\tilde{z})]$ and $\mathcal{H}_0(\tilde{z}) = \begin{bmatrix} \psi_1(\tilde{z}) \\ \psi_2(\tilde{z}) \\ \vdots \\ \psi_n(\tilde{z}) \end{bmatrix}$.

If $\gamma \geq 0$, $\tilde{z} + \kappa_1^{k+1} \tilde{z}_n \in \mathcal{F}_{\mathcal{H}}, 0 < \tilde{\lambda}(t + \kappa_1^{k+1} \tilde{u}_1) < e$, then $k = k + 1$ and go to Step 5.

else if $\gamma < 0, \mu(\tilde{z}+\kappa_1^{k+1}\tilde{z}_n) < \mu(\tilde{z}+\kappa_1^k\tilde{z}_n), \tilde{z}+\kappa_1^{k+1}\tilde{z}_n \in \mathcal{F}_{\mathcal{H}}, 0 < \tilde{\lambda}(t+\kappa_1^{k+1}\tilde{u}_1) < e$, then $k = k+1$ and go to Step 5. else reset $c_2 = 0$, and go to Step 6.

Step 5: If $\kappa_1^k > \kappa_2$, then set $k = k-1$, $c_2 = c_2 + 1$ and go to Step 6, else go to Step 4.

Step 6: Compute $\begin{bmatrix} \tilde{z}_p \\ \tilde{t}_p \end{bmatrix} = \begin{bmatrix} \tilde{z} \\ t \end{bmatrix} + \kappa_1^k \begin{bmatrix} \tilde{z}_n \\ \tilde{u}_1 \end{bmatrix}$,
$\begin{bmatrix} \tilde{z}_c \\ \tilde{t}_c \end{bmatrix} = \begin{bmatrix} \tilde{z}_p \\ \tilde{t}_p \end{bmatrix} - [J_{\mathcal{H}}(\tilde{z}_p, \tilde{\lambda}(\tilde{t}_p))^+ H(\tilde{z}_p, \tilde{\lambda}(\tilde{t}_p))]$, where $[J_{\mathcal{H}}(\tilde{z}_p, \tilde{\lambda}(\tilde{t}_p))^+]$ is the Moore-Penrose inverse. Set $\tilde{u}_s = \tilde{\lambda}(\tilde{t}_c)$. If $(\tilde{z}_c, \tilde{u}_s) \in \tilde{\mathcal{F}}_{\mathcal{H}}$, go to Step 9, else $k = k-1$ and go to Step 7.

Step 7: Calculate $a = \min(\kappa_1^k, \|\tilde{z} - \tilde{z}_c\|)$. If $a \leq \eta_2$, then go to Step 8 else go back to Step 5.

Step 8: If $\|\tilde{u}_s\| \leq \epsilon_1$, then stop with a solution else $i = i-1, i_c = i_c+1$, $\tilde{z}^{(0)} = \tilde{z}_c$ and go to Step 1.

Step 9: Set $\begin{bmatrix} \tilde{z} \\ t \end{bmatrix} = \begin{bmatrix} \tilde{z}_c \\ \tilde{t}_c \end{bmatrix}$. If $\frac{\|\tilde{u}_s\|}{\sqrt{n}} \leq \epsilon_1$, then stop with solution else $i = i+1$ and go to Step 2.

We show that the positive predictor direction depends on the sign of determinant of $J_{\mathcal{H}}(\tilde{z}^{(0)})$.

THEOREM 8.4.3. *If the path $\Gamma_{\tilde{z}}^{(0)}$ is smooth, then the positive predictor direction $\tilde{\tau}^{(0)}$ at the initial point $\tilde{z}^{(0)}$ satisfies* $sgn(\det(\begin{bmatrix} \mathcal{DH}_{(\tilde{z},\tilde{\lambda})}(\tilde{z}^{(0)}, 1) \\ e\tilde{\tau}^{(0)^T} \end{bmatrix})) = (-1)^n sgn(\det(J_{\mathcal{H}}(\tilde{z}^{(0)})))$, *where* $J_{\mathcal{H}}(\tilde{z}^{(0)}) = \frac{\partial\psi(\tilde{z}^{(0)})}{\partial\tilde{z}^{(0)}}$.

Proof. From Equation (8.4.1) we have $\mathcal{H}(\tilde{z}, \tilde{z}^{(0)}, \tilde{\lambda}) =$

$$\begin{bmatrix} \psi_1(\tilde{z}) - \lambda_1\psi_1(\tilde{z}^{(0)}) \\ \psi_2(\tilde{z}) - \lambda_2\psi_2(\tilde{z}^{(0)}) \\ \vdots \\ \psi_n(\tilde{z}) - \lambda_n\psi_n(\tilde{z}^{(0)}) \end{bmatrix} = 0.$$

Now $\mathcal{DH}_{(\tilde{z},\tilde{\lambda})}(\tilde{z}=\tilde{z}^{(0)},\tilde{\lambda}=e)=\begin{bmatrix} J_{\mathcal{H}}\tilde{z}^{(0)} & \Psi(\tilde{z}^{(0)})\end{bmatrix}$,
where $J_{\mathcal{H}}(\tilde{z}^{(0)})=\frac{\partial\psi(\tilde{z}^{(0)})}{\partial\tilde{z}^{(0)}}$ and
$\Psi(\tilde{z}^{(0)})=\begin{bmatrix} -\psi_1(\tilde{z}^{(0)}) & 0 & 0 & \cdots & 0 \\ 0 & -\psi_2(\tilde{z}^{(0)}) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\psi_n(\tilde{z}^{(0)}) \end{bmatrix}$. Let positive predictor direction be $\tau^{(0)}=\begin{bmatrix}\tilde{t}\\ -e\end{bmatrix}=\begin{bmatrix}(\tilde{R}_1^{(0)})^{(-1)}\tilde{R}_2^{(0)}e \\ -e\end{bmatrix}$, where $\tilde{R}_1^{(0)}=J_{\mathcal{H}}(\tilde{z}^{(0)})$ and $\tilde{R}_2^{(0)}=\Psi(\tilde{z}^{(0)})$. Here $\det(\tilde{R}_1^{(0)})\neq 0$. Therefore

$\det(\begin{bmatrix}\mathcal{DH}_{(\tilde{z},\tilde{\lambda})}(\tilde{z}^{(0)},e)\\ e\tilde{\tau}^{(0)^T}\end{bmatrix})=\det(\begin{bmatrix}\tilde{R}_1^{(0)} & \tilde{R}_2^{(0)} \\ ee^T(\tilde{R}_2^{(0)})^T(\tilde{R}_1^{(0)})^{(-T)} & -ee^T\end{bmatrix})$

$=\det(\begin{bmatrix}\tilde{R}_1^{(0)} & \tilde{R}_2^{(0)} \\ 0 & -ee^T-ee^T(\tilde{R}_2^{(0)})^T(\tilde{R}_1^{(0)})^{(-T)}(\tilde{R}_1^{(0)})^{(-1)}\tilde{R}_2^{(0)}\end{bmatrix})$

$=\det(\tilde{R}_1^{(0)})\det(-ee^T-ee^T(\tilde{R}_2^{(0)})^T(\tilde{R}_1^{(0)})^{(-T)}(\tilde{R}_1^{(0)})^{(-1)}\tilde{R}_2^{(0)})$

$=(-1)^n\det(\tilde{R}_1^{(0)})\det(ee^T(I+\mathcal{A}^T\mathcal{A}))$, where $\mathcal{A}=(\tilde{R}_1^{(0)})^{(-1)}\tilde{R}_2^{(0)}$.

