Unification of Different Cosmological Ages in Modified Gravity Theories and Study of Warm Inflation

by

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Unification of Different Cosmological Ages in Modified Gravity Theories and Study of Warm Inflation**" submitted by **Sri Akash Bose** who got his name registered on 08.08.2019 [Index No. 6/19/Maths./26] for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under my supervision and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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DECLARATION BY THE AUTHOR

I hereby declare that the thesis is based on my own work carried out at the Department of Mathematics, Jadavpur University, Kolkata - 700032, India. Also, I declare that no part of it has not been submitted for any degree/diploma/fellowship/some other qualification at any other university or institution.

The author has produced all the figures presented in this thesis using Maple, Mathematica, and CLASS software. The thesis has been checked several times with extreme care to free it from all discrepancies and typos. Even then vigilant readers may find some mistakes, and several portions of this thesis may seem unwarranted, mistaken, or incorrect. The author takes the sole responsibility for these unwanted errors which have resulted from his inadequate knowledge of the subject or escaped his notice.

Finally, I state that, to the best of my knowledge, all the assistance taken to prepare this thesis has been properly cited and acknowledged.

Akash Bose. 19.01.2023

Akash Bose

To

My parents

Ratna Bose

and

Debasish Bose

ABSTRACT

The thesis consists of nine chapters and it is divided into two parts. The first part consists of five chapters and it is related to the various cosmological solutions in different modified gravity theories. The first chapter contains an introduction to relativistic cosmology while the next four chapters deal with the research work in various modified gravity theories. The second part consists of three chapters and it deals with warm inflation. The first chapter in this part (i.e. chapter 6) gives an overview of inflation in a nutshell. Then the following two chapters (i.e. chapter 7 & 8) contain the research work on warm inflation. Finally, a brief discussion and future prospect of the research work has been presented in chapter 9. Briefly, this thesis describes different cosmological eras in various modified gravity theories together with a continuous cosmic evolution of the universe.

Recent observations have shown that our universe is going through an accelerated expanding phase. To support this observation, cosmologists have proposed various modified gravity theories. In chapter 2 & 3, two such modified gravity theories, namely Einstein-Cartan-kibble-Sciama (ECKS) gravity, and $f(T)$ gravity theory, have been extensively studied. In both these gravity theories, it has been shown that a continuous cosmic evolution from inflationary era to present time accelerating phase is possible. In chapter 4, the emergent scenario in Hořava-Lifsitz (HL) gravity has been studied. The nature of the fluid required to describe the emergent phase in HL gravity has been examined. Also, the thermodynamic analysis of these models has been presented in these chapters. In chapter 5, a cosmological model in $f(R,T)$ gravity model has been proposed. The detailed cosmological solutions of the model have been studied and different model parameters are estimated through the observational data.

In the early epoch, there was a huge expansion (inflation) of the universe which was fully satisfactory with observations. The idea of reheating phase which is required after the end of cold inflation to make smooth transition to matter dominated phase has been modified by introducing the idea of warm inflation where the inflationary dynamics is governed by the interaction between radiation and inflaton field. In chapter 7, the warm inflationary scenario has been studied in terms of particle creation mechanism in the context of non-equilibrium thermodynamics. In chapter 8, a detailed study of warm inflation in fractal gravity has been presented. Here various models have been presented as well as both the weak and strong dissipative regimes have been studied. Finally, the thesis ends with brief discussions and guidelines for possible future work in chapter 9.

LIST OF PUBLICATIONS

The work of this thesis has been carried out at the Department of Mathematics, Jadavpur University, Kolkata- 700032, India. The thesis is based on the following published papers:

- Chapter 2 has been published as *"Homogeneous and isotropic space-time, modified torsion field and complete cosmic scenario"*, **A. Bose** and S. Chakraborty, **Eur. Phys. J. C 80**, no.3, 205 (2020), DOI : 10.1140/epjc/s10052-020-7771-7.

- Chapter 3 has been published as *"Cosmic evolution in $f(T)$ gravity theory"*, **A. Bose** and S. Chakraborty, **Mod. Phys. Lett. A 35**, no.36, 2050296 (2020), DOI : 10.1142/S021773232050296X.

- Chapter 4 has been published as *"Does emergent scenario in Hořava–Lifshitz gravity demand a ghost field?"*, **A. Bose** and S. Chakraborty, **Phys. Dark Univ. 30**, 100740 (2020), DOI : 10.1016/j.dark.2020.100740.

- Chapter 5 has been published as *"Analytic Solutions and Observational support: A study of $f(R,T)$ gravity with $f(R,T) = R + h(T)$"*, **A. Bose**, G. Sardar, and S. Chakraborty, **Phys. Dark Univ. 37**, 101087 (2022), DOI : 10.1016/j.dark.2022.101087.

- Chapter 7 has been published as *"Is warm inflation quasi-stable?"*, **A. Bose** and S. Chakraborty, **Phys. Dark Univ. 35**, 100938 (2022), DOI : 10.1016/j.dark.2021.100938.

- Chapter 8 has been published as *"Does fractal universe favour warm inflation: Observational support?"*, **A. Bose** and S. Chakraborty, **Nucl. Phys. B 978**, 115767 (2022), DOI : 10.1016/j.nuclphysb.2022.115767.

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Akash Bose

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CHAPTER 1

INTRODUCTION

1.1 Prelude

মহাবিশ্বে মহাকাশে মহাকাল মাঝে
আমি মানব একাকী ভ্রমি বিস্ময়ে
— রবীন্দ্রনাথ ঠাকুর

In light of the aforementioned lyrics of the Nobel Prize-winning poet, one may say that, among all the living creatures in the world, Man is lonely traversing this universe with awe. Since ancient times, they have been fascinated by looking at the night sky where thousands of luminous objects move throughout years and years which leads to the idea of the geocentric universe. But Man is the most intelligent living being in the world. They believe in logic as well as inventions, and as a result, science progresses. The discovery of the telescope by Galileo (1609), the ground-breaking step in cosmology and astronomy, gave us the idea of the heliocentric universe. After that, both theoretical and experimental physics moves forward. The next remarkable thing was Newton's law of gravity (1687) which nicely describes planetary motion and numerous other things. But the problem arose when a series of experiments to find out the existence of the luminiferous aether, a supposed medium permeating space that was thought to be the carrier of light waves, had been performed (Michelson–Morley experiment (1887)). They were astounded to find no significant difference between the speed of light in the direction of movement through the presumed aether, and the speed at right angles. Everybody concluded that they did not have a sophisticated setup. Then Einstein came up with his Special Theory of Relativity (1905) which solved this problem. But at that time he could not afford the gravitational force for his relativity theory. 10 years later he eventually overcame this and introduced his General Theory of Relativity (1915) which is one of the cornerstones of Modern cosmology.

1.2 Ingredients of Modern Cosmology

The word "cosmology" was first used in 1656 and is derived from the Greek words "kosmos" which means "world" and "logia" which means "study of". Theoretical astrophysicist David N. Spergel has portrayed cosmology as "historical science". Modern cosmology is based on three basic ingredients:

1.2.1 Cosmological Principle

According to the cosmological principle, the universe is homogeneous and isotropic on large scale (i.e. in cosmic scale: $\approx$ 100 Mpc[1]). Homogeneity means there is no preferred point in space-time i.e. if we consider a part of the universe, then the number of galaxies in that part is in the vicinity of the number in another part with the same volume at any given time. Isotropy means that all the directions are identical i.e. the space looks similar regardless of in what direction we look. This implies that space-time must be spherically symmetric. The greatest support in favour of isotropy is Cosmic Microwave Background Radiation (CMBR) according to which the universe is currently in a bath of thermal radiation having 2.73 K with anisotropy $\mathcal{O}(10^{-5})$. On the other hand, the homogeneity of the universe is partially supported by the counts of the galaxies and the linearity of the Hubble law.

1.2.2 Weyl's Postulate

Weyl's postulate states that the cosmic fluid is such a fluid that it has a unique direction of flow i.e. it has a unique velocity. Particles move in a time-like path (since the time-like path does not intersect each other). This property of the particle is valid only for the perfect fluid. Hence, the cosmic particle should be perfect fluid in nature.

1.2.3 Einstein's General theory of Relativity (GR)

Einstein's GR is nothing but a relation between the geometry of space-time and the matter component of the Universe. The Einstein field equation is given by

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

where $\kappa = 8\pi G$ is the gravitational constant (assuming velocity of light c to be unity), G is the Newton's gravitational constant having value $G = 6.673\times10^{-11}$ Nm^{2}kg^{-1}, $T_{\mu\nu}$ is the Energy-momentum tensor of the matter component and the Einstein tensor $G_{\mu\nu}$ describes the geometry of the space-time which is defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{1.2}$$

[1] 1 Mpc $\approx 3.26\times10^{6}$ light years $\approx 3.08\times10^{19}$ Km

where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar respectively. $R_{\mu\nu}$ is obtained by contracting the Riemann curvature tensor $R^{\sigma}{}_{\lambda\mu\nu}$ and its further contraction gives the Ricci scalar as

$$R_{\mu\nu} = R^{\sigma}{}_{\sigma\mu\nu} \quad \text{and} \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{1.3}$$

The above field equation (1.1) can be obtained by varying the Einstein-Hilbert action given by

$$\mathcal{S} = \frac{1}{2\kappa} \int R\sqrt{-g}\mathrm{d}^4x + \int \sqrt{-g}\mathrm{d}^4x \mathcal{L}_M \tag{1.4}$$

with respect to the metric tensor $g_{\mu\nu}$.

1.3 Modern Cosmology

According to the cosmological principle, the space-time is characterized by the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right], \tag{1.5}$$

where $a(t)$ is the scale factor with cosmic time t. Here (r, θ, ϕ) are co-moving spherical polar co-ordinates, K is the constant of curvature of the spatial part of the space-time which may have values 0, +1, −1 for flat, closed, and open universe respectively. This is termed as Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time [1, 2, 3, 4].

Based on Weyl's Postulate, the energy-momentum tensor of cosmic fluid (perfect fluid) is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \text{ with } u_\mu u^\mu = -1, \tag{1.6}$$

where ρ and p are the energy density and thermodynamic pressure of the cosmic fluid respectively and u^μ is the four-velocity of the fluid.

Using equations (1.1), (1.5), and (1.6), Einstein's field equations for FLRW line element can be written in the form of, so-called Friedmann equations as

$$3\left(H^2 + \frac{K}{a^2}\right) = 8\pi G\rho, \tag{1.7}$$

$$2\left(\dot{H} - \frac{K}{a^2}\right) = -8\pi G(p+\rho). \tag{1.8}$$

where $H = \dfrac{\dot{a}}{a}$ is termed as Hubble parameter and 'overdot' represents the differentiation with respect to cosmic time t.

From the energy conservation equation, the energy-momentum tensor is conserved i.e.

$$T^{\mu}_{\nu;\mu} = 0, \tag{1.9}$$

which gives

$$\dot{\rho} + 3H(p + \rho) = 0. \tag{1.10}$$

It can be shown that equations (1.7), (1.8), and (1.10) are not linearly independent. Rather, any one of these three equations can be obtained from the remaining two.

Equation (1.7) can also be written as

$$\Omega(t) = 1 + \frac{K}{a^2 H^2}, \tag{1.11}$$

where $\Omega(t) = \dfrac{\rho}{\rho_c}$ is the dimensionless density parameter, with $\rho_c = \dfrac{3H^2}{8\pi G}$, the critical density. The spatial geometry of the Universe is determined by the matter distribution as follows:

$$\Omega(t) > 1 \implies \rho > \rho_c \implies K = +1. \tag{1.12}$$
$$\Omega(t) = 1 \implies \rho = \rho_c \implies K = 0. \tag{1.13}$$
$$\Omega(t) < 1 \implies \rho < \rho_c \implies K = -1. \tag{1.14}$$

Observational results from CMB indicate that at present $\Omega(t) \simeq 1$ [5], i.e. the geometry of the present universe is nearly spatially flat. So, without any loss of generality, one may assume the spatially flat ($K = 0$) space-time for simplicity.

From equations (1.7) and (1.8), one may write the amount of acceleration as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho), \tag{1.15}$$

This equation is known as the Raychaudhuri equation in FLRW space-time. In this context, one can define the deceleration parameter

$$q \equiv \frac{a\ddot{a}}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right), \tag{1.16}$$

which characterizes the dynamics of the universe. The sign of the above two parameters indicates the nature of the expansion of the universe.

(i) $\ddot{a} > 0 \implies q < 0 \implies (3p + \rho) < 0 \implies$ universe accelerates.

(ii) $\ddot{a} < 0 \implies q > 0 \implies (3p + \rho) > 0 \implies$ universe decelerates.

Hence, if the universe is assumed to be filled with perfect fluid; then Strong Energy Condition (SEC) ($\equiv (3p+\rho) > 0$) satisfies, implying the universe decelerates. Now assuming a perfect fluid with constant barotropic equation of state

$$p = \omega\rho, \tag{1.17}$$

the cosmological solution for the flat ($K = 0$) FLRW model can be written as

$$\rho = \rho_0 a^{-3(1+\omega)}, \tag{1.18}$$

$$a = a_0 \left(t - t_0\right)^{\frac{2}{3(1+\omega)}}, \tag{1.19}$$

$$H = \frac{2}{3(1+\omega)(t-t_0)}, \tag{1.20}$$

where ρ_0, t_0, a_0 are integration constants. One must note that $\omega \neq -1$ for this solution.

Different values of ω represent different cosmic eras as follows:

(i) $\omega = 0$ (Dust era): $\rho \propto a^{-3}$, $a \propto (t-t_0)^{\frac{2}{3}}$, $H = \dfrac{2}{3(t-t_0)}$.

(ii) $\omega = \dfrac{1}{3}$ (Radiation era): $\rho \propto a^{-4}$, $a \propto (t-t_0)^{\frac{1}{2}}$, $H = \dfrac{1}{2(t-t_0)}$.

(ii) $\omega = 1$ (Stiff fluid era): $\rho \propto a^{-6}$, $a \propto (t-t_0)^{\frac{1}{3}}$, $H = \dfrac{1}{3(t-t_0)}$.

1.4 Late-time acceleration

In the late '90s, two independent research teams, namely, "High-redshift Supernovae Search Team" (HSST) and "Supernovae Cosmology Project Team" (SCPT) were investigating type Ia Supernovae (SNIa). Both the teams, Riess *et al.* [6] from HSST, and Perlmutter *et al.* [7] from SCPT independently reported that our Universe has been going through an accelerated expanding phase as their conclusion of the investigations. After that, Cosmic microwave background radiation (CMBR) [5, 8, 9, 10, 11, 12, 13, 14, 15], Large Scale Structure (LSS) [16, 17, 18, 19], Baryon Acoustic Oscillation (BAO) [20, 21, 22], weak lensing [23, 24, 25], galaxy cluster number counts [26], gravitational waves detection [27, 28] confirmed this present acceleration of the Universe. The standard cosmology in the framework of GR cannot match the overwhelming abundance of observational evidences of cosmic speedup. The cosmologists are trying to explain this observed phenomenon in two ways:

(i) Modification of matter components,

(ii) Modification of Einstein gravity.

1.4.1 Acceleration due to matter modification

To accommodate the observational results, cosmologists have modified Weyl's postulate, i.e. they have considered that the cosmic fluid is no longer the perfect fluid; rather they consider such a fluid which violates the SEC (i.e. $3p + \rho < 0$) so that the universe accelerates which is termed as exotic matter (dark energy). For such a fluid with barotropic equation of state ω, one has

$$\omega < -\frac{1}{3}. \tag{1.21}$$

1.4.1.1 Cosmological constant

Cosmological constant Λ is the simplest dark energy candidate. Einstein himself introduced this to counterbalance the effect of gravity and achieve a static universe, a notion that was the prevalent idea at the time. But later when Edwin Hubble (1929) discovered that the universe is expanding, Einstein rejected this (1931) by remarking "the biggest blunder in my entire life". Much later it was reinterpreted as the vacuum energy in quantum mechanics. Finally, after 1998, it was again brought back to explain the accelerating phase [29, 30]. In general, a positive constant value has been chosen for Λ.

Introducing Λ, the action can be modified as

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^4x(R - 2\Lambda) + \int \sqrt{-g}\mathrm{d}^4x\mathcal{L}_M, \tag{1.22}$$

and the modified Einstein field equation can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + \frac{\Lambda}{8\pi G}g_{\mu\nu}\right). \tag{1.23}$$

Now in the background of FLRW space-time, the Friedmann equations are modified as

$$3\left(H^2 + \frac{K}{a^2}\right) = 8\pi G\rho + \Lambda, \tag{1.24}$$

$$2\left(\dot{H} - \frac{K}{a^2}\right) = -8\pi G(p + \rho). \tag{1.25}$$

and the acceleration equation can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho) + \frac{\Lambda}{3}. \tag{1.26}$$

Equation (1.26) shows that the cosmological constant acts as a repulsive force against gravity. Hence, one may write the energy density (ρ_Λ) and pressure (p_Λ) of the cosmological constant as

$$\rho_\Lambda = \frac{\Lambda}{8\pi G},\ p_\Lambda = -\frac{\Lambda}{8\pi G},\ \text{i.e., } p_\Lambda = -\rho_\Lambda, \tag{1.27}$$

which shows that in this case the equation of state parameter

$$\omega_\Lambda = -1, \tag{1.28}$$

i.e. it violates SEC. Thus, the cosmological constant may be a viable candidate for dark energy. Also, it has been found that Λ together with cold dark matter, i.e., ΛCDM model is consistent with the most number of observations. Although recently, it was shown that the ΛCDM model may also suffer from the age problem [31]. Now two major problems of the cosmological constant will be discussed.

Fine-tuning or the cosmological constant problem

In 1960's, Zeldovich pointed out if one assumes the source of the cosmological constant to be the vacuum energy density, then the cosmological constant suffers from the fine-tuning problem [32, 33]. From observations, it is known that Λ is of the order of the present value of the Hubble parameter H_0, i.e.,

$$\Lambda \approx H_0^2 \simeq 10^{-84}\ \text{GeV}^2. \tag{1.29}$$

The equation (1.29) corresponds to a critical energy density

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \simeq 10^{-47}\ \text{GeV}^4. \tag{1.30}$$

On the other hand, according to particle physics, the vacuum energy density (ρ_{vac}) can be evaluated by the summation of zero point energies of quantum fields with mass m and momentum l, given by

$$\rho_{\text{vac}} = \frac{1}{4\pi^2}\int_0^\infty l^2\sqrt{l^2+m^2}\ \mathrm{d}l, \tag{1.31}$$

which ensures a diverging nature for high frequency. Now considering a cut-off at Planck scales, one gets $\rho_{\text{vac}} \simeq 10^{76}$ GeV^4, which is 10^{123} times larger than the observed value $\rho_\Lambda \simeq 10^{-47}$ GeV^4. The discrepancy between the predicted value and the observed values of the cosmological constant has been called "the worst theoretical prediction in the history of physics". This is termed as the cosmological constant problem.

Coincidence problem

It is very difficult to explain why the matter density is of the same order with the vacuum energy density around the present time, albeit the two components evolve differently with the cosmic time. It seems that we are living at an exceptionally special moment in cosmic history. In other words, one can say why the accelerating Universe just started now. Why are we so privileged? Both in the very early Universe and in the far-future Universe, these densities differ by significant orders of magnitude. This so-called "Cosmological Coincidence Problem", was first formulated in Steinhardt's contribution to the proceedings of a conference celebrating the 250th anniversary of Princeton University. Since then, "the coincidence problem" has become a common jargon in cosmological literature [34]. The matter density $\rho_m \propto (1+z)^3$ coincides with the vacuum energy density Ω_Λ^0 at

$$z_{\rm coin} = \left(\frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}\right)^{\frac{1}{3}}, \tag{1.32}$$

where $\Omega_\Lambda^0 = 0.7$ amounts to $z_{\rm coin} \simeq 0.3$, at which the accelerating Universe begins.

1.4.1.2 Scalar field models

The cosmological constant corresponds to a fluid with a constant equation of state $\omega = -1$, and it suffers from fine-tuning and coincidence problems. To conquer these two problems, cosmologists started to ponder on the alternative dark energy candidates. They considered time-varying equation of state parameter for the dark energy which can evolve slowly to zero. So it gives us an answer to the cosmological constant problem. Thus dynamical dark energy models were introduced. The simplest models of dynamical dark energy are based on the idea of a classical scalar field, which generally appear in particle physics and string theory. So far cosmologists have proposed several scalar field dark energy models, namely, Quintessence, K-essence, Tachyon, Phantom, etc. We shall give a brief description of these models in the following:

A. Quintessence

The term "quintessence" was coined by Caldwell *et al.* in 1998. Quintessence is characterized by a canonical scalar field ϕ with a potential $V(\phi)$ minimally coupled to gravity [35, 36, 37, 38]. The action for Quintessence is described by

$$\mathcal{S} = \int \sqrt{-g}\,\mathrm{d}^4x \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{1.33}$$

By varying the action (1.33) with respect to $g^{\mu\nu}$, the energy-momentum tensor for the field can be obtained as

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}}{\delta g^{\mu\nu}} \\ &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] \end{aligned} \tag{1.34}$$

In the background of flat $(K=0)$ FLRW space-time, the pressure p_ϕ and energy density ρ_ϕ of the scalar field are given by

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.35}$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.36}$$

which gives the equation of state parameter (ω_ϕ) for quintessence field as

$$\omega_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{1.37}$$

Now for the flat Universe, equations (1.7) and (1.8) take the form

$$3H^2 = 8\pi G\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{1.38}$$

$$2\dot{H} = -8\pi G\dot{\phi}^2 \tag{1.39}$$

Further, by varying the action with respect to ϕ, one has

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \tag{1.40}$$

Equation (1.40) can be obtained either by performing some mathematical calculations combining the equations (1.38) and (1.39) or from the energy conservation equation for the scalar field

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0 \tag{1.41}$$

The acceleration equation (1.15) for the scalar field is given by

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V(\phi)\right) \tag{1.42}$$

So, to understand the present accelerating phase, one must have $\dot{\phi}^2 < V(\phi)$, which means, a flat potential ensures that the scalar field will slowly roll. In literature, there are many quintessence potentials, and they are classified into the following models [39]:

- ***Freezing models:***

In these models, the scalar field ϕ was moving fast in the early universe, then "freezes" at some later time to realize its equation of state ω_ϕ close to -1. For such a potential, it is well known that the quintessence field exhibits a tracking behavior where it traces the equation of state of the dominant background fluid (e.g., radiation and matter). This is why the freezing type is sometimes called the "tracker model". The following potentials are the representatives of this model:

(i) $V(\phi) = M^{4+n}\phi^{-n}$, $(n > 0)$. [36, 40]

(ii) $V(\phi) = M^{4+n}\phi^{-n}\exp\left(\frac{\alpha\phi^2}{M_{Pl}^2}\right)$, $(n > 0)$.[41]

Here, $M_{Pl}^2 = (8\pi G)^{-1}$, and, M is a constant having the dimension of mass, also, α is another constant.

- ***Thawing models:***

In these models, a quintessence field almost stays somewhere on the potential in the early times, during which it behaves like a cosmological constant $\omega_\phi \simeq -1$. At later times, the quintessence field starts to "thaw" or move, that is, starts to roll down the potential. The situation is quite similar to that of the inflationary dynamics. As in the case of inflation, typical potentials of this type are (positive) power law and hilltop type potentials as in chaotic and new inflation models, respectively. The following potentials belong to this group:

(i) $V(\phi) = V_0 + M^{4-n}\phi^{-n}$, $(n > 0)$. [42, 43]

(ii) $V(\phi) = M^4 \cos^2\left(\frac{\phi}{f}\right)$ [44]

B: K-essence:

In this model, the present acceleration of the universe has been described by modifying the kinetic energy term of the scalar field. Originally, K-inflation i.e., the inflation driven by the kinetic energy, was introduced by Armendariz-Picon *et al.* [45] to explain the inflationary scenario in the early universe. Later, Chiba *et al.* [46] applied this method to dark energy. Further, Armendariz-Picon *et al.* extended this analysis to a more general Lagrangian [47, 48], and termed it as "K-essence".

K-essence is described by a scalar field ϕ minimally coupled with a non-canonical kinetic energy term X. The action for such models is given by

$$\mathcal{S} = \int \sqrt{-g}\ \mathrm{d}^4x\ F(\phi, X) \tag{1.43}$$

where $X = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi$, and, F is any function of ϕ and X. Now we shall discuss the following subclass of the kinetic K-essence having the action

$$\mathcal{S} = \int \sqrt{-g}\ \mathrm{d}^4x\ F(X) \tag{1.44}$$

where $F = F(X)$ is independent of ϕ. Further, ϕ is assumed to be smooth, and hence, we have $X = \frac{\dot{\phi}^2}{2} \geq 0$. Also, assuming that our Universe is described by the FLRW line element, the energy-momentum tensor of the K-essence is given by

$$T_{\mu\nu} = F_X\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}F \tag{1.45}$$

Now the energy-momentum tensor of K-essence is that of a perfect fluid, $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$, with the four-velocity $u_\mu = \frac{\partial_\mu\phi}{\sqrt{2X}}$, where pressure and energy density of this model are given by

$$p = F, \tag{1.46}$$
$$\rho = 2XF_X - F \tag{1.47}$$

Since the energy density is always positive, hence $2XF_X - F > 0$. So the equation of state for K-essence is given by

$$\omega_X = \frac{p}{\rho} = \frac{F}{2XF_X - F} \tag{1.48}$$

which shows that the equation of state ω_X is close to -1 as long as $|2XF_X| \ll |F|$ is satisfied. Further, depending on the nature of F, one has the following scenarios in K-essence [49, 50]

$$F > 0 \implies \omega_X > 0 \tag{1.49}$$
$$F < 0, \text{ and } F_X > 0 \implies \omega_X > -1 \tag{1.50}$$
$$F < 0, \text{ and } F_X < 0 \implies \omega_X < -1 \tag{1.51}$$

Further, one may introduce a special type of K-essence model known as "Ghost condensate model" having the Lagrangian density

$$F = -X + \frac{X^2}{M^4} \tag{1.52}$$

where M is constant with the dimension of mass. The equation of state for K-essence in this model takes the form

$$\omega_X = \frac{1 - \frac{X}{M^4}}{1 - \frac{3X}{M^4}} \tag{1.53}$$

which shows for $\frac{1}{2} < \frac{X}{M^4} < \frac{2}{3}$, we have $-1 < \omega_X < -\frac{1}{3}$. Moreover, for $\frac{X}{M^4} = \frac{1}{2}$, the de Sitter solution ($\omega_X = -1$) is obtained.

C. Tachyon field

Nowadays, some interesting cosmological consequences have been found in rolling tachyon condensates, a class of string theories. Sen [51, 52, 53] showed that the decay of D-branes produces a pressureless gas with finite energy density which is almost identical to classical dust. The interesting feature of the rolling tachyon is the equation of state lies between -1 and 0 [54]. Therefore, one can construct viable models for inflation [55, 56], as well as, dark energy [57, 58, 59, 60, 61, 62, 63] by choosing suitably the potential.

In string theories, there are unstable D-branes known as non-Bogomol'nyi-Prasad-Sommerfield (BPS) D-branes. These unstable branes are characterized by having a single tachyon mode of a negative mass living on their world-volume. Now considering the dynamics of the tachyon on a non-BPS D3-brane, the effective Lagrangian for the tachyon with potential $V(\phi)$ is given by [52, 64]

$$\mathcal{S} = -\int \mathrm{d}^4x \; V(\phi)\sqrt{-\det\left(g_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi\right)} \tag{1.54}$$

where the effective potential $V(\phi)$ is given by

$$V(\phi) = \frac{V_0}{\cosh\left(\frac{\beta\phi}{2}\right)} \tag{1.55}$$

For bosonic string $\beta = 1$ and for the non-BPS D-brane in the super-string theory $\beta = \sqrt{2}$.

In open string theory, the tachyon starts to roll down from the top of the potential located at $\phi = 0$, and evolves toward a ground state at $|\phi| \to \infty$. Now, varying the action (1.54) the energy-momentum tensor can be obtained as

$$T_{\mu\nu} = \frac{V(\phi)\partial_\mu\phi\partial_\nu\phi}{\sqrt{1 + g^{\sigma\lambda}\partial_\sigma\phi\partial_\lambda\phi}} - g_{\mu\nu}V(\phi)\sqrt{1 + g^{\sigma\lambda}\partial_\sigma\phi\partial_\lambda\phi} \tag{1.56}$$

In a flat FLRW Universe, the pressure p_ϕ and energy density ρ_ϕ are given by

$$p_\phi = -V(\phi)\sqrt{1-\dot{\phi}^2} \tag{1.57}$$

$$\rho_\phi = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}} \tag{1.58}$$

which gives the equation of state parameter for the tachyon field as

$$\omega_\phi = \dot{\phi}^2 - 1 \tag{1.59}$$

Further, for flat FLRW Universe, the equations (1.7) and (1.8) become

$$3H^2 = 8\pi G \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}} \tag{1.60}$$

$$2\dot{H} = 8\pi G \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}\dot{\phi}^2 \tag{1.61}$$

and the energy conservation equation takes the form

$$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \tag{1.62}$$

It can be shown that the acceleration happens if $\dot{\phi}^2 - 1 < -\dfrac{1}{3}$ i.e., $\dot{\phi}^2 < \dfrac{2}{3}$.

D. Phantom field

From recent observations it is indicated that the equation of state parameter crosses the '-1' boundary [5, 65]. The region where the equation of state crosses '-1' i.e. $\omega < -1$ is called phantom region and behind this, which force drives the equation of state parameter to go below -1 is known as Phantom (ghost) dark energy. Specific models in brane-world or Brans-Dike scalar-tensor gravity can lead to Phantom energy. Scalar field with negative kinetic energy (phantom fields) is the well-known example of phantom dark energy. At first in Hoyle's version of steady state theory, Phantom fields were introduced. Later, the Phantom field theory was reformulated and modified by Hoyle and Narlikar.

The action for the Phantom field is given by [66, 67]

$$\mathcal{S} = \int \sqrt{-g}\, \mathrm{d}^4x \left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{1.63}$$

where the kinetic term is of opposite sign of an ordinary scalar field in (1.33). The pressure p_ϕ and energy density ρ_ϕ of the scalar field are given by

$$p_\phi = -\frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.64}$$

$$\rho_\phi = -\frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.65}$$

and the equation of state is given by

$$\omega_\phi = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)} \tag{1.66}$$

Thus to get $\omega_\phi < -1$, one has $\dot{\phi}^2 < 2V(\phi)$. In the presence of phantom energy, the universe accelerates so quickly that the universe reaches to big rip singularity.

1.4.1.3 Chaplygin gas and its generalization

Without considering different properties of equation of state parameter, cosmologists attempted to unify dark matter and dark energy by a single scalar field or a single fluid from the beginning of the accelerated universe. The idea of unification came after the introduction of Chaplygin gas, after the name of a Russian Mathematician, Physicist, and Engineer, Sergey Chaplygin (1869-1942) who found a similar behavior in aerodynamical studies. After that, this concept was generalized from Chaplygin to generalized Chaplygin to modified Chaplygin gas which can unify the dark sectors of dark energy and dark matter by a single scalar field [68, 69, 70, 71].

- **Chaplygin gas:**

The equation of state for Chaplygin gas (CG) [68] is

$$p = -\frac{A}{\rho} \tag{1.67}$$

where $A > 0$. Now, substituting the equation of state in the conservation equation (1.10), and then integrating, we obtain the energy density of the Chaplygin gas as

$$\rho = \sqrt{A + \frac{B}{a^6}} \tag{1.68}$$

where B is the integration constant. Now, we will discuss some interesting features corresponding to Chaplygin gas in early and late time era.

Early Universe: In the early epoch of the Universe, the scale factor is very small, hence the first term within the square root is negligible to the second term. So one has

$$\rho \propto a^{-3} \tag{1.69}$$

Late Universe: In the late time Universe, the scale factor is very large. So, the second term under the square root becomes insignificant. Hence one has,

$$\rho \approx \sqrt{A} = -p \implies p = -\rho \tag{1.70}$$

Eq. (1.69) shows that, at early time, Chaplygin gas is almost like a pressureless dust, whereas in late-time, Chaplygin gas behaves as a cosmological constant.

Further, if one associates a potential together with an ordinary scalar field ϕ in the context of Chaplygin gas, then the pressure and energy density of the scalar field can be written as

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = -\frac{A}{\sqrt{A + \frac{B}{a^6}}} \tag{1.71}$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \sqrt{A + \frac{B}{a^6}} \tag{1.72}$$

Solving equations (1.71) and (1.72) and using the Friedmann equations in flat Universe, one has the expressions of the potential and scalar field in terms of functions of scale factor as

$$\phi = \frac{1}{\sqrt{3}} \sinh^{-1}\left(\sqrt{\frac{B}{A}\frac{1}{a^3}}\right) \tag{1.73}$$

$$V(\phi) = \frac{1}{2}\left[\sqrt{A + \frac{B}{a^6}} + \frac{A}{\sqrt{A + \frac{B}{a^6}}}\right] \tag{1.74}$$

and using the relation of scalar field the potential can be explicitly written as

$$V(\phi) = \frac{\sqrt{A}}{2}\left(\cosh\sqrt{3}\phi + \frac{1}{\cosh\sqrt{3}\phi}\right) \tag{1.75}$$

Hence one may conclude that Chaplygin gas is equivalent to a minimally scalar field with this potential.

