DOCTORAL THESIS

Qualitative Analysis of Different Attack Pattern of Whitefly on Jatropha Curcas Plant Growth and Control of Mosaic Disease

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March, 2023

JADAVPUR UNIVERSITY
FACULTY OF SCIENCES

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

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Kolkata-700032, West Bengal, India

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "QUALITATIVE ANALYSIS OF DIFFERENT ATTACK PATTERN OF WHITEFLY ON JATROPHA CURCAS PLANT GROWTH AND CONTROL OF MOSAIC DISEASE" submitted by Mrs. Roshmi Das who got her name registered on 22/11/2016 (Index No. 218/16/Maths/25) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of Prof. Ashis Kumar Sarkar and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

....
(Signature of the Supervisor date with official seal)

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ABSTRACT

As the human civilisation is upgrading day by day the natural resources of energy face a crisis. To solve this problem we are searching for the elective vitality sources in a situation inviting way. To provide an affordable solution of shrinkage of fossil fuel we pay our attention to a very essential as well as wonder plant Jatropha curcas. Jatropha curcas is such a significant plant the seeds of which plant contains 37% oil that can be utilized to obtain a superior nature of biodiesel. So the plant is economically very important. This plant is also used for medicinal purpose.

In Mathematical Biology we also study the non-linear mathematical models which are based on various realistic phenomenon. The results of these study is very significant for understanding the actual dynamical behavior regarding effect of attack pattern of herbivore to the plant, renewable resource management , effect of growth pattern of the plant , pest control , permanent coexistence of all the species etc.. Mathematical ecology deals with the interaction between the living organisms with each other and their natural environment.

In my research work our concern goes to Jatropha curcas plant .This plant is easily effected by the mosaic virus through the vector whitefly. This attack affects the plant very badly. To protect the plant from the virus attack applying insecticide is very helpful. Mathematically it is done by applying control theory.

Mathematically exponential growth of plants gives the unstability where as logistic growth gives stable steady state of the system. Theoretical results shows that applying control theory for spraying insecticide the system can be stabilised.
Likewise the growth pattern the attack pattern of whitefly also plays an important role for the disease dynamics. Different probability distribution like Binomial, Poisson and Negative-binomial distribution which biologically express the regular, random and aggregated attack pattern of whitefly are also used in my research work to determine the effect of different attack pattern. It gives us interesting results like stable coexistence, periodic oscillation, Hopf bifurcation etc. depending upon the different parameter values. Persistence and permanence are also performed to ensure the permanent coexistence of all the species.

Besides the continuous-time model we also chosen discrete time model by introducing Mickens non-standard finite difference scheme ($NSFD$) as well as Euler's discrete time system. Comparing all of them we observed that discrete time system gives better approximation of the solution as well as the disease dynamics than the continuous counter part.

All the results so obtained are verified by numerical simulations.

Keywords : Jatropha Curcas, whitefly, Mosaic disease, Random attack pattern, Regular attack pattern, Aggregated attack pattern, control theory, discrete time system.

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Dedicated to my family...

1 Introduction

Mathematics is such a subject that left its impression on solving regular smaller problems to very complicated larger real world problems. To turn real life problems into mathematical problems and then solving it by the way of Mathematics, the genesis of Mathematical modeling took place. In this process we express the real life problems usually in the form of equations from which we can better understand the original problem and can solve these problems using Mathematical tools. Not only this but also we can discover new features about the problem. One of the very popular application of Mathematical modeling is weather forecast. The simplification process in constructing a model is very much important, but also hard. A very essential quote from Albert Einstein is:" A model should be as simple as possible, and no simpler." The derivation that comes out from an analytical calculation can give a clear view about the role of different parameter values of the system.

1.1 Mathematical Biology

Mathematical Biology which is also known as Biomathematics is such a branch of Biology which works as a bridge between Mathematics and Biology. It uses Mathematical models based on living organisms and other natural phenomenon to analyze the structure, growth or decay and moreover the behavior of that biological system. Biomathematics is very helpful in both theoretical as well as practical research where we use the techniques and tools of applied Mathematics.

1.2 Ecology

Ecology (Gazi, Das, 2010) is the scientific study of living organisms with respect to each other and their surrounding environment. When a group of the same class of species occupy a particular space then we call it the population which has some unique characteristics like growth rate, age classification, mortality etc. Mathematical modeling of ecological system is described using some state variables. By the possible admissible values of these state variables the system can be better understood and the ecological problem can be solved. This is the prime goal of Mathematical modeling of ecological systems.

1.3 Population dynamics

Population dynamics is the kind of Mathematics in which different populations and their variations according to their size, age, time and space(Desharnais,2005) etc. are studied as dynamical systems. Four main key variables of simplified population models are birth, immigration, emigration and death. In absence of immigration and emigration the rate of change in the number of individuals in a population can be described as:

$$\frac{dN}{dT} = B - D$$

where 'N' is the total number of individuals in the specific experimental population, 'B' is the number of birth and 'D' is the number of deaths per individual in a particular experiment. Later it was transformed into logistic equation:

$$\frac{dN}{dT} = \alpha N(1 - \frac{N}{k})$$

Where 'N' is the biomass density ,'α' is the maximum per capita rate of change, 'k' is the carrying capacity of the population.

In my work I have used both the logistic and exponential type growth. Population dynamics also deals with the influence of biological and environmental issues. Population dynamics is a very crucial area of Mathematical Biology and mathematical epidemiology.

1.4 Jatropha Curcas plant

Jatropha curcas (Ye Meng Li Caiyan et al.,2009; Basir F.A. et al.,2016, Chowdhury Jahangir et al., 2019, P.K. Roy, X Z Li et al.,2015) is a wonder plant which belongs to the spurge family, Euphorbiaceae. This plant originally found in American tropics, most likely Mexico and central America. Now it is cultivated world-wide. It is often called the physic nut, poison nut or purging nut.

It is a semi-evergreen shrub that attains a height upto 6 meters(20 feet). The leaves are green to pale green. Fruits are usually produced in winter although in presence of good moisture and sufficiently high temperature it is possible to obtain several crops during the year. Cultivation of this plant is not difficult as it can grow even in sandy or salty soils or in deserts.

The seeds of this plant contains (27-40)% (average 37%) oil that can be used to obtain a high quality of bio-diesel fuel that is usable in a standard diesel engine. So with the upgradation of the human civilization the necessity of alternative fuel is also increasing. Jatropha curcas plant can be a better solution to this problem.

1.5 Mosaic disease and Whitefly

This plant should be protected carefully as it is easily affected by the mosaic virus (Begomo virus) causing mosaic disease (Raj S.K. Snehi et al.,2008; Pal Pallav et al. 2014). Mosaic virus is one kind of plant virus that causes the leaves of plants with a spotted and speckled look. More symptoms of this disease are severe mosaic, mottling, blistering of leaves, yellowing of leaves, sap drainage, decreased leaf size, hindering of infected plants. It mainly affects the fruits by extensively reducing the quality and quantity of the fruits.

The mosaic virus spreading mainly depends on the vector whitefly. The number of inhabitants in whitefly is constrained by temperature and rainfall. Heavy rainfall creates an obstruction for the growth of whiteflies. In this disease the mosaic virus passes from an infected whitefly to a susceptible plant and vice-versa. The spread of the virus mainly depends on the plant thickness. The transmission of the disease spread when various infected whiteflies fed on the host plants through massive flux of saliva. Whiteflies are very much productive. If once they get conventional to contaminate the host plant, they will voluntarily roam and try to attack the near vegetation. Normally whiteflies need 3 hours feeding time to procure the virus and a latent phase of 8 hours. It requires 10 minutes time to contaminate the young leaves . Symptoms seem to be appeared after a latent period of 3-5 weeks. After acquisition of the mosaic virus adult whiteflies can infect the host plants within 48 hours.