Hence $sgn(\det(\begin{bmatrix}\mathcal{DH}_{(\tilde{z},\tilde{\lambda})}(\tilde{z}^{(0)},e)\\ e\tilde{\tau}^{(0)^T}\end{bmatrix}))=(-1)^n sgn(\det(J_{\mathcal{H}}(\tilde{z}^{(0)})))$. ■

8.5 Modified Newton Method

Now we introduce the modified Newton method to solve the system of nonlinear equations. Suppose $f:\mathbb{R}\to\mathbb{R}$. Then the equation $f(x)=0$ can be solved by Newton method with the iterative process $x^{k+1}=x^k-f'(x^k)^{-1}f(x^k)$. Suppose $g:\mathbb{R}^n\to\mathbb{R}^n$. Then the system of nonlinear equations $g(x)=0$ can be solved by Newton method with the iterative process $x^{k+1}=x^k-J_g(x^k)^{-1}g(x^k)$, where J_g is the Jacobian of the function g. For details see [176], [177], [178]. The algorithm of the modified Newton method is given below.

8.5.1 Algorithm: Modified Newton Method

Step 0: Give the initial approximation z^0 and a very small positive number e. Compute $\mathcal{J} = \psi'(z)$, the Jacobian of $\psi(z)$ with respect to z. Set $k = 0$.

Step 1: For k-th iteration compute the followings:

$$\begin{aligned}
y^k &= z^k - \frac{1}{2}[\psi'(z^k) + diag(t_i\psi_i(z^k))]^{-1}\psi(z^k);\\
x^k &= z^k - \frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}\\
&\qquad [\psi'(z^k) + \psi'(y^k)]\psi(z^k);\\
w^k &= x^k - [\{\psi'(x^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}\\
&\qquad [\psi'(x^k) + \psi'(y^k)]\psi(x^k);\\
z^{k+1} &= w^k - [\psi'(w^k) + diag(\eta_i\{\psi_i(w^k)\}^2)]^{-1}\psi(w^k),
\end{aligned} \tag{8.5.1}$$

where $\text{sgn}(t_i\psi_i(z^k)) = \text{sgn}(\frac{\partial\psi_i}{\partial z_i}(z^k))$,
$\text{sgn}(\lambda_i) = \text{sgn}(\frac{\partial\psi_i}{\partial z_i}(z^k))$,
$\text{sgn}(\mu_i) = \text{sgn}(\frac{\partial\psi_i}{\partial x_i}(x^k))$,
$\text{sgn}(\eta_i) = \text{sgn}(\frac{\partial\psi_i}{\partial w_i}(w^k))$.

Step 2: Compute $n_1 = \|\psi(z)\| = \sqrt{\sum_{i=1}^{n}\{\psi_i(z)\}^2}$.

Step 3: If $n_1 < e$, then z^{k+1} is the required solution of $\psi(z) = 0$, otherwise set $k = k + 1$ and go to step 1.

8.5.2 Order of Convergence

THEOREM 8.5.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ has a root $z^* \in D \subseteq \mathbb{R}^n$, where D is an open convex set. Assume that $\psi(z)$ is three times Fre'chet differentiable in some neighborhood N of the root (z^*). If for all $z \in N$, $diag(t_i\psi_i(z^k), diag(\lambda_i\{\psi_i(z^k)\}^2, diag(\mu_i\{\psi_i(x^k)\}^2$ and $diag(\eta_i\{\psi_i(w^k)\}^2$ are nonsingular, then the method defined by 8.5.1 is of seventh-order convergence.*

Proof. Let $e^k = z^k - z^*$. Now using Taylor series expansion we have,
$\psi(z^*) = \psi(z^k) + \psi'(z^k)(z^* - z^k) + \frac{1}{2}\psi''(z^k)(z^* - z^k)^2 + o(\|e^k\|^3)$
$\implies \psi(z^k) = \psi'(z^k)e^k - \frac{1}{2}\psi''(z^k)(e^k)^2 + o(\|e^k\|^3)$.
Let $d^k = y^k - z^k = -\frac{1}{2}[\psi'(z^k) + diag(t_i\psi_i(z^k))]^{-1}\psi(z^k)$.
Now using Taylor series expansion we have,
$\psi(y^k) = \psi(z^k) + \psi'(z^k)d^k + \frac{1}{2}\psi''(z^k)(d^k)^2 + o(\|d^k\|^3)$.
Now $d^k = y^k - z^k = -\frac{1}{2}[\psi'(z^k) + diag(t_i\psi_i(z^k))]^{-1}\psi(z^k)$
$= -\frac{1}{2}[\psi'(z^k) + diag(t_i\psi_i(z^k))]^{-1}\psi'(z^k)e^k + o(\|e^k\|^2)$
$= -\frac{1}{2}[\psi'(z^k) + diag(t_i\psi_i(z^k))]^{-1}[\psi'(z^k) + diag(t_i\psi_i(z^k))]e^k +$
$\frac{1}{2}[\psi'(z^k) + diag(t_i\psi_i(z^k))]^{-1}[diag(t_i\psi_i(z^k))]e^k + o(\|e^k\|^2)$
$= -\frac{1}{2}e^k + o(\|e^k\|^2)$, as $diag(t_i\psi_i(z^k))]e^k = o(\|e^k\|^2)$.
Hence $\psi(y^k) = \psi(z^k) - \frac{1}{2}\psi'(z^k)e^k + o(\|e^k\|^2) = \psi'(z^k)e^k - \frac{1}{2}\psi'(z^k)e^k + o(\|e^k\|^2) = \frac{1}{2}\psi'(z^k)e^k + o(\|e^k\|^2)$ and $\psi'(y^k) = \frac{1}{2}\psi'(z^k) + o(\|e^k\|)$.
Let $a^k = x^k - z^*$. Hence $x^k = z^k - \frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}[\psi'(z^k) + \psi'(y^k)]\psi(z^k)$ implies that
$a^k = e^k - \frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}[\psi'(z^k) + \psi'(y^k)]\psi(z^k)$
$\implies a^k = [\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}\{[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]e^k - \frac{1}{2}[\psi'(z^k) + \psi'(y^k)]\psi(z^k)\}$
$= [\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}\{[diag(\lambda_i\{\psi_i(z^k)\}^2)]e^k + \{\psi'(z^k)\}^2e^k + \{\psi'(y^k)\}^2e^k - \frac{1}{2}[\psi'(z^k) + \psi'(y^k)]\psi(z^k)\} = o(\|e^k\|^3)$, as $diag(\lambda_i\{\psi_i(z^k)\}^2)e^k = o(\|e^k\|^3)$.
Again using Taylor series expansion we have,
$\psi(x^k) = \psi(z^k) + \psi'(z^k)c^k + \frac{1}{2}\psi''(z^k)(c^k)^2 + o(\|c^k\|^3)$,where
$c^k = x^k - z^k = -\frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}[\psi'(z^k) + \psi'(y^k)]\psi(z^k) = -\frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}[\psi'(z^k) + \psi'(y^k)][\psi'(z^k)e^k] + o(\|e^k\|^2) = -\frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}[\{\psi'(z^k)\}^2e^k + \psi'(y^k)\psi'(z^k)e^k] + o(\|e^k\|^2) = -\frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2+$

$diag(\lambda_i\{\psi_i(z^k)\}^2)]e^k + \frac{1}{2}[\{\psi'(z^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2)]^{-1}[\{\psi'(y^k)\}^2 + diag(\lambda_i\{\psi_i(z^k)\}^2) - \psi'(y^k)\psi'(z^k)]e^k + o(\|e^k\|^2) = -\frac{1}{2}e^k + o(\|e^k\|^2)$.
Hence $\psi(x^k) = \psi(z^k) - \frac{1}{2}\psi'(z^k)e^k + o(\|e^k\|^2) = \psi'(z^k)e^k - \frac{1}{2}\psi'(z^k)e^k + o(\|e^k\|^2) = \frac{1}{2}\psi'(z^k)e^k + o(\|e^k\|^2)$ and $\psi'(x^k) = \frac{1}{2}\psi'(z^k) + o(\|e^k\|)$.