- **Generalized Chaplygin gas:**

The equation of state for generalized Chaplygin gas (GCG) [70] is

$$p = -\frac{A}{\rho^{\alpha}} \tag{1.76}$$

where $A > 0$, and, $0 < \alpha < 1$. From the latest observational data, it can be shown that at 95% confidence level [72], α lies between the interval $(0, 0.2)$. Now, inserting, the equation of state of GCG in the continuity equation (1.10), we have the energy density of GCG as

$$\rho = \left[A + \frac{B}{a^{3(1+\alpha)}}\right]^{\frac{1}{1+\alpha}} \tag{1.77}$$

where B is the integration constant. The behaviour of GCG in early and late Universe are as follows:

Early Universe: In this phase, the scale factor is very small, hence, the energy density of GCG reduces to

$$\rho \propto a^{-3} \tag{1.78}$$

i.e, in early universe, GCG behaves as dust.

Late Universe: In this phase, the scale factor is very small, thus, we have,

$$\rho \approx A^{\frac{1}{1+\alpha}} \approx -p \tag{1.79}$$

which shows that in late time universe GCG behaves as a cosmological constant.

- **Modified Chaplygin gas:**

The equation of state for modified Chaplygin gas (MCG) [71] is given by

$$p = A\rho - \frac{B}{\rho^{\alpha}} \tag{1.80}$$

where $A, B > 0$ and $0 \leq \alpha \leq 1$. The energy density of MCG takes the form

$$\rho = \left[\frac{B}{A+1} + \frac{C}{a^{3(A+1)(1+\alpha)}}\right]^{\frac{1}{1+\alpha}} \tag{1.81}$$

where C is the integration constant. We have the following behaviour of MCG in early and late universe.

Early Universe: Since in early universe the scale factor is very small, the energy density can be approximated as

$$\rho \propto a^{-3(1+A)} \tag{1.82}$$

which is almost similar to the universe which is dominated by some fluid having equation of state $p = A\rho$.

Late universe: In late universe, the scale factor is very large, as a result,

$$\rho \simeq \left(\frac{B}{A+1}\right)^{\frac{1}{1+\alpha}} \simeq -p \tag{1.83}$$

which corresponds to the Universe with a cosmological constant. Thus MCG can describe that the Universe evolves from radiation era to the present accelerated phase for $A = \frac{1}{3}$. Further, if MCG can be considered as a scalar field together with a potential $V(\phi)$, then one has the relations for scalar field and potential as

$$\phi = \frac{2}{\sqrt{3}(1+A)(1+\alpha)} \sinh^{-1}\left(\sqrt{\frac{C(1+A)}{B}} a^{-\frac{3}{2}(1+A)(1+\alpha)}\right) \tag{1.84}$$

$$V(\phi) = \frac{1-A}{2}\left(\frac{B}{A+1}\right)^{\frac{1}{1+\alpha}} \cosh^{\frac{2}{1+\alpha}}\left(\frac{1}{2}\sqrt{3(1+A)}(1+\alpha)\phi\right) + \frac{B}{2}\left(\frac{B}{A+1}\right)^{-\frac{\alpha}{1+\alpha}} \cosh^{-\frac{2\alpha}{1+\alpha}}\left(\frac{1}{2}\sqrt{3(1+A)}(1+\alpha)\phi\right) \tag{1.85}$$

Thus, one can conclude that the above three models can be considered as the unified models for dark matter and dark energy.

1.4.2 Acceleration due to gravity modification

So far we have discussed dark energy as a modification of the energy-momentum tensor in Einstein's field equation. But in this section, we will discuss the present accelerated phase in a different way. Here we have modified the gravitational sector compared to the General Relativity. The gravitational action of Einstein's general relativity is given by

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^4x\ R + \int \sqrt{-g}\mathrm{d}^4x L_M\left(g_{\mu\nu}, \Psi_M\right) \tag{1.86}$$

where $\kappa = 8\pi G$, L_M is the matter Lagrangian depending on the metric tensor $g_{\mu\nu}$, and the matter field Ψ_M, R is the Ricci scalar, $g = \det(g_{\mu\nu})$. In modified gravity theories, changing the action suitably, one may try to describe the present accelerating phase of the Universe. We shall now describe some relevant modified gravity theories.

1.4.2.1 $f(R)$ gravity

The simplest modification in General Relativity is the $f(R)$ gravity theory where $f(R)$ is any arbitrary function of Ricci scalar R. The action for $f(R)$ gravity is given by [73, 211]

$$\mathcal{S} = \frac{1}{2\kappa}\int\sqrt{-g}\mathrm{d}^4x\ f(R) + \int\sqrt{-g}\mathrm{d}^4x L_M\left(g_{\mu\nu},\Psi_M\right) \tag{1.87}$$

Now we shall study $f(R)$ gravity theory in two different approaches, namely, in metric and Palatini formalisms.

- ***$f(R)$ gravity in Metric formalism***

In metric formalism, it is assumed that the affine connections $\Gamma^{\alpha}_{\beta\gamma}$ are the usual metric connections in terms of $g_{\mu\nu}$. Now, varying the action (1.87) with respect to $g_{\mu\nu}$, one obtains the field equations as

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa T^M_{\mu\nu} \tag{1.88}$$

where $F(R) \equiv \dfrac{\mathrm{d}f(R)}{\mathrm{d}r}$, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the D'ALembert's operator, and the energy-momentum tensor of the matter field $T^M_{\mu\nu}$ is defined by

$$T^M_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{\mu\nu}} \tag{1.89}$$

which satisfies the continuity equation

$$\nabla^\mu T^M_{\mu\nu} = 0 \tag{1.90}$$

Further, considering the trace of the equation (1.88), one has,

$$\begin{aligned} 3\Box F(R) + RF(R) - 2f(R) &= \kappa g^{\mu\nu}T^M_{\mu\nu} \\ &= \kappa T \end{aligned} \tag{1.91}$$

In particular, if one considers $f(R) = R$, then one can retrieve the Einstein gravity. In general, $\Box F(R) \neq 0$, which corresponds to some propagating degrees of freedom $\phi = F(R)$, and the dynamics of the scalar field, namely, 'scalaron' ϕ is characterized by the equation (1.91).

So, the field equation (1.88) is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\left(T^{M}_{\mu\nu} + T^{eff}_{\mu\nu}\right) \tag{1.92}$$

where $T^{eff}_{\mu\nu}$ is termed as the effective energy-momentum tensor and defined as

$$\kappa T^{eff}_{\mu\nu} \equiv \frac{1}{2}(f(R) - R)g_{\mu\nu} + \nabla_\mu\nabla_\nu F(R) - g_{\mu\nu}\Box F(R) + (1 - F(R))R_{\mu\nu} \tag{1.93}$$

Since $\nabla^\mu G_{\mu\nu} = 0$, and, $\nabla^\mu T^{M}_{\mu\nu} = 0$, we have

$$\nabla^\mu T^{eff}_{\mu\nu} = 0 \tag{1.94}$$

which shows that the continuity equation for the effective energy-momentum tensor also holds.

Now, considering the flat FLRW space-time, the line element can be written in Cartesian coordinate as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{1.95}$$

and the Ricci scalar R is given by

$$R = 6\left(\dot{H} + 2H^2\right) \tag{1.96}$$

The energy-momentum tensor for the perfect fluid takes the form

$$T^{M}_{\mu\nu} = (p_M + \rho_M)u_\mu u_\nu + p_M g_{\mu\nu} \tag{1.97}$$

where p_M and ρ_M are the pressure and energy density of the perfect fluid respectively, u_μ is the four-velocity of the perfect fluid with a barotropic equation of state $\omega_M = \dfrac{p_M}{\rho_M}$.

Hence, the field equations for flat FLRW space-time is given by

$$3H^2 = \frac{\kappa}{f'}\left[\rho_M + \frac{1}{2}(Rf' - f) - 3H\dot{R}f''\right] \tag{1.98}$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'}\left[p_M + \dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right] \tag{1.99}$$

In vacuo, the field equations can be rewritten as

$$3H^2 = \kappa\rho_{eff} \tag{1.100}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(3p_{eff} + \rho_{eff}) \tag{1.101}$$

where the effective pressure p_{eff} and effective energy density ρ_{eff} are respectively given by,

$$p_{eff} = \frac{1}{f'}\left[\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right] \tag{1.102}$$

$$\rho_{eff} = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - 3H\dot{R}f''\right] \tag{1.103}$$

The equation of state parameter of the effective fluid is given by

$$\omega_{eff} = \frac{2\dot{R}^2 f''' + 4H\dot{R}f'' + 2\ddot{R}f'' + f - Rf'}{Rf' - f - 6H\dot{R}f''} \tag{1.104}$$

So to obtain accelerating universe, one must have $\omega_{eff} < -\frac{1}{3}$. Since, $\rho_{eff} > 0$, hence p_{eff} determines the sign of ω_{eff}.

- ***$f(R)$ gravity in Palatini formalism***

Italian Mathematician Attilio Palatini proposed another approach to obtain the field equations in $f(R)$ gravity. In this formalism, the affine connection $\Gamma^{\alpha}_{\beta\gamma}$ and the metric tensor $g_{\mu\nu}$ are treated as independent variables. Now, varying the action (1.87) with respect to the metric tensor, the field equations are obtained as

$$F(R)R_{\mu\nu}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T^M_{\mu\nu} \tag{1.105}$$

Here it should be noted that $R_{\mu\nu}(\Gamma)$, Ricci tensor corresponding to the affine connection, is different from the usual $R_{\mu\nu}(g)$, Ricci tensor corresponding to the metric connection. Now, taking the trace of the equation (1.105), one has

$$F(R)R - 2f(R) = \kappa T \tag{1.106}$$

where $R \equiv R(T) = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ is the Ricci scalar in Palatini formalism and it is different from the Ricci scalar in metric formalism. The explicit form $R_{\mu\nu}(T)$ is given by

$$R(T) = R(g) + \frac{3}{2[f'(R(T))]^2}\nabla_\mu f'(R(T))\nabla^\mu f'(R(T)) + \frac{3}{f'(R(T))}\Box f'(R(T)) \tag{1.107}$$

Now, varying the action (1.87) with respect to the connection, and using the relation (1.107), one has

$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \frac{\kappa}{F}T_{\mu\nu} - \frac{1}{2F}(FR(T) - f(R))g_{\mu\nu} + \frac{1}{F}\left(\nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F\right) - \frac{3}{2F^2}\left(\partial_\mu F\partial_\nu F - \frac{1}{2}g_{\mu\nu}(\nabla F)^2\right) \tag{1.108}$$

If one considers, $f(R) = R$, then the field equations (1.108) in Palatini formalism are identical to the field equations (1.88) of the metric formalism. The difference between these two formalisms arises for those $f(R)$ gravity model which contain nonlinear terms in R. In that case, the kinetic term $\Box F$ is present in metric formalism while that kind of term is absent in Palatini formalism. Consequently, the oscillatory mode appears in the metric formalism, which doesn't exist in Palatini formalism.

1.4.2.2 $f(T)$ gravity

Around 1920's, Einstein himself introduced torsion as a gravitating interaction term, to unify gravity and electromagnetism over Weitzenbock non-Riemannian manifold. As a result, the Levi-Civita connection is replaced by Weitzenbock connection in the underlying Riemann-Cartan space-time. In $f(T)$ gravity, the dynamical objects are the four linearly independent vierbein (tetrad) fields that form the orthogonal bases for the tangent space at each point of space-time. These vierbeins are parallel vector fields, that give the theory the descriptor "teleparallel". Now, in analogy to $f(R)$ gravity, teleparallel gravity has been generalized by replacing torsion scalar T with a generic function $f(T)$ and Linder coined it as $f(T)$ gravity. The action for $f(T)$ gravity is given by

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^4x\ f(T) + \int \sqrt{-g}L_M \mathrm{d}^4x \tag{1.109}$$

where T is the torsion scalar, $f(T)$ is a differentiable function of torsion and L_M is the matter Lagrangian. The torsion scalar is defined as

$$T = S_\sigma{}^{\mu\nu}T^\sigma{}_{\mu\nu} \tag{1.110}$$

where super-potential, $S_\sigma{}^{\mu\nu}$, is given by

$$S_\sigma{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\sigma + \delta^\mu_\sigma\ T^{\alpha\nu}{}_\alpha - \delta^\nu_\sigma\ T^{\alpha\mu}{}_\alpha\right), \tag{1.111}$$

the contortion tensor, $K^{\mu\nu}{}_\sigma$, is given by

$$K^{\mu\nu}{}_\sigma = -\frac{1}{2}\left(T^{\mu\nu}{}_\sigma - T^{\nu\mu}{}_\sigma - T_\sigma{}^{\mu\nu}\right) \tag{1.112}$$

and the torsion tensor, $T^\sigma{}_{\mu\nu}$ is defined as

$$T^\sigma{}_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu} = e^\sigma_A\left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu\right) \tag{1.113}$$

where the Weitzenböck connection $\Gamma^\sigma_{\mu\nu}$ is defined as $\Gamma^\sigma_{\mu\nu} = e^\sigma_A\partial_\nu e^A_\mu$. In teleparallel gravity, orthogonal tetrad components $e_A(x^\mu)$ are considered as dynamical variables and geometrically they form an orthonormal basis for the tangent space at each point x^μ of the manifold i.e.

$$e_A e_B = \eta_{AB} = diag(+1, -1, -1, -1) \tag{1.114}$$

Further, in a co-ordinate basis one may write $e_A = e^\mu_A\partial_\mu$ where e^μ_A are the components of e_A, with $\mu = 0, 1, 2, 3$ and $A = 0, 1, 2, 3$. It is to be noted that capital letters refer to the tangent space while Greek indices label coordinates on the manifold. Hence the metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x) = \eta_{AB}e^A_\mu(x)e^B_\nu(x) \tag{1.115}$$

Now, varying the action (1.109), the field equation reads

$$\left[e^{-1}\partial_\mu(eS_A{}^{\mu\nu}) - e_A^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}\right] f_T + S_A{}^{\mu\nu}\partial_\mu(T)f_{TT} + \frac{1}{4}e_A^\nu f(T) = \frac{\kappa}{2}e_A^\rho T_\rho{}^\nu \tag{1.116}$$

where $|e| = \det\left(e_\mu^A\right) = \sqrt{-g}$ and suffix T denotes the differentiation with respect to torsion scalar T. Now, assuming flat, homogeneous and isotropic FLRW space-time having line element

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \tag{1.117}$$

the Modified Friedmann equations and the continuity equation become

$$F - 2Tf_T = 2\kappa\rho \tag{1.118}$$
$$-8\dot{H}Tf_{TT} + 2(T - 2\dot{H})f_T - f = 2\kappa p \tag{1.119}$$
$$\dot{\rho} + 3H(p + \rho) = 0 \tag{1.120}$$

with $T = -6H^2$. Here one must note that for $f(T) = T$, the equations reduce to the usual Friedmann equations. Now, the modified Friedmann equations (1.118) and (1.119) can be rewritten as

$$3H^2 = \kappa(\rho + \rho_{eff}) \tag{1.121}$$
$$2\dot{H} + 3H^2 = -\kappa(p + p_{eff}) \tag{1.122}$$

where

$$2\kappa\rho_{eff} = 2Tf_T - T - f \tag{1.123}$$
$$2\kappa p_{eff} = 8\dot{H}Tf_{TT} - 2(T - 2\dot{H})f_T + f + T - 4\dot{H} \tag{1.124}$$

are the energy density and pressure of the effective fluid due to the contribution of torsion. The equation of state of the effective fluid is given by

$$\omega_{eff} = -1 + \frac{8\dot{H}Tf_{TT} + 4\dot{H}f_T - 4\dot{H}}{2Tf_T - T - f} \tag{1.125}$$

So the generic function $f(T)$ determines whether the universe is accelerating or not.

1.4.2.3 $f(R,T)$ gravity

Harko *et al.* have made a further extension to $f(R)$ gravity by choosing the Lagrangian density as an arbitrary function $f(R,T)$ with T, the trace of the energy-momentum tensor.

The action for $f(R,T)$ gravity is given by

$$\mathcal{S} = \frac{1}{2\kappa}\int\sqrt{-g}\mathrm{d}^4x\ f(R,T) + \int\sqrt{-g}L_M\mathrm{d}^4x \tag{1.126}$$

where $T = T_{\mu\nu}g^{\mu\nu}$ is the trace of the energy-momentum tensor $T_{\mu\nu}$, obtained from the matter Lagrangian density as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{1.127}$$

Further, if $\mathcal{L}_m$ depends only on $g_{\mu\nu}$ but not its derivatives, then the above form for $T_{\mu\nu}$ simplifies to

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}}. \tag{1.128}$$

Now variation of the above action (1.126), the field equation for $f(R,T)$ gravity theory can be written as

$$f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R - \frac{1}{2}g_{\mu\nu}f(R,T) = \kappa T_{\mu\nu} - (T_{\mu\nu} + \Theta_{\mu\nu}) f_T \tag{1.129}$$

with

$$\Theta_{\mu\nu} = g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = -2T_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m - 2g^{\alpha\beta}\frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}} \tag{1.130}$$

where $f_R \equiv \frac{\partial f(R,T)}{\partial R}$, $f_T \equiv \frac{\partial f(R,T)}{\partial T}$. It should be noted that if one choose $f(R,T) = f(R)$, then one recovers $f(R)$ gravity. Further, one may recover GR if $f(R,T) = R$ while ΛCDM model will be recovered if $R + 2\Lambda$ (Λ, a cosmological constant) with matter in the form of dust i.e. $L_m = \rho$.

1.4.2.4 Scalar-Tensor Theories

Scalar-tensor theory is one of the alternatives for the accelerating universe, where the gravitational action contains both metric and a scalar field [75]. Brans-Dicke theory [76] is a particular type of Scalar-tensor theory with constant coupling parameter. Though it has a long history in cosmology, however, in the recent time it has been studied by many authors [77, 78, 79, 80, 81] in the perspective of present acceleration and to resolve the fine-tuning and the coincidence problem.

The action for the scalar-tensor theories is given by

$$\mathcal{S} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[f(\phi,R) - \xi(\phi)(\nabla\phi)^2\right] + \int \sqrt{-g}d^4x L_M\left(g_{\mu\nu},\Psi_M\right) \tag{1.131}$$

where f is any arbitrary function of the scalar field ϕ and the Ricci scalar R, ξ is the function of ϕ only. Now various modified gravity theories can be developed and recovered by choosing $f(\phi,R)$ and $\xi(\phi)$ suitably.

If $f(\phi, R) = f(R)$ and $\xi(\phi) = 0$ is chosen, the action (1.131) recovers $f(R)$ gravity. In case of Brans-Dicke theory, $f(\phi, R) = \phi R$, and $\xi(\phi) = \frac{\omega_{BD}}{\phi}$, where ω_{BD} is called the Brans-Dicke parameter. Also $f(\phi, R) = 2Re^{-\phi} - U(\phi)$ corresponds the dilation gravity arising from low energy effective string theory [82].

We will now concentrate on salient features of Brans-Dicke theory. The action in BD theory is given by

$$\mathcal{S}_{BD} = \int \sqrt{-g}\mathrm{d}^4x \left[\frac{1}{2\kappa}\phi R - \frac{\omega_{BD}}{2\phi}(\nabla\phi)^2 - U(\phi)\right] + \int \sqrt{-g}\mathrm{d}^4x L_M\left(g_{\mu\nu}, \Psi_M\right) \tag{1.132}$$

Now, taking the variation of action (1.132) with respect to $g_{\mu\nu}$ and ϕ, one gets

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{\phi}T_{\mu\nu} - \frac{1}{\phi}g_{\mu\nu}U(\phi) + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) \\ + \frac{\omega_{BD}}{\phi^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right] \end{aligned} \tag{1.133}$$

and

$$(3 + 2\omega_{BD})\Box\phi + 4U(\phi) - 2\phi U_{,\phi} = g^{\mu\nu}T_{\mu\nu} \tag{1.134}$$

In this context, it should be mentioned that one can find a relation between BD theory and the $f(R)$ theory in both metric and Palatini formalisms. So to find out a relation, one has to consider the following correspondence

$$\phi = f(R), \quad U(\phi) = \frac{1}{2}(R(\phi)F - f(R(\phi))) \tag{1.135}$$

where $R = R(g)$ in the 'metric' case, and $R = R(T)$ in case of 'Palatini' formalism. Comparing the equations (1.133) and (1.134) with the equations (1.88) and (1.91), respectively, one may note that, the $f(R)$ gravity in metric formalism is equivalent to Brans-Dicke theory with $\omega_{BD} = 0$. Similarly, comparing the equations (1.133) and (1.134) with the equations (1.105) and (1.106) respectively, one obtains the $f(R)$ theory in Palatini formalism with $\omega_{BD} = -\frac{3}{2}$.

1.4.2.5 Gauss-Bonnet gravity

There is another way to modify gravity so that the accelerating universe can be realized. The way is to combine the Ricci scalar and the Riemann tensor, namely, the Gauss-Bonnet (GB) term [83, 84, 85, 86]. The GB is topologically invariant which gives the dynamics of the four dimensions along a dynamically evolving scalar field which is coupled with it.

The action for GB gravity can be written as

$$\mathcal{S}_{GB} = \int \sqrt{-g}\mathrm{d}^4x \left[\frac{1}{2}F(\phi)R - \frac{1}{2}\psi(\phi)(\nabla(\phi))^2 - V(\phi) - f(\phi)\tilde{G}\right] \tag{1.136}$$

where $\tilde{G}$, the GB is defined by

$$\tilde{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \tag{1.137}$$

There is another class of general GB theories with a self-coupling of the term

$$S_{GB} = \int \sqrt{-g}\mathrm{d}^4x \left[\frac{1}{2}R + f(\tilde{G})\right] \tag{1.138}$$

where $f(\tilde{G})$ is a function of the GB term defined in Eq. (1.137). In this context, the most important point is $f_{\tilde{G}\tilde{G}} > 0$ for stability of late-time de Sitter solution, and matter/radiation solutions.

1.4.2.6 Higher dimensional gravity

Higher dimensions can be one of the candidates explaining the recent accelerating universe. The idea is, our observable universe is a four dimensional (1+3) "brane" embedded in a higher dimensional space-time (1+3+d) called the"bulk", with d, the number of extra spatial dimensions. In this model, all matter fields are confined into the brane, whereas the graviton is free to propagate in the brane as well as bulk. The brane scenario has been presented by Randall and Sundram (RS) [87, 88].

Apart from these theories, there are many other theories which may work as alternative to Einstein's general relativity theory e.g. Fractal gravity [266, 267, 268, 269, 270, 271, 272], Hořava-Lifshitz gravity [186, 163, 194, 195, 196, 197, 198, 184], Rastall gravity [89, 90, 91, 92, 93], Dvali-Gabadadze-Porrati gravity [94, 95], Lanczos-Lovelock gravity [96, 97, 98, 99], $f(G)$ gravity [100, 101], $f(T, B)$ gravity [102, 103, 104], $f(Q, T)$ gravity [105, 106] etc.

However, there are two common questions that may come to mind, even though there is no alternative way to accommodate the observational data without introducing either dark energy or modified gravity theories. These are listed below.

a) How does the exotic matter suddenly come into scenario as the dominant fluid, producing the late-time accelerated expansion?

b) Why is General Relativity suddenly in need of modification while it nicely predicts several observational results with high accuracy and describes the evolution of the Universe until the matter dominated era?

1.5 Thermodynamics and Cosmology

Thermodynamics plays an important crucial role to understand the evolution of the universe. In this section, it will be discussed how thermodynamic quantities play a crucial role in the evolution of the universe.

Thermodynamics is a branch of physics dealing with heat, work, and temperature, and their relation to energy, entropy, and the physical properties of matter and radiation. To characterize a thermodynamic system completely, one needs some macroscopic parameters of the system along with the properties of the boundary and the interactions of the boundary with its surroundings. The boundaries need not be impenetrable and may permit the matter or energy in either direction. A thermodynamical system is said to be isolated if it does not exchange mass and energy with its surroundings. A closed system can exchange energy, but not matter with its surroundings while an open system can exchange both matter and energy. We will now describe the fundamental laws of thermodynamics as follows:

- **Zeroth Law of thermodynamics**

If two systems are separately in thermal equilibrium with a third, then they are also in thermal equilibrium with each other.

Suppose A and B are two systems which are separately in thermal equilibrium with another system C. We use the symbol $\sim$ to represent the thermal equilibrium between the systems. So $A \sim C$ (or, $C \sim A$), and, $B \sim C$ (or, $C \sim B$). Then the Zeroth law states that $A \sim B$ i.e., A and B are in thermal equilibrium. Mathematically, Zeroth law states that the thermal equilibrium is a transitive property as:

$$A \sim C \;\; \text{and} \;\; C \sim B \implies A \sim B \tag{1.139}$$

- **First law of thermodynamics**

According to the idea underlying the science of thermodynamics, the energy contained in a system is a specific function of its state and can only be changed when the state of the system is itself altered. When such a change in state occurs, it is important for thermodynamic reasons to distinguish two different modes of transfer by which the energy content (internal energy) may be affected, namely the heat flow and the performance of work. Recognizing these two possibilities, the first law of thermodynamics states the principle of the conservation of energy as

$$\mathrm{d}E = \mathrm{d}Q + \mathrm{d}W \tag{1.140}$$

where $\mathrm{d}E$ is the increase in internal energy corresponding to some given change in state, $\mathrm{d}Q$ is the heat flow into the system from the surroundings, and $\mathrm{d}W$ is the work done by the system on the surroundings when a particular process takes place that leads to the given change in state. This equation may be regarded as the principle of conservation of energy. Let p and V be the pressure and volume of the system respectively. Then if one considers the transformation between the system and its surroundings occurs so slowly that the system always stays nearly in equilibrium, i.e., the process is quasi-static or adiabatic, thus one has

$$\mathrm{d}W = -p\mathrm{d}V \tag{1.141}$$

Hence, the Eq. (1.140) can be written as

$$\mathrm{d}E = \mathrm{d}Q - p\mathrm{d}V \tag{1.142}$$

• Second law of thermodynamics

To study the second law of thermodynamics, one has to introduce another quantity related to thermodynamics, namely, entropy of the system which is a definite function of its state. The second law of thermodynamics establishes a relationship between the change in entropy content of the system and the nature of the process that causes the change in state.

Before going into details, let us define the reversible and irreversible process. A reversible process is one that is carried out in such a way that, at the end of the process, both the system and its surroundings may be returned to its initial state without causing any changes in the system or its surrounds. The process which does not satisfy the above conditions is said to be irreversible. As dissipation occurs in all natural processes, so they are irreversible in nature [107].

The entropy (S) of a thermodynamic system was first introduced by German physicist Rudolf Clausius (1865) and he defined it as the quotient of an infinitesimal amount of heat to the instantaneous temperature, i.e.,

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} \tag{1.143}$$

where $\mathrm{d}Q$ is the heat flow into the system from the surroundings, T is the temperature of the thermodynamic system and $\mathrm{d}S$ represents change in entropy of the system. Clausius argued that a gas which expands into vacuum cannot be brought back to its original state without a decrease in entropy. Thus, he concluded that the energy of the Universe is constant and the entropy of the Universe tends to a maximum.

However, The second law of thermodynamics can be written in various equivalent ways.

• **Clausius statement:** There exists no thermodynamic transformation whose sole effect is to transfer heat from a colder reservoir to a warmer reservoir.

• **Kelvin statement:** There exists no thermodynamic transformation whose sole effect is to extract heat from a reservoir and to convert that heat entirely into work.

Mathematically, the second law of thermodynamics states

$$\mathrm{d}S \geq 0 \tag{1.144}$$

which indicates that the entropy of a thermodynamic system never decreases, either it is constant or it increases. Now, using the definition of entropy (1.143), the 1st law of thermodynamics (1.142) can be written as

$$T\mathrm{d}S = \mathrm{d}E + p\mathrm{d}V \tag{1.145}$$

Now, considering the fact that, flat FLRW metric describes our universe, i.e.,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right] \tag{1.146}$$

We shall now describe the thermodynamics of a perfect fluid. Let p and ρ be the pressure and energy density of the perfect fluid. Then the energy-momentum tensor of the perfect fluid is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \tag{1.147}$$

where u^μ is the four-velocity of the perfect fluid particles. The particle flow vector (N^μ) is defined as

$$N^\mu = nu^\mu \tag{1.148}$$

where n is the particle number density. Assuming the conservation of the particle number of the fluid, i.e., no unbalanced creation or annihilation of the particles, we have the following equation

$$\nabla_\mu N^\mu \equiv \dot{n} + 3Hn = 0 \tag{1.149}$$

and the conservation of energy-momentum tensor gives

$$\nabla_\mu T^{\mu\nu} = 0 \implies \dot{\rho} + 3H(p + \rho) = 0 \tag{1.150}$$

Now if one considers that there are N number of particles having internal energy E in a comoving volume V, then the temperature of the perfect fluid is related by the Gibb's equation

$$T\mathrm{d}s = \mathrm{d}q = \mathrm{d}\left(\frac{\rho}{n}\right) + \mathrm{d}\left(\frac{1}{n}\right) \tag{1.151}$$

where s is the entropy per particle, $n = \frac{N}{V}$, and $\mathrm{d}q = \frac{\mathrm{d}Q}{N}$ is the heat flow per unit particle.

So far perfect fluid dynamics in equilibrium thermodynamics has been discussed. Perfect fluids in equilibrium can't generate entropy since their dynamics is reversible and without dissipation. Nature, on the other hand, says something totally different.

For many processes in cosmology and astrophysics, a perfect fluid model is adequate. However, real fluids behave irreversibly and there are many events in cosmology and astrophysics that can't be explained by this perfect fluid dynamics in equilibrium thermodynamics, but all of them are quite nicely explained and understood by introducing disability process. Thus relativistic theory of dissipative fluids was introduced for a better understanding.

To understand this issue, one needs to bring non-equilibrium and irreversible thermodynamics. Eckart was the first to extend irreversible thermodynamics in 1940. The fundamental hypothesis of the classical theory of non-equilibrium thermodynamics is the local equilibrium hypothesis, stating that each macroscopic point of the system remains in stable equilibrium. The theory demands an entropy production.

For a dissipative fluid, the particle flow vector N^μ has the same form as in (1.148) but the energy-momentum tensor takes the form

$$T^{\mu\nu} = (\rho + p + \Pi)u^\mu u^\nu + (p + \Pi)g^{\mu\nu} \tag{1.152}$$

where the effective pressure (p_{eff}) of the fluid is $p_{\text{eff}} = p + \Pi$. Here Π, the dissipative term is known as the bulk viscous pressure. Due to gravitationally induced particle productions, or annihilation, Π comes into scenario. Now, due to creation/annihilation of the particles, conservation of N^μ gives

$$\nabla_\mu N^\mu \equiv \dot{n} + 3Hn = \Gamma n \tag{1.153}$$

where Γ is called the particle creation rate due to non-equilibrium thermodynamics. $\Gamma \gtrless 0$ according to the creation (or, annihilation) of the particles.

Now, in the context of flat FLRW Universe, the Friedmann equations can be written as

$$3H^2 = \kappa\rho \tag{1.154}$$
$$2\dot{H} = -\kappa(p + \rho + \Pi) \tag{1.155}$$

Further, the Bianchi identity $T^{\mu\nu}_{;\nu} = 0$ gives the conservation relation

$$\dot{\rho} + 3H(p + \rho + \Pi) = 0 \tag{1.156}$$

Now using the conservation equation (1.153) in Gibbs equation (1.151) the entropy variation can be written as

$$nT\dot{s} = -3H\Gamma - \Gamma(\rho + p) \tag{1.157}$$

For adiabatic condition, the above equation can be simplified as

$$\Pi = -\frac{\Gamma}{3H}(\rho + p) \tag{1.158}$$

which shows that the dissipative pressure is entirely characterized by the particle creation rate for the adiabatic thermodynamical system. Further, considering the barotropic nature of the cosmic fluid, one can rewrite the evolution equation in terms of the particle creation rate Γ

$$\frac{\Gamma}{3H} = 1 + \frac{2}{3\gamma}\frac{\dot{H}}{H^2} \tag{1.159}$$

1.5.1 Unified Cosmic Evolution

In the very early universe, most of the particle creation effectively takes place and from thermodynamic viewpoint [108], there should be a maximal entropy production rate (i.e., maximal particle creation rate) at the beginning of the expansion so that the universe evolves from non-equilibrium thermodynamical state to equilibrium epoch as the Universe expands. Initially, there should be a regular (true) vacuum for radiation, i.e., $\rho \to 0$ as $a \to 0$. Also, in the very early Universe one should have $\Gamma > H$ so that the created radiation acts as a thermalized heat bath and subsequently the creation rate becomes slower than the expansion rate, making the creation of the particles dynamically insignificant.