1.6 Functional responses

In ecology functional response carries different interpretation. In our study functional response is relevant to the rate at which the whiteflies attack the Jatropha curcas plant. Following C.S. Holling functional responses are generally classified into three categories namely Holling type-I, Holling type-II and Holling type-III for different kinds of species.

In my work I have used the Holling type-I and Holling type-II functional response. Holling type-I is the simplest of all the three functional response mentioned above. The curve defined by this response function is one dimensional, linear that passes through the origin and is unbounded whereas the Holling-type-II response function is non-linear and bounded, graphically represented as rectangular hyperbola.

Holling type-I function can be mathematically represented as:

$$f(x) = \alpha x$$

Holling type-II function can be mathematically represented as:

$$f(x) = \frac{\alpha x}{1+\alpha\beta x}$$

where f denotes the intake rate and x is plant density. α is the rate at which whiteflies consume per unit plant density i.e. the attack rate. β is the average time spent.

Moreover functional response (Freedman,1979,1980a; Holling,1965 ; May,2001) is very important to determine the system's dynamic stability, the responses to the environmental issues and all others. As based on different attack function we obtain different dynamical behavior.

1.7 Probability distributions

In our model we have assumed that the whiteflies are distributed over the Jatropha Curcas plant according to probability distribution (Sarkar A.K., Roy A.B.,1989) like Poisson distribution, Binomial distribution, Negative-binomial distribution.

In probability distribution , Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval types like distance, area, time, volume etc. Biologically Poisson distribution reveals the random attack pattern.

In probability theory Binomial distribution is the discrete probability distribution with parameters 'n' and 'p' where 'n' is the number of independent experiments. It gives the Boolean valued outcomes like success (p) or failure $(1 - p = q)$. A single trial i.e. for $n = 1$ is the Binomial distribution is called Bernoulli distribution. Biologically Binomial distribution is expressed as regular attack pattern.

Negative-binomial distribution is also discrete probability distribution that expresses the number of failures in a sequence of independent identically distributed experiments before a specific number of success. Here negative-binomial distribution parameters are r,p where $p \in [0, 1]$. Biologically it expresses aggregated attack pattern.

1.8 Control theory

Control theory is a field of applied Mathematics in which certain physical processes and systems are to be controlled. Control theory is also relevant to classical area of Mathematics such as the calculus of variation and the theory of differential equation. Control theory (Khan et al.,2020; Khan et al., 2017; Jana et al. 2022; Hassel M,Varley G.1969 ; Khatua Anupam and Kar Tapan. 2020) can be applied in a broad area. In any optimization process, control theory can be used. In our model we used it by controlling the mosaic disease of the Jatropha curcas plant. We here optimized the objective function using Pontryagin Minimum principle.

1.9 Pontryagin Minimum Principle and optimality condition

A control Hamiltonian function H can be constructed as:

$H(u(t), z(t), \lambda(t), t) = L(u(t), z(t), t) + \lambda^t(t) f(u(t), z(t), t)$.............................(1)

where z(t) is the optimal control. u(t) is the corresponding optimal state. $\lambda(t)$ is the lagrange multiplier.

Then the Pontryagin Minimum Principle states that there exists a continuous function λ, known as an adjoint function, which is the solution of the adjoint equation:

$\dot{\lambda} = -H_u(u(t), z(t), \lambda(t), t)$...................................(2)

along with the initial conditions of λ. Now Pontryagin Minimum Principle states that the optimal control $z(t)$ and corresponding optimal state $u(t)$ and $\lambda(t)$ must minimize the Hamiltonian so that,

$H(u(t), z(t), \lambda(t), t) \leq H(u(t), z^*(t), \lambda(t), t)$..............................(3)

while the adjoint equation (2) is satisfied. So, for a feasible trajectory that satisfies the minimum principle condition. Equation (3) implies that Hamiltonian is minimum at the optimal control $z(t)$ such that

$H_z(u(t), z(t), \lambda(t), t) = 0$......................(4)

This is the necessary condition for optimality.

1.10 Euler's discrete time system and Mickens NSFD scheme

In continuous time system variation of time is not found, but in discrete time system (R.E.Mickens,1989, R.E.Mickens 2010) the variables vary with time and these changes are predominantly discontinuous. The state of variables varies only at a discrete set of points in time. As variation of time is considered for discrete time system , the results so obtained are more realistic than the continuous counter part.

In standard Euler forward scheme the system can be discretized as:

$x_{t+1} = x_t + f_t \Delta t$

where $x_t = x(t)$, $x_{t+1} = x(t + \Delta t)$ and $f_t = f(x_t, t)$

It requires very small step-size to obtain the accurate results.

NSFD scheme is general set of methods in Numerical Analysis by which we can obtain the numerical solutions as well as dynamical behavior of the system of differential equations by constructing a discrete model. In my research work I have performed both these Euler forward scheme and

Mickens non-standard finite difference scheme to compare these systems with their continuous counterpart.

1.11 Objective of the thesis

In the present study our concern goes to the wonder plant Jatropha Curcas. We have discussed previously that despite of many importance of this plant, the main drawback is that the plant is easily affected by the mosaic virus. So our main goal is to reduce the effect of the mosaic disease. In this study it is also our aim to find out some of the parameters that may affect the disease dynamics. In different models we have taken different growth patterns of the Jatropha curcas plant to find the effect of growth pattern of the plant on the disease dynamics. Besides this we also tried to find out the effect of attack pattern of the vector of the disease i.e. the whitefly. Without this we have compared the continuous and discrete time systems to find which one is more effective to better understand the disease dynamics.

1.12 Highlights of the Thesis

The thesis covers seven chapters. The content of each chapter is introduced herein to give the reader an idea on the topic concerned.

Chapter 1: It is the introductory chapter where we have discussed different topics that is relevant to this thesis. The objective of the thesis is also included here.

Chapter 2: We here formulated two mathematical models based on Jatropha Curcas plant and whitefly interaction. Here we assumed that whiteflies are distributed over the plants according to Poisson distribution. The growth of the plant is assumed to be logistic in some cases and exponential in other cases. The attack function of whitefly is assumed as Holling type-I functional response .

Chapter 3: Here we formulated two models like the chapter 2 but taken functional response as Holling type- II function.

Chapter 4: In this chapter we have taken the whitefly attack function as Holling type-II function and the plant is assumed to follow logistic growth function. Most importantly we here introduced control theory to control the mosaic disease of the plant. The persistence and permanence of the system is also discussed.

Chapter 5: In this chapter we have compared different attack pattern of whitefly. Here we studied the cases in which whiteflies are distributed over the plants according to Binomial, Poisson, Negative binomial distribution, biologically that interprets regular, random and aggregated attack pattern of whitefly. We have shown a possible way of controlling the mosaic disease here.

Chapter 6: We here formulated a three dimensional model regarding Jatropha curcas plant and whitefly interaction. Here we have taken two classes of the plant i.e. the healthy plant and the infected plant. The effect of whitefly attack on both the population is shown here. Pontryagin minimum principle is applied to get the optimal solution of the control problem. Permanent coexistence is ensured by checking the persistence and permanence of the system.

Chapter 7: This chapter is based on the comparison between continuous time system and discrete time system. In discrete time system we have taken Euler′s forward scheme and Micken′s non-standard finite difference scheme. Then all the three are compared to each other. Here we have assumed that distribution of whiteflies over the plants follow negative-binomial distribution which ecologically reflects the aggregated attack pattern. Persistence and permanence of the system is discussed also in this chapter.