Let $b^k = w^k - z^* = x^k - z^* - \frac{1}{2}[\{\psi'(x^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[\psi'(x^k) + \psi'(y^k)]\psi(x^k) = a^k - \frac{1}{2}[\{\psi'(x^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[\psi'(x^k) + \psi'(y^k)]\psi(x^k)$
$\Longrightarrow$ $b^k = [\{\psi'(x^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}\{[\{\psi'(x^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]a^k - \frac{1}{2}[\psi'(x^k) + \psi'(y^k)]\psi(x^k)\} = o(\|e^k\|^5)$, as $diag(\mu_i\{\psi_i(x^k)\}^2)a^k = o(\|e^k\|^5)$. Now using Taylor series expansion we have,
$\psi(w^k) = \psi(z^k) + \psi'(z^k)m^k + \frac{1}{2}\psi''(z^k)(m^k)^2 + o(\|m^k\|^3)$, where $m^k = w^k - z^k$.
Let $n^k = w^k - x^k = -\frac{1}{2}[\{\psi'(x^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[\psi'(x^k) + \psi'(y^k)]\psi(x^k) = -\frac{1}{2}[\{\psi'(x^k)\}^2 + \{\psi'(y^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[\frac{1}{2}\psi'(z^k) + \frac{1}{2}\psi'(z^k)][\frac{1}{2}\psi'(z^k)e^k] + o(\|e^k\|^2) = -\frac{1}{4}[\frac{1}{4}\{\psi'(z^k)\}^2 + \frac{1}{4}\{\psi'(z^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[\{\psi'(z^k)\}^2e^k] + o(\|e^k\|^2) = -\frac{1}{8}[\{\psi'(z^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[\{\psi'(z^k)\}^2e^k] + o(\|e^k\|^2) = -\frac{1}{8}[\{\psi'(z^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[\{\psi'(z^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]e^k + \frac{1}{8}[\{\psi'(z^k)\}^2 + diag(\mu_i\{\psi_i(x^k)\}^2)]^{-1}[diag(\mu_i\{\psi_i(x^k)\}^2)e^k] + o(\|e^k\|^2) = -\frac{1}{8}e^k + o(\|e^k\|^2)$.
Hence $m^k = w^k - z^k = w^k - x^k + x^k - z^k = n^k + c^k = -\frac{5}{8}e^k + o(\|e^k\|^2)$. Therefore $\psi(w^k) = \psi(z^k) - \frac{5}{8}\psi'(z^k)e^k + o(\|e^k\|^2)$.
Now $z^{k+1} - z^* = w^k - z^* - [\psi'(w^k) + diag(\eta_i\{\psi_i(w^k)\}^2]^{-1}\psi(w^k)$
$\Longrightarrow$ $e^{k+1} = b^k - [\psi'(w^k) + diag(\eta_i\{\psi_i(w^k)\}^2]^{-1}\psi(w^k) = [\psi'(w^k) + diag(\eta_i\{\psi_i(w^k)\}^2]^{-1}\{[\psi'(w^k) + diag(\eta_i\{\psi_i(w^k)\}^2]b^k - \psi(w^k)\} = [\psi'(w^k) + diag(\eta_i\{\psi_i(w^k)\}^2]^{-1}\{\psi'(w^k)b^k + diag(\eta_i\{\psi_i(w^k)\}^2b^k - \psi(w^k)\} = o(\|e^k\|^7)$, as $diag(\eta_i\{\psi_i(w^k)\}^2b^k = o(\|e^k\|^7)$. Therefore the introduced modified Newton method has seventh order of convergency.

■

REMARK 8.5.3. *The proposed modified Newton method converges to the solution for suitable initial point . Following this method we can obtain the solution of oligopolistic market equilibrium problem by solving the system of nonlinear equations (8.3.2).*

8.6 Numerical Illustration

Consider an oligopoly with five firms, each with a total cost function of the form [59]:

$$c_i(Q_i) = n_i Q_i + \frac{\beta_i}{\beta_i + 1} L_i^{-\frac{1}{\beta_i}} Q_i^{\frac{\beta_i+1}{\beta_i}} \tag{8.6.1}$$

The demand function is given by:

$$\tilde{Q} = 5000P^{-1.1}, \quad P(\tilde{Q}) = 5000^{1/1.1}\tilde{Q}^{-1/1.1}. \tag{8.6.2}$$

The parameters of the Equation (8.6.1) for the five firms are given below:

firm i	n_i	L_i	β_i
1	10	5	1.2
2	8	5	1.1
3	6	5	1
4	4	5	0.8
5	2	5	0.6

Table 8.1: Value of parameters for five firms

To solve this problem using the continuation method with multiple parameter

8.4.2, we first take the initial point $\tilde{z}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\tilde{\lambda}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and set $i = 0, i_c = 0, \eta_1 = 10^{-12}, \kappa_1 = \sqrt{2}, \kappa_2 = 9000$, $\epsilon_1 = 10^{-18}$. After 51 iterations we obtain the result $\tilde{z} = \begin{bmatrix} 36.9325 \\ 41.8181 \\ 43.7066 \\ 42.6592 \\ 39.1789 \end{bmatrix}$, $\tilde{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Here the stopping criteria depends on $\|\tilde{\lambda}\| < \epsilon_1$. For $\epsilon_1 = 10^{-15}$, the values of system of nonlinear equations $\psi(\tilde{z})$ are zero upto 8th decimal place , for $\epsilon_1 = 10^{-17}$, the values of system of nonlinear equations $\psi(\tilde{z})$ are zero upto 9th decimal place but for $\epsilon_1 = 10^{-18}$, the values of system of nonlinear equations $\psi(\tilde{z})$ are zero upto 10th decimal place.

Now to solve this oligopoly problem by modified Newton's method 8.5.1, first take the initial point $z_0 = \begin{bmatrix} 50 \\ 50 \\ 50 \\ 50 \\ 50 \end{bmatrix}$. Set $e = 10^{-10}$. After 21 iterations we obtain the solution $z = \begin{bmatrix} 36.9325 \\ 41.8181 \\ 43.7066 \\ 42.6592 \\ 39.1789 \end{bmatrix}$.

In this study, we establish two broad approaches to find the Nash equilibrium of oligopolistic market namely, continuation method and modified Newton method.

In this context we also establish a continuation method based on vector parameter to find the Nash equilibrium point. In addition we develop a modified Newton method to find the solution of system of nonlinear equations which provides the solution of oligopoly market. For this method, we prove that the order of convergence is seven.

Bibliography

[1] Berman, Abraham, and Robert J. Plemmons, Nonnegative matrices in the mathematical sciences. Society for Industrial and Applied Mathematics, 1994.

[2] A. Matei and Y. Tillé, Maximal and Minimal Sample Co-ordination, *Sankhya: The Indian Journal of Statistics,* 67 590-612, 2006.

[3] A. J. Hoffman, On simple linear programming problems, *Proceedings of Symposia in Pure Mathematics, Convexity,* VII 317-327, 1963.

[4] A. K. Das, R. Jana and Deepmala, On the Convergence of an Iterative Method for Solving Linear Complementarity Problem with WGPSBD Matrix, Accepted for publication in *Thai Journal of Mathematics,* 2017.

[5] A. K. Das, R. Jana and Deepmala, Finiteness of Criss-Cross Method in Complementarity Problem. Mathematics and Computing, eds: Debasis Giri et al., Springer, Singapore, pp. 170-180 2017.

[6] Ramaswamy. Chandrasekaran, A special case of the complementary pivot problem. Opsearch 7, 263-268, 1970.

[7] A. K. Das, Properties of some matrix classes based on principal pivot transform, *Annals of Operations Research,* 243(1-2), 375-382, 2016.

[8] A.K. Das, Deepmala, R. Jana, Some Aspects on Solving Transportation Problem. Communicated to International Journal of Applied and Computational Mathematics, 2016.

[9] R. W. Cottle and J. A. -Ferland, Matrix-theoretic criteria for the quasiconvexity and pseudoconvexity of quadratic functions, Dept. of Operations Research, Stanford University, Technical Report No. 70-6, 1970.

[10] A. W. Tucker, Principal pivot transforms of square matrices, *SIAM Review,* 5, 305, 1963.