Now, Gunzig *et al.* show [109], the simplest choice that satisfies these conditions is $\Gamma = 3\mu_1 \frac{H^2}{H_1}$ [110], i.e., the particle creation rate is proportional to the energy density, where H_1 has the dimension of Hubble parameter (i.e., reciprocal of time). Thus the dimension of the μ_1 is $M^{-1}T^{-1}$.

The simplest choice of the particle creation rate for the intermediate decelerating phase is $\Gamma \propto H$. One may note that this choice of Γ does not satisfy the third thermodynamical requirement at the early universe.

Further, the thermodynamical requirements for the late time accelerating phase can be obtained by modifying that of the early epoch. At the beginning of the late-time accelerated expansion, there should be minimum entropy production rate and the universe becomes thermodynamically non-equilibrium again. The late-time false vacuum should have $\rho \to 0$ as $a \to \infty$. The creation rate should be faster than the expansion rate. The simplest choice of Γ that satisfies the above requirements for the late time accelerating phase is $\Gamma \propto \frac{1}{H}$.

If two time instants $t = t_1$ and $t = t_2$ have been chosen as the time of transitions from acceleration to deceleration (in the early phase) and again from deceleration to acceleration (in the late era), then the particle creation rates can be denoted as

- $\Gamma = 3\mu_1 \frac{H^2}{H_1}$ for $t \leq t_1$ (called phase I)

- $\Gamma = 3\mu_2 H$ for $t_1 \leq t \leq t_2$ (called phase II)

- $\Gamma = 3\mu_3 \frac{H_2}{H}$ for $t \geq t_2$ (called phase III)

Thus the equation (1.159) together with the above choices of Γ can be considered as the determining equation for the cosmic evolution.

CHAPTER 2

COMPLETE COSMIC EVOLUTION IN EINSTEIN-CARTAN-KIBBLE-SCIAMA GRAVITY

2.1 Prelude

In 1922, Élie Joseph Cartan extended Einstein's gravity by interpreting intrinsic angular momentum (i.e, spin) of the matter in terms of torsion (known as Einstein-Cartan theory (ECT)) [111, 112, 113, 114]. Later in the 1960s introduction of spin of the matter to GR [115, 116], showed ECT to be the simplest classical modification of Einstein's theory [117].

In ECT, space-time contains asymmetric affine (non-Riemannian) connection which characterizes torsion. As a result, the gravitational pull is characterized by the metric tensor as well as the independent torsion field. Geometrically, the curvature tends to bend space-time while torsion twists it. In other words, due to the curvature of space-time, parallel transport of a vector along a closed loop depends on the path while torsion may even impede the formulation of the loop. Further, from the physical viewpoint, matter causes the curvature of the space-time while intrinsic angular momentum of matter characterizes the torsion.

On the other hand, the torsion field does not allow the space-time to be maximally symmetric (i.e. homogeneous and isotropic), the common choice for standard cosmology. Rather, it introduces anisotropic degrees of freedom. To overcome this unusual circumstance, a typical vectorial form of the torsion field has been proposed in [118] in which the space-time torsion fields [119, 120, 121] and the associated matter spin are entirely governed by a homogeneous scalar function. Further, this modified torsion field preserves the symmetry of the corresponding Ricci curvature tensor (in FLRW model) and, thus, the symmetry of the Einstein tensor and energy-momentum tensor. Phenomenologically, torsion plays the role of spatial curvature and has an input in the terms of cosmological constant/ dark energy. So it is speculated that torsion may be responsible for accelerated expansion. However, it is observationally speculated that torsion influences primordial nucleosynthesis, and hence the torsion field can be constrained from the observational viewpoint.

From another viewpoint, the effect of torsion in space-time can be interpreted as the intrinsic angular momentum of fermionic (i.e. spin) particles [122]. Geometrically, it is connected to the asymmetric affine connection of the space-time manifold. As a result, matter field acts as a source of torsion and thereby enriches the cosmic descriptions. Further, the well-known Einstein-Cartan-Kibble-Sciama (ECKS) gravity theory is very useful in terms of the invariance of local gauge with respect to the Poincare group [123, 124]. However, it should be mentioned that although there are suggestions for some experiments, there is no observational evidence supporting the existence of the torsion field.

2.2 Torsion in FLRW cosmology

The Einstein-Cartan theory is based on the asymmetric affine connection of space-time and one can define the torsion tensor as the antisymmetric part of the affine connection

$$S^a_{bc} = \Gamma^a_{[bc]}, \tag{2.1}$$

that vanishes in absence of torsion. Since the metric tensor is covariantly constant (i.e, $\nabla_c g_{ab} = 0$), a generalized connection can be decomposed into a symmetric and an antisymmetric part as

$$\Gamma^a_{bc} = \tilde{\Gamma}^a_{bc} + K^a_{bc}, \tag{2.2}$$

where the symmetric part $\tilde{\Gamma}^a_{bc}$ is the usual Christoffel symbol and the antisymmetric part K^a_{bc} is termed as the contortion tensor with $K_{abc} = K_{[ab]c}$ which is related to the torsion tensor as

$$K_{abc} = S_{abc} + 2S_{(bc)a}. \tag{2.3}$$

Thus torsion can be considered as a connecting tool between the intrinsic angular momentum (spin) of the matter and the geometry of space-time.

Due to the antisymmetric nature of the torsion tensor, it can be defined as

$$S_a \equiv S^b{}_{ab} \left(= -S^b{}_{ba}\right), \tag{2.4}$$

and consequently, for the contortion tensor one has

$$K^b{}_{ab} = 2S_a = -K_{ab}{}^b. \tag{2.5}$$

Now considering the FLRW line element (1.5) and assuming that the Universe is consisting of perfect fluid with barotropic equation of state given by $p = \omega\rho$ and $\omega = \gamma - 1$, the torsion tensor and the contortion tensor can be expressed as

$$S_{abc} = 2\phi h_{a[b}u_{c]}, \tag{2.6}$$
$$K_{abc} = 4\phi u_{[a}h_{b]c}, \tag{2.7}$$

where $\phi = \phi(t)$ is a scalar function, h_{ab} is the metric of the three-space and u_a is the four-velocity field along the tangent to a congruence of time-like curves. For spatially homogeneous and isotropic FLRW space-time torsion vector is completely characterized by the torsion tensor (and hence the contortion tensor). For this space-time, the torsion vector takes the form [125, 126, 127, 128]

$$S_a = -3\phi u_a, \tag{2.8}$$

The torsion vector is a time-like vector [129] and (the sign of) the scalar function ϕ indicates the relative orientation between the torsion and the four-velocity (i.e, torsion vector is future-directed for $\phi < 0$, while it is past directed for $\phi > 0$).

Further, the matter conservation equation in FLRW model can be written as [125]

$$T^{b}_{a;b} = -4\phi T_{ab}u^{b}, \tag{2.9}$$

which for perfect fluid has the explicit form

$$\dot{\rho} + 3(H + 2\phi)(\rho + p) = 4\phi\rho, \tag{2.10}$$

where ρ and p are the energy density and thermodynamic pressure of the perfect fluid respectively. Hence, the modified Friedmann equations due to torsion in the background of homogeneous and isotropic flat FLRW space-time can be written as

$$3H^2 = \kappa\rho - 12\phi^2 - 12H\phi, \tag{2.11}$$
$$2\dot{H} = -\kappa(\rho + p) - 4\dot{\phi} + 8\phi^2 + 4H\phi. \tag{2.12}$$

2.3 Different cosmic eras in torsion gravity

In this section, several cosmological solutions have been presented with proper choices of torsion scalar function. Throughout this section, the torsion scalar function is chosen in a typical (but general) form as

$$\phi = -\lambda(a)H, \tag{2.13}$$

where $\lambda(a)$ is an arbitrary function of the scale factor. Although the above choice is purely phenomenological, there is some reason behind this choice. From equation (2.11), one may note that if there is no matter (i.e., $\rho = 0$), $\phi = -2H$ i.e., $\phi \propto H$. So generalizing it, the above choice (2.13) has been made. The modified Friedmann equations (2.11) and (2.12) with the above choice (2.13) for ϕ give the cosmic evolution equation as,

$$\frac{2\dot{H}}{3H^2} = \frac{4a\lambda'}{3(1-2\lambda)} + 2\lambda\left(\gamma - \frac{2}{3}\right) - \gamma. \tag{2.14}$$

where ' $'$ ' denotes differentiation with respect to scale factor a.

2.3.1 Emergent scenario: non-singular solution

In this section, it will be examined whether it is possible to have an emergent scenario (non-singular cosmological solution) for this gravity. Now choosing λ in such a way that it satisfies

$$\frac{2a\lambda'}{(1-2\lambda)} + 3\lambda\left(\gamma - \frac{2}{3}\right) = \frac{\mu}{H}, \tag{2.15}$$

where μ is an arbitrary constant and consequently equation (2.14) takes the form

$$\dot{H} = \mu H - \frac{3\gamma}{2}H^2, \tag{2.16}$$

Now depending on the signs of μ the possible solutions are as follows.

Case 1, $\mu > 0$,

$$\begin{aligned} \frac{H}{H_0} &= \frac{\mu}{\frac{3\gamma}{2}H_0 - \left(\frac{3\gamma}{2}H_0 - \mu\right)e^{-\mu(t-t_0)}}, \\ \frac{a}{a_0} &= \left[1 + \frac{3\gamma H_0}{2\mu}\left(e^{\mu(t-t_0)} - 1\right)\right]^{\frac{2}{3\gamma}}. \end{aligned} \tag{2.17}$$

Case 2, $\mu = 0$,

$$\begin{aligned} \frac{H_0}{H} &= 1 + \frac{3\gamma}{2}H_0(t-t_0), \\ \frac{a}{a_0} &= \left[1 + \frac{3\gamma}{2}H_0(t-t_0)\right]^{\frac{2}{3\gamma}}. \end{aligned} \tag{2.18}$$

Case 3, $\mu < 0$,

$$\begin{aligned} \frac{H}{H_0} &= \frac{|\mu|}{\left(|\mu| + \frac{3\gamma}{2}H_0\right)e^{|\mu|(t-t_0)} - \frac{3\gamma}{2}H_0}, \\ \frac{a}{a_0} &= \left[1 + \frac{3\gamma H_0}{2|\mu|}\left(1 - e^{-|\mu|(t-t_0)}\right)\right]^{\frac{2}{3\gamma}}, \end{aligned} \tag{2.19}$$

In the above solutions λ_0, t_0, a_0, and H_0 are integration constants with $\lambda = \lambda_0$, $a = a_0$, $H = H_0$ at $t = t_0$.

Note that equation (2.17) represents big bang singularity for $\mu < \frac{3\gamma}{2}H_0$, while for $\mu > \frac{3\gamma}{2}H_0$ the solution (2.17) represents the emergent scenario of the universe and $\mu = \frac{3\gamma}{2}H_0$ represents only the inflationary era (i.e. the exponential expansion). The asymptotic limits for the emergent scenario are given by

(i) $a \to a_0 \left(1 - \frac{3\gamma H_0}{2\mu}\right)^{\frac{2}{3\gamma}}$, $H \to 0$ as $t \to -\infty$,

(ii) $a \sim a_0 \left(1 - \frac{3\gamma H_0}{2\mu}\right)^{\frac{2}{3\gamma}}$, $H \sim 0$ as $t \ll t_0$,

(iii) $a \sim a_0 \left(\frac{3\gamma H_0}{2\mu}\right)^{\frac{2}{3\gamma}} e^{\frac{2\mu}{3\gamma}(t-t_0)}$, $H \sim \frac{2\mu}{3\gamma}$ as $t \gg t_0$.

and the parameter λ can be explicitly written (for $\mu = 3\gamma H_0$) as

$$\frac{1}{1-2\lambda} = \frac{aH}{H_0}\left[\frac{1}{a_0(1-2\lambda_0)} + \left(\frac{3\gamma}{2} - 1\right) H_0 \int_{t_0}^{t} \frac{dt}{a(t)}\right]$$

Also (2.18) and (2.19) represent the big bang singularity and the big bang singularity occurs at the time $t = t_s$ given by,

$$t_s = \begin{cases} t_0 + \frac{1}{\mu}\ln\left|1 - \frac{2\mu}{3\gamma H_0}\right| & \text{for the equation (2.17)}, \\ t_0 - \frac{2}{3\gamma H_0} & \text{for the equation (2.18)}, \\ t_0 - \frac{1}{\mu}\ln\left|\frac{H_0}{H_0 - \frac{2\mu}{3\gamma}}\right| & \text{for the equation (2.19)}. \end{cases}$$

2.3.2 Continuous cosmic evolution in torsion gravity

Let t_1 be the time instant in which the universe evolves from the inflationary era to the matter-dominated era. Similarly, $t_2(> t_1)$ is the time instant at which the universe transits into the late-time accelerated era [130].

Inflationary era ($t < t_1$):

The choice for λ is

$$\frac{1}{1-2\lambda} = 1 + \frac{H}{H_1}\left[\frac{a}{a_1}\left(\frac{1}{1-2\lambda_1} - 1 + \frac{3\gamma\mu_1}{2}\right) - \frac{3\gamma\mu_1}{2}\right],$$

and the cosmic solution can be written as

$$\frac{H_1}{H} = \mu_1 + (1-\mu_1)\left(\frac{a}{a_1}\right)^{\frac{3\gamma}{2}}, \tag{2.20}$$

$$\text{i.e,}\ H = H_1\left[\mu_1\left\{LambertW\left(\frac{1-\mu_1}{\mu_1}\exp\left\{\frac{2(1-\mu_1)+3\gamma H_1(t-t_1)}{2\mu_1}\right\}\right)+1\right\}\right]^{-1},$$

$$a = a_1\left[\frac{\mu_1}{1-\mu_1}LambertW\left(\frac{1-\mu_1}{\mu_1}\exp\left\{\frac{2(1-\mu_1)+3\gamma H_1(t-t_1)}{2\mu_1}\right\}\right)\right]^{\frac{2}{3\gamma}},$$

$$q = \frac{3\gamma}{2} - 1 - \mu_1\frac{H}{H_1}, \tag{2.21}$$

where $LambertW(x)e^{LambertW(x)} = x$.

Matter-dominated era $(t_1 < t < t_2)$:

The choice for λ is

$$\frac{1}{1-2\lambda} = \frac{3\gamma-2}{3\gamma(1-\mu_2)-2} + \frac{aH}{a_1H_1}\left[\frac{1}{1-2\lambda_1} - \frac{3\gamma-2}{3\gamma(1-\mu_2)-2}\right],$$

and the cosmic solution is in the form of

$$H = H_1\left(\frac{a}{a_1}\right)^{-\frac{3\gamma}{2}(1-\mu_2)}, \tag{2.22}$$

$$\text{i.e,}\ H = H_1\left[1+\frac{3\gamma}{2}H_1(1-\mu_2)(t-t_1)\right]^{-1},$$

$$a = a_1\left[1+\frac{3\gamma}{2}H_1(1-\mu_2)(t-t_1)\right]^{\frac{2}{3\gamma(1-\mu_2)}},$$

$$q = \frac{3\gamma}{2} - 1 - \mu_2. \tag{2.23}$$

Late-time accelerated era $(t > t_2)$:

The choice for λ is

$$\frac{1}{1-2\lambda} = \frac{aH}{H_2}\left[\frac{1}{a_2(1-2\lambda_2)} - \left(\frac{3\gamma}{2}-1\right)H_2\int_{t_2}^{t}\frac{dt}{a(t)}\right],$$

and the cosmic solution is given by

$$H = H_2\left[\mu_2 + (1-\mu_2)\left(\frac{a}{a_2}\right)^{-3\gamma}\right]^{\frac{1}{2}}, \tag{2.24}$$

$$\text{i.e,}\ H = \sqrt{\mu_2}H_2\coth\left\{\frac{3\gamma}{2}\sqrt{\mu_2}H_2(t-t_i)\right\}$$

$$a = a_2\left[\sqrt{\frac{1-\mu_2}{\mu_2}}\sinh\left\{\frac{3\gamma}{2}\sqrt{\mu_2}H_2(t-t_i)\right\}\right]^{\frac{2}{3\gamma}},$$

$$q = \frac{3\gamma}{2} - 1 - \mu_2 \left(\frac{H_2}{H}\right)^2. \tag{2.25}$$

where t_i is the integration constant. Here (λ_1, a_1, H_1) and (λ_2, a_2, H_2) are the values of parameter λ, scale factor, and Hubble parameter respectively at the transition points $t = t_1$ and $t = t_2$.

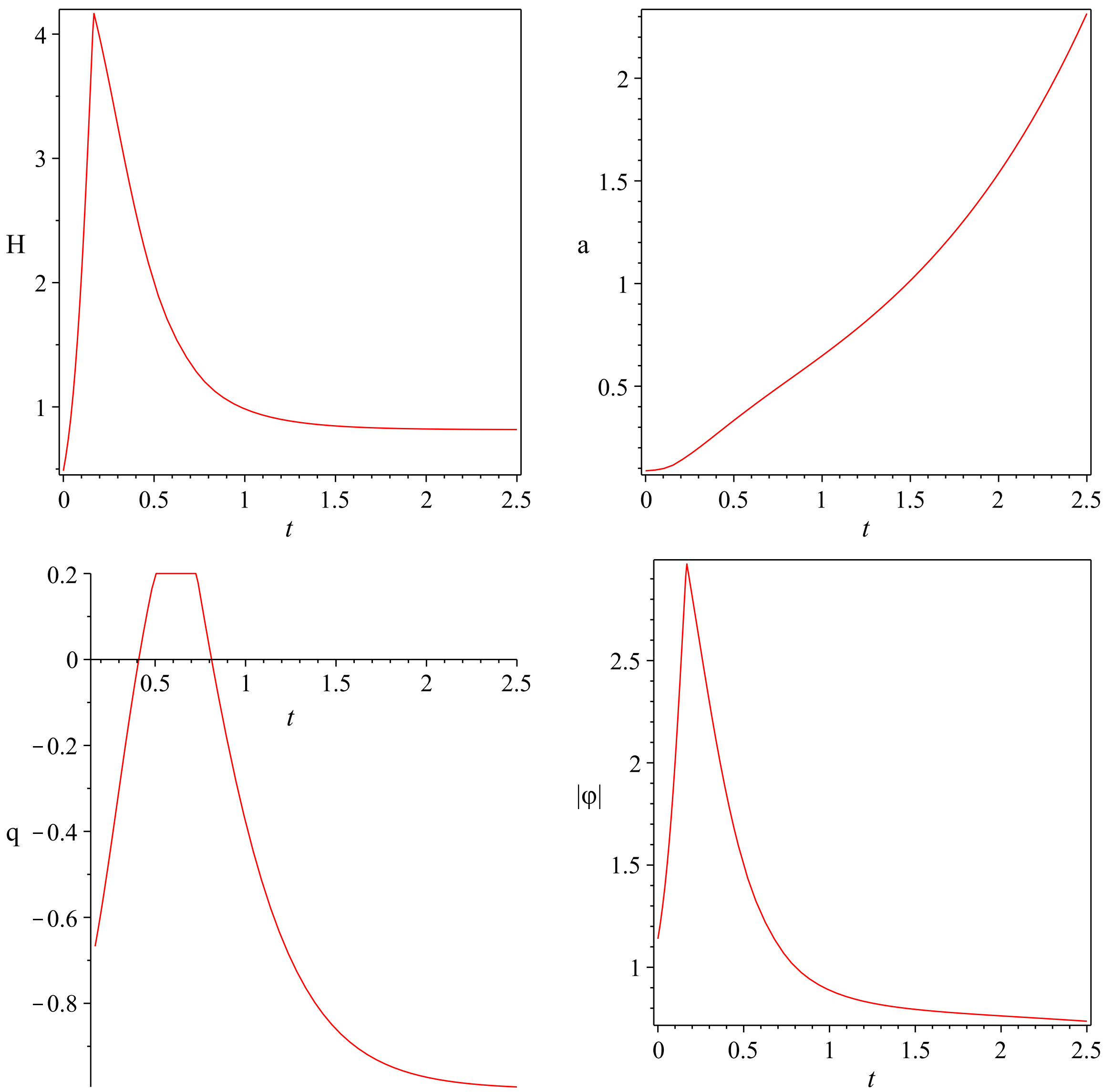

Figure 2.1: The evolution of Hubble parameter $H(t)$ (upper left panel), scale factor $a(t)$ (upper right panel), deceleration parameter q (lower left panel), the modulus of torsion $|\phi|$ (lower right panel) with respect to cosmic time

Now it will be examined whether this cosmic evolution across these three phases is continuous or not. Then continuity of the deceleration parameter at the transition time $t = t_1$ and $t = t_2$, gives

$$\mu_1 = \mu_2.$$

Continuity of physical parameters (i.e Hubble parameter, scale factor) at $t = t_1$ is obvious while the continuity at $t = t_2$ demands

$$\frac{3\gamma}{2}(1-\mu_2)(t_2-t_1) = \frac{1}{H_2} - \frac{1}{H_1} \quad , \quad \left(\frac{a_1}{a_2}\right)^{\frac{3\gamma(1-\mu_2)}{2}} = \frac{H_2}{H_1}, \tag{2.26}$$

$$\frac{1}{1-2\lambda_2} = \frac{3\gamma-2}{3\gamma(1-\mu_2)-2} + \frac{a_2 H_2}{a_1 H_1}\left[\frac{1}{1-2\lambda_1} - \frac{3\gamma-2}{3\gamma(1-\mu_2)-2}\right], \tag{2.27}$$

$$\sinh(\eta_2) = \sqrt{\frac{\mu_2}{1-\mu_2}}, \tag{2.28}$$

where

$$\eta_2 = \frac{3\gamma}{2}\sqrt{\mu_2}H_2(t_2 - t_i).$$

Also, let $t = t_0(< t_1)$ be the time instant in which the universe makes a transition from the emergent scenario to inflationary era. Then from the continuity across the transition time $t = t_0$, one has

$$\frac{1}{1-2\lambda_0} = 1 + \frac{H_0}{H_1}\left[\frac{a_0}{a_1}\left(\frac{1}{1-2\lambda_1} - 1 + \frac{3\gamma\mu_1}{2}\right) - \frac{3\gamma\mu_1}{2}\right],$$
$$\frac{H_0}{H_1} = \mu_1 + (1-\mu_1)\left(\frac{a_0}{a_1}\right)^{\frac{3\gamma}{2}}, \tag{2.29}$$

where λ_0, H_0 and a_0 are the values of parameter λ, Hubble parameter, and scale factor at $t = t_0$ respectively. Also the smoothness of physical parameters H, a, q, and $|\phi|$ have been presented graphically in Figure 2.1 for the parameter values $\mu_1 = 0.4$, $\gamma = \frac{4}{3}$, $H_1 = 2$, $a_1 = \frac{1}{3}$, $H_2 = 1.29$, $t_1 = 0.5$, $t_0 = \frac{1}{6}$, $\lambda_1 = 0.75$ and $\mu = 3\gamma H_0$.

2.4 Torsion gravity and Einstein gravity: A correspondence

In this section the thermodynamics of the gravity theory with torsion has been discussed in FLRW model. Equations (2.11) and (2.12) can be written as

$$\begin{aligned} 3H^2 &= \kappa\rho - 12\phi^2 - 12H\phi \\ &= \kappa(\rho + \rho_e) = \kappa\rho_T, \end{aligned} \tag{2.30}$$

$$\begin{aligned} 2\dot{H} &= -\kappa\gamma\rho - 4\dot{\phi} + 8\phi^2 + 4H\phi \\ &= -\kappa\left[(p+\rho) + (p_e + \rho_e)\right] = -\kappa(p_T + \rho_T), \end{aligned} \tag{2.31}$$

where ρ_e, p_e are energy density and thermodynamic pressure of the effective fluid particles respectively and are defined as,

$$\kappa\rho_e = -12\phi^2 - 12H\phi, \quad (2.32)$$
$$\kappa p_e = 4\phi^2 + 8H\phi + 4\dot{\phi}. \quad (2.33)$$

From the field equations (2.30) and (2.31) due to Bianchi identity one has,

$$\dot{\rho_T} + 3H(\rho_T + p_T) = 0. \quad (2.34)$$

From equations (2.10) and (2.34) one have the individual matter conservation equations as,

$$\dot{\rho} + 3H(\rho + p) = Q, \quad (2.35)$$
$$\dot{\rho_e} + 3H(\rho_e + p_e) = -Q. \quad (2.36)$$

Thus the modified Friedmann equations can be interpreted as Friedmann equations in Einstein gravity for an interacting two-fluid system of which one is the usual normal fluid under consideration and the other is the effective fluid and the interaction term is given by $Q = -2(\rho + 3p)\phi$.

For interacting two-fluid system, the interacting term Q should be positive as the energy is transferred to the usual fluid. In this case $Q > 0 \implies \phi < 0$.

One can write the above conservation equations (2.35) and (2.36) in terms of state parameter as

$$\omega^{(eff)} = \omega + 2(1 + 3\omega)\frac{\phi}{3H}, \quad (2.37)$$
$$\omega_e^{(eff)} = \omega_e - 2(1 + 3\omega)\frac{\phi}{3H}r, \quad (2.38)$$

where $\omega_e = \dfrac{p_e}{\rho_e}$ is equation of state parameter of effective fluid and $r = \dfrac{\rho}{\rho_e}$ is coincidence parameter.

Equations (2.35) and (2.36) can also be written as

$$\dot{\rho} + 3H(\rho + p + p_c) = 0, \quad (2.39)$$
$$\dot{\rho_e} + 3H(\rho_e + p_e + p_{ce}) = 0, \quad (2.40)$$

with

$$p_c = \frac{2\phi}{3H}(\rho + 3p) = -p_{ce}, \quad (2.41)$$

where p_c, p_{ce} are the dissipative pressure of the fluid components.

In non-equilibrium thermodynamics this dissipative pressure may be caused by the particle creation process. So the particle number conservation equations take the form,

$$\dot{n} + 3Hn = \Gamma n, \tag{2.42}$$
$$\dot{n_e} + 3Hn_e = \Gamma_e n_e. \tag{2.43}$$

Here n denotes the normal fluid particle density and n_e represents the number density of effective fluid particles.

If one assumes the non-equilibrium thermodynamic process to be adiabatic then the dissipative pressures are related to the particle creation rate linearly as [131],

$$\begin{aligned} p_c &= -\frac{\Gamma}{3H}(p+\rho), \\ p_{ce} &= \frac{\Gamma_e}{3H}(p_e+\rho_e). \end{aligned} \tag{2.44}$$

Comparing equations (2.41) and (2.44) one has,

$$\Gamma = -2\phi\frac{\rho+3p}{\rho+p}, \tag{2.45}$$
$$\Gamma_e = 2\phi\frac{\rho+3p}{\rho_e+p_e}. \tag{2.46}$$

Thus the particle creation rate is directly related to ϕ. It is also clear that $\Gamma > 0$ i.e, usual fluid particles are created and $\Gamma_e < 0$ i.e, effective particles are annihilated. Also $p_{ct} = p_c + p_{ce} = 0$ so the particle creation rate for resulting fluid particles vanishes identically. Hence resulting fluid forms a closed system.

Further, due to the particle creation mechanism, there is an energy transfer between the two fluid systems. So these two systems may have different temperatures.

Using Euler's thermodynamical equation, the evolution of the temperature of the individual fluid is given by [132],

$$\frac{\dot{T}}{T} = -3H\left(\omega^{(eff)} + \frac{\Gamma}{3H}\right) + \frac{\dot{\omega}}{1+\omega}, \tag{2.47}$$
$$\frac{\dot{T}_e}{T_e} = -3H\left(\omega_e^{(eff)} - \frac{\Gamma_e}{3H}\right) + \frac{\dot{\omega}_e}{1+\omega_e}, \tag{2.48}$$

where $\omega^{(eff)}$ and $\omega_e^{(eff)}$ defined in equations (2.37) and (2.38) can be written in terms of particle creation rate as,

$$\omega^{(eff)} = \omega - \frac{\Gamma}{3H}(1+\omega), \tag{2.49}$$
$$\omega_e^{(eff)} = \omega_e + \frac{\Gamma_e}{3H}(1+\omega). \tag{2.50}$$

Integrating equations (2.47) and (2.48) one has,

$$T = T_0(1+\omega)\exp\left[-3\int_{a_0}^{a}\omega\left(1-\frac{\Gamma}{3H}\right)\frac{da}{a}\right], \tag{2.51}$$

$$T_e = T_0(1+\omega_e)\exp\left[-3\int_{a_0}^{a}\omega_e\left(1+\frac{\Gamma_e}{3H}\right)\frac{da}{a}\right], \tag{2.52}$$

where T_0 is the common temperature of the two fluids in equilibrium phase and a_0 is the scale factor in the equilibrium state. In particular, using equations (2.45) and (2.51), the temperature of the normal fluid for constant ω can be written explicitly in the following form,

$$T = T_0(1+\omega)\left(\frac{a}{a_0}\right)^{-3\omega}\exp\left[-2\int_{t_0}^{t}\frac{\omega(1+3\omega)}{1+\omega}\phi dt\right]. \tag{2.53}$$

In general, at very early phases of the evolution of the universe $T_e < T$, and then when the cosmic fluid attains equilibrium i.e. $a = a_0$, one has $T = T_e = T_0$. In the next phase of evolution of universe, one has $a > a_0$ and $T_e > T$ because energy flows from effective fluid to the usual fluid continuously and hence the thermodynamical equilibrium is violated. Now, from thermodynamical consideration, equilibrium temperature T_0 can be considered as the (modified) Hawking temperature [133] i.e,

$$T_0 = \left.\frac{H^2 R_h}{2\pi}\right|_{a=a_0}, \tag{2.54}$$

where R_h is the geometric radius of the horizon, bounding the universe.

2.5 Conclusion

This chapter presents a detailed cosmological study of gravity with torsion. At first, it is examined whether a non-singular universe model is possible or not, and it is found that emergent scenario is possible for a particular choice of torsion field. Also, it is possible to have a complete cosmic evolution starting from inflationary era to present late-time accelerated era through the matter-dominated era for suitable evolution of torsion scalar function. Further, it is shown that the modified Friedmann equations for this gravity may be regarded as the Friedmann equations for Einstein gravity with interacting two-fluid system of which one is the usual fluid and the other is effective fluid. In course of cosmic evolution, the former is created and the latter is annihilated. Also from thermodynamical consideration, the temperatures of individual fluid particles are evaluated and are found to be distinct. Lastly, different choices of hypothetical fluids component give rise to different cosmic solutions.

CHAPTER 3

COSMOLOGICAL EVOLUTION IN $f(T)$ GRAVITY

3.1 Prelude

In teleparallel gravity, instead of curvature, torsion represents the gravitational interaction [134, 135, 136, 137]. In fact, Einstein suggested this model initially to unify electromagnetism and gravity over Weitzenböck non-Riemannian manifold. As a result, in the underlying Riemann-Cartan space-time, the Levi-Civita connection is replaced by Weitzenböck connection. Hence pure geometric nature of gravitational interaction is no longer there rather torsion acts as a force, and gravity may be interpreted as a gauge theory of the translation group [138]. It is to be noted that although there are conceptual directions between these two (i.e. GR and teleparallel) gravity theories, still they have equivalent dynamics at least at all the classical level.

Further, a generalization to teleparallel gravity has been developed [139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149] by substituting torsion scalar T with a generic function $f(\mathrm{T})$. Linder proposed the term $f(\mathrm{T})$ gravity to describe this modified gravity theory. Consequently, there is no equivalence (with GR) at the classical level i.e. distinct dynamics comes into scenario. Moreover, there are two significant differences between these two theories namely (i) The field equations in $f(\mathrm{T})$ gravity remain second order while one has fourth order equations in $f(R)$ [150], (ii) $f(\mathrm{T})$ gravity does not satisfy local Lorentz invariance (which is satisfied by $f(R)$ gravity) so that all 16 components of the vierbien are independent and hence six of them cannot be fixed by a gauge choice [151]. The dynamical object in this model is the four linearly independent vierbein (tetrad) fields that form the orthogonal bases for the tangent space at each point of space-time. The vierbeins are parallel vector fields, giving the theory the descriptor "teleparallel". The advantage is that the torsion tensor is formed entirely from products of the first derivatives of the tetrad. Inflation may be avoided by using this modified teleparallel gravity [139].

3.2 Brief review of $f(\mathrm{T})$ gravity

The action for $f(\mathrm{T})$ gravity can be written as ($\kappa = 8\pi G = 1$) [152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165]

$$\mathcal{A} = \frac{1}{2}\int d^4x \, |e| \, \left[T + f(\mathrm{T}) + L_m\right] \tag{3.1}$$

where T is torsion scalar, $f(\mathrm{T})$ is a differentiable function of torsion, $|e| = \det\left(e^A_\mu\right) = \sqrt{-g}$, and L_m is the matter Lagrangian.