Lastly, we concluded our study as discussed in different chapters and also proposed an outline for future scope.

2 Effect of Growth Functions on Jatropha Curcas Plant with Random Attack Pattern of Whitefly: A Mathematical Study

[Chapter based on the paper published in Global Journal of Pure and Applied Mathematics, ISSN 0973-1768 Volume 16, Number 1(2020), pp. 27-38]

2.1 Introduction

In this chapter first we introduce the Jatropha curcas plant. Jatropha curcas plant is one of the wonder plant with economic potentiality and ecological applications in various aspects. The plant produces seeds with oil (biodiesel) that can be combusted as fuel without being refined. Jatropha curcas is a species of flowering plant in the spurge family, Euphorbiaceae and popularly called as physic nut. This plant has been introduced to Africa and Asia and is now cultivated world-wide (Pandey et al., 2012). .

Jatropha curcas is a semi-evergreen shrub or small tree with large green to pale green leaves. Normally it grows between (3-5) meter in height but attains a height upto (8-10) meter under favourable conditions. It is a drought resistant plant which grows even in poor soil. The tree can be grown in dry and infertile conditions and can be cultivated also in rough, sandy and salty soils. It has low plantation cost and the first harvesting is made just after 18 months. It grows quickly and lives producing seeds for 50 years. The most successful cultivation occurs in the drier regions of the tropics with annual rainfall of (300-1000) mm. It occurs mainly at lower altitudes (0-500) meter in areas with average annual temperature well above 20 °C but it can also grow at higher altitudes, low nutrient and tolerates slight frost. Fruits are produced in winter or it may produce several crops during the year if soil moisture is good and temperature is sufficiently high. The seeds become mature when the capsule changes from green to yellow, after two to four months. It's life span is around 45-50 years. Seed production ranges from about 0.4 tonnes to 12 tonnes per hectare per year. The seeds of this plant contain 37% oil that can be used to obtain a better quality of biodiesel (Sahoo et al., 2009). .

Jatropha curcas is naturally infected by *Begomovirus*. The symptoms of mosaic disease (Guin, 2016, Narayana, Shankarappa, Govindappa, Prameela, Rao and Rangaswamy, 2006 and Sahoo, Kumar, Sharma and Naik, 2009) are severe mosaic , mottling, blistering of leaves, yellowing of leaves, reduced leaf size, stunting of diseased plants. It mainly attacks its fruits, considerably reducing the production and quality of seeds. The mosaic virus is carried through infected whiteflies (Gao et al., 2010 ; Holt et al., 1997)..

The population of whitefly is controlled by temperature and rainfall. Heavy rainfall creates an obstruction for the growth of whiteflies. In this disease the mosaic virus passes from an infected whitefly to a susceptible plant and vice-versa. The spread of the virus is highly dependent on the plant density. A single whitefly is adequate to infect the host plants but transmission of the disease spread when numerous infected whiteflies feed on the host plants through massive flux of saliva. As a result host plant (Jatropha curcas) faces leaf damage and sap drainage due to such feeding. Whiteflies are tremendously productive, if once they get conventional on any part of the plants they will voluntarily roam and try to attack any other immediate vegetation n (Narayana D.S.A. et al., 2006 ; Venturino et al., 2016). . Normally they need 3 hours feeding time to procure the virus and a latent phase of 8 hours. It requires 10 minutes time to contaminate the young leaves. Symptoms seem to be appeared after a latent period of 3-5 weeks. Moreover the infected whiteflies inject the virus to the plant with the infection being more likely if more insects attack the same plant . After acquisition of the mosaic

virus adult whiteflies can infect the host plants within 48 hours. In this chapter we will show the dynamics due to different growth function of the plant Jatropha Curcas and the effect of random attack pattern of whitefly on the plant (Sarkar and Roy , 1989). .
It is observed that if the Jatropha curcas plant grows exponentially then it shows fragile behaviour and it can be stabilize by considering logistic growth of the plant. The result so obtained is verified by numerical simulation.

2.2 Statement of the model

In our model we have considered that v whiteflies are distributed over x plants in such a way that some plants are whitefly free and others have 1,2,..........,i whiteflies per plant. Thus we have:

$$\sum_{i=1}^{x} i = v$$

We here assumed that the whiteflies are distributed over x plants according to a probability distribution so that the proportion of plants with i whiteflies is p(i). So the number of plants with i whiteflies is p(i)x. If the intrinsic plant loss-rate per whitefly is f then the loss-rate of plants with i whiteflies will be fip(i)x. Therefore the total loss-rate of plants is

$$fx\sum_{i=0}^{\infty} ip(i)$$

Here $\sum ip(i)$ is the mean number of whitefly per plant and v/x is the expectation of i. So the loss rate due to whitefly consumption is fv. The loss of whiteflies occur in the following ways.
e=natural mortality of whitefly.
b=natural mortality of the host plant.
f=by their killing the host plant.
This self induced mortality occurs at a rate $fi^2p(i)x$. So for the whole plant population it is

$$fx\sum_{i=0}^{\infty} i^2p(i)$$

The term $\sum i^2p(i)$ is the expectation of i^2. We have chosen here the poisson distribution which ecologically reflects random attack pattern. Here whitefly-inflicted losses through the plant death are $fxE(i^2)$. For Poisson distribution we have $E(i^2) = \frac{v}{x} + (\frac{v}{x})^2$ (Sarkar and Roy , 1989).
We have chosen two different types of growth function of the plant (Jatropha curcas) population. In the first model we have chosen the growth of plant population in exponential form and in the second model the growth of of plant population is assumed to be in logistic form. The attack pattern of whitefly on the plant is taken as holling type-1 function in both cases . Here r is growth rate of the whitefly, k is the carrying capacity.

Based on the above assumptions the first model is formulated as:

2.3 Model 2.1

Assuming that the plant and whitefly follow the exponential growth the model takes the figure:

$$\begin{aligned} \frac{dx}{dt} &= rx - axv - fv \\ \frac{dv}{dt} &= v[cx - (e+b+f) - \frac{fv}{x}] \end{aligned} \tag{2.1}$$

where $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$.
For mathematical simplicity we consider the following transformation:

$x = \frac{rX}{c}, v = \frac{r^2V}{cf}, t = \frac{\tau}{r}$.

Based on the transformation the model becomes:

$$\begin{aligned} \frac{dX}{d\tau} &= X - \alpha XV - V \\ \frac{dV}{d\tau} &= V[X - \beta - \frac{V}{X}] \end{aligned} \tag{2.2}$$

Where $\alpha = \frac{ar}{c^2 f}, \beta = \frac{b+e+f}{r}$

2.3.1 *Equilibria*

The steady state of the system is obtained by setting $\frac{dX}{d\tau} = 0, \frac{dV}{d\tau} = 0$ and solving the equations :

$X - \alpha XV - V = 0$

$X - \beta - \frac{V}{X} = 0$

We have seen that the system has only one equilibrium point i.e. the interior equilibrium point $E(X^*, V^*)$. To solve $E(X^*, V^*)$ we have a quadratic equation which has atleast one positive real root. Therefore $E(X^*, V^*)$ exists.