[11] B. Martos, Subdefinite matrices and quadratic forms, *SIAM Journal on Applied Mathematics,* 17(6), 1215-1223, 1969.

[12] B. Dantzig, E. Eisenberg and R. W. Cottle, Symmetric Dual Nonlinear Programs, *Pacific Journal of Mathematics,* 16, 809-812, 1965.

[13] B. Kheirfam, An Infeasible Interior-point Method for Monotone LCP Based on a Kernel Function With Full-Newton Step, *Southeast Asian Bulletin of Mathematics,* 40(5), 707-22, 2016.

[14] S. K. Neogy and A. K. Das, Principal pivot transforms of some classes of matrices, to appear in *Linear Algebra and its Applications.*

[15] B. Martos, *Nonlinear Programming: Theory and Methods,* North-Holland, Amsterdam, 1975.

[16] B. Kheirfam and N. Mahdavi-Amiri, A new interior-point algorithm based on modified Nesterov-Todd direction for symmetric cone linear complementarity problem, *Optimization Letters,* 8(3), 1017-29, 2014.

[17] B. C. Eaves, The linear complementarity problem, *Management science,* 17(9), 612-634, 1971.

[18] J. F. Nash, Noncooperative games, *Ann. of Math.,* 54, 286-295, 1951.

[19] P. C. Jones, Even more with the Lemke complementarity algorithm, *Mathematical Programming,* 35, 239-242, 1986.

[20] C. C. Gonzaga and L. A. Carlos, A primal affine scaling algorithm for linearly constrained convex programs. Report, Federal University of Rio de Janeiro, 1990.

[21] C. C. Gonzaga, Large step path-following methods for linear programming, part I: barrier function method, *SIAM Journal on Optimization,* 1(2), 268-279, 1991.

[22] C. E. Lemke, Bimatrix equilibrium points and mathematical programming, *Management Science,* 11, 681-689, 1965.

[23] C. E. Lemke and J. T. Howson, Equilibrium points of bimatrix games, *SIAM Journal on Applied Mathematics,* 12, 413-423, 1964.

[24] C. R. Glassey, A quadratic network optimization model for equilibrium single commodity trade flows, *Mathematical Programming,* 14, 98-107, 1978.

[25] C. W. Cryer, The method of Christopherson for solving free boundary problems for infinite journal bearings by means of finite differences, *Mathematics of Computation,* 25, 435-443, 1971.

[26] D. Hertog, C. Roos and T. Terlaky, The linear complementarity problem, sufficient matrices, and the criss-cross method, *Linear Algebra and Its Applications,* 187, 1-14, 1993.

[27] Deepmala, A. K. Das and R. Jana, On existence of coincidence solutions for a generalized system of functional equations. *Gazi University Journal of Science,* 30(4),442-452, 2017.

[28] D. Den Hertog, Interior point approach to linear, quadratic and convex programming: algorithms and complexity (Vol. 277). Springer Science & Business Media, 2012.

[29] E. Klafszky and T. Terlaky, Some generalizations of the criss-cross method for quadratic programming, *Optimization,* 24(1-2),127-139, 1992.

[30] E. R. Barnes, A variation on Karmarkar's algorithm for solving linear programming problems, *Mathematical Programming,* 36(2) (1986) 174-182.

[31] E. de Klerk, Primal-Dual Affine-Scaling Methods, *Aspects of Semidefinite Programming: Interior Point Algorithms and Selected Applications,* 95-113,2002.

[32] F. Flores-Bazán and R. López, Characterizing Q-matrices beyond L-matrices, *Journal of Optimization Theory and Applications,* 127(2),447-457,2005.

[33] G. S. R. Murthy, T. Parthasarathy and B. Sriparna, On the Solution Sets of Linear Complementarity Problems, *SIAM Journal on Matrix Analysis and Applications,* 21,1229-1235, 2000.

[34] G. S. R. Murthy, T. Parthasarathy and B. Sriparna, Some recent results on the linear complementarity problem, *SIAM Journal on Matrix Analysis and Applications,* 19 898-905, 1998.

[35] G. S. R. Murthy and T. Parthasarathy, Fully copositive matrices, *Mathematical Programming,* 82(3), 401-411, 1998.

[36] Zhang, Li-Li, and Zhi-Ru Ren, A modified modulus-based multigrid method for linear complementarity problems arising from free boundary problems. Applied Numerical Mathematics 164, 89-100, 2021.

[37] Yu, Qian, Chongchao Huang, and Xianjia Wang, A combined homotopy interior point method for the linear complementarity problem. Applied mathematics and computation 179, no. 2, 696-701, 2006.

[38] Xu, Junyan, Qinghuai Liu, and Zhuang Miao, A Infeasible Interior point homotopy method for solving linear complementarity problem. In 2010 3rd International Conference on Advanced Computer Theory and Engineering (ICACTE), vol. 1, pp. V1-417. IEEE, 2010.

[39] Wang, Xiuyu, and Xingwu Jiang, A homotopy method for solving the horizontal linear complementarity problem. Computational and Applied Mathematics 33, no. 1 1-11,2014.

[40] Zhao, Xue, Shugong Zhang, and Qinghuai Liu, A combined homotopy interior point method for the linear complementarity problem. JOURNAL OF INFORMATION & COMPUTATIONAL SCIENCE 7, no. 7, 1589-1594, 2010.

[41] G. Q. Wang, C. J. Yu and K. L. Teo, A full-Newton step feasible interior-point algorithm for $\mathrm{P}_*(\kappa)$-linear complementarity problems, *Journal of Global Optimization,* 59(1), 81-99, 2014.

[42] S. K. Neogy and A. K. Das, On almost type classes of matrices with Q-property, revised version submitted in *Linear and Multilinear Algebra.*

[43] H. Väliaho, Almost copositive matrices, *Linear Algebra and Its applications,* 116, 121-134, 1989.

[44] Hannu. Väliaho, $P^{\star}$-matrices are just sufficient. *Linear Algebra and Its Applications,* 239, 103-108, 1996.

[45] H. Väliaho, Criteria for sufficient matrices. *Linear algebra and its applications,* 233, 109-129, 1996.

[46] den Hertog, Dick, Cornelis Roos, and Tamás Terlaky. The linear complimentarity problem, sufficient matrices, and the criss-cross method. *Linear Algebra and Its Applications,* 187, 1-14, 1993.

[47] H. W. Kuhn, The Hungarian method for the assignment problem, *Naval Research Logistics Quarterly,* 2,83-97, 1955.

[48] H. W. Kuhn and A. W. Tucker, Nonlinear Programming. Proceedings 2nd Berkley symposium on Mathematical Statistics and Probablity, J. Neyman(Ed.), University of California Press, 1951.

[49] Chow, Shui Nee, John Mallet-Paret, and James A. Yorke, Finding zeroes of maps: homotopy methods that are constructive with probability one. Mathematics of Computation 32, no. 143 887-899, 1978.

[50] I. Zang, E. U. Choo and M. Avriel, A note on functions whose local minima are global, *Journal of Optimization Theory and Applications,* 18(4),555-559, 1976.

[51] I. Kaplansky, A contribution to von Neumann's theory of games, *Annals of Mathematics,* 46,474-479, 1945.

[52] J. Renegar, A polynomial-time algorithm based on Newton's method for linear programming, *Mathematical Programming,* 40(1-3),59-93,1988.

[53] J. Gondzio, Interior point methods 25 years later, *European Journal of Operational Research,* 218(3),587-601, 2012.

[54] J. Egerváry, Matrixok Kombinatorikus Tulajdonságairól. Mat. es Fiz. Lapok, 38, 6-28 Translation by H. W. Kuhn, On combinatorial properties of matrices, George Washington University Logistic Papers, 11,1955.

[55] J. J. M. Evers, More with the Lemke complementarity algorithm, *Mathematical Programming,* 15,214-219, 1978.