Varying the above action (3.1) the modified Einstein field equations become

$$\begin{aligned} e^{-1}\partial_\mu(ee^\rho_A S_\rho{}^{\mu\nu})[1+f_T] + e^\rho_A S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT} &- [1+f_T]e^\lambda_A T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu} \\ &+ \frac{1}{4}e^\nu_A[T+f(\mathrm{T})] = 4\pi G e^\rho_A T_\rho{}^\nu \end{aligned} \tag{3.2}$$

Then for FLRW model the modified Friedmann equations become

$$3H^2 = \frac{1}{2f_T+1}\left[\rho - \frac{f}{2}\right], \tag{3.3}$$

$$2\dot{H} = -\frac{(p+\rho)}{1+f_T+2Tf_{TT}}. \tag{3.4}$$

where p and ρ are respectively the thermodynamic pressure and density of the matter fluid having the conservation equation

$$\dot{\rho} + 3H(p+\rho) = 0 \tag{3.5}$$

It is to be noted that for various choices of $f(\mathrm{T})$, one has different gravity theories and it is presented in the following table.

Choice of $f(\mathrm{T})$	Gravity theory
0	Einstein gravity
Nonzero Constant	Einstein gravity with Cosmological Constant
$f(\mathrm{T}) = \alpha T$, $\alpha \neq 0$, a constant	Einstein gravity with the reconstruction of gravitational constant
General $f(\mathrm{T})$	Modified gravity theory

Assuming fluid to be barotropic in nature with equation of state $p = \omega\rho = (\gamma-1)\rho$, $0 \leq \gamma \leq 2$ and eliminating ρ from equations (3.4) and (3.5), one has the differential equation for cosmic evolution as,

$$2\dot{H}\left(1+f_T+2Tf_{TT}\right) = \frac{\gamma}{2}\left[T(2f_T+1) - f\right]. \tag{3.6}$$

3.3 Emergent scenario in $f(\mathrm{T})$ gravity

In the context of cosmic evolution the most common idea of the beginning of the Universe is the standard big bang model which evolves from a singularity. To solve this problem of singularity, cosmologists propose the idea of emergent scenario of the Universe which is a non-singular model and the cosmological solution represents Einstein static model in past infinity.

In cosmic evolution to have an emergent scenario, the evolution of Hubble parameter is given by the differential equation [166, 167, 269]

$$\dot{H} = \alpha H + \lambda H^2 \tag{3.7}$$

with parameters α and λ are restricted as $\lambda < 0$ and $\alpha > |\lambda H_0|$ for emergent scenario. Comparing equation (3.7) with the evolution equation (3.6) in $f(\mathrm{T})$ cosmology, $f(\mathrm{T})$ is chosen as

$$f(\mathrm{T}) = b + c\sqrt{|T|} - T + d\sqrt{|T|}\ln(|T|) \tag{3.8}$$

where b, c, and d are arbitrary constants and hence

$$\dot{H} = \beta H - 3\gamma H^2 \tag{3.9}$$

where $\beta = \dfrac{\gamma b\sqrt{6}}{4d} = \gamma\mu,\ \mu = \dfrac{b\sqrt{6}}{4d}$. Thus this choice of $f(\mathrm{T})$ in equation (3.8) is purely phenomenological with the motivation of obtaining a solution for emergent scenario. One may note that since this choice of $f(\mathrm{T})$ contains terms such as square root of T and the logarithm of T, perturbations must be problematic around emergence of space-time when T tends to zero.

Depending on the sign of β (i.e. μ), various solutions may be obtained.

If $\beta > 0$ (i.e. $\mu > 0$), then the cosmological solutions are given by

$$\begin{aligned} \frac{H}{H_0} &= \frac{\beta}{3\gamma H_0 - (3\gamma H_0 - \beta)\, e^{-\beta(t-t_0)}}, \\ \frac{a}{a_0} &= \left[1 + \frac{3\gamma H_0}{\beta}\left(e^{\beta(t-t_0)} - 1\right)\right]^{\frac{1}{3\gamma}}. \end{aligned} \tag{3.10}$$

It may be noted that equation (3.10) represents big bang solution for $\beta < 3\gamma H_0$ (i.e. $\mu < 3H_0$) while equation (3.10) represents the emergent solution for $\beta > 3\gamma H_0$ (i.e. $\mu > 3H_0$) and $\beta = 3\gamma H_0$ (i.e. $\mu = 3H_0$) gives the inflationary solution (i.e. the exponential expansion). In particular, if one considers

$$\beta = 6\gamma H_0 \ \text{ i.e. } \ \mu = 6H_0$$

the cosmological solutions are in the following form

$$\frac{H}{H_0} = 2 - \left(\frac{a}{a_0}\right)^{-3\gamma} \tag{3.11}$$

$$\begin{aligned} \text{i.e.}\quad \frac{H}{H_0} &= \frac{2}{1+e^{-6\gamma H_0(t-t_0)}}, \\ \frac{a}{a_0} &= \left[\frac{1+e^{6\gamma H_0(t-t_0)}}{2}\right]^{\frac{1}{3\gamma}}, \\ \text{and}\quad \frac{\rho}{\rho_0} &= \left(\frac{a}{a_0}\right)^{-3\gamma}. \end{aligned} \tag{3.12}$$

For emergent solution represented by equation (3.12) one has the following asymptotic behavior

(i) $a \to a_0 2^{-\frac{1}{3\gamma}}$, $H \to 0$ as $t \to -\infty$,

(ii) $a \sim a_0 2^{-\frac{1}{3\gamma}}$, $H \sim 0$ as $t \ll t_0$,

(iii) $a \sim a_0 2^{-\frac{1}{3\gamma}} e^{2H_0(t-t_0)}$, $H \sim 2H_0$ as $t \gg t_0$.

Also $\beta = 0$ (i.e. $b = 0$) and $\beta(= -\nu^2) < 0$ (i.e. $\mu < 0$) represent big bang solutions and the solutions are given by

$$\begin{aligned} \frac{H_0}{H} &= 1 + 3\gamma H_0(t-t_0), \\ \frac{a}{a_0} &= \left[1 + 3\gamma H_0(t-t_0)\right]^{\frac{1}{3\gamma}}. \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \frac{H}{H_0} &= \frac{\nu^2}{(\nu^2 + 3\gamma H_0)\, e^{\nu^2(t-t_0)} - 3\gamma H_0}, \\ \frac{a}{a_0} &= \left[1 + \frac{3\gamma H_0}{\nu^2}\left(1 - e^{-\nu^2(t-t_0)}\right)\right]^{\frac{1}{3\gamma}}, \end{aligned} \tag{3.14}$$

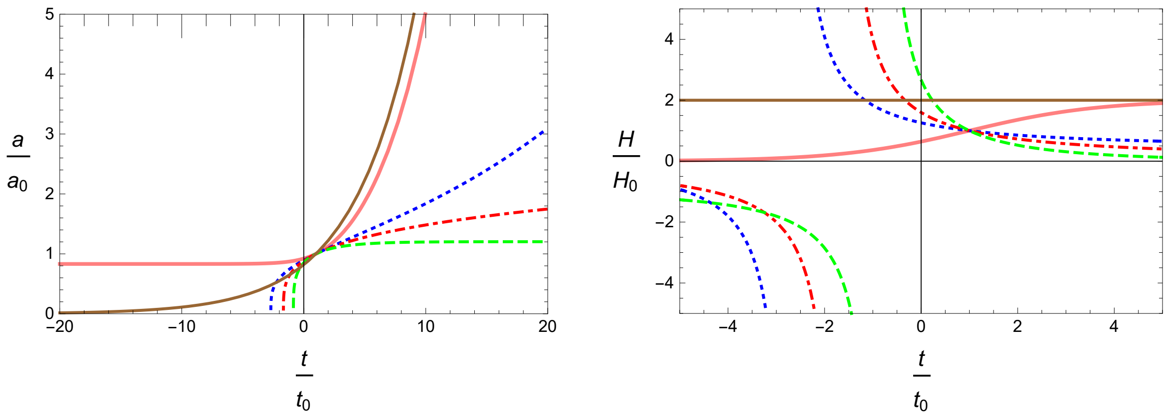

Figure 3.1: Evolution of scale factor $a(t)$ (left panel) and Hubble parameter $H(t)$ (right panel) with respect to cosmic time t in early Universe.

where t_0, a_0, H_0 and ρ_0 are integration constants, $a = a_0$, $H = H_0$, $\rho = \rho_0 \left(= \sqrt{6} d H_0\right)$ at $t = t_0$. Hence it is possible to have an emergent scenario of the Universe in f(T) gravity theory. Also, the cosmic evolution described above (in equations (3.10)-(3.14)) have been shown graphically in Figure 3.1 for the parameter values $H_0 = 1$, $a_0 = 0.1$, $t_0 = 0.1$. Here Pink line, blue line, brown line, red line, and green line represent the solutions for equation (3.12), equation (3.10) with $\beta = \dfrac{3\gamma H_0}{2}$ and $\beta = 3\gamma H_0$, equation (3.13), and equation (3.14) with $\beta = -3\gamma H_0$ respectively.

3.4 Equivalence between f(T) gravity and Einstein gravity

The modified Friedmann equations in f(T) gravity given by equations (3.3) and (3.4) can be written in the following forms

$$\begin{aligned} 3H^2 &= \rho - \frac{f}{2} + T f_T \\ &= (\rho + \rho_e) = \rho_T, \qquad (3.15) \\ 2\dot{H} &= -\gamma\rho - 2\dot{H}(f_T + 2T f_{TT}) \\ &= -\left[(p + \rho) + (p_e + \rho_e)\right] = -(p_T + \rho_T), \qquad (3.16) \end{aligned}$$

where ρ_e, p_e are the energy density and thermodynamic pressure of the effective fluid respectively and are given by,

$$\rho_e = T f_T - \frac{f}{2}, \qquad (3.17)$$

$$p_e = 2\dot{H}(f_T + 2T f_{TT}) - T f_T + \frac{f}{2}. \qquad (3.18)$$

From the field equations (3.15) and (3.16) due to Bianchi identity one has,

$$\dot{\rho_T} + 3H(\rho_T + p_T) = 0, \qquad (3.19)$$

From equations (3.5) and (3.19) one has the individual matter conservation equations as,

$$\dot{\rho} + 3H(\rho + p) = 0, \qquad (3.20)$$

$$\dot{\rho_e} + 3H(\rho_e + p_e) = 0. \qquad (3.21)$$

Thus the modified Friedmann equations in f(T) gravity can be interpreted as Friedmann equations in Einstein gravity for an non-interacting two fluid system of which one is the usual normal fluid under consideration and the other is the effective fluid.

Further using Euler's thermodynamical equation, the evolution of the temperature of the individual fluid are given by [132],

$$\frac{\dot{T}}{T} = -3H\omega + \frac{\dot{\omega}}{1+\omega}, \tag{3.22}$$

$$\frac{\dot{T}_e}{T_e} = -3H\omega_e + \frac{\dot{\omega}_e}{1+\omega_e}, \tag{3.23}$$

Integrating equations (3.22) and (3.23) one has,

$$T = T_0(1+\omega)\exp\left[-3\int_{a_0}^{a}\omega\frac{da}{a}\right], \tag{3.24}$$

$$T_e = T_{0e}(1+\omega_e)\exp\left[-3\int_{a_0}^{a}\omega_e\frac{da}{a}\right], \tag{3.25}$$

where T_0 and T_{0e} are constants of integration. Here a_0 is the value of the scale factor at the equilibrium era with equilibrium temperature

$$T_E = T_0(1+\omega_0) = T_{0e}(1+\omega_{0e})$$

where ω_0 and ω_{0e} are respectively the values of the equation of state parameters ω and ω_e at the equilibrium epoch. In particular, using equations (3.24), the temperature of the normal fluid for constant ω can be written explicitly in the following form,

$$T = T_0(1+\omega)\left(\frac{a}{a_0}\right)^{-3\omega}. \tag{3.26}$$

In general, at very early phases of the evolution of the universe $T_e < T$, and then with the evolution of the Universe T decreases while T_e increases, and then a state of thermal equilibrium will occur at $T = T_e = T_E$ at $a = a_0$. In the next phase of evolution of universe, one has $a > a_0$ and $T_e > T$ because energy flows from effective fluid to the usual fluid continuously and hence the thermodynamical equilibrium is violated. Figure 3.2 shows the variation of the temperature T (red line) and T_e (blue line) with respect to the scale factor $a(t)$ and in the figure, the equilibrium temperature has been identified.

Now, from thermodynamical consideration, equilibrium temperature T_E can be considered as the (modified) Hawking temperature [133] i.e,

$$T_E = \left.\frac{H^2R_h}{2\pi}\right|_{a=a_0}, \tag{3.27}$$

where R_h is the geometric radius of the horizon, bounding the universe.

Therefore, $f(\mathrm{T})$ gravity is equivalent to Einstein gravity with non-interacting two-fluid system having different temperatures, and the above (modified) Hawking temperature gives the common temperature in equilibrium prescription.

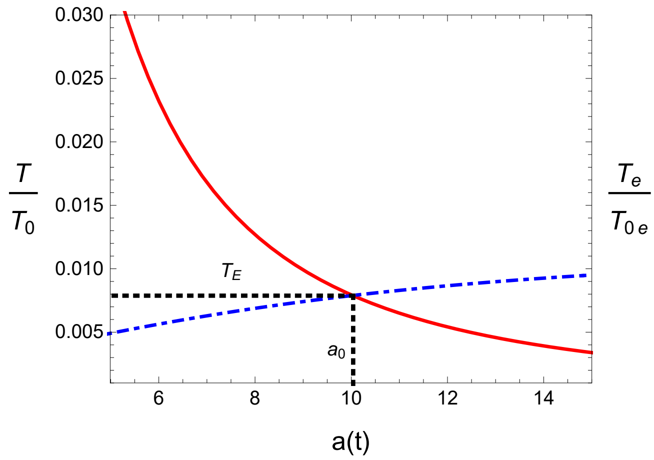

Figure 3.2: The variation of the temperature of usual fluid T and of effective fluid T_e

3.5 Continuous cosmic evolution in $f(\mathrm{T})$ cosmology

Now after emergent scenario the Universe evolves through three phases, namely, inflationary era, radiation and matter-dominated era, and present time accelerating phase. Chakraborty *et al.* show continuous cosmic evolution in the background of Einstein gravity with particle creation mechanism [130]. Depending on the choices of the particle creation rate (which are motivated from thermodynamical viewpoint) it has been shown that there is a continuous cosmic evolution from inflation to the present accelerating era. In this section, the possibility of such type of continuous cosmic evolution in $f(\mathrm{T})$ cosmology will be examined. Since there is no physical reason for choosing any form of $f(\mathrm{T})$, the following $f(\mathrm{T})$ will be chosen phenomenologically for different eras comparing with the solution equations (in different eras) with that equation (3.6). Let $t = t_1$ and $t = t_2(> t_1)$ be the time instants when the Universe evolves from the inflationary era to the matter-dominated era and then to the late-time accelerated phase.

For inflationary era $(t < t_1)$:

Choice for $f(\mathrm{T})$ is

$$f(\mathrm{T}) = \frac{\sqrt{6}H_1(\alpha_1\sqrt{|T|} + \alpha_2 T) + \mu_1|T|^{\frac{3}{2}}}{\mu_1\sqrt{|T|} - \sqrt{6}H_1}$$

and the cosmological solutions are given by

$$\frac{H_1}{H} = \mu_1 + (1-\mu_1)\left(\frac{a}{a_1}\right)^{\frac{3\gamma}{2}}, \tag{3.28}$$

$$a = a_1\left[\frac{\mu_1}{1-\mu_1} LambertW\left(\frac{1-\mu_1}{\mu_1}\exp\left\{\frac{2(1-\mu_1)+3\gamma H_1(t-t_1)}{2\mu_1}\right\}\right)\right]^{\frac{2}{3\gamma}},$$

$$\rho = \frac{H_1^2}{2(\mu_1-1)^2}\left(6-6\alpha_2+\frac{\sqrt{6}\alpha_1\mu_1}{H_1}\right)\left(\frac{a}{a_1}\right)^{-3\gamma}, \tag{3.29}$$

$$q = \frac{3\gamma}{2}-1-\mu_1\frac{H}{H_1}, \tag{3.30}$$

where $LambertW(x)e^{LambertW(x)} = x$ and α_1, α_2 are arbitrary constants. Here H_1, and a_1 are the values of the Hubble parameter, and scale factor at $t = t_1$ respectively.

For matter-dominated era ($t_1 < t < t_2$):

Choice for f(T) is

$$f(\mathrm{T}) = \beta_1\sqrt{|T|}+\beta_2|T|^{\frac{1}{1-\mu_2}}-T$$

and the cosmological solutions are given by

$$H = H_1\left(\frac{a}{a_1}\right)^{-\frac{3\gamma}{2}(1-\mu_2)}, \tag{3.31}$$

$$a = a_1\left[1+\frac{3\gamma}{2}H_1(1-\mu_2)(t-t_1)\right]^{\frac{2}{3\gamma(1-\mu_2)}},$$

$$\rho = \frac{\beta_2}{2}\left(\frac{\mu_2+1}{\mu_2-1}\right)(6H_1^2)^{\frac{1}{1-\mu_2}}\left(\frac{a}{a_1}\right)^{-3\gamma}, \tag{3.32}$$

$$q = \frac{3\gamma}{2}-1-\mu_2, \tag{3.33}$$

where β_1, and β_2 are arbitrary constants.

For present time accelerated era ($t > t_2$):

Choice for f(T) is

$$f(\mathrm{T}) = \delta_1\sqrt{|T|}+\delta_2 T-6(\delta_2+1)\mu_2 H_2^2$$

and the cosmological solutions are given by

$$H = H_2\left[\mu_2+(1-\mu_2)\left(\frac{a}{a_2}\right)^{-3\gamma}\right]^{\frac{1}{2}}, \tag{3.34}$$

$$a = a_2\left[\sqrt{\frac{1-\mu_2}{\mu_2}}\sinh\left\{\frac{3\gamma}{2}\sqrt{\mu_2}H_2(t-t_i)\right\}\right]^{\frac{2}{3\gamma}},$$

$$\rho = 3\,(\delta_2+1)\,(1-\mu_2)H_2^2\left(\frac{a}{a_2}\right)^{-3\gamma}, \tag{3.35}$$

$$q = \frac{3\gamma}{2}-1-\mu_2\left(\frac{H_2}{H}\right)^2, \tag{3.36}$$

where H_2, a_2 are the values of the Hubble parameter, scale factor at $t = t_2$ respectively and δ_1, δ_2 are arbitrary constants.

Now smoothness of the deceleration parameter demands $\mu_1 = \mu_2$. Also the continuity of the Hubble parameter, scale factor at $t = t_1$ is obvious and continuity across $t = t_2$ leads to the following conditions

$$\frac{3\gamma}{2}(1-\mu_2)(t_2-t_1) = \frac{1}{H_2} - \frac{1}{H_1} \quad , \quad \left(\frac{a_1}{a_2}\right)^{\frac{3\gamma(1-\mu_2)}{2}} = \frac{H_2}{H_1}, \tag{3.37}$$

$$\sinh(\eta_2) = \sqrt{\frac{\mu_2}{1-\mu_2}}, \tag{3.38}$$

where

$$\eta_2 = \frac{3\gamma}{2}\sqrt{\mu_2}H_2(t_2 - t_i).$$

Further, let $t = t_0 (< t_1)$ be the time instant in which the universe makes a transition from the emergent scenario to the inflationary era. Then from the continuity across the transition time $t = t_0$, one has

$$\frac{H_1}{H_0} = \mu_1 + (1-\mu_1)\left(\frac{a_0}{a_1}\right)^{\frac{3\gamma}{2}}, \tag{3.39}$$

Now, the continuity of f(T) and the energy density across all the transition points $t_0,\ t_1,\ t_2$ provides the following conditions

$$dH_0\left(\frac{a_0}{a_1}\right)^{3\gamma} = \left(\sqrt{6}(1-\alpha_2) + \frac{\alpha_1\mu_1}{H_1}\right)\frac{H_1^2}{2(\mu_1-1)^2} \tag{3.40}$$

$$\left(6 - 6\alpha_2 + \frac{\sqrt{6}\alpha_1\mu_1}{H_1}\right) = \beta_2\left(\mu_2+1\right)\left(\mu_1-1\right)\left(6H_1^{2\mu_2}\right)^{\frac{1}{1-\mu_2}} \tag{3.41}$$

$$\beta_2\left(\frac{\mu_2+1}{\mu_2-1}\right)\left(6H_1^2\right)^{\frac{1}{1-\mu_2}} = 6\left(\delta_2+1\right)\left(1-\mu_2\right)H_2^2\left(\frac{a_2}{a_1}\right)^{3\gamma} \tag{3.42}$$

$$4d + c + d\ln\left(6H_0^2\right) = \frac{\alpha_1 + (1-\alpha_2)\sqrt{6}H_0}{\frac{\mu_1}{H_1}H_0 - 1} \tag{3.43}$$

$$\frac{\alpha_1 + (1-\alpha_2)\sqrt{6}H_1}{\mu_1 - 1} = \beta_1 + \beta_2\left(\sqrt{6}H_1\right)^{\frac{1+\mu_2}{1-\mu_2}} \tag{3.44}$$

$$\beta_1 + \beta_2\left(\sqrt{6}H_2\right)^{\frac{1+\mu_2}{1-\mu_2}} = \delta_1 - \sqrt{6}(\delta_2+1)(\mu_2+1)H_2 \tag{3.45}$$

Hence, these six equations (3.40)-(3.45) determines 8 unknown arbitrary constants, namely, d, c, α_1, α_2, β_1, β_2, δ_1, δ_2. So one must note that 2 arbitrary constants always be free. However, these constants follow some restrictions as

$$d > 0,\ \sqrt{6}(1-\alpha_2) + \frac{\alpha_1\mu_1}{H_1} > 0,\ \beta_2 < 0,\ \delta_2 > -1.$$

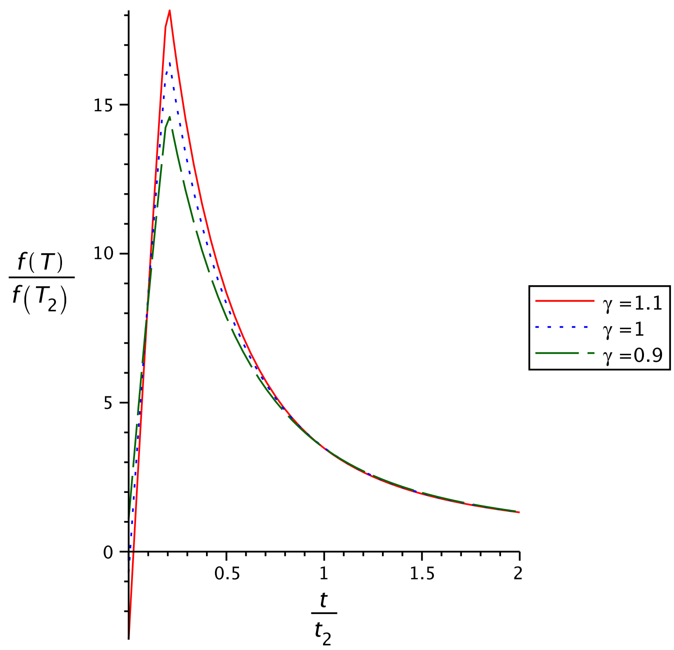

Figure 3.3: Evolution of f(T) with cosmic time t for various choices of γ.

Finally, the continuity of f(T) with respect to cosmic time t is shown graphically in Figure 3.3 with the choices and for the choices: $\mu_1 = 0.25$, $H_0 = 1$, $a_0 = 1$, $t_0 = .2$, $t_2 = 2.5$, $H_1 = 0.6$, $T_2 = -6H_2^2$, $\alpha_1 = 0.3$ and $\alpha_2 = -0.5$.

3.6 Conclusion

This chapter presents a detailed cosmological analysis in f(T) gravity theory. Firstly, it is examined whether a non-singular model of the universe is possible, and it is found that an emergent scenario is possible for a specific choice of f(T) only at the level of background cosmology. It is also shown that the f(T) gravity model has a correspondence with the Einstein gravity for non-interacting two-fluid system, one of which is the usual normal fluid and the other is the effective fluid, due to the extra geometric term in the f(T) gravity. Also from thermodynamical consideration, the temperature of both the fluid particles is evaluated. Finally, it is possible to have a complete cosmic evolution starting from the emergent scenario to present late-time accelerating era through the inflationary epoch and matter-dominated era. Thus both physical and geometrical parameters involved in the solutions are found to be continuous throughout the evolution. Therefore, it can be concluded that one may consider f(T) gravity theory as an alternative to GR in the context of continuous cosmic evolution as well as to resolve the dark energy problem.

CHAPTER 4

EMERGENT SCENARIO IN HOŘAVA-LIFSHITZ GRAVITY

4.1 Prelude

The Big Bang singularity is a consequence of Einstein gravity being a classical field theory. To overcome this singularity cosmologists proposed bouncing universe, emergent universe which are very much relevant in the context of inflationary models. An emergent Universe [167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182] is a model of the Universe without any time-like singularity, existing throughout the time axis (i.e. $-\infty < t < +\infty$) and static in nature in the past infinity (t$\rightarrow -\infty$).

In this non-singular model, the initial Big Bang singularity is replaced by an Einstein static era (with $a = a_0 \neq 0$) so that physical quantities namely energy density, pressure, etc. are all finite at the beginning. Subsequently, the Universe expands smoothly (known as pre-inflationary era) and enters the inflationary epoch. Then as usual in standard cosmology, there is a phase of reheating and gradually the Universe approaches the usual classical thermal radiation-dominated era of evolution. Table 4.1 shows the behaviour of the scale factor and Hubble parameter in the above eras of evolution [183].

From observational viewpoint, cosmologists are giving much attention to the modified gravity theories over recent years due to geometric matter components, which may be considered among the most natural choices for dark energy. In this section, such a modified gravity theory namely Hořava-Lifshitz (HL) gravity will be studied for constructing an emergent model of the Universe. The HL gravity theory is a power-counting renormalization theory with consistent ultra-violet behavior. Also, it provides Einstein gravity as a critical point. Various versions of HL gravity have been distin-

Table 4.1: Nature of the scale factor and Hubble parameter at various cosmic eras

Name of the epoch	Scale factor	Hubble parameter
Emergent era	$a = a_0 (\neq 0)$	$H = 0$
Pre-inflationary era	'a' increases from a_0 to a_i ($> a_0$)	H increases from zero to a finite value H_0
Inflationary era	'a' increases exponentially with $a_i < a < a_f$	$H = H_0$
Radiation era	$a > a_f$	H decreases gradually from H_0

guished in [184]. Also, the dynamics and the viability of these versions have been discussed in this review. Further, Luongo *et al.* have investigated the limits imposed by observations on the minimal paradigm of HL in the framework of homogeneous and isotropic space-time [185]. They have found statistical discrepancies in the model, ruling out HL paradigm at the level of background cosmology. The validity of HL cosmology both at early and late times is yet to be analyzed and they have speculated that by adding new extra terms into the Lagrangian one may have viable HL gravity model.

4.2 Basic equations in Hořava-Lifshitz gravity

Using the detailed balance condition the gravitational action integral for HL gravity can be written as [186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198]

$$A_{HL} = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}N\left[\frac{2}{\kappa}(k_{ij}k^{ij} - \lambda k^2) + \frac{\kappa}{2w^4}C_{ij}C^{ij} - \frac{\kappa\mu}{2w^2}\frac{\epsilon^{ijk}}{\sqrt{g}}R_{il}\nabla_j R^l_k \right.$$
$$\left. + \frac{\kappa\mu^2}{8}R_{ij}R^{ij} - \frac{\kappa\mu^2}{8(3\lambda-1)}\left(\frac{1-4\lambda}{4}R^2 + \Lambda R - 3\Lambda^2\right)\right] \quad (4.1)$$

where N is the Lapse function, k_{ij} is the extrinsic curvature tensor, ϵ^{ijk} is the totally antisymmetric tensor, λ is a dimensionless constant, the quantities w and μ are purely constants and C^{ij}, the Cotton tensor is given by [198]

$$C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}}\nabla_k\left(R^j_l - \frac{1}{4}R\delta^j_l\right) \quad (4.2)$$

Now varying the action (4.1) with respect to the homogeneous and isotropic FLRW metric, the modified Einstein field equations for perfect fluid in the background of Hořava-Lifshitz gravity obtained as (choosing $\kappa = 8\pi G = 1$, $N = 1$)[194]

$$3H^2 = \frac{1}{2(3\lambda-1)}\rho - \frac{3\mu^2}{16(3\lambda-1)^2}\left(\Lambda - \frac{K}{a^2}\right)^2 \quad (4.3)$$

$$2\dot{H} + 3H^2 = -\frac{1}{2(3\lambda-1)}p - \frac{\mu^2}{16(3\lambda-1)^2}\left(\Lambda - \frac{K}{a^2}\right)\left(3\Lambda + \frac{K}{a^2}\right) \quad (4.4)$$

where K, the curvature scalar may have the values 0, +1, −1 for flat, closed, and open model while the continuity equation for the perfect fluid is

$$\dot{\rho} + 3H(p + \rho) = 0 \tag{4.5}$$

with barotropic equation of state $p = \omega\rho$.

Now eliminating ρ from the field equations (4.3) and (4.4) one gets the cosmic evolution equation as

$$2\dot{H} = -3(1+\omega)H^2 - \frac{\mu^2}{16(3\lambda - 1)^2}\left(\Lambda - \frac{K}{a^2}\right)\left[3\Lambda(1+\omega) + \frac{K}{a^2}(1 - 3\omega)\right] \tag{4.6}$$

4.3 Emergent scenario in Hořava-Lifshitz gravity

In order to examine the possibility of a singularity free cosmological solution (i.e. emergent scenario) in HL gravity one has to consider the following asymptotic behavior of the Hubble parameter and scale factor as [167]

$$\begin{array}{ll} \text{(i)} & a \to a_0 \;,\; H \to 0 \text{ as } t \to -\infty \;; \\ \text{(ii)} & a \simeq a_0 \;,\; H \simeq 0 \text{ for } t \ll t_0 \;; \\ \text{(iii)} & a \simeq a_0 \exp\{H_0(t - t_0)\} \;,\; H \simeq H_0 \text{ for } t \gg t_0 \;. \end{array} \tag{4.7}$$

where a_0, a constant is the tiny value of the scale factor during emergent phase, and similarly $H_0(>0)$ is the value of the Hubble parameter at $t = t_0$.

From the above asymptotic behavior of the scale factor, one has the explicit form of the scale factor as [167],[178]

$$a = a_0\left[1 + e^{\frac{H_0}{\alpha}(t - t_0)}\right]^{\alpha} \tag{4.8}$$

so that

$$H = H_0\left[1 - \left(\frac{a_0}{a}\right)^{\frac{1}{\alpha}}\right], \tag{4.9}$$

and

$$\alpha a\frac{\mathrm{d}H}{\mathrm{d}a} = H_0\left(1 - \frac{H}{H_0}\right) \tag{4.10}$$

Now using the evolution equation (4.10) for the Hubble parameter into equation (4.6) one has

$$-(1+\omega) = \frac{2aH\frac{\mathrm{d}H}{\mathrm{d}a} + \frac{\mu^2 K}{4a^2(3\lambda - 1)^2}\left(\Lambda - \frac{K}{a^2}\right)}{3H^2 + \frac{3\mu^2}{16(3\lambda - 1)^2}\left(\Lambda - \frac{K}{a^2}\right)^2} \tag{4.11}$$

and the energy density of the perfect fluid has the expression

$$\rho = 6H^2(3\lambda - 1) + \frac{3\mu^2}{8(3\lambda-1)^2}\left(\Lambda - \frac{K}{a^2}\right)^2 \tag{4.12}$$

From the above expressions for equation of state parameter and energy density, one may conclude that,

1. For the flat model (i.e. $K = 0$) of the Universe, to have a singularity free era of evolution at the early phase for the HL gravity, the perfect fluid is at the phantom barrier ($\omega = -1$) i.e. behaves as cosmological constant. However, phantom fluid ($\omega < -1$) is needed for the pre-inflationary era.

2. For the closed model (i.e. $K = +1$) of the Universe, if $a_0 > \frac{1}{\Lambda}$ and also $a_0 > l_{pl}$ (Planck length) (to avoid quantum gravity effect) i.e. $a_0 > \min\left(\frac{1}{\Lambda}, l_{pl}\right)$ then the phantom perfect fluid is needed both for the emergent era and pre-inflationary epoch. Here the cosmological constant plays no role to characterize the nature of the perfect fluid.

3. For the open model (i.e. $K = -1$) of the Universe, normal perfect fluid describes both emergent as well as pre-inflationary phases of evolution.