2.3.2 *Dynamic behavior*

At $E(X^*, V^*)$ the characteristic equation is,

$\lambda^2 + \lambda(\frac{V^*}{X^*} - \frac{1}{(1+\alpha X^*)}) + \frac{V^*}{X^*} + X^* - \frac{V^*}{X^*(1+\alpha X^*)} = 0$ which can be written as,
$\lambda^2 + A\lambda + B = 0$

Where $A = \frac{V^*}{X^*} - \frac{1}{(1+\alpha X^*)} = 0$
$B = \frac{V^*}{X^*} + X^* - \frac{V^*}{X^*(1+\alpha X^*)} = \frac{\alpha V^*}{1+\alpha X^*} + X^* > 0$
Therefore the roots are purely imaginary and $E(X^*, V^*)$ is a centre which shows the fragile behavior of the system (2.2).

2.4 Model 2.2

Assuming that the plant follows logistic growth then the model becomes:

$$\begin{aligned} \frac{dx}{dt} &= rx(1 - \frac{x}{k}) - axv - fv \\ \frac{dv}{dt} &= v[cx - (e + b + f) - \frac{fv}{x}] \end{aligned} \tag{2.3}$$

where $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$.

Here x_0 is the initial plant population density and v_0 is the initial whitefly density.
For mathematical simplicity we consider the following transformation.
$x = kX, v = \frac{ck^2V}{f}, t = \frac{\tau}{ck}$

The transformed equation is,

$$\begin{aligned} \frac{dX}{d\tau} &= \alpha X(1 - X) - \beta XV - V \\ \frac{dV}{d\tau} &= V[X - \gamma - \frac{V}{X}] \end{aligned} \tag{2.4}$$

where $\alpha = \frac{r}{ck}, \beta = \frac{ak}{f}, \gamma = \frac{b+e+f}{ck}$

2.4.1 *Solution Properties*

Lemma 2.1 :

The solutions of (2.4) are positive.

Proof:
Since $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$, we have $X(0) = X_0 > 0$ and $V(0) = V_0 > 0$. Suppose $X(\tau)$ is not positive for all $\tau \geq 0$. Since $X_0 > 0$ then there exist τ_0 with $X(\tau_0) = 0$ and $X(\tau) > 0$ for $0 \leq \tau \leq \tau_0$. For $0 \leq \tau < \tau_0$
$\frac{\dot{X(\tau)}}{X(\tau)} = \alpha(1 - X) - \beta V - \frac{V}{X} > -\alpha X - \beta V - \frac{V}{X}$
$X(\tau_0) > X_0 exp[-\int_0^{\tau_0} V(\eta)/X(\eta)d\eta] > 0$
This is a contradiction and hence $X(\tau)$ is positive for all $\tau \geq 0$. Similarly it can be shown that $V(\tau)$ is also positive for all $\tau \geq 0$.

2.4.2 *Equilibria*

The equilibrium point is obtained by setting $\frac{dX}{d\tau} = 0$ and $\frac{dV}{d\tau} = 0$ and solving the equations of (2.4):

$$\alpha X(1 - X) - \beta XV - V = 0$$

$$X - \gamma - \frac{V}{X} = 0$$

We have seen that the system has two equilibrium points i.e. $E_1(X, 0) = (1, 0)$ which is the whitefly free equilibrium and $E_2(X^*, V^*)$ which is the interior equilibrium.

Here $V^* = \frac{\alpha X^*(1-X^*)}{\beta X^*+1}$

$$X^* = \frac{-(\alpha+1-\beta\gamma)\pm\sqrt{(\alpha+1-\beta\gamma)^2+4\beta(\alpha+\gamma)}}{2\beta}$$

For feasibility of X^* we have chosen the positive sign.

2.4.3 *Dynamic behavior*

From the variational matrix we obtained the behavior of different equilibrium points of the system. E_1 is saddle as $\gamma < 1$. The characteristic equation for $E_2(X^*, V^*)$ is given by,
$\lambda^2 + \alpha X^* \lambda - \frac{V^{*2}}{X^{*2}} + \alpha V^* + \alpha X^*(1 - X^*) + \frac{\alpha V^*(1-X^*)}{X^*} = 0$
which can be written as,
$\lambda^2 + A\lambda + B = 0$
where $A = \alpha X^* > 0$

$$B = \frac{\alpha(1-X^*)[\alpha X^*+\alpha\beta X^{*2}+\alpha X^*+\beta^2 X^{*3}+\beta X^{*2}+X^*+\alpha\beta X^*]}{\beta X^*+1} > 0.$$

Therefore $E_2(X^*, V^*)$ is locally asymptotically stable (Ali N., Chakravarty S.,2015)equilibrium if $\gamma < 1$.

2.4.4 *Global Stability*

Lemma 2.2 :

The XV subsystem is globally asymptotically stable.

Proof:
$H(X,V) = \frac{1}{XV}$, then $H > 0$ if $X > 0$ and $V > 0$.
$h_1(X,V) = \alpha X(1-X) - \beta XV - V$
$h_2(X,V) = V(X - \gamma - \frac{V}{X})$
Therefore, $\nabla(X,V) = \frac{\partial(h_1 H)}{\partial X} + \frac{\partial(h_2 H)}{\partial V}$

$$= \frac{\partial[\frac{\alpha(1-X)}{V} - \beta - \frac{1}{X}]}{\partial X} + \frac{\partial[1 - \frac{\gamma}{X} - \frac{V}{X^2}]}{\partial V} = -\frac{\alpha}{V} < 0$$

Hence by Bendixon-Dulac criteria $E_2(X^*, V^*)$ is globally asymptotically stable in the positive XV plane (Konar et al., 1999 ; Cheng K,H Su S, Lin S, 1982) . This completes the proof of the lemma.

2.4.5 Persistence and permanence of the system

Theorem

The system (2.4) is permanent if $\gamma < 1$.

Proof:
The index theorem states that the system with dissipativeness assumption has at least one saturated equilibrium. If all these saturated equilibria are regular, then the sum of their indices is +1. From the lemma 2.1, the system is dissipative and so there exists atleast one saturated equilibrium and the sum of their indices is +1 if they are regular (Konar et al., 1999).. The permanence of the system implies that none of the boundary fixed points are saturated. Hence the interior fixed point exists and must be saturated. Therefore all the eigenvalues are negative or have negative real parts, which we have shown before.

We now construct the average Lyapunov function . In our model, we consider the average Lyapunov function as $\sigma(X) = X^{r_1}.V^{r_2}$
where $r_i > 0$, i=1,2.
Let, $\psi(X) = \frac{\dot{\sigma}(X)}{\sigma(X)}$
$= r_1\frac{\dot{X}}{X} + r_2\frac{\dot{V}}{V}$
$= r_1(\alpha(1-X) - \beta V - \frac{V}{X}) + r_2(X - \gamma - \frac{V}{X})$
If $\psi(X) > 0$ for the ω -limit sets of trajectories initiated in $\mathbb{R}^3_+$, then the trajectories more away from the boundary and the system (2.4) is permanent. It is evident that there is no periodic trajectory. Hence if there exist $r_1 > 0$ such that $\Psi(E_1) > 0$,then (2.4) is permanent.
Therefore for $E_1(1,0)$, $\psi(X) = r_2(1-\gamma) > 0$ should be satisfied for atleast one positive vector $r = (r_1, r_2, r_3)$ since $\gamma < 1$
Hence the system (2.4) is uniformly persistent(or permanent) if $\gamma < 1$.