[56] M. Fiedler and V. Pták, Some generalizations of positive definiteness and monotonicity, *Numer. Math.,* 9,163-172, 1966.

[57] A. Dutta and A. K. Das, Solution of oligopoly market equilibrium problem using modified newton method. arXiv preprint arXiv:2209.07966, 2022.

[58] A. Dutta and A. K. Das, Solution of nonlinear system of equations through homotopy path. arXiv preprint arXiv:2209.01005, 2022.

[59] Murphy, Frederic H., Hanif D. Sherali, and Allen L. Soyster, A mathematical programming approach for determining oligopolistic market equilibrium. Mathematical Programming 24, no. 1, 92-106, 1982.

[60] K. G. Murty, On the number of solutions to the linear complementarity problem and spanning properties of complementarity cones, *Linear Algebra and its Applications,* 5,65-108, 1972.

[61] J. S. Pang and S. C. Lee, A parametric linear complementarity technique for the computation of equilibrium prices in a single commodity spatial model, *Mathematical Programming*, 20, 81-102, 1981.

[62] J. A. Filar and T. A. Schultz, Bilinear programming and structured stochastic games, *JOTA,* 53, 85-104,1987.

[63] J. A. Filar and O. J. Vrieze, *Competitive Markov Decision Processes,* Springer, New York, 1997.

for the solution of the finite-dimensional contact problem, *U.S.S.R. Computational Mathematics and Mathematical Physics* 7,210-214,1967.

[64] J. S. Pang, On Q-matrices, *Mathematical Programming,* 17(1),243-247, 1979.

[65] J. S. Pang, Iterative descent algorithms for a row sufficient linear complementarity problem, *SIAM Journal on Matrix Analysis and Applications,* 12(4),611-624, 1991.

[66] J. von Neumann, A certain zero-sum two-person game equivalent to the optimal assignment problem, *Contributions to the Theory of Games,* 2,5-12, 1953.

[67] Harker, Patrick T., and Jong-Shi Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. Mathematical programming 48, no. 1-3 , 161-220, 1990.

[68] K. G. Ramamurthy, A polynomial algorithm for testing the nonnegativity of principal minors of Z-matrices, *Linear Algebra and its Applications,* 83,39-47, 1986.

[69] Murty, Katta G., and Feng-Tien Yu., Linear complementarity, linear and nonlinear programming. Vol. 3. Berlin: Heldermann, 1988.

[70] Jong-Shi. Pang,Complementarity problems. In Handbook of global optimization, pp. 271-338. Springer, Boston, MA, 1995.

[71] N. H. JOSEPHY, Newton's Method for Generalized Equations, Technical summary report 1965, Mathematics Research Center, University of Wisconsin, Madison, WI, 1979.

[72] Harker, Patrick T., and Baichun Xiao, Newton's method for the nonlinear complementarity problem: A B-differentiable equation approach. Mathematical Programming 48, 339-357, 1990.

[73] George. Isac,Complementarity problems. Springer, 2006.

[74] L. G. Khachiyan, Polynomial algorithms in linear programming, *USSR Computational Mathematics and Mathematical Physics,* 20(1),53-72, 1980.

[75] L. Li, J. Y. Tao, M El Ghami, X. Z. Cai and G. Q. Wang, A new parametric kernel function with a trigonometric barrier term for $P_{(\kappa)}$-linear complementarity problems, *Pacific Journal of Optimization,* 13(2),255-78, 2017.

[76] A. Dutta, R. Jana, and A. K. Das, On Column Competent Matrices and Linear Complementarity Problem. In Proceedings of the Seventh International Conference on Mathematics and Computing, pp. 615-625. Springer, Singapore, 2022.

[77] Markham, L. Thomas, Nonnegative matrices whose inverses are M-matrices. *Proceedings of the American Mathematical Society 36,* no. 2 326-330, 1972.

[78] M. Aganagic and R. W. Cottle, A constructive characterization of Q_0-matrices with nonnegative principal minors, *Mathematical Programming,* 37(2), 223-231, 1987.

[79] M. Galewski, A note on invex problems with nonnegative variable, *European Journal of Operational Research,* 163 (2), 565-568,2005.

[80] M. Kojima, N. Megiddo and Y. Ye, An interior point potential reduction algorithm for the linear complementarity problem, *Mathematical Programming,* 54(1-3),267-279, 1992.

[81] M. Cao and M.C. Ferris, P_c-matrices and the linear complementarity problem, *Linear Algebra and Its Applications,* 246,299-312, 1996.

[82] M. C. Ferris and J. S. Pang, Engineering and economic applications of complementarity problems, *Siam Review,* 39(4),669-713, 1997.

[83] M. Fiedler and V. Pták, On matrices with non-positive off-diagonal elements and positive principal minors, *Czechoslovak Mathematical Journal,* 12(3),382-400, 1962.

[84] Chen, Jein-Shan, and Shaohua Pan, A family of NCP functions and a descent method for the nonlinear complementarity problem. Computational Optimization and Applications 40, no. 3 389-404, 2008.

[85] Ding, Jundi, and Hongyou Yin, A new homotopy method for nonlinear complementarity problems. NUMERICAL MATHEMATICS-ENGLISH SERIES- 16, no. 2,155, 2007.

[86] Kojima, Masakazu, Shinji Mizuno, and Toshihito Noma, Limiting behavior of trajectories generated by a continuation method for monotone complementarity problems. Mathematics of Operations Research 15, no. 4, 662-675, 1990.

[87] Xu, Qing, and Chuangyin Dang, A new homotopy method for solving non-linear complementarity problems. Optimization 57, no. 5,681-689, 2008.

[88] Cordero, Alicia, José L. Hueso, Eulalia Martínez, and Juan R. Torregrosa, Increasing the convergence order of an iterative method for nonlinear systems. Applied Mathematics Letters 25, no. 12, 2369-2374, 2012.

[89] Kojima, Masakazu, Nimrod Megiddo, and Toshihito Noma, Homotopy continuation methods for nonlinear complementarity problems. Mathematics of Operations Research 16, no. 4, 754-774, 1991.

[90] M. J. Tsatsomeros, Principal pivot transforms: properties and applications, *Linear Algebra and Its Applications,* 307,151-165, 2000.

[91] M. J. Todd and Y. Ye, A centered projective algorithm for linear programming, *Mathematics of Operations Research,* 15(3),508-529, 1990.

[92] M. S. Bazaraa, H. D. Sherali and C. M. Shetty, Nonlinear programming: Theory and Algorithms. John Wiley & Sons, 2013.

[93] M. S. Gowda, Affine pseudomonotone mappings and the linear complementarity problem, *SIAM Journal on Matrix Analysis and Application,* 11(3), 373-380,1990.

[94] M. S. Gowda, Pseudomonotone and copositive star matrices, *Linear Algebra and Its Applications,* 113, 107-118, 1989.

[95] M. S. Gowda and J.S. Pang, On solution stability of the linear complementarity problem, *Mathematics of Operations Research,* 17(1),77-83, 1992.

[96] N. Megiddo and M. Shub, Boundary behavior of interior point algorithms in linear programming, *Mathematics of Operations Research,* 14(1),97-146, 1989.

[97] N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, 1978.

[98] N. Karmarkar, A new polynomial-time algorithm for linear programming, *In Proceedings of The Sixteenth Annual ACM Symposium on Theory of Computing ACM,* 302-311, 1984.

[99] Olvi. Mangasarian, and Chunhui Chen., A class of smoothing functions for nonlinear and mixed complementarity problems, 1994.

[100] M. S. Gowda, Applications of degree theory to linear complementarity problems, *Mathematics of Operations Research* 18, 868-879, 1993.

[101] Q. Zhang, Optimality conditions and duality for semi-infinite programming involving B-arcwise connected functions, *Journal of Global Optimization,* 45(4), 615-629, 2009.

[102] R. Bellman, Dynamic Programming and a new formalism in the calculus of variations, *Proceedings of The National Academy of Sciences,* 40, 231-235,1954.