Further using (4.9) and (4.10) one has the energy density and equation of state parameter explicitly in terms of the scale factor 'a' as

$$\rho(a) = 6H_0^2(3\lambda-1)\left[1-\left(\frac{a_0}{a}\right)^{\frac{1}{\alpha}}\right]^2 + \frac{3\mu^2}{8(3\lambda-1)^2}\left(\Lambda - \frac{K}{a^2}\right)^2 \tag{4.13}$$

$$-3(1+\omega) = \frac{\frac{2H_0^2}{\alpha}\left[1-\left(\frac{a_0}{a}\right)^{\frac{1}{\alpha}}\right]\left(\frac{a_0}{a}\right)^{\frac{1}{\alpha}} + \frac{\mu^2 K}{4a^2(3\lambda-1)^2}\left(\Lambda - \frac{K}{a^2}\right)}{H_0^2(3\lambda-1)\left[1-\left(\frac{a_0}{a}\right)^{\frac{1}{\alpha}}\right]^2 + \frac{\mu^2}{16(3\lambda-1)^2}\left(\Lambda - \frac{K}{a^2}\right)^2} \tag{4.14}$$

Figures 4.1 and 4.2 show the variation of energy density ρ (in equation (4.13)) and the equation of state parameter (in equation (4.14)) with the scale factor for the choices (i) $K = 0,\ \Lambda \neq 0$, (ii) $K \neq 0,\ \Lambda = 0$, (iii) $K \neq 0,\ \Lambda \neq 0$ respectively, for the choice of the parameters $a_0 = 0.4$, $H_0 = .01$, $\alpha = 4$, $\mu = 2$.

From the field theoretic viewpoint, the perfect fluid has an analogous scalar field description. The action integral for the present gravity model of a self-interacting scalar field takes the form

$$A_m = \int \mathrm{d}t \left[\frac{3\lambda - 1}{4}\dot{\phi}^2 - V(\phi)\right] a^3 \tag{4.15}$$

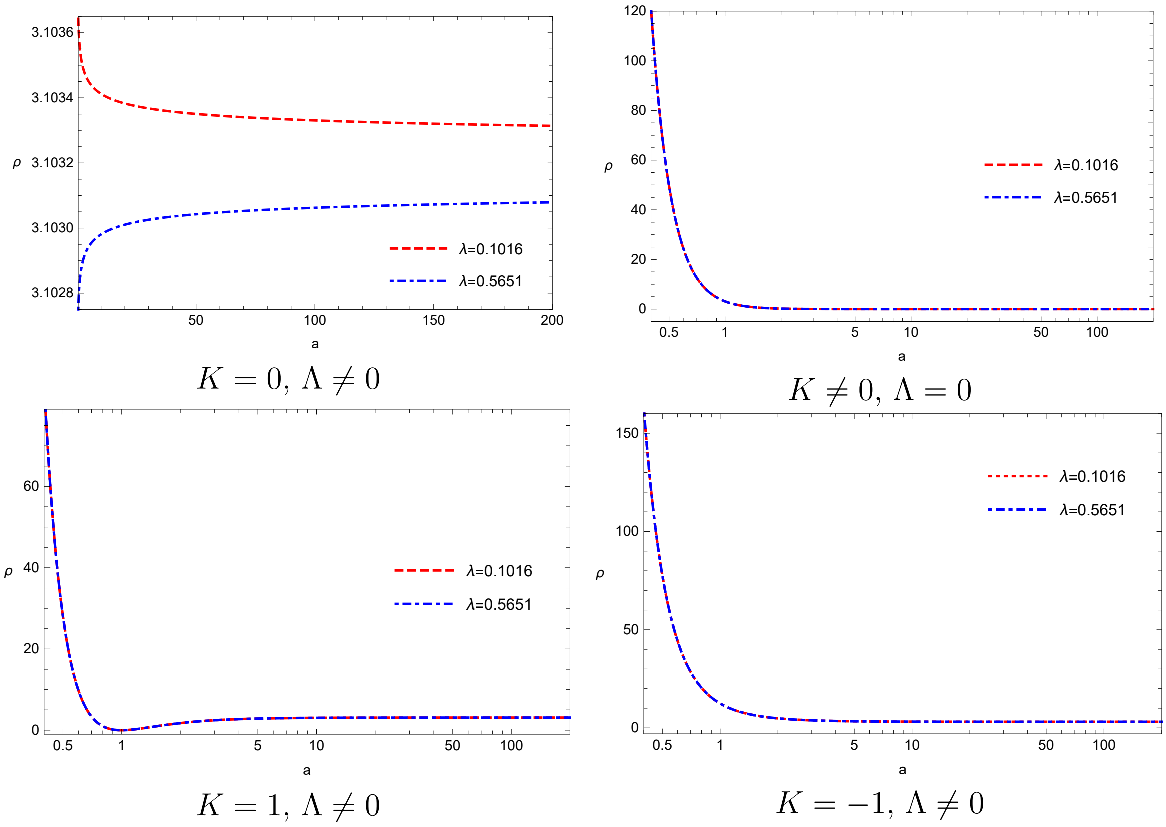

Figure 4.1: Evolution of energy density ρ for various models

The pressure and energy density of the perfect fluid are related to the scalar field as

$$p = \frac{3\lambda - 1}{4}\dot{\phi}^2 - V(\phi)$$

$$\text{and} \quad \rho = \frac{3\lambda - 1}{4}\dot{\phi}^2 + V(\phi) \tag{4.16}$$

Further, for the flat case (i.e. $K = 0$) a typical analytic form of the scalar field and the potential as a function of scalar field has the following expression:

$$\phi = -4i\sqrt{2\alpha}\sin^{-1}\left(\left(\frac{a_0}{a}\right)^{\frac{1}{2\alpha}}\right) \tag{4.17}$$

$$V(\phi) = 2(3\lambda - 1)H_0^2\Bigg[3\cosh^4\left(\frac{\phi}{4\sqrt{2\alpha}}\right) - \frac{1}{\alpha}\cosh^2\left(\frac{\phi}{4\sqrt{2\alpha}}\right)\sinh^2\left(\frac{\phi}{4\sqrt{2\alpha}}\right) + \frac{3\mu^2\Lambda^2}{16(3\lambda - 1)^2H_0^2}\Bigg] \tag{4.18}$$

with graphical representation in Figure 4.3. It should be noted that 'cosh' type potential is appropriate for the inflationary era according to Planck data [199].

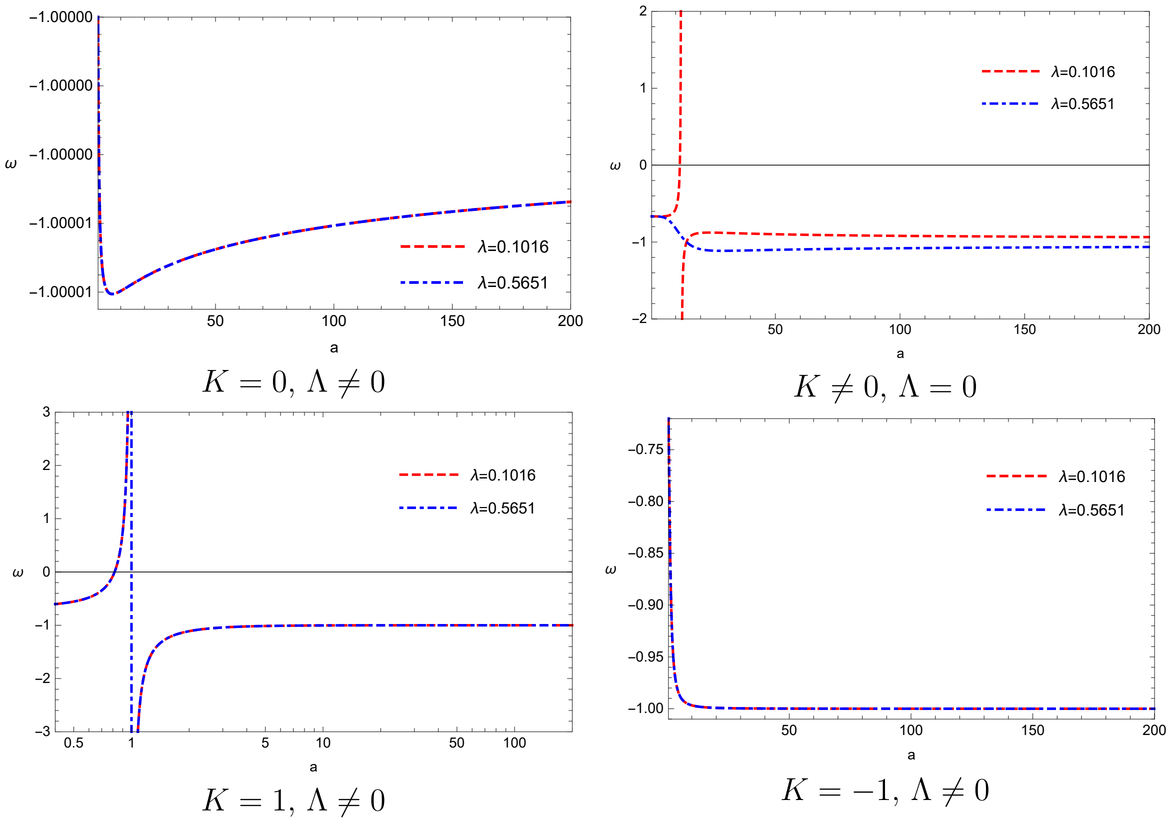

Figure 4.2: Evolution of equation of state parameter ω for various models

4.4 Thermodynamical Analysis

In this section, the thermodynamical behaviour of this non-singular era of evolution will be analyzed. In particular, the validity of the generalized second law of thermodynamics (GSLT) and thermodynamical equilibrium (TE) for the emergent universe bounded by apparent horizon (event horizon does not exist in emergent era) will be examined.

The FLRW metric can have the following (2+2)-decomposition

$$\mathrm{d}s^2 = h_{ij}(x^i)\mathrm{d}x^i\mathrm{d}x^j + R^2\mathrm{d}\Omega_2^2 \tag{4.19}$$

where

$$h_{ij} = diag\left(-1, \frac{a^2}{1-Kr^2}\right);\ i,j = 0,1$$

is the metric on the 2-space, normal the 2-sphere and $R = ar$ is the area radius, a scalar in the normal 2D space. Another scalar

$$\chi(x) = h^{ij}(x)\partial_i R \partial_j R = 1 - \left(H^2 + \frac{K}{a^2}\right)R^2 \tag{4.20}$$

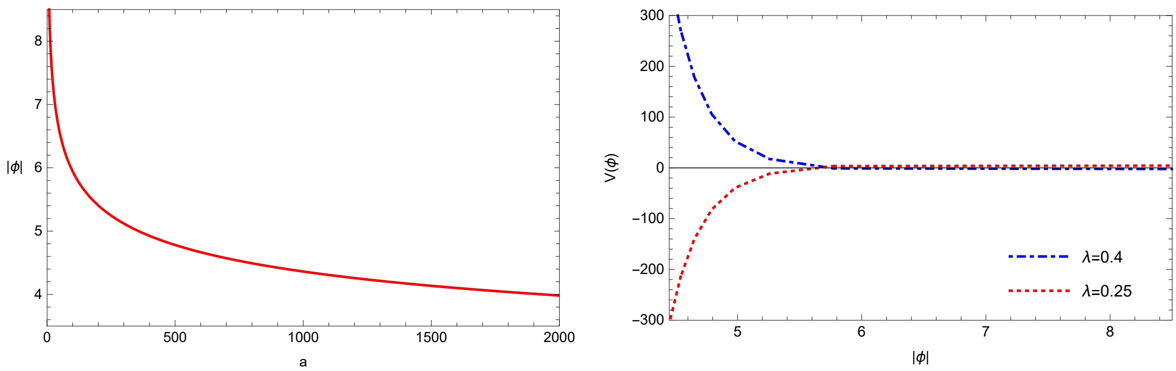

Figure 4.3: Evolution of scalar field and Potential for flat universe

on the normal (r,t)-plane defined the surface gravity (on the horizon) as

$$K_H = -\frac{1}{2}\frac{\partial \chi}{\partial R}\bigg|_{R=R_H} = R_H\left(H^2 + \frac{K}{a^2}\right) \tag{4.21}$$

and $\chi = 0$ defines a null surface known as apparent horizon having area radius

$$R_A = \frac{1}{\sqrt{H^2 + \frac{K}{a^2}}} \tag{4.22}$$

Hence the usual Hawking temperature on the horizon gives [133]

$$T_H = \frac{|K_H|}{2\pi} = \frac{R_H}{2\pi R_A^2} \tag{4.23}$$

If S_H and S_f denote the entropy of the horizon and that of the cosmic fluid bounded by the horizon then GSLT and TE demand [130]

$$\text{(i) } \dot{S_H} + \dot{S_f} > 0 \quad \text{and} \quad \text{(ii) } \ddot{S_H} + \ddot{S_f} < 0 \tag{4.24}$$

Now, the entropy on a horizon (having area radius R_H) can be determined from the first law of thermodynamics (i.e. Clausius relation)

$$T_H \mathrm{d}S_H = \delta Q = -\mathrm{d}E = 4\pi R_H^2(\rho + p)H dt \tag{4.25}$$

as

$$\dot{S_H} = \frac{4\pi R_H^2 H(\rho + p)}{T_H} \tag{4.26}$$

On the other hand, using Gibbs relation

$$T_f \mathrm{d}S_f = \mathrm{d}E + p\mathrm{d}V \ , \tag{4.27}$$

the entropy variation of the cosmic fluid has the explicit expression

$$\dot{S_f} = \frac{4\pi R_H^2(\rho + p)}{T_f}\left(\dot{R_H} - HR_H\right) \tag{4.28}$$

where $V = \frac{4}{3}\pi R_H^3$ is the volume of the cosmic fluid, $E = \rho V$ is the total energy and T_f is the temperature of the fluid. Hence the total entropy variation is given by

$$\dot{S}_T = \dot{S}_H + \dot{S}_f = \frac{4\pi R_H^2(p+\rho)}{T_f}\Big(\dot{R}_H - H(R_H - 1)\Big) \tag{4.29}$$

where for equilibrium configuration $T_f = T_H$.

Using the 2nd Friedmann equation

$$\left(\dot{H} - \frac{K}{a^2}\right) = -4\pi(p+\rho) \tag{4.30}$$

and equation (4.27) for the Horizon temperature, the total entropy variation takes the form;

$$\dot{S}_T = -2R_A^2 R_H \left(\dot{H} - \frac{K}{a^2}\right)\left[\dot{R}_H - H(R_H - 1)\right] \tag{4.31}$$

Again differentiating equation 4.31 one gets

$$\begin{aligned}\ddot{S}_T &= -2\Bigg[\left(2R_A\dot{R}_A R_H + R_A^2\dot{R}_H\right)\left(\dot{H} - \frac{K}{a^2}\right)\left\{\dot{R}_H - H\left(R_H - 1\right)\right\} \\ &\quad + R_A^2 R_H\left(\ddot{H} + \frac{2k}{a^2}H\right)\left\{\dot{R}_H - H\left(R_H - 1\right)\right\} \\ &\quad + R_A^2 R_H\left(\dot{H} - \frac{k}{a^2}\right)\left\{\ddot{R}_H - \dot{H}\left(R_H - 1\right) - \dot{R}_H H\right\}\Bigg]\end{aligned} \tag{4.32}$$

As R_A is constant, $H = 0$, $\dot{H} = 0$ at the emergent era, so the expressions for total entropy variation take the form (horizon is distinct from apparent horizon)

$$\begin{aligned}\dot{S}_T &= \frac{R_A^2 k}{a_0^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(R_H^2\right) \\ \text{and}\quad \ddot{S}_T &= \frac{R_A^2 k}{a_0^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(R_H^2\right)\end{aligned} \tag{4.33}$$

So the validity of GSLT and thermodynamical equilibrium can be constrained in Table 4.2. It is shown that both GSLT and thermodynamical equilibrium is satisfied for the closed model (i.e. $K = +1$) if the area radius increases in a decelerated way while for the open model (i.e. $K = -1$) if the area radius decreases in an accelerated way.

In the emergent era, the area radius at the apparent horizon is given by

$$R_A = \begin{cases} \dfrac{a_0}{\sqrt{|K|}} & , \ K \neq 0 \\ 0 & , \ K = 0 \end{cases}$$

Table 4.2: Condition for GSLT and thermal equilibrium

Curvature scalar	GSLT $\left(\dot{S}_T > 0\right)$	Thermodynamical equilibrium $\left(\ddot{S}_T < 0\right)$
$K = 0$	not satisfied	no
$K = +1$	$\frac{\mathrm{d}}{\mathrm{d}t}(R_H^2) > 0$	$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(R_H^2) < 0$
$K = -1$	$\frac{\mathrm{d}}{\mathrm{d}t}(R_H^2) < 0$	$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(R_H^2) > 0$

Thus both $\dot{S}_T$ and $\ddot{S}_T$ vanish for Universe bounded by apparent horizon in the emergent era. Hence GSLT is not obeyed and one cannot predict thermodynamical equilibrium in non-singular emergent phase for Universe bounded by the apparent horizon.

4.5 Conclusion

This chapter deals with a detailed study of the emergent phase of evolution in the context of HL gravity theory. In the background of homogeneous and isotropic space-time, the occurrence of emergent era restricts the perfect fluid as exotic type both for flat and closed model while for open model normal perfect fluid is sufficient for non-singular epoch. In equivalent scalar field description, the emergent scenario demands ghost type scalar field for flat and closed model; and normal scalar field for the open model of the Universe. However, it should be noted that this scalar field description of the barotropic fluid model is not unique in general [200] and even it may encounter few problems [201]. Further, Baryogenesis occurs independently of any emergent model [202] and it is independent of any extended/modified theory of gravity [203]. Finally, from thermodynamical analysis, it is found that GSLT is not valid and there is no thermodynamical equilibrium for this static phase at the apparent horizon while no definite conclusion is possible at any other horizon.

CHAPTER 5

COSMOLOGICAL SOLUTION OF MINIMALLY COUPLED POWER-LAW $f(R,T)$ GRAVITY MODEL

5.1 Prelude

In the context of Einstein gravity, a natural replacement of the Einstein-Hilbert action is by an arbitrary function of R. This modified gravity theory is the well-known $f(R)$ gravity theory. This modified gravity not only explains the late-time cosmic acceleration [204] but also satisfies local gravitational tests [205, 206, 207, 208, 209, 210, 211]. Harko *et al.* have extended this modified gravity theory [212] by choosing the Lagrangian density as an arbitrary function $f(R,T)$ where T is the trace of the energy-momentum tensor. The introduction of the matter part in the gravity Lagrangian is justified by the Quantum effect (known as conformal anomaly). Further, due to the coupling in matter and geometry, the gravity model obviously depends on the source term, and consequently, the test particles do not follow the geodesic path (as there is a hypothetical force term perpendicular to the four-velocity). Due to the highly complicated form of the field equation, there is a simple choice of $f(R,T)$ in an unorthodox manner ($f(R,T) = R + h(T)$) [213], keeping in mind the path of the test particle will be along the geodesic. Further, this particular choice of $f(R,T)$ is not possible for electromagnetic fields while if one considers the matter as a perfect fluid with constant equation of state then $h(T)$ turns out to be a power law in T where the power of T depends on the equation of state parameter.

5.2 $f(R,T)$ gravity: A brief description

In this chapter, a particular choice namely $f(R,T) = R + h(T)$ is considered so that the field equation (1.129) simplifies to [213]

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - h'(T)(T_{\mu\nu} + \Theta_{\mu\nu}) + \frac{1}{2}h(T)g_{\mu\nu}. \tag{5.1}$$

If the divergence of equation (5.1) is considered with conservation of energy-momentum tensor (i.e, $\nabla_\mu T^\mu_\nu = 0$) then one obtains

$$(T_{\mu\nu} + \Theta_{\mu\nu})\nabla^\mu h'(T) + h'(T)\nabla^\mu \Theta_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\nabla^\mu h(T) = 0 \tag{5.2}$$

Thus one may note that $h(T)$ is not arbitrary, rather it depends on the choice of the matter field. It is to be mentioned that for this choice of $f(R,T)$, it is not possible to consider the electromagnetic field as the matter field.

In the context of perfect fluid, the energy-momentum tensor has the following form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \tag{5.3}$$

with matter Lagrangian as $\mathcal{L}_m = -p$. Here p and ρ are the thermodynamic pressure and energy density of the perfect fluid with restrictions on the four-velocity u^μ as

$$u_\mu u^\mu = 1 \; ; \quad u^\mu \nabla_\nu u_\mu = 0.$$

The symmetric (0,2) tensor $\Theta_{\mu\nu}$ simplifies to

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - p g_{\mu\nu}. \tag{5.4}$$

Using this form of $\Theta_{\mu\nu}$ in equation (5.2) one obtains

$$(T_{\mu\nu} + p g_{\mu\nu})\nabla^\mu h'(T) + h'(T) g_{\mu\nu}\nabla^\mu p + \frac{1}{2}g_{\mu\nu}\nabla^\mu h(T) = 0. \tag{5.5}$$

Moreover, if the perfect fluid is assumed to have barotropic equation of state $p = \omega\rho$, ω a constant, then for flat FLRW space-time $h(T)$ reduces to the power law form as [213]

$$h(T) = h_0 T^\alpha \tag{5.6}$$

with $\alpha = \dfrac{1+3\omega}{2(1+\omega)}$, h_0 is integration constant and $\omega \neq -1, \pm\dfrac{1}{3}$.

5.3 Cosmology in $f(R,T)$ gravity theory

In the background of homogeneous and isotropic space-time with equation (5.6) as the choice for $h(T)$, the field equations in $f(R,T)$ gravity theory can be written as

$$3H^2 = \rho + h_0(1-3\omega)^{\alpha-1}\rho^\alpha \tag{5.7}$$

$$\text{and} \quad 2\dot{H} + 3H^2 = -p + \frac{1}{2}h_0(1-3\omega)^\alpha\rho^\alpha \tag{5.8}$$

with matter field conservation equation

$$\dot{\rho} + 3H(p+\rho) = 0. \tag{5.9}$$

Due to constant equation of state, equation (5.9) can be integrated to give

$$\rho = \rho_0(1+z)^{3(1+\omega)}\ , \qquad \rho_0, \text{ an integration constant} \tag{5.10}$$

where the redshift parameter z is defined as $\frac{a_0}{a} = (1+z)$. Using equation (5.10) in equation (5.7) and solving for the redshift parameter one gets

$$t = t_0 + \frac{2}{\sqrt{3\rho_0}}\frac{(1+z)^{-\frac{3(1+\omega)}{2}}}{(1+\omega)}\ {}_2F_1\left(\frac{1}{2}, \beta-1, \beta, x\right) \tag{5.11}$$

where $\beta = \frac{2}{1-\omega}$, $x = -h_0\left[\rho_0(1-3\omega)\right]^{\frac{\omega-1}{2(1+\omega)}}(1+z)^{\frac{3(\omega-1)}{2}}$. t_0 is an integration constant and ${}_2F_1$ is the usual confluent hypergeometric function. Also, the Hubble parameter and the measure of acceleration can be expressed in terms of the redshift parameter as

$$\begin{aligned} H^2 &= \frac{\rho_0}{3}(1+z)^{3(1+\omega)}(1-x) \\ &= \frac{\rho_0}{3}\left[(1+z)^{3(1+\omega)} + h_0(1-3\omega)^{\frac{\omega-1}{2(1+\omega)}}\rho_0^{\frac{\omega-1}{2(1+\omega)}}(1+z)^{\frac{3(1-\omega)}{2}}\right] \end{aligned} \tag{5.12}$$

and

$$\frac{\ddot{a}}{a} = -\frac{\rho_0}{6}(1+3\omega)(1+z)^{3(1+\omega)} + \frac{h_0}{12}(1-9\omega)(1-3\omega)^{\frac{\omega-1}{2(1+\omega)}}\rho_0^{\frac{1+3\omega}{2(1+\omega)}}(1+z)^{\frac{3(1+3\omega)}{2}}. \tag{5.13}$$

It should be noted that for a viable cosmological solution, one must have $\omega < \frac{1}{3}$. Also if $h_0 < 0$, it can be shown that the Hubble parameter vanishes for a certain value of scale factor and it does not exist after that. Hence the Universe is bounded. So to obtain an unbounded cosmological solution, h_0 must be positive and consequently, $\ddot{a}$ may be positive or negative for different choices of h_0, ρ_0, and ω. For unbounded universe i.e, $h_0 > 0$, acceleration or deceleration depends on the equation of the state parameter in the following manner

$$\begin{aligned} \frac{1}{9} \le \omega < \frac{1}{3} &\quad : \text{ decelerating model } (\ddot{a} < 0) \\ -\frac{1}{3} < \omega < \frac{1}{9} &\quad : \text{ no definite conclusion, depends on } \frac{\rho_0}{h_0} \\ \omega \le -\frac{1}{3} &\quad : \text{ accelerating model } (\ddot{a} > 0) \end{aligned}$$

Further, the modified Friedmann equations (5.7) and (5.8) in $f(R,T)$ gravity theory are equivalent to evolution equations in Einstein gravity with non-interacting two-fluid system – both of which are perfect fluid. The first one is the usual matter field with constant equation of state ω while the second effective perfect fluid has energy density and thermodynamic pressure as $\rho_e = h_0\,(1-3\omega)^{\alpha-1}\,\rho^\alpha$ and $p_e = -\frac{1}{2}h_0\,(1-3\omega)^\alpha\,\rho^\alpha$. So the effective perfect fluid has barotropic equation of state $\omega_e = -\frac{1}{2}(1-3\omega)$. Further, the modified field equation can be written as Einstein field equation with a perfect fluid as

$$3H^2 = \rho_T, \text{ and } 2\dot{H} = -\rho_T(1+\omega_T) \tag{5.14}$$

where the variable equation of state parameter is given by

$$\omega_T = \frac{\omega - \frac{1}{2}h_0(1-3\omega)^{\frac{3\omega+1}{2(\omega+1)}}\rho_0^{\frac{\omega-1}{2(\omega+1)}}(1+z)^{\frac{3}{2}(\omega-1)}}{1 + h_0(1-3\omega)^{\frac{\omega-1}{2(\omega+1)}}\rho_0^{\frac{\omega-1}{2(\omega+1)}}(1+z)^{\frac{3}{2}(\omega-1)}}. \tag{5.15}$$

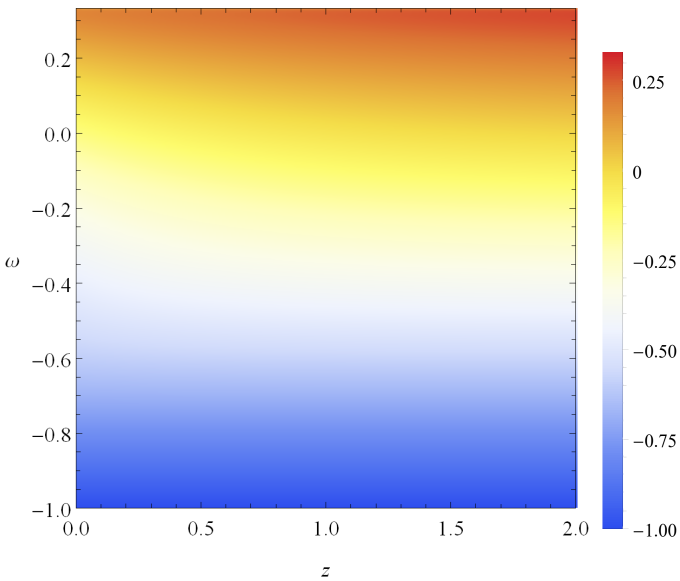

Figure 5.1: Density plot of ω_T with respect to ω and z.

The variation of ω_T with the variation of ω and redshift parameter z has been shown in Figure 5.1 for the choice $\rho_0 = 2$ and $h_0 = 0.9$.

The present $f(R,T)$ gravity model or equivalently two-fluid Einstein gravity model cannot describe the warm inflationary scenario for the following reasons: (i) There is no interaction between the two fluids, and as a result, dissipative or friction term is absent in the matter evolution equations, (ii) Neither the matter fluid nor the effective fluid can have radiation equation of state. Moreover, the present cosmological model is an equilibrium thermodynamical prescription due to the non-existence of any dissipative pressure in both the fluids. Further, in the present effective Einstein gravity with two fluids, the effective fluid will always be exotic in nature (i.e, DE) provided $\omega < \frac{1}{9}$ while it will be a normal fluid for $\frac{1}{9} < \omega < \frac{1}{3}$. Also from Figure 5.2, it can be concluded that the present $f(R,T)$ model can describe the evolution of the universe from the decelerating phase to the present accelerating era for suitable choices of ω and $\frac{\rho_0}{h_0}$. Lastly, the cosmological parameters namely the scale factor, Hubble parameter, and acceleration parameter are shown graphically in Figure 5.3 for $\omega = .25$ (dashed line), $\omega = -.1$ (dot-dashed line), and $\omega = -.5$ (dotted line) with $\rho_0 = 2$ and $h_0 = 0.9$.

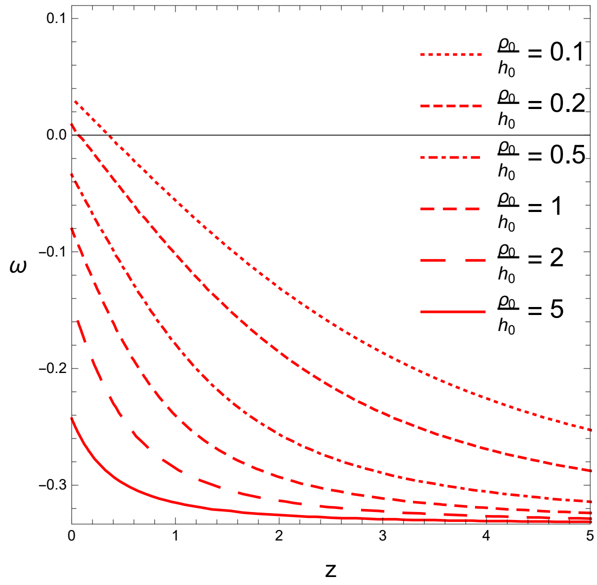

Figure 5.2: $\ddot{a} = 0$ is plotted in $\omega - z$ plane for various choices of $\frac{\rho_0}{h_0}$.

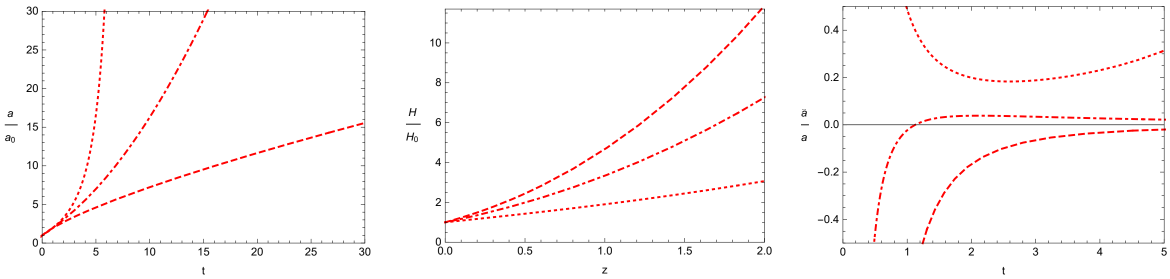

Figure 5.3: Scale factor (left panel), Hubble parameter (middle panel), acceleration of the Universe (right panel) are plotted for various choices of ω

5.4 Field-theoretic descriptions

To describe the present $f(R,T)$ cosmological model from field theoretical point of view, a scalar field ϕ having self-interacting potential $V(\phi)$ is introduced to describe the effective fluid. So the energy density ρ_ϕ and pressure p_ϕ of the scalar field are given by

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) = h_0(1-3\omega)^{\alpha-1}\rho^\alpha \tag{5.16}$$

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) = -\frac{1}{2}h_0(1-3\omega)^\alpha\rho^\alpha \tag{5.17}$$

i.e,

$$\dot{\phi}^2 = \frac{1}{2}h_0(1-3\omega)^{\alpha-1}(1+3\omega)\rho^\alpha \quad ; \quad V(\phi) = \frac{3}{4}h_0(1-3\omega)^{\alpha-1}(1-\omega)\rho^\alpha. \tag{5.18}$$

Using the solution for ρ from equation (5.10) into equation (5.18) and integrating ϕ has the explicit solution

$$\phi = \phi_0 \sinh^{-1}(\mu a^r) \tag{5.19}$$

and the potential function takes the form

$$V(\phi) = V_0 a^y \tag{5.20}$$

or in term of the scalar field

$$V(\phi) = V_1\left[\sinh\left(\frac{\phi}{\phi_0}\right)\right]^s \tag{5.21}$$

where $\phi_0 = \frac{2}{1-\omega}\sqrt{\frac{2(1+3\omega)}{3}}$, $\mu = \sqrt{h_0}\,[(1-3\omega)\rho_0]^{\frac{\alpha-1}{2}}$, $r = \frac{3}{4}(1-\omega)$, $s = \frac{2(1+3\omega)}{\omega-1}$, $V_0 = \frac{3}{4}h_0\,(1-3\omega)^{\alpha-1}\,(1-\omega)\,\rho_0^\alpha$, $V_1 = \frac{3\,(1-\omega)\,h_0^{\frac{2(1+\omega)}{1-\omega}}}{4(1-3\omega)}$, $y = -\frac{3}{2}(1+3\omega)$.