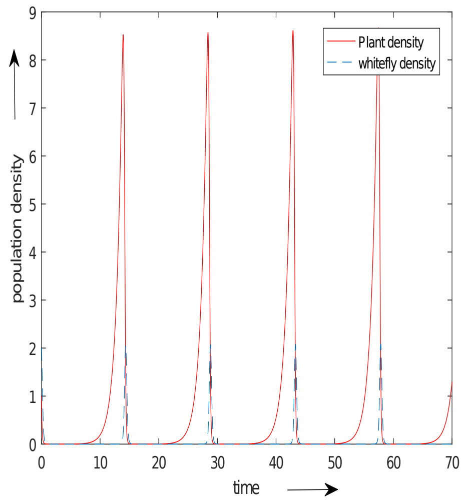

FIGURE 2.1: Variation of plant-herbivore densities with time in model 2.1 for $\alpha = 4.17$ and $\beta = 0.14$.

This completes the proof of the theorem.

2.5 Numerical simulations of model 2.1 and model 2.2

We here used ode23 solver for numerical simulations using MATLAB 2017a. Keeping in mind all the feasibility criteria the numerical values are chosen for different parameter values. The equilibrium point corresponding to the parameter values for model 2.1 is (0.475330834 , 0.159393085) and the equilibrium point corresponding to the parameter values for the model 2.2 is (0.390263193,0.103522461). The numerical results also supports the theoretical findings of the model 2.1 and model 2.2.

Here we observed large amplitude oscillations that indicates the unstable condition for both the populations.

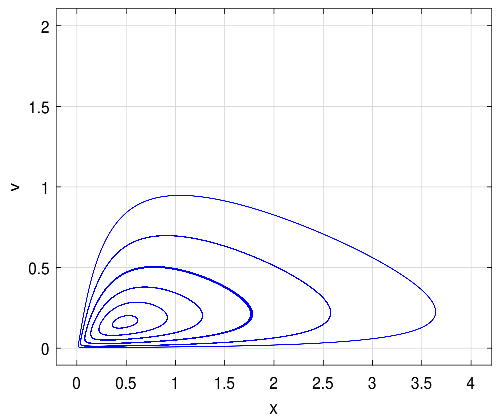

FIGURE 2.2: Variation of plant-herbivore densities with time in model 2.1 for $\alpha = 4.17$ and $\beta = 0.14$.
This shows the phase portrait in the XV-plane which shows the fragile behavior.

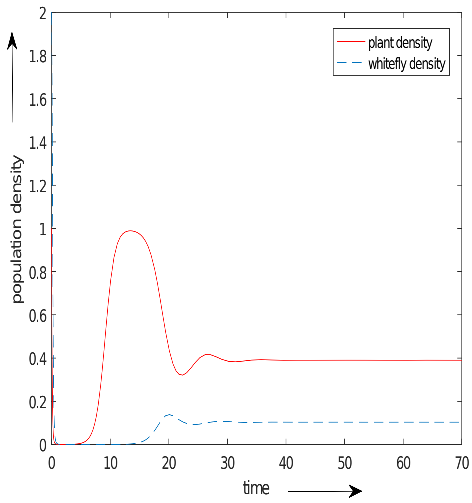

FIGURE 2.3: Globally asymptotically stable steady-state in model 2.2 for $\alpha = 1.25$, $\beta = 4.8$ and $\gamma = 0.125$.
This shows the stable behavior as time increases for both the populations.

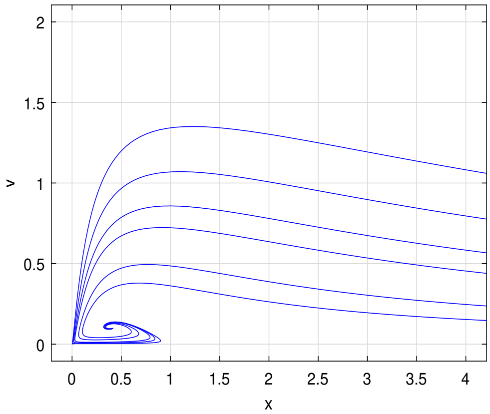

FIGURE 2.4: Globally asymptotically stable steady-state in model 2.2 for $\alpha = 1.25$, $\beta = 4.8$ and $\gamma = 0.125$.
This shows the phase portrait for global asymptotic behavior in the XV-plane.

2.6 Conclusions

These two models represents the description of interaction between Jatropha curcas plant and the whitefly. Here we made a comparative study between two different growth functions of the Jatropha curcas plant population with random attack pattern of whitefly using poisson distribution. From the study it is revealed that if the plant grows logistically then the effect of whitefly can not destabilize the system but if the plant growth is exponential, then it shows a fragile behavior. Therefore growth function of the plant (Jatropha curcas) plays an important role for the stability of plant-herbivore system. Our numerical results also reflects the same phenomena.

3 Comparison Between Different Growth Functions of The Jatropha Curcas Plant with Random Attack Pattern of Whitefly

[Chapter based on the paper published in Global Journal of Engineering Science and Researches, ISSN 2348-8034 7(9): September2020,pp.16-26]

3.1 Introduction

With the continuation of the previous chapter we now formulate another model to make a comparison between different growth functions of the plant where we have taken the Holling type -II (Pal Pallav et al. 2011) functional response for the attack pattern of whitefly and parameters represents the same meaning. We have chosen two different types of growth function of the plant(Jatropha curcas) population (Pandey et al., 2012). In the first model we have chosen the growth of the plant population in logistic form and in the second model exponential growth is assumed. .

3.2 Model 3.1

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{k}) - \frac{axv}{k+x} \\ \frac{dv}{dt} &= v[\frac{cx}{k+x} - (e+b+f) - \frac{fv}{x}] \end{aligned} \tag{3.1}$$

with the initial conditions,
$x(0) = x_0 > 0, v(0) = v_0 > 0$
Here x_0 is the initial plant population density and v_0 is the initial whitefly density.
For mathematical simplicity we consider the following transformation,
$x = kX, v = \frac{kcv}{f}, t = \frac{\tau}{c}$.
The transformed equation is,

$$\begin{aligned} \frac{dX}{d\tau} &= \alpha X(1-X) - \frac{\beta XV}{1+X} \\ \frac{dV}{d\tau} &= V[\frac{X}{1+X} - \gamma - \frac{V}{X}] \end{aligned} \tag{3.2}$$

where $\alpha = \frac{r}{c}, \beta = \frac{a}{f}, \gamma = \frac{b+e+f}{c}$.

3.2.1 Solution properties

lemma :

The solution of (3.2) are positive.

Proof:
since $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$, we have $X(0) = X_0 > 0$ and $V(0) = V_0 > 0$.Suppose $X(\tau)$ is not positive for all $\tau \geq 0$. Since $X_0 > 0$ then there exist τ_0 with $X(\tau_0) = 0$ and $X(\tau) > 0$ for $0 \leq \tau \leq \tau_0$. For $0 \leq \tau \leq \tau_0$
$\frac{\dot{X(\tau)}}{X(\tau)} = \alpha(1 - X) - \frac{\beta V}{1+X} > -\alpha X - \frac{\beta V}{1+X}$
$X(\tau_0) > X_0 exp[-\int_0^{\tau_0} V(\eta)/X(\eta)d\eta] > 0$
This is a contradiction and hence $X(\tau)$ is positive for all $\tau \geq 0$. Similarly it can be shown that $V(\tau)$ is also positive for all $\tau \geq 0$.

3.2.2 Equilibria

The equilibrium points are obtained by setting $\frac{dX}{d\tau} = 0$ and $\frac{dV}{d\tau} = 0$ and solving the equations

$\alpha(1 - X) - \frac{\beta V}{1+X} = 0$ and
$\frac{X}{1+X} - \gamma - \frac{V}{X} = 0$.