[103] R. Howe and R. Stone, Linear complementarity and the degree of mappings, in (B. C. Eaves, F. J. Gould, H-O. Peitgen, and M. J. Todd eds.) *Homotopy Methods and Global Convergence*, Plenum, New York, 179-124,1983.

[104] R. Burkard, B. Klinz and R. Rudolf, Perspectives of Monge Properties in Optimization, *Discrete Applied Mathematics,* 70(2), 95-16,1996.

[105] R. J. Vanderbei, Affine-scaling for linear programs with free variables, *Mathematical Programming,* 43(1-3),31-44,1989.

[106] R. D. Doverspike, Some perturbation results for the linear complementarity problem, *Mathematical Programming,* 23, 181-192,1982.

[107] R. L. Graves, A principal pivoting algorithm for linear and quadratic programming, *Operations Research,* 15, 482-494, 1967.

[108] R. Jana, A. K. Das and S. Sinha, On Processability of Lemke's Algorithm, Accepted for publication in *Applications and Applied Mathematics: An International Journal(AAM),* 2018.

[109] Abbott, James P., and Richard P. Brent, A note on continuation methods for the solution of nonlinear equations. The ANZIAM Journal 20, no. 2, 157-164, 1977.

[110] R. W. Cottle, The principal pivoting method revisited, *Mathematical Programming,* 48(1), 369-385, 1990.

[111] Jong-Shi. Pang, Two characterization theorems in complementarity theory. Operations research letters 7, no. 1,27-31, 1988.

[112] R. W. Cottle and S. M. Guu, Two characterizations of sufficient matrices, *Linear Algebra and Its applications,* 170,65-74, 1992.

[113] R. W. Cottle, J. S. Pang and R. E. Stone, The linear complementarity problem, Society for Industrial and Applied Mathematics, 2009.

[114] R. W. Cottle, The principal pivoting method revisited, *Mathematical Programming,* 48, 369-385, 1990.

[115] R. W. Cottle, J. S. Pang and V. Venkateswaran, Sufficient matrices and the linear complementarity problem, *Linear Algebra and Its Applications,* 114/115, 231-249, 1989.

[116] M. W. Jeter and W. C. Pye, Some remarks on copositive Q-matrices and on a conjecture of Pang, *Industrial Mathematics,* 35, 75-80, 1985.

[117] S. R. Mohan, S. K. Neogy, and A. K. Das, A note on linear complementarity problems and multiple objective programming, *Mathematical Programming,* 100(2), 339-344, 2004.

[118] R. W. Cottle and R. E. Stone, On the uniqueness of solutions to linear complementarity problems, *Mathematical Programming,* 27, 191-213, 1983.

[119] R. W. Cottle, G. J. Habetler and C. E. Lemke, On classes of copositive matrices, *Linear Algebra and its Applications,* 3, 295-310, 1970.

[120] S. R. Mohan, S. K. Neogy and A. K. Das, On the classes of fully copositive and fully semimonotone matrices, *Linear Algebra and Its Applications,* 323(1), 87-97, 2001.

[121] S. Xu, Notes on sufficient matrices, *Linear Algebra and Its Applications,* 191,1-13, 1993.

[122] S. Zionts, The criss-cross method for solving linear programming problems, *Management Science,* 15(7),426-445, 1969.

[123] R. Jana, A. Dutta, and A. K. Das, More on hidden Z-matrices and linear complementarity problem. Linear and Multilinear Algebra 69, no. 6, 1151-1160, 2021.

[124] R. Jana, A. K. Das, and A. Dutta, On hidden Z-matrix and interior point algorithm. Opsearch 56, no. 4, 1108-1116, 2019.

[125] N. Eagambaram and S. R. Mohan, On strongly degenerate complementary cones and solution rays, *Mathematical Programming,* 44, 77-83, 1989.

[126] N. Eagambaram and S. R. Mohan, On some classes of linear complementarity problems with matrices of order n and rank $(n-1)$, *Mathematics of Operations Research,* 15, 243-257, 1991.

[127] T. H. Chu, On semimonotone matrices with nonnegative principal minors, *Linear algebra and Its applications,* 367,147-154, 2003.

[128] Y. Ye and P. M. Pardalos, A class of linear complementarity problems solvable in polynomial time, *Linear Algebra and Its Applications,* 152, 3-17, 1991.

[129] Y. Ye, Interior point algorithms: theory and analysis (Vol. 44). John Wiley & Sons, 2011.

[130] Y. Ye, An O (n^3L) potential reduction algorithm for linear programming, *Mathematical Programming,* 50(1-3),239-258, 1991.

[131] C. B. Garcia, Some classes of matrices in linear complementarity theory, *Mathematical Programming,* 5,299-310, 1973.

[132] Ingleton, Aubrey William, A probelm in linear inequalities. Proceedings of the London Mathematical Society 3, no. 1 , 519-536, 1966.

[133] Samelson, Hans, Robert M. Thrall, and Oscar Wesler. A partition theorem for Euclidean n-space. Proceedings of the American Mathematical Society 9, no. 5, 805-807, 1958.

[134] Dipti Dubey, and S. K. Neogy, On hidden Z-matrices and the linear complementarity problem. Linear Algebra and its Applications 496, 81-100, 2016.

[135] Olvi L. Mangasarian, Linear complementarity problems solvable by a single linear program. Mathematical Programming 10, no. 1,263-270, 1976.

[136] Jong-Shi. Pang, Hidden Z-matrices with positive principal minors. Linear Algebra and Its Applications 23,201-215, 1979.

[137] A. Dutta, and A. K. Das, Tracing homotopy path for the solution of nonlinear complementarity Problem. arXiv preprint arXiv:2209.00382, 2022.

[138] Teresa H. Chu, Generalizations of the hidden Minkowski property. Linear algebra and its applications 429, no. 8-9 ,2076-2088, 2008.

[139] Song. Xu, On local w-uniqueness of solutions to linear complementarity problem. Linear algebra and its applications 290, no. 1-3, 23-29, 1999.

[140] Kostreva, Michael M., and Malgorzata M. Wiecek, Linear complementarity problems and multiple objective programming. Mathematical Programming 60, no. 1, 349-359, 1993.

[141] Jong-Shi. Pang, Inexact Newton methods for the nonlinear complementarity problem. Mathematical Programming 36, no. 1, 54-71, 1986.

[142] S. Karamardian, Complementarity problems over cones with monotone and pseudomonotone maps, *J. Optim. Theory Appl.,* 18,445-454, 1976.

[143] Pang, Jong-Shi, and Steven A. Gabriel, NE/SQP: A robust algorithm for the nonlinear complementarity problem. Mathematical Programming 60, no. 1, 295-337, 1993.

[144] Stepan. Karamardian, The nonlinear complementarity problem with applications, Part 1. Journal of Optimization Theory and Applications 4, no. 2, 87-98, 1969.

[145] S. Karamardian,The nonlinear complementarity problem with applications, Part 2. Journal of Optimization Theory and Applications 4, no. 3, 167-181, 1969.

[146] Layne T. Watson, Solving the nonlinear complementarity problem by a homotopy method. SIAM Journal on Control and Optimization 17, no. 1, 36-46, 1979.

[147] Jiang, Houyuan, and Liqun Qi, A new nonsmooth equations approach to nonlinear complementarity problems. SIAM Journal on Control and Optimization 35, no. 1 178-193, 1997.

[148] De Luca, Tecla, Francisco Facchinei, and Christian Kanzow, A semismooth equation approach to the solution of nonlinear complementarity problems. Mathematical programming 75, no. 3, 407-439, 1996.

[149] Facchinei, Francisco, and Christian Kanzow, A nonsmooth inexact Newton method for the solution of large-scale nonlinear complementarity problems. Mathematical Programming 76, no. 3, 493-512, 1997.

[150] Kanzow, Christian, and Heiko Pieper, Jacobian smoothing methods for nonlinear complementarity problems. SIAM Journal on Optimization 9, no. 2, 342-373, 1999.