The variation of the potential over scalar field has been presented in Figure 5.4 for the choice $h_0 = 0.9$, $\rho_0 = 2$, $\omega = -0.1$.

Further, eliminating ρ between the equations in (5.18) one obtains

$$\frac{\dot{\phi}^2}{V(\phi)} = \frac{2}{3}\frac{(1+3\omega)}{(1-\omega)}$$

$$\text{i.e.,} \int \frac{d\phi}{\sqrt{V(\phi)}} = \sqrt{\frac{2(1+3\omega)}{3(1-\omega)}}(t-t_0) \tag{5.22}$$

So the above field theoretic description is possible for $-\frac{1}{3} < \omega < 1$. In particular, for some known potential we have the explicit form of the scalar field as

$$\text{(i) } V(\phi) = V_0\phi^{-2(n-1)} \quad , \quad \phi^n = n\sqrt{\frac{2V_0(1+3\omega)}{3(1-\omega)}}(t-t_0)$$

$$\text{(ii) } V(\phi) = V_0\,\text{sech}^2\phi \quad , \quad \phi = \sinh^{-1}\left[\sqrt{\frac{2V_0(1+3\omega)}{3(1-\omega)}}(t-t_0)\right]$$

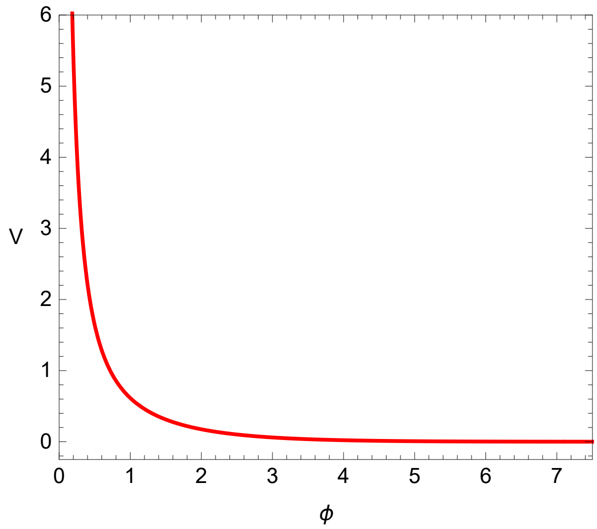

Figure 5.4: Evolution of the potential $V(\phi)$

From the expression of $\dot{\phi}^2$ in equation (5.18), it is easy to see that the equivalent scalar field in field theoretic description will be a real scalar field provided either (i) $h_0 > 0$ and $\omega > -\frac{1}{3}$ i.e, normal fluid in an unbounded universe, or (ii) $h_0 < 0$ and $\omega < -\frac{1}{3}$ i.e, exotic fluid in the bounded universe.

Thus an unbounded model of the universe will be ever accelerating if the equivalent scalar field is of ghost nature while the unbounded universe will be ever decelerating or undergoes a transition from deceleration to acceleration for the equivalent scalar field to be real.

5.5 An equivalent notion of modified Chaplygin gas and perfect fluid in $f(R,T)$ gravity theory

This section presents a nice equivalence between the $f(R,T)$ gravity theory with the typical choice of the $f(R,T)$ function (in the present work) and Einstein gravity. The modified Einstein field equations in $f(R,T)$ gravity theory for perfect fluid in FLRW space-time geometry are given by equations (5.7) and (5.8) with $\omega = \frac{p}{\rho}$ as the equation of state parameter. Now one may assume there exist a modified Chaplygin gas (MCG) having equation of state

$$p = \gamma\rho - \Gamma\rho^{\alpha} \tag{5.23}$$

with γ, Γ and α are constant parameters of the Chaplygin gas given by

$$\gamma = \frac{1}{2}(1-3\omega),\ \ \Gamma = \frac{1}{2}|h_0|(1-3\omega)^{\alpha}.$$

If the above modified field equations are considered as equivalent Einstein field equations then one may write

$$\begin{aligned} 3H^2 &= \rho_e &= \frac{p}{\gamma} \\ 2\dot{H} + 3H^2 &= -p_e &= -\gamma\rho \end{aligned} \tag{5.24}$$

Thus the effective single fluid in Einstein gravity has the equation of state $p_e = \omega_e \rho_e$ where $\omega_e = \frac{\gamma^2}{\omega}$, a constant. So one may conclude that a modified Chaplygin gas fluid in $f(R,T)$ gravity theory is equivalent to a perfect fluid in Einstein gravity with constant equation of state $\frac{\gamma^2}{\omega}$. Therefore, the present choice of the $f(R,T)$ function shows that the equivalence between the modified gravity theory and Einstein gravity is just an exchange of a perfect fluid with constant equation of state to the well-known modified chaplygin gas.

5.6 Numerical Analysis and Observational Constraint

Here the goal is to constraint the cosmological parameters analyzing the observational data sets. In order to do so, the pressure of the matter component and effective fluid can be rewritten as

$$p_m \equiv p = \omega\rho \equiv w0_b \rho_m \tag{5.25}$$

$$p_{fld} \equiv p_e = -\frac{1}{2}(1-3\omega)\rho_e \equiv -\frac{1}{2}(1-3w0_b)\rho_{fld} \tag{5.26}$$

where the corresponding energy density for two components can be rewritten as

$$\rho_m \equiv \rho = \rho_0(1+z)^{3(1+\omega)} \equiv \Omega_m H0^2 (1+z)^{3(1+w0_b)} \tag{5.27}$$

$$\rho_{fld} \equiv \rho_e = h_0(1-3\omega)^{\alpha-1}\rho^{\alpha} \equiv \Omega 0_{fld} H0^2 (1+z)^{\frac{3}{2}(1+3w0_b)} \tag{5.28}$$

and the public version of the CLASS Boltzmann code has been modified to include the dark energy sector as effective fluid and for baryonic matter and corresponding cold dark matter (Eqns. (5.25)-(5.28)). The MCMC code Montepython3.5 [216] has been used to estimate the relevant cosmological parameters.

In order to analyze and make a comparison, two cosmological datasets as dataset I (Pantheon [19], BAO (BOSS DR12 [217], $SMALLZ-2014$ [218]) and HST [219]) and dataset II (Pantheon [19], HST [219]) have been used. In both cases, a PLANCK18 prior has been imposed.

The flat priors on the base cosmological parameters have been chosen as follows: the baryon density $100\omega_b = [1.9, 2.5]$; cold dark matter density $\omega_{cdm} = [0.095, 0.145]$; Hubble parameter $H0 = [60, 80]$ $km\ s^{-1}Mpc^{-1}$; and a wide range of flat prior has been chosen for $w0_b = [-1, 1]$.

In Table 5.1 the constraints on the various cosmological parameters have been enlisted and Figure 5.5 shows the posterior distribution of those parameters. Here for both the datasets PLANCK18 prior has been considered.

	Dataset I		Dataset II	
Param	best-fit	mean$\pm\sigma$	best-fit	mean$\pm\sigma$
$100\ \omega_b$	2.244	$2.242^{+0.045}_{-0.046}$	2.252	$2.249^{+0.045}_{-0.046}$
ω_{cdm}	0.1158	$0.1162^{+0.0023}_{-0.0023}$	0.118	$0.1177^{+0.0023}_{-0.0023}$
$w0_b$	0.02035	$0.02012^{+0.0022}_{-0.002}$	-0.1722	$-0.1706^{+0.021}_{-0.021}$
$H0$	75.77	$75.71^{+1.7}_{-1.6}$	73.11	$73.18^{+1.7}_{-1.8}$
M	-19.06	$-19.07^{+0.049}_{-0.044}$	-19.24	$-19.24^{+0.053}_{-0.054}$
$\Omega0_{fld}$	0.7591	$0.7577^{+0.012}_{-0.01}$	0.737	$0.7376^{+0.014}_{-0.013}$
Ω_m	0.2407	$0.2422^{+0.01}_{-0.012}$	0.2629	$0.2623^{+0.013}_{-0.014}$
$\chi^2_{\min}$	1130		1029	

Table 5.1: Best-fit values of the relevant parameters both for dataset I & II

From the above numerical analysis and the observed data it is found that the equation of state parameter turns out to be $w0_b = 0.02035$ (for DataSet I), $w0_b = -0.1722$ (for DataSet II) which is consistent with the theoretical prediction $\left(-\frac{1}{3} < \omega < \frac{1}{9}\right)$ in the present work for the $f(R,T)$ modified gravity model. Moreover, the above observational data analysis shows a transition of the model from the decelerated era of expansion to the present accelerated expansion era as inferred by the theoretic prediction. Now using the best-fit values of the parameters from Table 5.1, ρ_0 and h_0 can be estimated as

Best fit	ρ_0	h_0
Dataset I	1381.88	98.47
Dataset II	1405.22	637.392

Hence the choice of $f(R,T)$ becomes $f(R,T) = R + 98.47T^{0.52}$ (Dataset I) and $f(R,T) = R + 637.39T^{0.29}$ (Dataset II). So it may be noted that after adding the BAO data (Dataset I) the parameter $w0_b$ is estimated as higher value compared to Dataset II, consequently the value of $H0$ increases for Dataset I which is consistent with Figure 5.3. Further, it can be concluded that the trace of the energy-momentum tensor is more significant in the $f(R,T)$ choice when the BAO data is absent.

5.7 Conclusion

This chapter presents an extensive study of FLRW cosmology in $f(R,T)$ gravity with a suitable choice of the function $f(R,T)$. The matter field is chosen as perfect with the constant equation of state. The modified field equations are equivalent to

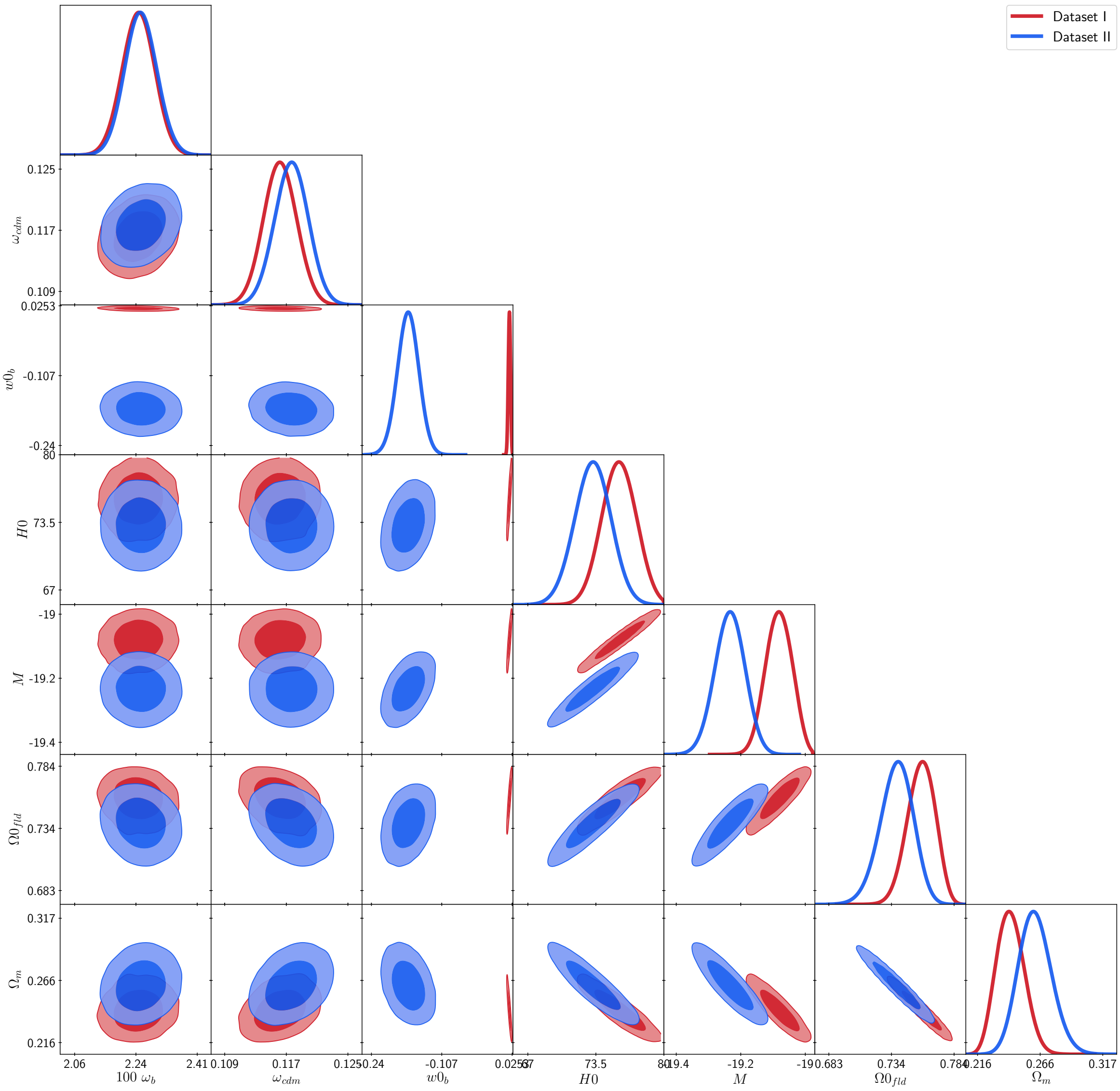

Figure 5.5: Posterior distribution for the cosmological parameters using the datasets I and datasets II.

the non-interacting two-fluid system in Einstein gravity. The effective fluid is also a perfect fluid with constant equation of state (depending on the state parameter of the given physical fluid). The detailed cosmological solution of this gravity model has been evaluated and the graphical representation of the cosmological parameters have been shown in this chapter. Also, the variation of equation of state parameter for the combined single fluid has been shown graphically in a 3D plot against redshift parameter and equation of state parameter for the physical fluid. Depending on the sign of a parameter (h_0) the present model may describe a bounded universe or an unbounded model of the universe. Due to the non-interacting nature of the two-fluid system, the model cannot describe the warm inflationary scenario and the present model is in thermodynamical equilibrium configuration. Depending on the choices of equation of state parameter of the usual fluid and the ratio $\left(\frac{\rho_0}{h_0}\right)$ the model may describe the evolution from decelerating phase to the present accelerating era. This result is in accord with the work of M. J. S. Houndjo [215]. There is a field-theoretic description, of the model with effective fluid described by a scalar field. From this field theoretic description it is found that an unbounded model of the Universe will be ever-accelerating if the nature of the equivalent scalar field is of ghost type while for a real scalar field, the universe will experience either ever decelerating phase or a transition from the deceleration to acceleration. Also, it is interesting to see that a modified Chaplygin gas in $f(R,T)$ gravity is equivalent to a perfect fluid in Einstein gravity. Thus with the transition from one gravity theory to the other, the physical fluid also makes a transition to another form of fluid. Moreover, from observational viewpoint, the parameters involved in the present model are estimated from Pantheon Data & BAO Data. It is found that the contribution of the matter part (by the trace term T) in the model function $f(R,T)$ is dominant when BAO data is not taken into account. Finally, both from the theoretical prediction and from the observational data analysis, one may infer that the present $f(R,T)$ gravity model describes the cosmic scenario from matter-dominated era to the present late time accelerating phase, it does not predict the early era of the universe.

CHAPTER 6

BRIEF REVIEW OF WARM INFLATION

6.1 Prelude

The theory of cosmic inflation is the most established candidate to explain the high levels of large scale homogeneity and isotropy in the universe, as well as the seeding mechanism of large scale structure (LSS). The basic concept of inflationary dynamics involves the evolution of a scalar field, namely, the inflaton [220, 221], so that its potential drives an accelerated expansion of the universe for around 10^{-34} seconds following the big bang singularity. The fluctuations in the Cosmic Microwave Background (CMB), recently measured by the Planck Mission with unprecedented accuracy [199], provide additional support for the inflationary theory by observing small fluctuations in the primordial density of the universe that grew into the observed LSS through gravitational instability [222].

However, there are two distinct dynamical concepts of inflation, namely, cold inflation and warm inflation, with cold inflation as the standard scenario. Inflaton is assumed to be an isolated system in cold inflation, with interactions between the inflaton and other fields are taken into account when making radiative corrections to the scalar potential. Any other initial component of energy density is redshifted, leaving the universe in a supercooled state. Consequently, once the inflation has finished, reheating is necessary for transition of the universe into the radiation-dominated expanding era.

Warm inflation [223, 224] is the realization that interaction between the inflaton and other fields results in the inflaton energy being dissipated to other dynamical degrees of freedom in addition to radiative corrections to the scalar potential. This typically happens because despite being heavy due to their coupling to the inflaton, the fields associated with it are unstable against decay into light degrees of freedom, acting as a medium of dissipating the inflaton energy into the light sector. This both dampens the motion of inflaton and enables particle creation, that in turn can form a thermal radiation bath.

6.2 Motivation for Inflation

The standard big bang cosmology faces several major problems due to the lack of understanding of initial conditions at the beginning of the universe. The historical motivation for inflation was concerned with the question of whether the initial conditions required for Hot Big Bang seem likely or not. The standard Big bang cosmology has two major issues, namely,

(1) Horizon Problem,

(2) Flatness Problem.

6.2.1 Horizon Problem

This problem is one of the puzzles of the standard Big bang cosmological model [225, 226, 227]. In a nutshell, it states why cosmic microwave background radiation (CMBR) is so isotropic throughout the universe. The standard Big bang model cannot provide any answer to it.

Usually, if two regions are very close to each other, they may have the same physical state (i.e. identical temperature and other physical parameters) so that information can be exchanged between them. Since any information or signal cannot travel faster than the speed of light, any regions that cannot be communicated by light signals are called causally disconnected (or isolated). The boundary of the causally connected region is referred as Horizon and it is analogous to the event horizon of a black hole. Therefore, information cannot be exchanged across regions beyond their horizons.

In the case of CMBR, these photons are now reaching us after traveling nearly the age of the universe. So these photons cannot reach beyond the horizon i.e. they cannot enter the region of the universe opposite to where they came from. However, surprisingly, observations show that the microwave background appears uniform throughout the universe to a very high degree of precision. Cosmologists have no answer to it in the framework of Big bang cosmology. Further, going against the expansion, it is found that during the formation of microwave background, regions a degree apart (in angular separation) were beyond the horizon of each other.

6.2.1.1 Conformal time, horizon

The causal structure of the FLRW space-time is characterized by the propagation of light, with photons, being massless, following null geodesics, $\mathrm{d}s^2 = 0$. The conformal time is defined as

$$\tau = \int \frac{\mathrm{d}t}{a(t)} \tag{6.1}$$

and consequently, the FLRW metric can be written as

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2 + \Psi_k(\chi^2)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right] \tag{6.2}$$

with

$$r^2 = \Psi_k(\chi^2) \equiv \begin{cases} \sinh^2\chi & , K = -1 \\ \chi^2 & , K = 0 \\ \sin^2\chi & , K = 1 \end{cases} \tag{6.3}$$

This expression for the FLRW metric can be simplified if one assumes the universe to be isotropic, and hence, the radial propagation of light can be determined by the two-dimensional line element

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2\right] \tag{6.4}$$

Considering the FLRW metric in this form it can be inferred that in conformal time the radial null-geodesics of light in the FLRW spacetime satisfies $\chi(\tau) = \pm\tau + \text{constant}$, which means the light will travel in straight lines at an angle of 45° in the $\tau - \chi$ plane. Comoving particle horizon is defined as the maximum distance that a particle can travel from an initial time t_i to a later time t_f, and is denoted by

$$\chi(t) = \tau - \tau_i = \int_{t_i}^{t_f} \frac{\mathrm{d}t}{a(t)} \tag{6.5}$$

In particular if $t_i = 0$ and $t_f = t$, the comoving particle horizon τ is given by

$$\tau = \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \frac{\mathrm{d}a}{Ha^2} = \int_0^a \frac{\mathrm{d}(\ln a)}{aH} \tag{6.6}$$

where $\dfrac{1}{aH}$ is the comoving Hubble radius.

If one takes the initial time t_i as the time of the initial singularity, so that $a(t_i) = 0$ at $t_i = 0$, then one can relate the comoving particle horizon to the physical particle horizon through the scale factor as

$$d_p(t) = a(t)\chi \tag{6.7}$$

So the standard big bang cosmology has an initial time t_i at some finite point in the past, i.e., the particle horizon is finite and thus puts a limit on the causally connected regions of the universe. Information sent at a certain moment τ will never be received by an observer in the future if that observer is beyond the event horizon. The event horizon in comoving coordinates is given by

$$\chi > \chi_{\text{event horizon}} = \int_{\tau}^{\tau_{\text{max}}} \mathrm{d}\tau = \tau_{\text{max}} - \tau \tag{6.8}$$

where τ_{max} is the maximum value of time (maybe finite or infinite). From this, the physical size of the event horizon can be obtained as

$$d_e(t) = a(t)\chi_{\text{event horizon}}. \tag{6.9}$$

6.2.2 Flatness Problem

Another problem in the standard Big bang model is termed as flatness problem [220]. This problem is related to the geometry of the universe, appearing as flat model. The matter density and expansion rate of the universe appear to be almost perfectly balanced throughout the evolution of the universe but any small changes at the beginning should have grown to a huge amount. Cosmologists have no answer why such small variation after the big bang would not increase with the expansion. It raises the question "why is the density of the universe so close to the critical density?" or equivalently, "why is the universe so flat?". From Einstein's field equations the energy density and curvature are related by (1.11)

$$\frac{\rho - \rho_c}{\rho_c} = \frac{K}{a^2 H^2}$$

It can be shown that if the density of the universe deviated (slightly larger) from critical density by a small amount around a billion years after Big bang then the universe would be re-collapsed by present era. Thus the curvature (or energy density) would so precisely be tuned at very early times so that at present the energy density is much closer to critical density. Any very small deviation from flatness after the first few seconds of the big bang may have either a re-collapsing model before even fusion could have started or the universe would have expanded to such an extent that it would be devoid of matter. So this heavily tuned density or curvature at early times raises a question mark about big bang theory.

The horizon and flatness problems of the standard cosmological model do not suggest that there is an anomaly in the Big Bang model itself, rather, the model is incomplete with some underlying dynamical process that accounts for these peculiar features. The standard cosmological model cannot predict the homogeneity and flatness of the universe, rather it must be assumed in the initial conditions, representing a shortcoming in the predictive power of the model. The most likely candidate explaining the dynamical origin of these initial conditions for the universe is the idea of inflation.

6.3 Idea of Inflation

It has been shown that the horizon and flatness problems of the standard cosmological model are brought on by the increasing comoving Hubble radius $\frac{1}{aH}$. This section will feature how inflation addresses to these issues. Many characteristics of the observable universe, including its spatial flatness and large-scale homogeneity, are understood to be explained by inflation, which is acknowledged as the leading, and best-developed idea. The idea of inflation was originally proposed in [220, 221] to reverse the behavior of the comoving Hubble radius, so that it decreases in the early universe. This corresponds to an accelerated expansion of space, and results in a universe that naturally tends towards high levels of homogeneity and spatial flatness. One may relate the pressure and acceleration of the universe to the Hubble radius as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \implies \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0 \implies \rho + 3p < 0 \tag{6.10}$$

Therefore, to have an inflationary epoch so that it can solve the horizon and flatness problems one can assume a decreasing Hubble sphere (i.e., shrinking Hubble radius). Now defining the slow-roll parameter $\epsilon_1 \equiv -\frac{\dot{H}}{H^2}$, one can write the acceleration equation as

$$\frac{\ddot{a}}{a} = (1 - \epsilon_1)H^2 \tag{6.11}$$

Hence accelerated expanding period is possible as long as

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}N_e} < 1 \tag{6.12}$$

where N_e is the number of e-folds of inflation, with $\frac{\mathrm{d}N_e}{\mathrm{d}t} = H$. The amount of inflation can be measured by the number of e-folds of accelerated expansion as

$$N_e = \ln\left(\frac{a_e}{a_i}\right) \tag{6.13}$$

where a_i and a_e are the scale factors at the beginning and end of inflation respectively. Further, the Friedmann equations that describe the evolution of the universe would allow an accelerated expansion for a stress-energy

$$p < -\frac{1}{3}\rho \tag{6.14}$$

From the above study, one can infer that any inflationary period involves a shrinking Hubble sphere, an accelerated expansion, and a negative pressure. One can define the comoving horizon as an integral of the comoving Hubble radius

$$\tau = \int_0^a \frac{\mathrm{d}(\ln a')}{a' H(a')} \tag{6.15}$$

The difference between the comoving Hubble radius $\frac{1}{aH}$ and the comoving horizon τ is that particles separated by distances greater than comoving Hubble radius are causally disconnected in the present, whereas particles separated by the comoving horizon have never been causally connected. If τ is much larger than $\frac{1}{aH}$ then particles outside the Hubble radius in the present, were causally connected in the past. This is possible if the comoving horizon receives the majority of its contribution in the early universe as the Hubble radius became larger in the past. If there is a phase of decreasing Hubble radius, there may have been a time when it was larger. This happens during inflation because the scale factor increases exponentially and the Hubble parameter remains approximately constant.

6.4 Consequences of Inflation

The major success of inflationary theory immediately after the big bang scenario is to resolve the horizon and flatness problems of Big bang cosmology.

The horizon problem is solved because huge inflationary expansion causes the causally connected region to become causally disconnected. Because of the tiny size of the universe, each part of the universe is causally connected to the other parts immediately after the big bang. However, due to huge expansion during the inflationary era, some distant parts of the universe appear to be isolated from each other.

The huge expansion in the inflationary era has led the universe to nearly flat geometry. All kinds of wrinkles and other abnormalities are automatically smoothed out by this exponential expansion. Besides resolving the above two problems of Big bang cosmology, inflation also provides the seeds for this structure formation of the universe. A very small energy variation during inflation (due to quantum uncertainty) causes the matter to accumulate, forming galaxies and clusters of galaxies. However, it is still unknown by which mechanism the huge expansions occur and then turns off. Most of the inflationary models deal with a scalar field (known as inflaton) having self-interacting potential. In any case, most of the cosmologists favour an inflationary era just after the Big bang scenario.

6.5 Kinematics of Inflation

It can be seen that a shrinking Hubble parameter can result in an inflationary epoch providing a solution to the horizon and flatness problems. This raises the question of what could be the possible physical mechanism for an accelerated expansion in the early universe. There are several reviews of inflationary models subject to particle physics [225, 228, 229]. A universe governed by a cosmological constant would expand forever at an exponential rate and is therefore ruled out. Hence it may be concluded that, for the universe to experience a radiation-dominated expanding phase once inflation is over, the vacuum energy density that drives inflation must be a dynamical quantity evolving with time. One can model this dynamical evolution with a scalar field $\phi \equiv \phi(t)$ with action

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{6.16}$$

In the background of FLRW metric, the energy density and thermodynamic pressure of the scalar field are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{6.17}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{6.18}$$

and the equation of state is

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{6.19}$$

From Equation (6.18) it can be seen that a scalar field can generate a negative pressure if the potential $V(\phi)$ dominates over the kinetic term $\frac{1}{2}\dot{\phi}^2$. Therefore the inflaton field is required to be "slowly rolling".

6.6 Inflationary Dynamics

Two different dynamical realizations of inflation have emerged as a result of the search for a realistic particle physics structure that drives slow-roll inflation. In the standard scenario, termed cold inflation (CI), the inflaton field is treated as an isolated system, with the interactions between the inflaton and other fields taken into account when making radiative corrections to the scalar potential. The other one is warm inflation (WI), originally proposed in [223, 224]. In WI, the interaction between the inflaton field and other fields results in the inflaton energy being dissipated, allowing particle creation to happen simultaneously with inflationary expansion. There are several warm inflationary models in literature [230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242]. In this section, some of the key equations related to the dynamics of inflation will be introduced.

6.6.1 Cold Inflation

In cold inflation, the evolution equation for the inflaton field is given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{6.20}$$

and the slow roll parameter is given by

$$\epsilon_1 = \frac{3}{2}(\omega_\phi + 1) = \frac{\dot{\phi}^2}{2H^2} \tag{6.21}$$

An accelerated expanding phase will be maintained until $\epsilon_1 < 1$. This can be realized with the de Sitter condition $p_\phi \to -\rho_\phi$, where $\epsilon_1 \to 0$ and the kinetic energy remains subdominant to the potential $\dot{\phi}^2 \ll V(\phi)$. Hence the first Friedmann equation can be written as

$$3H^2 m_p^2 \simeq V(\phi) \tag{6.22}$$

where $m_p = \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck mass. To have a sufficiently long accelerated expanding phase one requires the second derivative of the inflaton field ϕ remains to be small so that

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V'(\phi)|. \tag{6.23}$$

which requires the second slow roll parameter to be

$$\eta = -\frac{\ddot{H}}{2H\dot{H}} = -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon_1 - \frac{1}{2\epsilon_1}\frac{\mathrm{d}\epsilon_1}{\mathrm{d}N} < 1 \tag{6.24}$$

Using these slow roll approximations, the evolution equation (6.20) can be written as

$$3H\dot{\phi} + V'(\phi) = 0 \tag{6.25}$$

Further, the slow roll parameters can be written in terms of the potential as

$$\epsilon_\phi = \frac{1}{2}m_p^2\left(\frac{V'(\phi)}{V}\right)^2 < 1 \tag{6.26}$$

$$\eta_\phi = m_p^2\left(\frac{V''(\phi)}{V}\right) < 1 \tag{6.27}$$

Under these assumptions the background evolution can be written as

$$H^2 \approx \frac{1}{3m_p^2}V(\phi) \approx \text{constant} \tag{6.28}$$

$$\dot{\phi} \approx -\frac{V'(\phi)}{3H} \tag{6.29}$$

and space-time becomes approximately de Sitter with the scale factor $a(t) \sim e^{Ht}$. It may be noted that the slow-roll parameters depending on the potential can be related to the Hubble slow-roll parameters $\epsilon_1 = \epsilon_\phi$ and $\eta = \eta_\phi - \epsilon_\phi$, and inflation ends with the violation of the slow-roll, $\epsilon_1 = \epsilon_\phi \equiv 1$. Further, the 2nd Friedmann equation can be written as

$$2\frac{\partial H}{\partial \phi} = -\frac{1}{m_p^2}\dot{\phi} \tag{6.30}$$

Using the above relation, the 1st Friedmann equation can be written as

$$[H'(\phi)]^2 - \frac{3}{2m_p^2}H^2(\phi) = -\frac{1}{2m_p^4}V(\phi) \tag{6.31}$$

This is known as Hamilton Jacobi Equation. One can consider $H(\phi)$ rather than $V(\phi)$ because $H(\phi)$ is a fundamental quantity. The Hamilton-Jacobi formulation gives a smoother derivation in many inflationary results. It has many applications in inhomogeneous situation. The formulation can be obtained by considering time-dependent scalar field.

The number of e-folds of inflation can be calculated using the potential as

$$N(\phi) = \ln\frac{a_e}{a} = \int_t^{t_e} H\mathrm{d}t = \int_\phi^{\phi_e} \frac{H}{\dot{\phi}}\mathrm{d}\phi \approx \int_{\phi_e}^{\phi} \frac{V(\phi)}{V'(\phi)}\mathrm{d}\phi \tag{6.32}$$

Further, one can write it in terms of the slow-roll parameters as

$$N(\phi) = \int_{\phi_e}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_1}} \approx \int_{\phi_e}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_\phi}} \tag{6.33}$$

To have enough inflation so that it overcomes the horizon and flatness problems, one needs a period of $50-60$ e-folds. As mentioned earlier, in cold inflation the inflaton is treated as an isolated system and so any other initial component of energy density is rapidly redshifted away during the expansion, and the universe is left in a supercooled state. Hence, a separate reheating phase is necessary at the end of the inflationary epoch to transit the universe to the radiation-dominated phase [243]. It is possible if the inflaton oscillates coherently around the minimum potential at the end of inflation, then the inflaton acts as pressureless matter

$$\frac{\mathrm{d}\tilde{\rho}_\phi}{\mathrm{d}t} + 3H\tilde{\rho}_\phi = 0 \tag{6.34}$$

To decay the inflaton, it must be coupled to other degrees of freedom, and this decay of the inflaton field modifies the equation of motion for the inflaton energy density by including the dissipation coefficient Υ,

$$\frac{\mathrm{d}\tilde{\rho}_\phi}{\mathrm{d}t} + (3H + \Upsilon)\tilde{\rho}_\phi = 0 \tag{6.35}$$

This allows particles to create after inflation ends and the standard hot Big Bang evolution proceeds. The dissipation coefficient Υ coincides with the standard decay width of the field Γ_ϕ when the field oscillates about the minimum. This is usually negligible during inflation because $\Gamma_\phi < m_\phi \ll H$.