We have seen that there are two equilibrium points i.e. $E_1(X, 0) = (1, 0)$ which is the whitefly free equilibrium and $E_2(X^*, V^*)$ is the interior equilibrium. From the first equation we obtain V^* as a function of X^*, which is given by
$V^* = \frac{\alpha(1-X^*)(1+X^*)}{\beta}$
Clearly V^* is feasible as $X^* \leq 1$. Substituting this in the second equation we have a cubic equation as,
$\alpha X^{*3} + (\beta - \beta\gamma + \alpha)X^{*2} - (\alpha + \beta\gamma)X^* - \alpha = 0$
Since there is atleast one change of sign therefore by Descartes' rule of sign there exist atleast one positive X^*. Therefore (X^*, V^*) exists.

3.2.3 Stability

The equilibrium E_1 is stable if $\gamma > 0.5$ or saddle if $\gamma < 0.5$ as its eigen values are -α and$0.5 - \gamma$.
The characteristic equation for $E_2(X^*, V^*)$ is a quadratic equation which is as follows,
$\lambda^2 + \lambda(-\alpha + 2\alpha X^* + \frac{\beta V^*}{(1+X^*)^2} + \frac{V^*}{X^*}) - \frac{\alpha V^*}{X^*} + 2\alpha V^* + \frac{\beta V^{*2}}{X^*(1+X^*)^2} + \frac{\alpha X^*(1-X^*)}{(1+X^*)^2} + \frac{\beta V^{*2}}{X^*(1+X^*)} = 0$
Which can be written as
$\lambda^2 + A\lambda + B = 0$
Where,
$A = \frac{2\alpha X^{*2}}{(1+X^*)} + \frac{V^*}{X^*} > 0$
and
$B = \frac{\alpha^2(1-X^{*2})}{\beta} + \frac{\alpha^2(1-X^*)^2}{\beta X^*} + \frac{\alpha X^*(1-X^*)}{(1+X^*)^2} > 0$
Since $A > 0$,$B > 0$, $E_2(X^*, V^*)$ is locally asymptotically stable.

3.2.4 Global stability

Let us consider $H(X,V) = \frac{1}{XV}$
then $H > 0$ as $X > 0$ and $V > 0$

Let $h_1(X,V) = \alpha X(1-X) - \frac{\beta XV}{1+X}$
and $h_2(X,V) = V[\frac{X}{(1+X)} - \gamma - \frac{V}{X}]$

Therefore $\nabla(X,V) = \frac{\partial(h_1 H)}{\partial X} + \frac{\partial(h_2 H)}{\partial V}$
$= \frac{\partial \frac{\alpha(1-X)}{V} - \frac{\beta}{(1+X)}}{\partial X} + \frac{\partial \frac{1}{(1+X)} - \frac{\gamma}{X} - \frac{V}{X^2}}{\partial V}$
$= \frac{-2\beta X}{(1+X)^2(1-X)} - \frac{1}{X^2} < 0$

Hence by Bendixson-Dulac criteria E_2 is globally asymptotically stable in the positive XV-plane (Konar et al., 1999, Sarwardi Sahabuddin et al. ,2014)..

3.2.5 Persistence and Permanence of the system

Theorem 3.1

The system is permanent iff $\gamma < \frac{1}{2}$.

Proof:

The index theorem states that the system with dissipativeness assumption has atleast one saturated equilibrium. If all these saturated equilibria are regular, then the sum of their indices is +1. From the lemma 1 the system is dissipative and so there exists atleast one saturated equilibrium and the sum of their indices is +1 if they are regular. The permanence of the system implies that none of the boundary fixed points are saturated. Hence the interior fixed point exists and must be saturated (Konar et al., 1999).. Hence all the eigen values are negative or have negative real part,which is possible if $\gamma < \frac{1}{2}$.

We now construct the average Lyapunov function to prove the sufficient condition. In our model, we consider the average Lyapunov function as $\sigma(X) = X^{r_1}.V^{r_2}$ where $r_i > 0$ i=1,2.
Let,$\psi(X) = \frac{\dot{\sigma}(X)}{\sigma(X)}$
$= r_1\frac{\dot{X}}{X} + r_2\frac{\dot{V}}{V}$
$= r_1[\alpha(1-X) - \frac{\beta V}{(1+X)}] + r_2[\frac{X}{(1+X)} - \gamma - \frac{V}{X}]$

If $\psi(X) > 0$ for the ω -limit sets of trajectories initiated in $\mathbb{R}^3_+$,then the trajectories more away from the boundary and the system (3.1) is permanent. It is evident that there is no periodic trajectory.Hence if there exist $r_1 > 0$ such that $\Psi(E_1) > 0$,then (1) is permanent.
Therefore for $E_1(1,0)$, $\psi(X) = r_2(\frac{1}{2} - \gamma) > 0$

The inequality is evidently satisfied for atleast one positive $r = (r_1, r_2)$ if

$\gamma < \frac{1}{2}$. Hence the system is uniformly persistent(or permanent) if $\gamma < \frac{1}{2}$. This completes the proof of the theorem.

3.3 Model 3.2

In this model keeping all the parameters same as 3.1 but we have taken the exponential growth of the plant (Jatropha curcas). Now our model 3.2 is as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx - \frac{axv}{k+x} \\ \frac{dv}{dt} &= v[\frac{cx}{k+x} - (e+b+f) - \frac{fv}{x}] \end{aligned} \tag{3.3}$$

For convenience we have chosen the dimensionless form by taking,
$x = kX, v = \frac{kc}{a}V, t = \frac{\tau}{c}$

The dimension less form becomes:

$$\begin{aligned} \frac{dX}{d\tau} &= \alpha X - \frac{XV}{1+X} \\ \frac{dV}{d\tau} &= V[\frac{X}{1+X} - \beta - \frac{\gamma V}{X}] \end{aligned} \tag{3.4}$$

where $\alpha = \frac{r}{c}$, $\beta = \frac{e+f}{c}$, $\gamma = \frac{f}{a}$.

3.3.1 Equilibria

The equilibrium points can be obtained by setting $\frac{dX}{d\tau} = 0$, and $\frac{dV}{d\tau} = 0$ we here observed that there is only one equilibrium point $E(X^*, V^*)$ i.e. the interior equilibrium point.

From the first equation of (3.4) we get V^* in terms of X^* which is as follows:
$V^* = \alpha(1 + X^*)$
substituting this in the second equation X^* is obtained as:

$X^* = \frac{(\beta+2\alpha\gamma)\pm\sqrt{(\beta+2\alpha\gamma)^2+4(1-\beta-\alpha\gamma)\alpha\gamma}}{2(1-\beta-\alpha\gamma)}$

So X^* exists if $\beta + \alpha\gamma < 1$

3.3.2 Stability

The local behavior of the equilibrium point of the system is determined by the real parts of the eigenvalues of the Jacobian matrix at that point.
The characteristic equation is given by:
$\lambda^2 + \lambda(-\frac{\alpha X^*}{1+X} + \frac{\gamma V}{X}) - \frac{\alpha\gamma V}{1+X} + \frac{\alpha X}{(1+X)^2} + \frac{\gamma V^2}{X(1+X)} = 0$
Which can be written as:
$\lambda^2 + A\lambda + B = 0$

Where $A = \frac{X-\beta-\beta X-\alpha X}{1+X}$
and $B = -\frac{\alpha\gamma V}{1+X} + \frac{\alpha X}{(1+X)^2} + \frac{\gamma V^2}{X(1+X)} > 0$
This leads to the following results for $A > 0$ or $A = 0$ or $A < 0$:

Theorem 3.2:

If $\beta + \alpha\gamma < 1$ and $X^* - \beta - \beta X^* - \alpha X^* > 0$ then $E(X^*, V^*)$ is globally asymptotically stable.