[151] Chen, Jein-Shan, and Shaohua Pan, A family of NCP functions and a descent method for the nonlinear complementarity problem. Computational Optimization and Applications 40, no. 3,389-404, 2008.

[152] Olvi L. Mangasarian, and Mikhail V. Solodov, Nonlinear complementarity as unconstrained and constrained minimization. Mathematical Programming 62, no. 1-3, 277-297, 1993.

[153] Yamashita, N., and M. Fukushima, On stationary points of the implicit Lagrangian for nonlinear complementarity problems. Journal of Optimization Theory and Applications 84, 653-663, 1995.

[154] Kojima, Masakazu, Nimrod Megiddo, and Toshihito Noma, Homotopy continuation methods for nonlinear complementarity problems. Mathematics of Operations Research 16, no. 4, 754-774, 1991.

[155] Katta G. Murty, Note on a Bard-type scheme for solving the complementarity problem. Opsearch 11, no. 2-3, 123-130, 1974.

[156] Michael M. Kostreva, Block pivot methods for solving the complementarity problem. Linear Algebra and its applications 21, no. 3, 207-215, 1978.

[157] M. KOJIMA, T. NOMA, AND A. YOSHISE, Global convergence in interior-point algorithms, Math. Programming, 65, pp. 43–72, 1994.

[158] Paul. Tseng, An infeasible path-following method for monotone complementarity problems. SIAM Journal on Optimization 7, no. 2, 386-402, 1997.

[159] Christian. Kanzow, Nonlinear complementarity as unconstrained optimization. Journal of Optimization theory and applications 88, 139-155, 1996.

[160] Chen, Bintong, and Patrick T. Harker, Smooth approximations to nonlinear complementarity problems. SIAM Journal on Optimization 7, no. 2, 403-420, 1997.

[161] Augustin A. Cournot, Researches into the Mathematical Principles of the Theory of Wealth, English translation of French original. New York: Kelly, 1838.

[162] Nash Jr, Equilibrium points in n-person games. Proceedings of the national academy of sciences 36, no. 1,48-49, 1950 .

[163] John. Nash, Non-cooperative games. In Essays on Game Theory, pp. 22-33. Edward Elgar Publishing, 1996.

[164] Christian. Kanzow, Some equation-based methods for the nonlinear complementarity problem. Optimization Methods and Software 3, no. 4, 327-340, 1994.

[165] Gabay, D., and H. Moulin, On the uniqueness and stability of Nash equilibria in noncooperative games, Applied Stochastic Control in Econometrics and Management Science. A. Bensoussan, P. Kleindorfer, CS Tapiero , 271-294, 1980.

[166] Salant, Stephen W., Imperfect competition in the international energy market: a computerized Nash-Cournot model. Operations research 30, no. 2,252-280, 1982.

[167] Marcotte, Patrice, and Jean-Pierre Dussault, A note on a globally convergent Newton method for solving monotone variational inequalities. Operations Research Letters 6, no. 1, 35-42, 1987.

[168] M. FUKUSHIMA, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, Math. Programming, 58, pp.99-110, 1992.

[169] Auchmuty, Giles., Variational principles for variational inequalities. Numerical Functional Analysis and Optimization 10, no. 9-10, 863-874, 1989.

[170] Facchinei, Francisco, and Christian Kanzow, A nonsmooth inexact Newton method for the solution of large-scale nonlinear complementarity problems. Mathematical Programming 76, no. 3, 493-512,1997.

[171] Sayevand, K., and H. Jafari, On systems of nonlinear equations: some modified iteration formulas by the homotopy perturbation method with accelerated fourth- and fifth-order convergence. Applied Mathematical Modelling 40, no. 2, 1467-1476,2016.

[172] Noor, Muhammad Aslam, and Muhammad Waseem, Some iterative methods for solving a system of nonlinear equations. Computers & Mathematics with Applications 57, no. 1, 101-106, 2009.

[173] Andreas. Fischer, A special Newton-type optimization method. Optimization 24, no. 3-4 , 269-284, 1992.

[174] Jafari, Hossein, and Varsha Daftardar-Gejji, Revised Adomian decomposition method for solving a system of nonlinear equations. Applied Mathematics and Computation 175, no. 1, 1-7, 2006.

[175] Yamamura, Kiyotaka, Hitomi Kawata, and Ai Tokue, Interval solution of nonlinear equations using linear programming. BIT Numerical Mathematics 38, no. 1, 186-199, 1998.

[176] Homeier, Herbert HH., A modified Newton method with cubic convergence: the multivariate case. Journal of Computational and Applied Mathematics 169, no. 1 , 161-169, 2004.

[177] Kou, Jisheng, Yitian Li, and Xiuhua Wang, A modification of Newton method with third-order convergence. Applied Mathematics and Computation 181, no. 2, 1106-1111, 2006.

[178] Ehrlich, Louis W., A modified Newton method for polynomials. Communications of the ACM 10, no. 2, 107-108, 1967.

[179] A. Dutta, R. Jana, and A. K. Das, Bounded Homotopy Path Approach to Find the Solution of Linear Complementarity Problems. arXiv preprint arXiv:2108.06497, 2021.

[180] Hamming, Richard Wesley., Stable predictor-corrector methods for ordinary differential equations. Journal of the ACM (JACM) 6, no. 1, 37-47, 1959.

[181] A. Dutta, and A. K. Das, More on homotopy continuation method and discounted zero-sum stochastic game with ARAT structure. arXiv preprint arXiv:2203.08561, 2022.

[182] Eaves, B. Curtis, and Romesh Saigal, Homotopies for computation of fixed points on unbounded regions. Mathematical Programming 3, no. 1, 225-237, 1972.

[183] Layne T. Watson, Globally convergent homotopy methods: a tutorial. Applied Mathematics and Computation 31, 369-396, 1989.

[184] Watson, Layne T., and Raphael T. Haftka, Modern homotopy methods in optimization. Computer Methods in Applied Mechanics and Engineering 74, no. 3 , 289-305, 1989.

[185] Kojima, Masakazu, and Susumu Shindo, Extension of Newton and quasi-Newton methods to systems of PC^1 equations. Journal of the Operations Research Society of Japan 29, no. 4, 352-375, 1986.

[186] Olvi L. Mangasarian, Equivalence of the complementarity problem to a system of nonlinear equations. SIAM Journal on Applied Mathematics 31, no. 1, 89-92, 1976.

[187] Kojima, Masakazu, Nimrod Megiddo, Toshihito Noma, and Akiko Yoshise, A unified approach to interior point algorithms for linear complementarity problems: A summary. Operations Research Letters 10, no. 5, 247-254, 1991.

[188] Kappel, Nicholas W., and Layne T. Watson, Iterative algorithms for the linear complementarity problem. International journal of computer mathematics 19, no. 3-4, 273-297, 1986.

[189] Uwe. Schäfer, On the modulus algorithm for the linear complementarity problem. Operations Research Letters 32, no. 4,350-354, 2004.

[190] Hadjidimos, Apostolos, and M. Tzoumas, Nonstationary extrapolated modulus algorithms for the solution of the linear complementarity problem. Linear Algebra and its Applications 431, no. 1-2, 197-210, 2009.

[191] Hadjidimos, Apostolos, M. Lapidakis, and M. Tzoumas, On iterative solution for linear complementarity problem with an H_+-matrix. SIAM Journal on Matrix Analysis and Applications 33, no. 1 , 97-110, 2012.

[192] Prasenjit Mondal, S. Sinha, S. K. Neogy, and A. K. Das, On discounted AR–AT semi-Markov games and its complementarity formulations. International Journal of Game Theory 45, no. 3 , 567-583, 2016.

[193] Patrick T. Harker, A variational inequality approach for the determination of oligopolistic market equilibrium. Mathematical Programming 30, 105-111, 1984.

[194] Patrick T. Harker, Alternative models of spatial competition. Operations research 34, no. 3, 410-425, 1986.