6.6.2 Warm Inflation

Warm inflation (WI) is the realization that the interactions between the inflaton and other fields results in the inflaton energy being dissipated to other dynamical degrees of freedom [244, 245]. This suggests that particle creation can take place alongside an inflationary expansion until the scalar potential is the dominating component of energy density in the universe, with the ambient temperature greater than the Hubble scale. Radiation can naturally begin to dominate the energy density of the universe through this particle creation without the requirement for a distinct reheating phase of cold inflation.

The simplest warm inflationary scenario is one in which particles that are created thermalize faster than the Hubble expansion, resulting in a quasi-adiabatic and near-equilibrium evolution that modifies the equation of motion of the inflaton field by including the dissipation coefficient Υ which acts as a friction term. Assuming that the produced particles are lighter than the ambient temperature (i.e. relativistic) then the energy of the inflaton field is dissipated in a nearly-thermal radiation bath. The Friedman equations can be written as

$$3H^2 = (\rho_\phi + \rho_r)\,, \qquad 2\dot{H} = -(\rho_\phi + p_\phi) - (\rho_r + p_r) \tag{6.36}$$

where ρ_r and p_r are the energy density and thermodynamic pressure of the radiation field with $p_r = \dfrac{1}{3}\rho_r$. The total energy density $\rho = \rho_\phi + \rho_r$ and the continuity equation for the total energy density satisfies the standard relation $\dot{\rho} + 3H(\rho + p) = 0$.

During the warm inflationary scenario, the scalar and radiation components interact, and hence energy is transferred from the scalar field to radiation fluid which is described by the conservation equations

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\Upsilon\dot{\phi}^2, \qquad \dot{\rho}_r + 3H(\rho_r + p_r) = \Upsilon\dot{\phi}^2 \tag{6.37}$$

Assuming the evolution equations for the inflaton field and the radiation energy density, the above equations become

$$\ddot{\phi} + 3H\dot{\phi} + \Upsilon\dot{\phi} + V'(\phi) = 0 \tag{6.38}$$

$$\dot{\rho}_r + 4H\rho_r = \Upsilon\dot{\phi}^2 \tag{6.39}$$

where $\rho_r = \dfrac{\pi^2 g_*}{30}T^4$ with g_* light degrees of freedom and $\sigma = \dfrac{\pi^2 g_*}{30}$ is the Stefan-Boltzmann constant. Now assuming the system is in the slow-roll regime, equations (6.38) and (6.39) can be approximated as

$$\dot{\phi} \simeq -\frac{V'(\phi)}{3H(1+Q)} \tag{6.40}$$

$$\rho_r \simeq \frac{\Upsilon\dot{\phi}^2}{4H} \tag{6.41}$$

where the dissipative ratio $Q = \dfrac{\Upsilon}{3H}$ and consequently, the temperature of the thermal bath takes the form

$$T = \left[\frac{Q}{4\sigma(1+Q)^4}\frac{(V'(\phi))^2}{V}\right]^{\frac{1}{4}}. \tag{6.42}$$

In the context of WI, the first and second slow roll parameter can be defined as

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2(1+Q)}\left(\frac{V'(\phi)}{V}\right)^2 \equiv \epsilon_\phi \tag{6.43}$$

$$\eta_\phi = \frac{1}{(1+Q)}\left(\frac{V''(\phi)}{V}\right) \tag{6.44}$$

Another slow roll parameter can be defined as

$$\beta = \frac{1}{(1+Q)}\frac{V'\Upsilon'}{V\Upsilon} \tag{6.45}$$

Now during the warm inflationary scenario, there are two distinct regimes, namely, weak dissipative regime in which $Q \ll 1$ (i.e., $\Upsilon \ll 3H$) and strong dissipative regime where $Q \gg 1$ (i.e., $\Upsilon \gg 3H$). Further, the parameter Υ can be chosen as a constant (i.e., $\Upsilon \equiv \Upsilon_0$) or a function of the potential (i.e., $\Upsilon \equiv \Upsilon(\phi)$) or a function of the temperature of thermal bath T (i.e., $\Upsilon \equiv \Upsilon(T)$) or function of both (i.e., $\Upsilon \equiv \Upsilon(\phi, T)$). If $\Upsilon \equiv \Upsilon(\phi, T)$ has been chosen, then using the result of quantum field theory in curved space, Υ takes the form $\Upsilon = \Upsilon_0 \dfrac{T^m}{\phi^{m-1}}$.

Slow-roll inflation is sustained in the warm regime as long as $\eta, \epsilon_1 < (1+Q)$ is satisfied, which, in the strong dissipative regime is maintained without any fine-tuning of the scalar potential. The subdominance of the radiation energy density to the potential can be expressed as

$$\frac{\rho_r}{\rho_\phi} \simeq \frac{\epsilon_1}{2}\frac{Q}{(1+Q)} \leq 1. \tag{6.46}$$

Once the system enters the strong dissipative regime, Q becomes large and the radiation energy density may increase relative to the scalar potential, allowing radiation to provide a significant fraction of the total energy density of the universe when slow-roll inflation ends with $\epsilon_1 \sim 1$. Generally if inflation ends in the strong dissipative regime, radiation will dominate the energy density of the universe immediately after the end of slow-roll regime [230].

6.7 Inflationary observables

In WI due to the presence of the radiation field, the source of the density fluctuations is due to thermal fluctuation [223, 246] so that the scalar field fluctuations are dominated by thermal, rather than quantum [223, 224] in nature. As in WI the mixture of the scalar field and radiation is produced at the perturbative levels so the curvature and entropy perturbations coexist. But it has been shown [246] that during WI the entropy perturbations decay while the curvature (as adiabatic modes) perturbation survives [223, 224], so the power spectrum of the curvature perturbation (in slow roll approximation) is given by [247]

$$P_s \simeq \frac{H^3 T}{\dot{\phi}^2}\sqrt{1+Q} = \frac{HT}{2\epsilon_1}\sqrt{1+Q}. \tag{6.47}$$

Also the scalar spectral index defined as $n_s = \dfrac{\mathrm{d}\ln P_s}{\mathrm{d}\ln k}$ and can be written explicitly as [248]

$$n_s = 1 - \frac{(9Q+17)}{4(1+Q)}\epsilon_1 - \frac{(9Q+1)}{4(1+Q)}\beta + \frac{3}{2}\eta, \tag{6.48}$$

Since there is no coupling between the tensor perturbation with thermal background, the tensor modes have an equivalent amplitude as in cold inflation (i.e. the tensor spectrum $P_T = 8H^2$) and hence the tensor-to-scalar ratio r has the expression

$$r = \frac{P_T}{P_s} = \frac{16\epsilon_1}{(1+Q)^{\frac{3}{2}}}\frac{H}{T}. \tag{6.49}$$

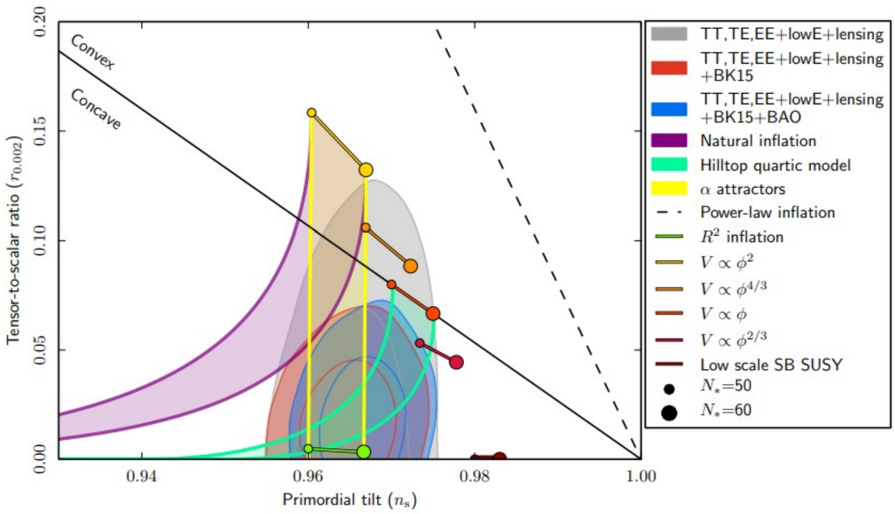

Figure 6.1: Marginalized joint 68% and 95% CL regions for n_s and r from Planck 2018 in combination with BK15 data, compared to theoretical predictions of selected models of inflation.

6.7.1 Planck constraints

The primordial density perturbations are the seeds that will grow into the LSS of the universe, as well as the perturbations in the temperature of the CMB. The recent measurements of the anisotropy of the CMB by the Planck mission support inflation, with observations pointing to inflationary models based on a slowly rolling scalar field which produces a primordial spectrum of density perturbations which is essentially adiabatic, Gaussian, and almost completely scale-invariant. The parameter space for viable models of inflation has been significantly reduced thanks to the Planck mission. According to the recent measurement by the Planck satellite [199]

$$P_s = 2.17 \times 10^{-9}, \tag{6.50}$$

$$n_s = 0.9649 \pm 0.0042. \tag{6.51}$$

The tensor mode has not yet been detected, although progress has been made to constrain the inflationary models further by excluding possible regions of the parameter space. The allowable regions of parameter space for inflationary models are typically determined by the n_s vs. r exclusion plot, shown in Figure 6.1.

CHAPTER 7

WARM INFLATION AND QUASI-STABLE CONDITION

7.1 Prelude

Warm inflationary scenario can be described from field theoretic point of view if the radiation field is in thermal equilibrium. Here the interaction between the inflaton and other fields is so significant that a quasi-stationary thermalized radiation bath is formed. The thermal fluctuations of this radiation bath is the primary source for the density fluctuations which is transported to the inflaton field as adiabatic curvature perturbation [246, 249]. Due to this dissipative effect, non-trivial dynamics in WI has significant effect on observational quantities such as tensor-to-scalar ratio (r), the spectral index (n_s), and the non-Gaussianity parameter (f_{NL}) [250], and hence WI can well be tested observationally. Further in WI, the radiation production takes place during the inflationary expansion (driven by inflaton) so there is no need for any reheating era. This is possible for the friction term in the evolution equation for the inflaton field and consequently, the scalar field dissipates into a thermal bath with other fields.

7.2 Non-equilibrium thermodynamics and quasi-stable condition in the context of warm inflation

Usually, in WI one can eliminate the reheating period considering the decay of the inflaton particle number, and radiation particles will be gradually produced during inflationary era. As a result, there will be a smooth transition from the inflationary era into the radiation-dominated era. So one naturally assumes that the above created (radiation) particles give rise to a thermal gas of radiation. The damping term in the evolution equation of the inflaton field is responsible for the creation of radiation particle. Hence in the prescription of non-equilibrium thermodynamics, the WI model can be considered as an open thermodynamical system with non-conservation of individual fluid particle number, i.e., $N^{\alpha}_{I;\alpha} \neq 0$

$$N^{\alpha}_{I;\alpha} \equiv \dot{n}_I + \Theta n_I = n_I \Pi_I \tag{7.1}$$

Here $\Theta = u^{\alpha}_{I;\alpha}$ is the fluid expansion, u^{α}_I is the fluid four-velocity, $N^{\alpha}_I = n_I u^{\alpha}_I$ is the particle flow vector, n_I is the particle number density, $\dot{n}_I = n_{I,\alpha} u^{\alpha}_I$, Π_I represents the rate of change of the number of particles in a comoving volume a^3 and suffix $I = (\phi, r)$ stands for the above two fluids. Here Π_I can take both signs; $\Pi_I > 0$ implies the creation of particles while $\Pi_I < 0$ indicates the annihilation of particles.

One may note that any nonzero interaction term Υ may act as an effective bulk viscous pressure so that the matter conservation equations (6.37) can be written as

$$\dot{\rho}_I + 3H(\rho_I + p_I + p^c_I) = 0, \tag{7.2}$$

with $p^c_{\phi} = Q\dot{\phi}^2 > 0$ and $p^c_r = -Q\dot{\phi}^2 < 0$ as the effective bulk viscous pressure.

So in the present interacting two-fluid system (where particle numbers are not conserved) Gibbs equation (1.151) takes the form [167]

$$T_I \mathrm{d}s_I = \mathrm{d}\left(\frac{\rho_I}{n_I}\right) + p_I \mathrm{d}\left(\frac{1}{n_I}\right) \tag{7.3}$$

for the I-th fluid and is the initial step to study non-equilibrium thermodynamics. Using the individual conservation equations for the fluid system and the particle numbers, the above Gibbs equation has the explicit form as

$$n_I T_I \dot{s}_I = -\Theta p^c_I - \Pi_I(\rho_I + p_I) \tag{7.4}$$

If one assumes the thermodynamical system to be isentropic (adiabatic) in nature, the entropy per particle remains constant and hence from equation (7.4) the bulk viscous pressure depends linearly on the particle creation rate as

$$p^c_I = -\frac{\Pi_I}{\Theta}(\rho_I + p_I) \tag{7.5}$$

So one may consider dissipative fluid as equivalent to a perfect fluid with non-constant particle number. However, in the adiabatic process, there is entropy variation both due to particle number variation as well as due to the enlargement of the phase space of the system.

In the second-order formulation of the non-equilibrium thermodynamics due to Israel and Stewart, the entropy flow vector of the I-th fluid is given by

$$S^{\alpha}_I = s_I N^{\alpha}_I - \frac{\tau p^{c2}_I}{2\zeta_I T_I} u^{\alpha}_I \tag{7.6}$$

where τ is the relaxation time and ζ_I is the coefficient of bulk viscosity of the I-th fluid. Now using non-conservation of particle number (i.e., equation (7.1) and the Gibbs equation (7.3) (or equation (7.4))) one gets

$$T_I S^{\alpha}_{I;\alpha} = -n_I \mu_I \Pi_I - p^c_I \left[\Theta + \frac{\tau \dot{p}^c_I}{\zeta_I} + \frac{1}{2} p^c_I T_I \left(\frac{\tau}{\zeta_I T} u^{\alpha}_I\right)_{;\alpha}\right] \tag{7.7}$$

where $\mu_I = \dfrac{\rho_I + p_I}{n_I} - T_I s_I$ is the chemical potential. So the choice of the generalized ansatz

$$\Theta + \frac{\tau \dot{p}^c_I}{\zeta_I} + \frac{1}{2} p^c_I T_I \left(\frac{\tau}{\zeta_I T} u^{\alpha}_I\right)_{;\alpha} + \frac{\mu_I n_I \Pi_I}{p^c_I} = -\frac{p^c_I}{\zeta_I} \tag{7.8}$$

guarantees the second law of thermodynamics i.e.

$$S^{\alpha}_{I;\alpha} = \frac{p^{c\,2}_I}{\zeta_I T_I} \geqslant 0$$

As a result, the effective viscous pressure p^c_I (of the I-th fluid) has the non-linear evolution equation as

$$T_I \frac{\mathrm{d}}{\mathrm{d}t}\left(p^{c2}_I\right) + 2p^{c2}_I + \zeta_I T p^{c2}_I \left(\frac{\tau}{\zeta_I T} u^{\alpha}_I\right)_{;\alpha} + 2\zeta_I p^c_I \Theta = -2\zeta \mu_I n_I \zeta_I \Pi_I \tag{7.9}$$

It should be noted that the chemical potential is responsible for the above non-linearity and hence it may act as an effective symmetry-breaking parameter in relativistic field theories.

The present second-order theory is also known as causal theory (non-vanishing relaxation time) compare to the first-order Eckart theory as non-causal (i.e. vanishing relaxation time). Physically, the difference between these two theories can be distinguished as follows: Here the effective bulk viscous pressure p^c_I appears due to particle creation rate Π_I. In causal theory, if Π_I disappears (i.e. switch off) then p^c_I decays to zero over the relaxation time τ while in non-causal theory this will happen instantaneously.

Now choosing the particle number density n and the temperature T as the basic thermodynamical variables the equation of state can be written in a general form as

$$\rho = \rho(n, T) \text{ and } p = p(n, T) \tag{7.10}$$

Using the above equation of state and the conservation equations (for fluids and number density) in the general thermodynamical relation

$$\frac{\partial \rho}{\partial n} = \frac{\rho + p}{n} - \frac{T}{n}\frac{\partial p}{\partial T} \tag{7.11}$$

the evolution equation for the temperature can be obtained as

$$\frac{\dot{T}_I}{T_I} = -\Theta\left[\frac{p_I^c}{T_I\left(\frac{\partial\rho_I}{\partial T_I}\right)} + \frac{\frac{\partial p_I}{\partial T_I}}{\frac{\partial\rho_I}{\partial T_I}}\right] + \Pi_I\left[\frac{\frac{\partial p_I}{\partial T_I}}{\frac{\partial\rho_I}{\partial T_I}} - \frac{(\rho_I + p_I)}{T_I\left(\frac{\partial\rho_I}{\partial T_I}\right)}\right] \tag{7.12}$$

which on simplification (using equation (7.4)) gives

$$\frac{\dot{T}_I}{T_I} = -\left(\Theta - \Pi_I\right)\frac{\frac{\partial p_I}{\partial T_I}}{\frac{\partial\rho_I}{\partial T_I}} + \frac{n_I\dot{s}_I}{\frac{\partial\rho_I}{\partial T_I}} \tag{7.13}$$

Further, if the thermodynamical system is assumed to be adiabatic in nature then equation (7.13) takes the form

$$\frac{\dot{T}_I}{T_I} = -\left(\Theta - \Pi_I\right)\frac{\partial p_I}{\partial\rho_I} \tag{7.14}$$

while the evolution of the other thermodynamical variables are given by

$$\begin{aligned} \dot{\rho}_I &= -(\Theta - \Pi_I)(\rho_I + p_I), \\ \dot{p}_I &= -c_s^2(\Theta - \Pi_I)(\rho_I + p_I), \\ \text{and} \quad \frac{\dot{n}_I}{n_I} &= -(\Theta - \Pi_I) \end{aligned} \tag{7.15}$$

where $c_s^2 = \left(\frac{\partial p_I}{\partial\rho_I}\right)_{\text{adia}}$, is the (square) adiabatic sound speed.

Now equation (7.5) can be explicitly written as

$$\begin{aligned} & p_\phi^c = -\frac{\Pi_\Phi}{\Theta}\left(\rho_\phi + p_\phi\right) \qquad \text{and} \qquad && p_r^c = -\frac{\Pi_r}{\Theta}\left(\rho_r + p_r\right) \\ \implies & Q\dot{\phi}^2\Theta = -\Pi_\Phi\dot{\phi}^2, && \implies -Q\dot{\phi}^2\Theta = -\Pi_r\cdot\frac{4}{3}\rho_r \\ \implies & \Pi_\Phi = -3HQ = -\Upsilon < 0, && \implies \Pi_r = \frac{3}{4}\frac{\Upsilon\dot{\phi}^2}{\rho_r} > 0 \end{aligned} \tag{7.16}$$

This implies inflaton particles annihilate and radiation particles create in accordance with WI scenario. If one assumes the radiation production during inflation to be quasi-stable in nature, i.e., $\dot{\rho}_r \ll H\rho_r, \Upsilon\dot{\phi}^2$, then the radiation energy conservation relation gives $\rho_r \approx \frac{\Upsilon\dot{\phi}^2}{4H}$ and equations (7.15) and (7.16) can be simplified as

$$\Pi_r \approx 3H, \qquad \Pi_\Phi = -3HQ \tag{7.17}$$

$$\frac{\dot{n}_r}{n_r} = -(\Theta - \Pi_r) \approx 0 \tag{7.18}$$

Hence radiation particle number is conserved. Further from equations (7.14) and (7.15), all the physical quantities (like temperature, energy density, pressure) of radiation particle is conserved. This is in contradiction to the warm inflationary scenario where the Universe should gradually be dominated by radiation. Also, particle number conservation is not in favor of non-equilibrium thermodynamics. This issue can be overcome either assuming the non-adiabatic nature of the thermodynamical system (i.e., equation (7.5) does not hold) or the radiation production process should not be quasi-stable in nature (i.e., $\rho_r \not\approx \frac{\Upsilon\dot{\phi}^2}{4H}$ and hence $\Pi_r \not\approx 3H$). So, it can be concluded that adiabatic thermodynamical analysis of WI does not provide quasi-stable radiation process.

However, such an exact adiabatic radiation production process never happens in any realistic analysis of WI. In fact, in the radiation evolution equation (6.37) the above conclusion would imply that the dissipation term, which acts as a source term in that equation, would exactly counterbalance the dilution term due to the expansion. But this can only hold approximately, at best in a zeroth-order in the slow-roll approximation. Also, the approximated relation in equation (6.41) receives corrections from slow-roll terms and it is, thus, not an exact expression (for details see [251]). It can be realized that near the end of WI, just before the radiation energy density overtakes the inflaton energy density, all quantities change quickly, which is just a consequence of the slow roll approximation no longer holding.

7.3 Warm inflation in modified gravity theories

In this section the consequence of quasi-stable scenario in the background of various modified gravity theories will be studied.

Any modified gravity theory can be regarded as the Einstein gravity with two-fluid system of which one is the usual fluid and the other one is hypothetical effective fluid whose energy density and pressure is given by the extra terms in the Friedmann equations of the corresponding modified gravity theory. In the context of WI, the effective fluid is chosen as the inflaton fluid while the usual fluid is considered as the radiation fluid. Depending on the nature of these two fluids, the modified gravity can be classified into two types.

(i) The fluids are non-interacting, i.e., the continuity equation of the corresponding modified gravity theory is given by $\dot{\rho} + 3H(\rho + p) = 0$ (for example $f(R)$ gravity, $f(T)$ gravity, and so on). Since there is no interaction, there is no energy transfer between these two fluids and consequently, non-equilibrium thermodynamics cannot come into the scenario. Hence there is no possibility of the production of radiation particles. So WI is not possible for those types of modified gravity theories.

(ii) The fluids are interacting, i.e, there is some extra terms on the right-hand side of the continuity equation (for example Einstein-Cartan-Kibble-Sciama (ECKS) gravity theory, fractal gravity, and so on). This extra term can be referred as interaction term

$\mathcal{I}$ and consequently the evolution equation of the individual fluid takes the form of equation (6.37). Now proceeding similarly, one can finally obtain the same conclusion $\dot{n}_r = 0$.

(For different modified gravity theories, the only difference is that this interaction term is characterized by the corresponding model-dependent parameters. For example, in ECKS gravity, the torsion scalar function $\varphi \equiv \varphi(t)$ characterizes the interaction term $\mathcal{I}(= -4\varphi\rho_r)$ [125] while for fractal gravity the interaction term $\left(-\frac{4}{3}\frac{\dot{v}}{v}\rho_r\right)$ is characterized by the fractal function $v \equiv v(t)$ [267].)

So the conclusion that the adiabatic thermodynamic prescription is not consistent with quasi-stable nature of radiation fluid in case of warm inflation, is independent of the choice of gravity theories. Now one can solve this problem by changing the nature of the inflaton fluid.

It should be noted that this problem is basically involved with equation (7.5).To obtain equation (7.5) from equation (7.4), one has to consider $\rho_I + p_I \neq 0$. As non-equilibrium thermodynamics is considered, so $p_I^c \neq 0$. Hence if one assumes $\rho_I + p_I = 0$, one cannot consider the adiabatic process in the background of non-equilibrium thermodynamics.

Therefore, this problem can be solved by considering the inflaton fluid as variable cosmological constant and consequently $\rho_\phi + p_\phi = 0$. So the Gibbs equation (7.4) for inflaton fluid takes the form

$$n_\phi T_\phi \dot{s}_\phi = -\Theta p_\phi^c \tag{7.19}$$

Further, since the universe is in equilibrium state as a mixture of radiation and inflaton fluid, $\dot{s}_r \neq 0$ and the Gibbs equation for radiation fluid can be written as

$$n_r T_r \dot{s}_r = -\Theta p_r^c - \frac{4}{3}\Pi_r \rho_r \tag{7.20}$$

Using the expression for dissipative pressure for radiation fluid and quasi-stable condition, the particle creation rate for radiation fluid can be written as

$$\Pi_r = 3H - \frac{3H n_r T_r \dot{s}_r}{\Upsilon \dot{\phi}^2} \tag{7.21}$$

So the radiation particle number evolution is given by

$$\frac{\dot{n}_r}{n_r^2} = -\frac{3H T_r \dot{s}_r}{\Upsilon \dot{\phi}^2} \neq 0 \tag{7.22}$$

Hence radiation particle number is not conserved. In fact as $\dot{s}_r < 0$, radiation particle is created. The particle number of radiation fluid can be written as

$$\frac{n_{r0}}{n_r} = 1 + n_{r0} \int \frac{3 T_r \dot{s}_r}{a \Upsilon \dot{\phi}^2} da \tag{7.23}$$

Thus, it is reasonable to assume as a hypothesis that if radiation is produced in WI exactly in the form of an adiabatic process, then a non-equilibrium thermodynamics analysis leads to a contradiction. However, such an exact adiabatic radiation production process never happens in any realistic analysis of WI. In the slow roll approximation, this holds at best at the zeroth order. Moreover, a variable cosmological constant may accommodate the quasi-stable process in WI with non-equilibrium thermodynamic description.

7.4 Conclusion

This chapter deals with non-equilibrium thermodynamics that is associated with the warm inflationary scenario. The premise is that an adiabatic radiation production process holds exactly i.e. the radiation dilution is exactly counterbalanced by a dissipation term. Under this hypothesis, it is found that radiation particle number, temperature, radiation energy density, and pressure are all conserved, which is contrary to the nature of the warm inflationary dynamics. However, this cannot happen in a WI realization or equivalently, this does not hold in any realistic model of WI. As far as the slow-roll conditions apply during WI the adiabatic process is only at best, an approximation valid up to slow-roll coefficients. When the slow-roll approximation starts to be violated, the stronger will become the variations in temperature, radiation, etc. as naturally happens at the end of inflation. Further, it is worth mentioning that when the slow-roll contributions are taken into account in the derivations, the above contradiction found is naturally resolved. Finally, it is shown that an alternative way to resolve this issue is to introduce a variable cosmological constant that accommodates the quasi-stable process in WI with non-equilibrium thermodynamic description.

CHAPTER 8

WARM INFLATION IN FRACTAL GRAVITY

8.1 Prelude

Inflationary paradigm is the best predictive description of the universe in the early eras just after the big bang. This scenario not only solves the problems of the standard big bang cosmology but also explains the origin of the CMB anisotropies and the large scale structure of the universe [252, 253]. On the other hand, recent CMB data discards several models of inflation and puts severe constraints on many other models [199] with a single scalar field (inflaton) model as the best option.

In the warm inflationary model [223, 242, 254] the interaction between the inflaton and other fields is very much dominated to produce a quasi-stationary thermalized radiation bath during inflation. So the thermal fluctuations in the radiation bath are the primary source of density fluctuations and it is transported to the inflaton field as adiabatic curvature perturbations [246, 249, 255, 256, 257]. Due to this non-trivial dissipative dynamics (and also stochastic effects), the observed inflationary parameters namely tensor-to-calar ratio (r), the scalar spectral index (n_s), and the non-Gaussianity parameters (f_{NL}) [250, 258, 259, 260, 261, 262] differ significantly from their values in CI. Further, in the warm inflationary scenario, it is possible to have a strong coupling between the inflaton field and other fields to have sufficient amount of radiation production, preserving the required flatness of the potential. As a result, the supercooling of the universe (observed in CI) is compensated by the radiation production and the universe makes a smooth transition from the accelerated era of expansion (inflationary epoch) to the radiation-dominated phase in WI without encountering any (pre) reheating era. Moreover, according to swampland conjectures [263, 264, 265], it is not possible to have de-Sitter vacua in string theory and also set very stringent constraints on inflation model-building leading to impossibility for CI while dissipation mechanism dominated WI can accommodate these conjectures.

8.2 Fractal gravity: an overview

The total action of the Einstein gravity in fractal space-time is given by

$$S = S_g + S_m, \tag{8.1}$$

where the gravitational part of the action is given by

$$S_g = \frac{1}{16\pi G}\int \mathrm{d}^4x\ v(x)\sqrt{-g}\left(R - \omega\partial_\mu v \partial^\mu v\right), \tag{8.2}$$

and S_m, the action of the matter part minimally coupled to gravity [266, 267, 268], is given by

$$S_m = \int \mathrm{d}^4x\ v(x)\sqrt{-g}\mathcal{L}_m. \tag{8.3}$$

Here g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, v is the fractal function and ω is the fractal parameter. The standard measure d^4x is replaced by a Lebesgue-Stieltjes measure $\mathrm{d}\mathfrak{g}(x)$. The scaling dimension of $\mathfrak{g}$ is -4α, where the parameter α $(0 < \alpha < 1)$ corresponds to the fraction of states preserved at a given time during the evolution of the system. Further, the measure v is not a scalar field (nor a dynamical object), but its profile is fixed a priori by the underlying geometry.

Now varying the above action (8.1), with respect to homogeneous, isotropic, and flat FLRW metric $g_{\mu\nu}$, the Friedmann equations in a fractal Universe can be obtained as (for convenience $8\pi G = 1$ is chosen)

$$3H^2 = \rho - 3H\frac{\dot{v}}{v} + \frac{\omega}{2}\dot{v}^2 \tag{8.4}$$

$$2\dot{H} + 3H^2 = -p - 2H\frac{\dot{v}}{v} - \frac{\omega}{2}\dot{v}^2 - \frac{\ddot{v}}{v} \tag{8.5}$$

where p and ρ are the pressure and energy density of the matter field, respectively. It should be noted that if the fractal function v is chosen as constant, then the standard Friedmann equations are recovered. The continuity equation in a fractal universe takes the form [269, 270, 271, 272]

$$\dot{\rho} + \left(3H + \frac{\dot{v}}{v}\right)(p+\rho) = 0 \tag{8.6}$$

The above modified Einstein field equations can be written as Einstein field equations with interacting two fluids [269, 271]

$$3H^2 = \rho + \rho_f$$
$$2\dot{H} + 3H^2 = -p - p_f$$

with conservation equations

$$\begin{aligned}\dot{\rho} + 3H(\rho + p) &= Q = -\frac{\dot{v}}{v}(\rho + p), \\ \dot{\rho_f} + 3H(\rho_f + p_f) &= -Q\end{aligned}$$

and $\rho_f = \frac{\omega}{2}\dot{v}^2 - 3H\frac{\dot{v}}{v}$, $p_f = 2H\frac{\dot{v}}{v} + \frac{\omega}{2}\dot{v}^2 + \frac{\ddot{v}}{v}$.

8.3 Warm inflation in fractal gravity

In the context of present warm inflationary scenario, (ρ_f, p_f) can be considered as the energy density and thermodynamic pressure for the inflaton field (i.e. $\rho_f = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, $p_f = p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$) while the usual field is chosen as radiation field i.e $\rho = \rho_r$, $p = p_r = \frac{1}{3}\rho_r$ [273, 274]. So the individual evolution equation of two fluids can be written as

$$\dot{\rho_r} + 4H\rho_r = \Upsilon\dot{\phi}^2 = -\frac{4}{3}\frac{\dot{v}}{v}\rho_r, \tag{8.7}$$

$$\dot{\rho_\phi} + 3H(\rho_\phi + p_\phi) = -\Upsilon\dot{\phi}^2 = \frac{4}{3}\frac{\dot{v}}{v}\rho_r. \tag{8.8}$$

One may note that the nonminimal interaction term Υ in equations (8.7) and (8.8) is not phenomenological; rather, it is a consequence of the continuity equation (8.6). In particular, in the present model, Υ depends on the time variation of the logarithm of v.

Assuming the quasi-stable production of the radiation component (i.e. $\dot{\rho_r} \ll H\rho_r$ and $\dot{\rho_r} \ll \Upsilon\dot{\phi}^2$), one can obtain the Hubble parameter by using equation (8.7) as

$$H = -\frac{1}{3}\frac{\dot{v}}{v} \tag{8.9}$$

and hence the potential can also be written in terms of fractal function as

$$V(\phi) = \frac{1}{3}\left(\frac{\dot{v}}{v}\right)^2 \tag{8.10}$$

Now, we choose various choices of the fractal function.

8.3.1 Model I : $v = v_0 t^{-\delta}$, $\delta = 4(1 - \alpha)$

This is the most common choice for the fractal function in literature [266, 269]. From equation (8.9), the Hubble parameter, scale factor, and the potential function (from equation (8.10)) is obtained as

$$H = \frac{\delta}{3}\frac{1}{t}, \qquad a = a_0 t^{\frac{\delta}{3}}, \qquad V(\phi) = \frac{\delta^2}{3}\frac{1}{t^2} \tag{8.11}$$

Thus the 1st slow roll parameter turns out to be constant as

$$\epsilon_1 = \frac{3}{\delta}$$

Hence for the power-law fractal function, there is no mechanism to halt the inflation, it continues forever.

8.3.2 Model II : $v = v_0 e^{-\delta t}$, $\delta = 4(1-\alpha)$

In literature, there is another form of fractal function, namely exponential form. In this case equation (8.9) leads to

$$H = \frac{\delta}{3}, \qquad a = a_0 \exp\left(\frac{\delta}{3}t\right), \qquad V(\phi) = \frac{\delta^2}{3}, \tag{8.12}$$

and consequently, the first slow roll parameter vanishes. So for this choice of the fractal function, the universe corresponds to de Sitter model.