Proof:

If possible let Γ be any periodic orbit around $E(X^*, V^*)$ in the positive XV- plane. Then ,

$$\Delta = \int_\Gamma div(\dot{X}, \dot{V})\, d\tau$$
$$= \int_\Gamma (\alpha - \frac{V}{(1+X)^2} + \frac{X}{(1+X)} - \beta - \frac{2\gamma V}{X})\, d\tau$$
$$= \int_\Gamma (\frac{\alpha X}{(1+X)} - \frac{\gamma V}{X})\, d\tau$$

Under the given assumption $E(X^*, V^*)$ is locally stable. Thus $\Delta < 0$. The Poincare criteria now implies that the postulated periodic orbit Γ is stable , which leads to a contradiction. Therefore, there is no periodic orbit around $E(X^*, V^*)$ in the positive XV plane and thus $E(X^*, V^*)$ is a global attractor. This completes the proof of the theorem.

Theorem 3.3:

If $\beta + \alpha\gamma < 1$ and $X^* - \beta - \beta X^* - \alpha X^* = 0$ then the system bifurcates into small amplitude periodic solutions near $E(X^*, V^*)$.

Proof:

To prove this theorem we can show that the conditions for a hopf bifurcations are satisfied. If $X^* - \beta - \beta X^* - \alpha X^* = 0$ and the two roots of the characteristic equation $\lambda^2 + A\lambda + B = 0$ are purely imaginary namely $\pm i\eta$.
where $\eta^2 = \frac{-\alpha V\gamma}{(1+X)} + \frac{\alpha X}{(1+X)^2} + \frac{\gamma V^2}{X(1+X)}$.

The necessary and sufficient condition for hopf bifurcation to occur is that there exist a $\gamma = \gamma^*$ such that
i)$X^* - \beta - \beta X^* - \alpha X^* = 0$ and

ii)$\frac{d(Real\lambda)}{d\gamma}\Big|_{\gamma=\gamma^*} \neq 0$

Hence all the conditions for a Hopf bifurcation are satisfied. This completes the proof of the theorem.

Theorem 3.4:

If $\beta + \alpha\gamma < 1$ and $X^* - \beta - \beta X^* - \alpha X^* < 0$ then there exists a stable limit cycle around $E(X^*, V^*)$ in the positive XV plane.

Proof:

Now if we decrease the value of $X^* - \beta - \beta X^* - \alpha X^*$ such that $X^* - \beta - \beta X^* - \alpha X^* < 0$ then $E(X^*, V^*)$ is locally unstable.

Again $\quad \Delta = \int_\Gamma div(\dot{X}, \dot{V})\, d\tau$

$= \int_\Gamma (\alpha - \frac{V}{(1+X)^2} + \frac{X}{(1+X)} - \beta - \frac{2\gamma V}{X})\, d\tau$

$= \int_\Gamma (\frac{\alpha X}{(1+X)} - \frac{\gamma V}{X})\, d\tau$

So, we can conclude that $\Delta > 0$ if $X^* - \beta - \beta X^* - \alpha X^* < 0$. Hence by Poincare criteria any periodic orbit is stable. Therefore there exists atleast one stable limit cycle around $E(X^*, V^*)$ in the positive XV plane.

3.4 Numerical simulation and discussions

To verify the theoretical results numerical simulations have been carried out using MATLAB-2016a. Here we have used MATLAB routine ODE23. In this numerical simulation we have used different admissible values of the system parameters to ensure our theoretical results. For the model 3.2, we have chosen a set of parameter values such as $\alpha = 0.75$, $\beta = 2$, $\gamma = 0.2$ that shows the local as well as global stability (see fig.3.1 and 3.2) which also ensures the theoretical results.The equilibrium point corresponding to this set of parameter values of model 3.2 is (0.7429,0.168037346). For the model 3.4, keeping in mind the feasibility criteria we have chosen the values of γ by using the following conditions,

i)$X^* - \beta - \beta X^* - \alpha X^* > 0$
ii)$X^* - \beta - \beta X^* - \alpha X^* = 0$
iii)$X^* - \beta - \beta X^* - \alpha X^* < 0$

For the set of parameter values $\alpha = 0.7$, $\beta = 0.1$, $\gamma = 0.35$ satisfying the condition $X^* - \beta - \beta X^* - \alpha X^* > 0$ the equilibrium point becomes (1.209914074,1.546939852). The corresponding figure shows locally steady state which leads to global stability around the equilibrium point. The corresponding phase portrait for the same set of parameter values also ensures the same by figure 3.3 and 3.4.

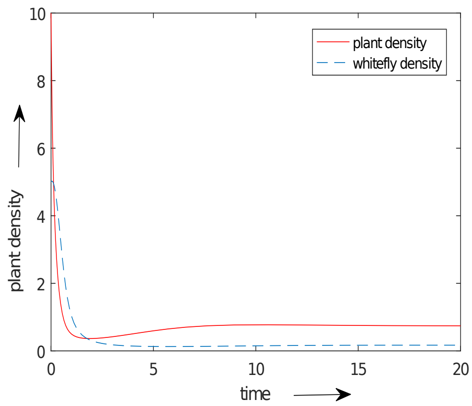

FIGURE 3.1: Variation of plant-herbivore densities with time in model 3.2 for $\alpha = 0.75$, $\beta = 2$, $\gamma = 0.2$. Here we observe local stability for the population with increasing time.

For the set of parameter values $\alpha = 0.7$, $\beta = 0.1$, $\gamma = 0.111111111$ satisfying the condition $X^* - \beta - \beta X^* - \alpha X^* = 0$ the equilibrium point becomes
(1.124650736,1.487255515) . The corresponding figure shows small amplitude oscillation which leads to Hopf bifurcation around the equilibrium point. The corresponding phase portrait of this Hopf bifurcation of the system (3.4) has been represented in the figure 3.5 and 3.6 for the same set of parameter values.

For the set of parameter values $\alpha = 0.7$, $\beta = 0.1$, $\gamma = 0.07$ satisfying the condition $X^* - \beta - \beta X^* - \alpha X^* < 0$ the equilibrium point is
(0.383003659,0.968102561) which locally shows the unstable behavior. It is observed that there is a large amplitude oscillation with increasing time for both the plant and whitefly which leads to limit-cycle. The corresponding phase portrait of this stable limit- cycle of the system (3.4) has been shown in the figure 3.7 and 3.8.
In the realistic situation we also observe the same phenomena.

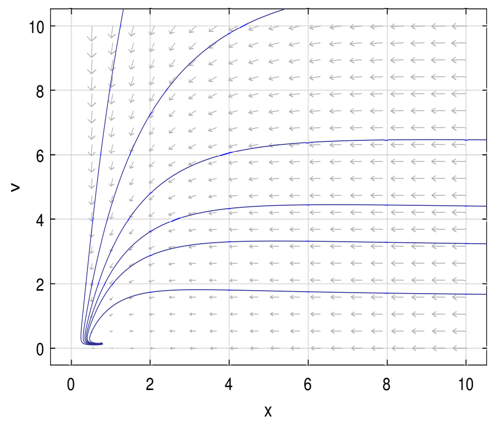

FIGURE 3.2: Variation of plant-herbivore densities in model 3.2 $\alpha = 0.75$, $\beta = 2$, $\gamma = 0.2$ .This shows the phase portrait in the XV plane which is globally asymptotically stable state of model 3.2.