[195] S. R. Mohan, S. K. Neogy and T. Parthasarathy, Linear complementarity and discounted polystochastic game when one player controls transitions, in *Complementarity and Variational Problems,* 284-294, 1997.

[196] S. R. Mohan, S. K. Neogy and T. Parthasarathy, Linear complementarity and the irreducible polystochastic game with the average cost criterion when one player

controls transitions, in *Game Theoretical Applications to Economics and Operations Research,* 153-170, 1997.

algorithm via a bimatrix game to a single controller non-zerosum stochastic game, *Math. Programming,* 59, 249-259,1993.

[197] S. R. Mohan and T. E. S. Raghavan, An algorithm for discounted switching control games, *OR Spektrum,* 9, 41-45,1987.

[198] S. K. Neogy, and A. K. Das, Linear complementarity and two classes of structured stochastic games. Operations Research with Economic and Industrial Applications: Emerging Trends, eds: SR Mohan and SK Neogy, Anamaya Publishers, New Delhi, India, 156-180, 2005.

[199] S. K. Neogy, A. K. Das, S. Sinha, and A. Gupta, On a mixture class of stochastic game with ordered field property. In Mathematical programming and game theory for decision making, pp. 451-477. 2008.

Pivoting algorithms for some classes of stochastic games: a survey, *International Game Theory Review,* 3,253-281,2001.

[200] S. R. Mohan, S. K. Neogy, T. Parthasarathy and S. Sinha, Vertical linear complementarity and discounted zero-sum stochastic games with ARAT structure, *Mathematical Programming Series A,* 637-648,1999.

[201] S. R. Mohan, S. K. Neogy, and A. K. Das, A note on linear complementarity problems and multiple objective programming. Mathematical programming 100, no. 2, 339-344, 2004.

[202] G. S. R. Murthy and T. Parthasarathy, Some properties of fully semimonotone matrices, *SIAM Journal on Matrix Analysis and Applications,* 16,1268-1286, 1995.

[203] C. Olech, T. Parthasarathy and G. Ravindran, Almost N-matrices in linear complementarity, *Linear Algebra and its Applications,* 145,107-125, 1991.

[204] Habetler, G. J., and M. M. Kostreva, On a direct algorithm for nonlinear complementarity problems. SIAM Journal on Control and Optimization 16, no. 3, 504-511, 1978.

[205] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[206] J. S. Pang, I. Kaneko, and W. P. Hallman, On the solution of some (parametric) linear complementarity problems with application to portfolio selection, structural engineering and actuarial graduation, *Mathematical Programming,* 16, 325-347, 1979.

[207] W. C. Pye, Almost P_0-matrices, *Mathematical Programming,* 57,439-444,1992.

[208] T. E. S. Raghavan and J. A. Filar, Algorithms for stochastic games, a survey, *Zietch. Oper. Res.,* 35, 437-472, 1991.

[209] T. E. S. Raghavan, S. H. Tijs and O. J. Vrieze, On stochastic games with additive reward and transition structure, *JOTA,* 47,375-392, 1985.

discounted switching controller stochastic games, *JOTA,* 73,89-99,1992.

[210] Zheng, Ning, and Jun-Feng Yin, Accelerated modulus-based matrix splitting iteration methods for linear complementarity problem. Numerical Algorithms 64, no. 2, 245-262, 2013.

[211] Zheng, Ning, and Jun-Feng Yin, Convergence of accelerated modulus-based matrix splitting iteration methods for linear complementarity problem with an H+-matrix. Journal of Computational and Applied Mathematics 260, 281-293, 2014.

[212] Zheng, Hua, Wen Li, and Seakweng Vong, A relaxation modulus-based matrix splitting iteration method for solving linear complementarity problems. Numerical Algorithms 74, no. 1, 137-152, 2017.

[213] Dai, Ping-Fan, Jicheng Li, Jianchao Bai, and Jinming Qiu, A preconditioned two-step modulus-based matrix splitting iteration method for linear complementarity problem. Applied Mathematics and Computation 348,542-551, 2019.

[214] Bai, Zhong-Zhi, On the convergence of the multisplitting methods for the linear complementarity problem. SIAM Journal on Matrix Analysis and Applications 21, no. 1, 67-78, 1999.

[215] Dong, Jun-Liang, and Mei-Qun Jiang, A modified modulus method for symmetric positive-definite linear complementarity problems. Numerical Linear Algebra with Applications 16, no. 2, 129-143, 2009.

[216] Cui, Lu-Bin, Cui-Xia Li, and Shi-Liang Wu, The relaxation convergence of multi-splitting AOR method for linear complementarity problem. Linear and Multilinear Algebra 69, no. 1, 40-47, 2021.

[217] Liu, Shumi, Hua Zheng, and Wen Li. A general accelerated modulus-based matrix splitting iteration method for solving linear complementarity problems. Calcolo 53, no. 2,189-199, 2016.

[218] L. S. Shapley, Stochastic games, *Proc. Nat. Acad. Sci. USA.,* 39,1095-1100, 1953.

[219] M. J. Sobel, Noncooperative stochastic games, *Ann. Math. Statist.,* 42, 1930-1935, 1971.

[220] R. E. Stone, Geometric aspects of the linear complementarity problem, Ph.D. Thesis, Department of Operations Research, Stanford University, 1981.

[221] H. Väliaho, Criteria for copositive matrices, *Linear Algebra and its Applications,* 81,19-34, 1986.

[222] H. Väliaho, Almost copositive matrices, *Linear Algebra and its Applications,* 116,121-134, 1989.

[223] H. Väliaho, Criteria for sufficient matrices, *Linear Algebra and its Applications,* 233,109-129, 1996.

[224] O. J. Vrieze, A finite algorithm for the switching controller stochastic game, *OR Spektrum,* 5,15-24, 1983.

[225] A. Dutta, and A. K. Das, On some properties of k-type block matrices in the context of complementarity problem. In Mathematics and Computing: ICMC 2022, Vellore, India, January 6–8, pp. 143-154. Singapore: Springer Nature Singapore, 2023.

[226] T. E. S. Raghavan, Stochastic games—An overview. Stochastic Games and Related Topics, 1-9, 1991.

[227] S. Sinha, Contribution to the theory of stochastic games. PhD diss., Indian Statistical Institute-Kolkata, 1989.

[228] Solan, Eilon, and Nicolas Vieille. Stochastic games. Proceedings of the National Academy of Sciences 112, no. 45, 13743-13746, 2015.

[229] Dantzig, G. B., and R. W. Cottle, Complementary pivot theory of mathematical programming. Mathematics of the decision sciences, part 1,115-136, 1968.

[230] C. B. Garcia, Some classes of matrices in linear complementarity theory. Mathematical Programming 5, no. 1, 299-310, 1973.

[231] Michael M. Kostreva, Direct algorithms for complementarity problems. PhD diss., Rensselaer Polytechnic Institute, Troy, NY, 1976.

[232] Goeree, Jacob K., and Charles A. Holt., Stochastic game theory: For playing games, not just for doing theory. Proceedings of the National Academy of sciences 96, no. 19, 10564-10567, 1999.

[233] Flesch, János, Frank Thuijsman, and Okko Jan Vrieze, Stochastic games with additive transitions. European Journal of Operational Research 179, no. 2, 483-497, 2007.

[234] Cottle, Richard W., and George B. Dantzig, A generalization of the linear complementarity problem. Journal of Combinatorial Theory 8, no. 1, 79-90, 1970.

[235] S. R. Mohan, S. K. Neogy, and T. Parthasarathy, Pivoting algorithms for some classes of stochastic games: A survey. International Game Theory Review 3, no. 02n03, 253-281, 2001.

[236] S. R. Mohan, S. K. Neogy, and R. Sridhar, The generalized linear complementarity problem revisited. Mathematical Programming 74, no. 2, 197-218, 1996.

[237] S. K. Neogy, A. K. Das, and Abhijit Gupta, Generalized principal pivot transforms, complementarity theory and their applications in stochastic games. Optimization Letters 6, no. 2, 339-356, 2012.