8.3.3 Model III : $v = v_0 e^{-\delta t^n}$, $\delta = 4(1-\alpha)$

One can consider this model as a nonperturbative extension of the measure $v = 1+t^n$ [267] (it should be noted that the constant 1 is necessary to recover GR). Moreover, expanding the measure of the present model, one gets $v \approx$ constant $+\, t^n$ which has the correct multiscaling. However, this would happen only at early times (not at late times) or when α is very close to 1 (which is not theoretically desirable). To obtain non-constant slow-roll parameters for the inflationary paradigm, the fractal function can be chosen in the above form phenomenologically by generalizing the choice for model II. Similarly from equations (8.9) and (8.10) one has

$$H = \frac{\delta n}{3} t^{n-1}, \qquad a = a_0 \exp\left(\frac{\delta}{3}t^n\right), \qquad V(\phi) = \frac{\delta^2 n^2}{3} t^{2n-2} \tag{8.13}$$

The slow roll parameter can also be obtained as

$$\epsilon_1 = \frac{3(1-n)}{\delta n} t^{-n} \tag{8.14}$$

One may note that the value of α should be close to 0 i.e., δ is close to 1; so that the first slow roll parameter ϵ_1 (given in (8.14)) becomes realistic (i.e., $\epsilon_1 \ll 1$). However, if α is close to 1, i.e., δ is close to 0, then ϵ_1 can no longer be very small, and hence slow roll approximation cannot be valid. Hence, it is not physically justified. Also, small value of α implies the fractal dimension should be small.

8.3.3.1 Weak dissipative regime

In the weak dissipative regime (WDR), i.e. $Q \ll 1$, the evolution equation of the scalar field is given by

$$\dot{\phi}^2 = -\frac{\dot{V}}{3H} = \frac{2\delta n(1-n)}{3} t^{n-2} \tag{8.15}$$

and on integration, it gives the scalar field as

$$\phi = \sqrt{\frac{8\delta(1-n)}{3n}}\, t^{\frac{n}{2}} = \phi_0 t^{\frac{n}{2}} \tag{8.16}$$

and the potential is obtained as

$$V(\phi) = \frac{\delta^2 n^2}{3} \left(\frac{8\delta(1-n)}{3n}\right)^{\frac{2(1-n)}{n}} \phi^{\frac{4(n-1)}{n}} = V_0 \phi^{\frac{4(n-1)}{n}} \tag{8.17}$$

One can write the Hubble parameter, and scale factor in terms of scalar field as

$$H(\phi) = H_0 \phi^{\frac{2(n-1)}{n}} \text{ with } H_0 = \frac{\delta n}{3}\left(\frac{8\delta(1-n)}{3n}\right)^{\frac{(1-n)}{n}} \tag{8.18}$$

$$a(\phi) = a_0 \exp\left(\frac{n}{8(1-n)}\phi^2\right) \tag{8.19}$$

It should be noted that $V_0 = 3H_0^2$. The slow roll parameters can be written as

$$\epsilon_1(\phi) = \frac{8(n-1)^2}{n^2\phi^2} \tag{8.20}$$

$$\eta(\phi) = \frac{4(n-1)(3n-4)}{n^2\phi^2} \tag{8.21}$$

At the end of inflation, $\epsilon_1(\phi_e) = 1$. Therefore, $\phi_e^2 = \dfrac{8(n-1)^2}{n^2}$. The number of e-folds is given by

$$N = -\int_{\phi_\star}^{\phi_e} \frac{V}{V'} d\phi = \frac{n}{8(1-n)}\left[\frac{8(n-1)^2}{n^2} - \phi_\star^2\right] \tag{8.22}$$

which consequently gives the scalar field at the time of horizon crossing as

$$\phi_\star^2 = \frac{8(n-1)^2}{n^2}\left[1 + \frac{n}{n-1}N\right] \tag{8.23}$$

Here suffixes 'e' and '$\star$' denote the time instant of end of inflation and crossing the horizon respectively. The slow roll parameters at the horizon crossing are given by

$$\epsilon_{1\star} = \left(1 + \frac{n}{n-1}N\right)^{-1} \tag{8.24}$$

$$\eta_\star = \frac{(3n-4)}{2(n-1)}\left(1 + \frac{n}{n-1}N\right)^{-1} \tag{8.25}$$

Case 1. $\Upsilon = \Upsilon_0 \phi^m$

The temperature of the radiation fluid can be expressed as

$$T = \left(\frac{\Upsilon_0(1-n)}{2\sigma}\right)^{\frac{1}{4}} \phi_0^{\frac{1}{2n}} \phi^{\frac{1}{4}(m-\frac{2}{n})} \tag{8.26}$$

The other slow roll parameter can be written as

$$\beta(\phi) = \frac{4m(n-1)}{n\phi^2} \tag{8.27}$$

and at the horizon crossing time it is given by

$$\beta_\star = \frac{mn}{2(n-1)}\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.28}$$

Substituting the above values of slow-roll parameters, the scalar spectral index n_s can be written as a function of number of e-folds as

$$n_{s_\star}(n,m,N) = 1 + \frac{n}{4(1-n)}\left(8+\frac{m}{2}-\frac{5}{n}\right)\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.29}$$

The tensor-to-scalar ratio at the same time can be written as

$$\begin{aligned} r_\star(n,m,N) &= \frac{16n\delta}{3}\left(\frac{2\sigma}{1-n}\right)^{\frac{1}{4}} \frac{1}{\Upsilon_0^{\frac{1}{4}}(n,m,N)}\left(\frac{3n}{8\delta(1-n)}\right)^{1-\frac{3}{4n}} \\ &\quad \left(\frac{8(n-1)^2}{n^2}\right)^{1-\frac{m}{8}-\frac{3}{4n}}\left(1+\frac{n}{n-1}N\right)^{-\frac{m}{8}-\frac{3}{4n}} \end{aligned} \tag{8.30}$$

Using the definition (6.47) of P_s, one can find out $\Upsilon_0(n,m,N)$ and the tensor-to-scalar ratio can be simplified to

$$r_\star(n,m,N) = \frac{8(1-n)^2}{P_s}\left(\frac{\delta n}{3(1-n)}\right)^{\frac{2}{n}}\left(1+\frac{n}{n-1}N\right)^{2-\frac{2}{n}} \tag{8.31}$$

Using the $r - n_s$ diagram of Planck-2018, one can plot a $n - m$ diagram as shown in Figure 8.1, where the dark blue colour indicates an area of (n,m) with the points (r, n_s) of the model in 68% CL while that with light blue colour stands for 95% CL.

Choosing some particular values of (n,m) from Figure 8.1, the evolution of the slow-roll parameters with respect to the number of e-folds have been shown in Figure 8.2. The figures show that it support the slow-roll approximations. The behaviour of the n_s and r have been represented in Figure 8.3. The ratio of temperature and Hubble parameter during inflation in WDR has been depicted in Figure 8.4. The figures 8.2-8.4 have been drawn choosing the data sets for (n,m) as $(.0145, 672.82)$ (solid line), $(.0142, 686.98)$ (dotted line) and $(.014, 696.76)$ (dot-dashed line).

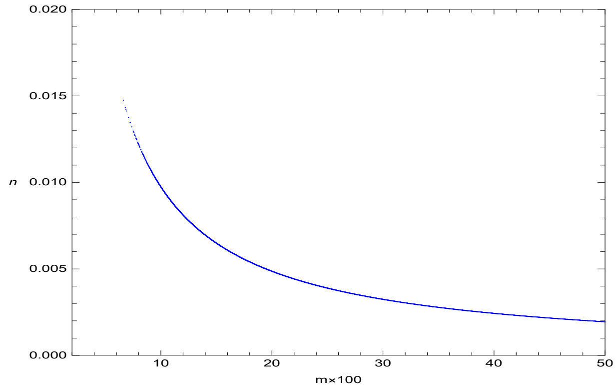

Figure 8.1: Permissible values of the parameters (n, m) of the fractal warm inflationary model in the weak dissipative regime case I.

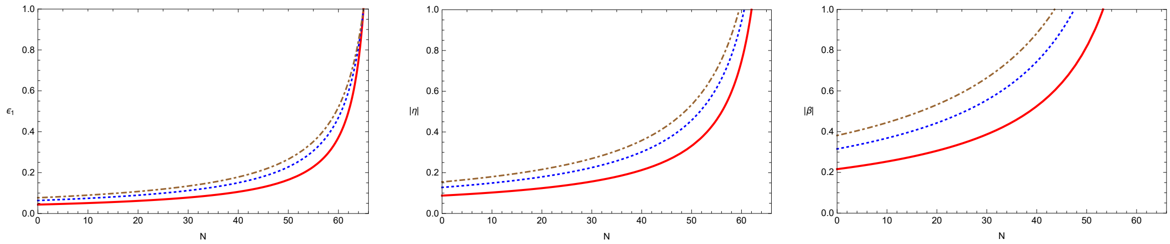

Figure 8.2: Evolution of slow roll parameters with respect to the number of e-folds

Case 2: $\Upsilon = \Upsilon_0 \dfrac{T^m}{\phi^{m-1}}$

The temperature can be written as

$$T^{m-4} = \frac{2\sigma}{\Upsilon_0(1-n)} \left(\frac{3n}{8\delta(1-n)}\right)^{\frac{1}{n}} \phi^{m-1+\frac{2}{n}} \tag{8.32}$$

Υ can be explicitly written as

$$\Upsilon = \Upsilon_0 \left(\frac{2\sigma}{\Upsilon_0(1-n)}\right)^{\frac{m}{m-4}} \left(\frac{3n}{8\delta(1-n)}\right)^{\frac{m}{n(m-4)}} \phi^{\frac{2(m-2n+2mn)}{n(m-4)}} \tag{8.33}$$

The slow roll parameter β can be obtained as

$$\beta(\phi) = \frac{4(n-1)}{n^2} \frac{(m-2n+2mn)}{(m-4)} \frac{1}{\phi^2} \tag{8.34}$$

The scalar spectral index at the horizon crossing time can be written as

$$n_{s_\star}(n, m, N) = 1 + \frac{9m + 66n - 18mn - 40}{8(m-4)(n-1)} \left(1 + \frac{n}{n-1}N\right)^{-1} \tag{8.35}$$

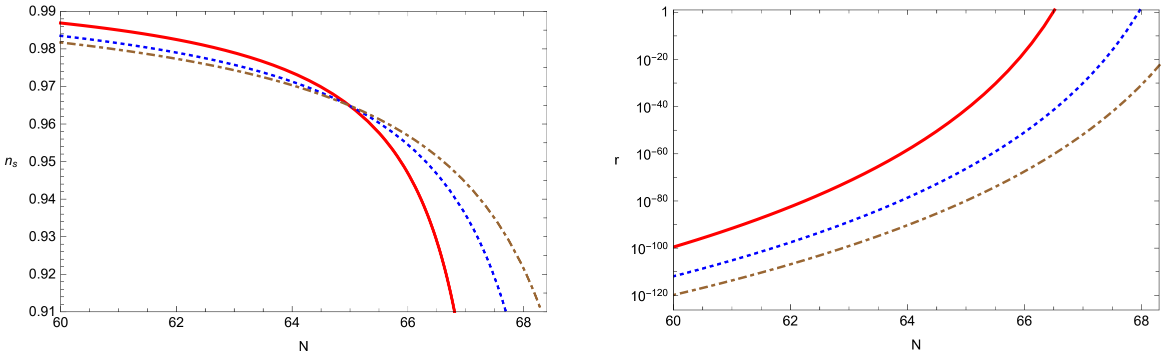

Figure 8.3: Variation of spectral index and tensor-to-scalar ratio with respect to number of e-folds

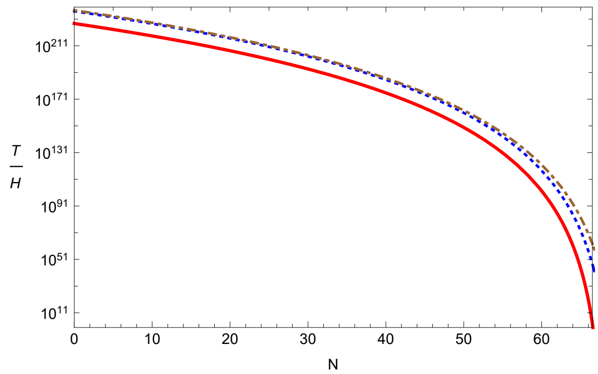

Figure 8.4: Ratio of temperature and Hubble parameter with respect to number of e-folds

The tensor to scalar ratio at that time can be written as

$$r_\star(n,m,N) = \frac{16n\delta}{3}\left(\frac{1-n}{2\sigma}\right)^{\frac{1}{m-4}}\Upsilon_0^{\frac{1}{m-4}}(n,m,N)\left(\frac{3n}{8\delta(1-n)}\right)^{\frac{mn-m-4n+3}{n(m-4)}} \left(\frac{8(n-1)^2}{n^2}\right)^{\frac{mn-2m-7n+6}{2n(m-4)}}\left(1+\frac{n}{n-1}N\right)^{\frac{n-2m-mn+6}{2n(m-4)}} \tag{8.36}$$

In Figure 8.5, $n-m$ diagram has been plotted using the $r-n_s$ diagram of Planck-2018 data as before. In the figure, the dark blue colour and light blue colour indicate an area of (n,m) in which the point (r,n_s) of the model stand at 68% and 95% CL respectively.

It can be shown that the equation (8.33) can also be written as

$$\Upsilon = \hat{\Upsilon}_0\phi^{\hat{m}}$$

which implies that case 2 can be reduced to case 1 with some modified values of n and m for WDR. So detailed study of graphical representation is similar and has not been done in this section.

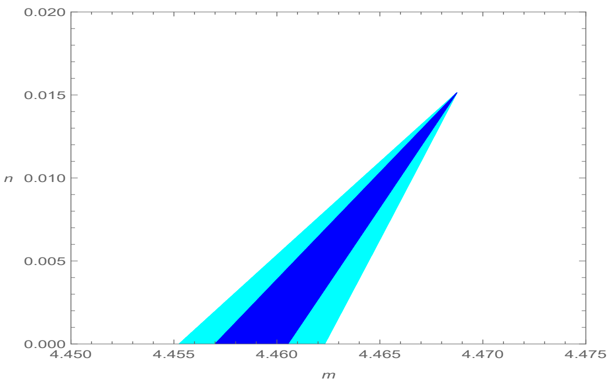

Figure 8.5: Location of (n, m) points in weak dissipative regime case 2 for fractal warm inflationary model

8.3.3.2 Strong dissipative regime

In the strong dissipative regime (SDR), the dissipative ratio is much larger than unity ($Q \gg 1$). Using this approximation, the equation (6.40) can be simplified for SDR as

$$\dot{\phi} = -\frac{1}{\Upsilon}\frac{\partial V(\phi)}{\partial \phi} \tag{8.37}$$

Case 1. $\Upsilon = \Upsilon_0 \phi^m$

For this choice of dissipation coefficient, one can write the evolution equation for the scalar field as

$$\dot{\phi}^2 = \frac{2\delta^2 n^2(1-n)}{3\Upsilon_0}\frac{t^{2n-3}}{\phi^m} \tag{8.38}$$

which subsequently on integration gives

$$\phi^{m+2} = \left(\frac{m+2}{2n-1}\right)^2 \frac{2\delta^2 n^2(1-n)}{3\Upsilon_0} t^{2n-1} = \phi_0 t^{2n-1} \tag{8.39}$$

Using the above relation, the potential, Hubble parameter, and scale factor can be written as

$$V(\phi) = \frac{\delta^2 n^2}{3}\left(\frac{\phi^{m+2}}{\phi_0}\right)^{\frac{2(n-1)}{(2n-1)}} \tag{8.40}$$

$$H(\phi) = \frac{\delta n}{3}\left(\frac{\phi^{m+2}}{\phi_0}\right)^{\frac{(n-1)}{(2n-1)}} \tag{8.41}$$

$$a(\phi) = a_0 \exp\left[\frac{\delta}{3}\left(\frac{\phi^{m+2}}{\phi_0}\right)^{\frac{n}{(2n-1)}}\right] \tag{8.42}$$

One can write the temperature of the radiation fluid as

$$T = \left[\frac{(1-n)n\delta}{3\sigma}\right]^{\frac{1}{4}} \left[\frac{\phi^{m+2}}{\phi_0}\right]^{\frac{n-2}{4(2n-1)}} \tag{8.43}$$

The slow roll parameters can be written as

$$\epsilon_1 = \frac{3(1-n)}{\delta n}\left(\frac{\phi^{m+2}}{\phi_0}\right)^{\frac{n}{(1-2n)}} \tag{8.44}$$

$$\eta = \frac{(2m-2n-2mn+3)}{(m+2)(1-n)}\epsilon_1 \tag{8.45}$$

$$\beta = \frac{m(1-2n)}{(m+2)(1-n)}\epsilon_1 \tag{8.46}$$

Inflation ends with $\epsilon_1(\phi_e) = 1$ which implies $\phi_e^{\frac{(m+2)n}{(2n-1)}} = \frac{3(1-n)}{\delta n}\phi_0^{\frac{n}{2n-1}}$. Using the notion of N, the scalar field at the horizon crossing can be explicitly written as

$$\phi_\star^{\frac{(m+2)n}{(2n-1)}} = \frac{3(1-n)}{\delta n}\phi_0^{\frac{n}{2n-1}}\left(1+\frac{n}{n-1}N\right)$$

The slow roll parameters at the horizon crossing can be expressed as

$$\epsilon_{1\star} = \left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.47}$$

$$\eta_\star = \frac{(2mn-2m+2n-3)}{(m+2)(n-1)}\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.48}$$

$$\beta_\star = \frac{m(2n-1)}{(m+2)(n-1)}\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.49}$$

Inserting the above parameters, the scalar spectral index at the same time can be obtained as

$$n_s(n,m,N) = 1 + \frac{3}{4}\frac{(2m-2n-5mn)}{(m+2)(n-1)}\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.50}$$

In Figure 8.6, at 68% and 95% CL, the points (r, n_s) of the model are indicated by dark blue colour and light blue colour respectively in the $n-m$ plot, considering $(r-n_s)$ diagram of Planck 2018 data.

Choosing particular values of (n,m) from Figure 8.6, the evolution of the slow roll parameters with respect to the number of e-folds have been plotted in Figure 8.7 and it matches with the slow roll approximations. The behaviour of the n_s and r have been represented in Figure 8.8. The figures 8.7-8.8 have been drawn considering the values for (n,m) as (.0145, .01715) (solid line), (.0142, .01779) (dotted line), and (.014, .01882) (dot-dashed line).

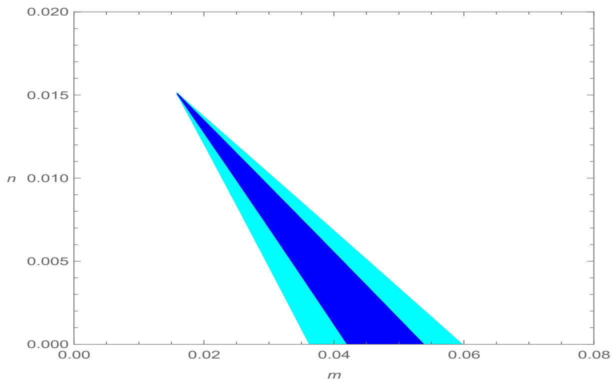

Figure 8.6: The location of the (n, m) points for the strong dissipative regime case 1

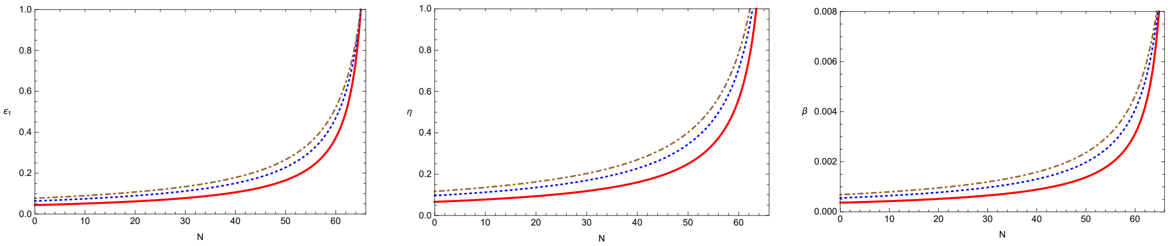

Figure 8.7: Variation of slow roll parameters with respect to number of e-folds

Case 2: $\Upsilon = \Upsilon_0 \dfrac{T^m}{\phi^{m-1}}$

The scalar field can be written as

$$\phi^{3-m} = \left[\frac{4(3-m)}{8n-4-m(n-2)}\right]^2 \frac{2\delta^2 n^2(1-n)}{3\Upsilon_0 \left(\frac{\delta n(1-n)}{2}\right)^{\frac{m}{4}}} t^{\frac{8n-4-m(n-2)}{4}} = \phi_0 t^{\frac{8n-4-m(n-2)}{4}} \tag{8.51}$$

The temperature can be written as

$$T = \left[\frac{\delta n(1-n)}{2}\right]^{\frac{1}{4}} \left[\frac{\phi^{3-m}}{\phi_0}\right]^{\frac{n-2}{8n-4-m(n-2)}} \tag{8.52}$$

So the dissipation coefficient Υ can be rewritten as

$$\Upsilon = \Upsilon_0 \left[\frac{\delta n(1-n)}{2}\right]^{\frac{m}{4}} \frac{\phi^{\frac{m(3-m)(n-2)}{8n-4-m(n-2)}-(m-1)}}{\phi_0^{\frac{m(n-2)}{8n-4-m(n-2)}}} \tag{8.53}$$

Using the above relation, the potential, Hubble parameter, and scale factor can be

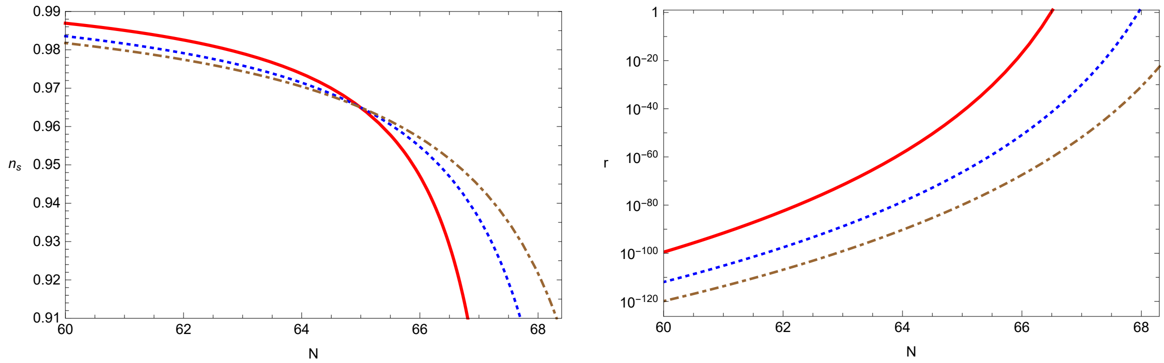

Figure 8.8: Spectral index and tensor-to-scalar ratio versus number of e-folds

written as

$$V(\phi) = \frac{\delta^2 n^2}{3}\left[\frac{\phi^{3-m}}{\phi_0}\right]^{\frac{8(n-1)}{8n-4-m(n-2)}} \tag{8.54}$$

$$H(\phi) = \frac{\delta n}{3}\left[\frac{\phi^{3-m}}{\phi_0}\right]^{\frac{4(n-1)}{8n-4-m(n-2)}} \tag{8.55}$$

$$a(\phi) = a_0 \exp\left(\frac{\delta}{3}\left[\frac{\phi^{3-m}}{\phi_0}\right]^{\frac{4n}{8n-4-m(n-2)}}\right) \tag{8.56}$$

The slow roll parameters can be written as

$$\epsilon_1 = \frac{3(1-n)}{\delta n}\frac{\phi^{\frac{4n(3-m)}{m(n-2)+4-8n}}}{\phi_0^{\frac{4n}{m(n-2)+4-8n}}} \tag{8.57}$$

$$\eta = \frac{3mn-2m-4n+8}{4(m-3)(1-n)}\epsilon_1 \tag{8.58}$$

$$\beta = \frac{8n-6mn-4}{4(3-m)(1-n)}\epsilon_1 \tag{8.59}$$

At the end of inflation $\epsilon_1(\phi_e) = 1$, hence $\phi_e^{\frac{4n(3-m)}{m(n-2)+4-8n}} = \frac{\delta n}{3(1-n)}\phi_0^{\frac{4n}{m(n-2)+4-8n}}$. The scalar field at horizon crossing time can be written as

$$\phi_\star^{\frac{4n(3-m)}{8n-4-m(n-2)}} = \frac{3(1-n)}{n\delta}\phi_0^{\frac{4n}{8n-4-m(n-2)}}\left(1+\frac{n}{n-1}N\right).$$

The slow roll parameter at the horizon crossing time can be written as

$$\epsilon_{1\star} = \left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.60}$$

$$\eta_\star = \frac{3mn-2m-4n+8}{4(m-3)(1-n)}\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.61}$$

$$\beta_\star = \frac{8n-6mn-4}{4(3-m)(1-n)}\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.62}$$

The scalar spectral index can be obtained as

$$n_{s_\star}(n,m,N) = 1 - \frac{3(m(27n+8)-26n-2)}{8(m-3)(1-n)}\left(1+\frac{n}{n-1}N\right)^{-1} \tag{8.63}$$

Figure 8.9 indicates the location of (n, m) points in the observational region for the strong dissipative regime case of the present fractal warm inflationary model, considering the numerical values of (r, n_s) parameters from Planck 2018 data set.

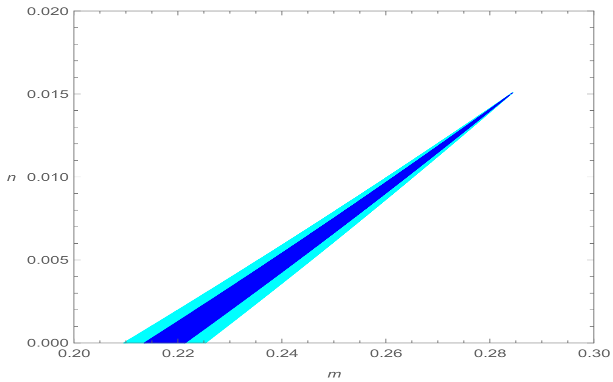

Figure 8.9: The location of the (n, m) points of the present model in the strong dissipative regime case 2

It can be shown that one can rewrite the equation (8.53) as

$$\Upsilon = \tilde{\Upsilon}_0 \phi^{\tilde{m}}$$

which implies that also in SDR, case 2 can be simplified to case 1 in SDR with some modified values of n and m. So further detailed study of graphical representation has not been done in this section.

8.4 Conclusion

In this chapter, warm inflationary scenario in the fractal gravity theory has been studied. For WI, the universe is assumed to be consisting of a scalar field (inflaton) and radiation which are interacting with each other. In the fractal gravity theory, the modified Friedmann equations can be considered as Friedmann equations of Einstein gravity with interacting two fluids in which one is the usual fractal fluid and the other is effective fluid. Without loss of generality, in the context of WI, the fractal fluid is chosen as radiation and the effective fluid has been considered as the inflaton field. Now incorporating the quasi-stable conditions and slow- roll approximations the Hubble parameter, potential function, and the scale factor have been expressed as a function of fractal function. Now different choices of fractal function have been considered so that they correctly describe the inflationary era. It is shown that inflation continues indefinitely for power law choice of fractal function while the exponential fractal function represents the de Sitter model. From the analysis, it is found that neither of the usual choices of the fractal function is favourable for WI. So a generalized form of exponential fractal function is chosen so that it describes the warm inflationary paradigm and makes a smooth transition to the radiation-dominated era.

For this choice of fractal function warm inflationary scenario have been discussed in both weak and strong dissipative regimes. It is shown that in WDR, the scalar field, potential, Hubble parameters, scale factor, number of e-folds, and the first two slow roll parameters can be expressed explicitly without choosing the dissipation coefficient. To obtain the expression for temperature and the third slow roll parameter one needs to choose the form of dissipation coefficient. At first, the dissipation coefficient has been chosen as a function of the potential alone. Hence the scalar spectral index and tensor-to-scalar ratio also can be expressed as function of model parameters (n, m). Using the $r - n_s$ diagram of Planck 2018 data, the allowable range for (n, m) has been found. The variations of slow roll parameters with respect to the number of e-folds have been shown graphically which match with the slow roll approximations. Next, the dissipation coefficient is chosen as a function of both temperature and scalar field. Though it is a function of both temperature and scalar field, it can be shown that using the temperature expression, the dissipation coefficient can be reduced to function of potential alone. As a consequence, a similar study as previously mentioned can be made with that choice for dissipation coefficient. Now for the SDR, one has to choose the form of dissipation coefficient from the very first step as the dissipation coefficient is dominant over the Hubble parameter.

From the above analysis, one can conclude that regardless of the choice of the dissipation coefficient, it can be reduced to a function of scalar field alone using the quasi-stable condition. Hence the theoretical models can be compared with observational data by constraining the arbitrary parameters in this approach. Though the initial conditions and the birth of our universe are not entirely known, however, this technique may make it possible to compare the predictions of the theoretical models of inflation in different modified gravity theories with cosmological observational data.

CHAPTER 9

SUMMARY AND FUTURE WORK

The present chapter gives an overview of the research works presented in the thesis and at the end, some future prospects have been mentioned.

In chapter 2, a detailed analysis of Einstein-Cartan-Kibble-Sciama gravity theory has been performed with the ratio of torsion scalar function and Hubble parameter to be a function of scale factor. It is found that for a particular form of the ratio, non-singular emergent solution is possible. The equivalence between interacting two-fluid model in Einstein gravity and ECKS gravity has been established and in the context of non-equilibrium thermodynamics ECKS gravity is shown to be equivalent with Einstein gravity with particle creation mechanism. For suitable choice of continuous form of the ratio, a complete cosmic scenario has been obtained starting from the inflationary era to the present accelerating phase.

An explicit cosmological investigation for generalized teleparallel gravity is presented in chapter 3. Here also, non-singular model of the universe is possible for some suitable choice of $f(T)$. Also similar to the previous chapter, equivalence between $f(T)$ gravity with non-interacting two-fluid model in Einstein gravity with particle creation has been established. Further, it is shown that a complete cosmic evolution from the inflationary era to present accelerating phase is possible by choosing the continuous function $f(T)$ suitably.

Chapter 4 investigates the emergent scenario in Hořava Lifshitz gravity. It is found that in the flat universe, although the usual perfect fluid describes the emergent phase, it is necessary to have phantom fluid for the pre-inflationary era. In the closed universe, phantom fluid is required both for the emergent and pre-inflationary eras while for the open universe, usual perfect fluid is sufficient for describing the emergent phase and pre-inflationary era.

Chapter 5 deals with the $f(R,T)$ cosmological model with $f(R,T) = R + h(T)$. It is shown that if $h(T)$ is chosen to be power law in T then the power is no longer an arbitrary constant, it is a function of the equation of state parameter. The detailed cosmological solutions have been presented in this chapter. It is shown that depending on the model parameter, the solutions represent closed and open models. Further, the solutions represent deceleration/ acceleration/ a transition from deceleration to acceleration of the universe according to different values of the equation of state parameter. Finally, these model parameters have been constrained by using observational data and it has been found that it supports a smooth transition from deceleration to the present accelerating phase. However, the model cannot support the warm inflationary phase.

In chapter 7, warm inflation has been studied from the viewpoint of non-equilibrium thermodynamics. It is found that if quasi-stable condition is assumed then the system cannot be adiabatic in nature. However, in the real scenario this does not hold because slow-roll approximation and quasi-stable conditions are not exact, they are approximations. Finally, it is shown that this issue can be solved by considering a dissipative fluid instead of the usual fluid.

Chapter 8 presents warm inflationary model in fractal gravity. The Friedmann equations in fractal gravity can be considered as the Friedmann equations with an interacting two-fluid system in Einstein gravity of which one is usual fractal fluid and the other is effective fluid, and due to this interaction, warm inflation is possible in fractal gravity. In the context of warm inflation, fractal fluid is chosen as radiation fluid and effective fluid is chosen as inflaton field. It is shown that power law and exponential law fractal function cannot explain the real scenario. So a new model has been proposed by generalizing the common models in the literature. Both weak and strong dissipative regimes have been studied for this model. Finally, the allowable regions of the model parameters have been found by using the model parameters.

The thesis discusses two important areas of cosmic evolution. In the first part unification of different cosmological ages in various modified gravity theories has been presented. For future work, it will be interesting to study the behavior of the hypothetical dark energy in modified gravity theories. Further, attempts will be made to identify modified gravity theories as Einstein gravity with non-equilibrium thermodynamical presentation. Moreover, it will be tried to evolve the cosmic fluid suitably instead of model parameters of modified gravity theory to find out the unified cosmic ages. In the second part, warm inflation has been studied. For future work, it will be interesting to constrain warm inflationary models from various observational data.

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