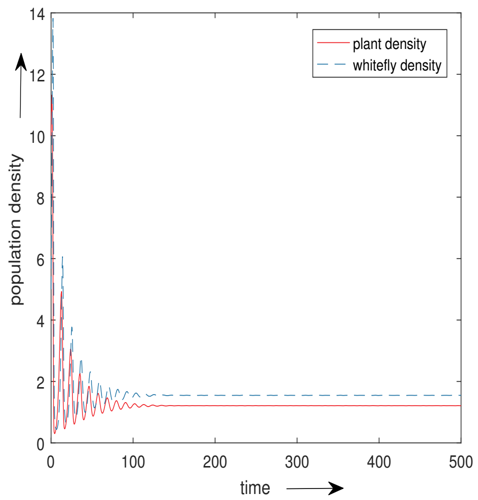

FIGURE 3.3: Locally asymptotically stable state for both the population for the set of parameter values $\alpha = 0.7$, $\beta = 0.1$, $\gamma = 0.35$ for model 3.4.

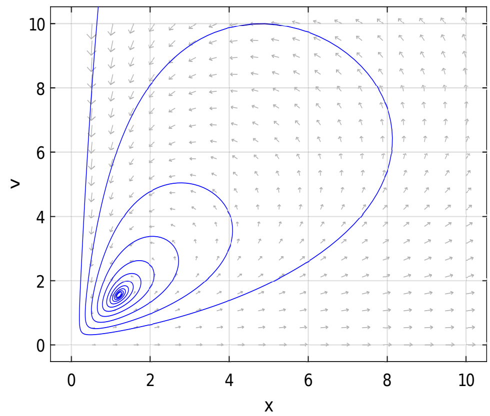

FIGURE 3.4: Phase portrait for the plant-herbivore system with the same parameter values $\alpha = 0.7$ $\beta = 0.1$, $\gamma = 0.35$ which shows global asymptotic stability of model 3.4.

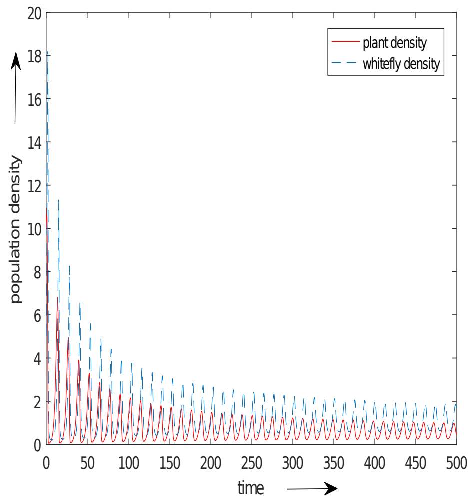

FIGURE 3.5: Small amplitude oscillations of both the population for the set of parameter values $\alpha = 0.7$, $\beta = 0.1$, $\gamma = 0.111111111$ of model 3.4.

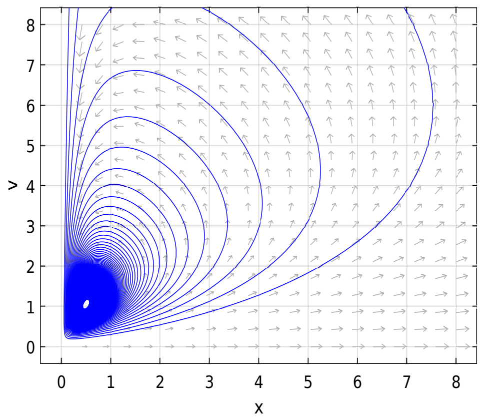

FIGURE 3.6: Hopf bifurcation for the parameter values $\alpha = 0.7$, $\beta = 0.1$, $\gamma = 0.111111111$ of model 3.4.

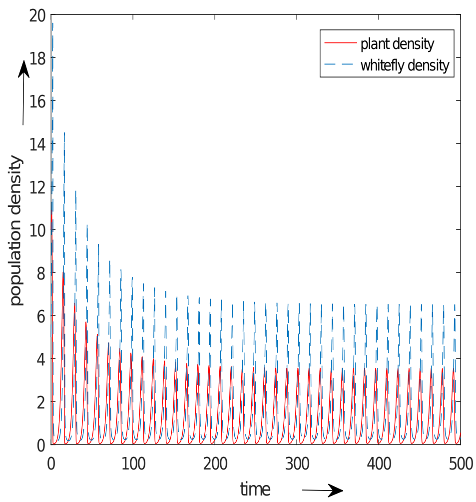

FIGURE 3.7: Large amplitude oscillations of both the population which indicates unstable condition as time increase for the set of parameter values $\alpha = 0.7$, $\beta = 0.1$, $\gamma = 0.07$ of model model 3.4.

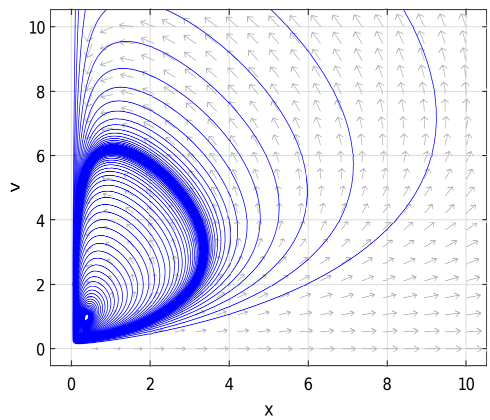

FIGURE 3.8: Stable limit-cycle for the parameter values $\alpha = 0.7, \beta = 0.1, \gamma = 0.07$.

3.5 Conclusions

This study is based on the interaction between Jatropha curcas plant and the vector whitefly. Here a comparison of two different growth function of the Jatropha curcas plant is represented with random attack pattern of the whitefly using poisson distribution. From our study it is explicit that if the plant grows logistically then the effect of whitefly cannot destabilize the system but if the plant growth is exponential then it shows three different types of behavior depending upon the different parameter values. It shows global stability for some parameter values, hopf bifurcation for some other parameter values and stable limit cycle for some another set of parameter values. Our numerical results also supports the same behavior.

4 An Effort for Controlling the Mosaic Disease of Jatropha Curcas Plant

[Chapter based on the paper published in Advances and Application in Mathematical Sciences Volume 21, Issue 11,September 2022, pp. 6437-6454]

4.1 Introduction

In the previous chapter we have already discussed about the Jatropha Curcas plant and its mosaic disease (Pandey et al., 2012; Sahoo et al., 2009; Gao et al., 2010 ; Holt et al., 1997) . Here we are interested to find out any solution for controlling this disease (Narayana D.S.A. et al., 2006 ; Venturino et al., 2016) . We here formulated a mathematical model regarding plant-herbivore interaction and applied control theory to control the mosaic disease of the Jatropha Curcs plant.

4.2 Statement of the model

In our model the mosaic virus (Begomovirus) which is responsible for the disease is taken implicitly by considering its vector whiteflies. The vector can be controlled by removing infected plant biomass, spraying insecticide etc. Here growth of the plant is considered in logistic form and the attack pattern is taken as Holling type-II function. Here ′x′ denotes the Jatropha Curcas plant population and ′v′ denotes the whitefly population. ′k′ is the carrying capacity, ′r′ is the growth rate of the plant.

The loss of whiteflies occur in the following ways.
d= natural mortality of whitefly.
e= natural mortality of the host plant.
f= by their killing the host plant.
Based on the above assumptions the model takes the form as given in 4.3,

4.3 Model

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{k}) - \frac{cxv}{a+x} \\ \frac{dv}{dt} &= v[\frac{cx}{a+x} - (e+d+f)] \end{aligned} \tag{4.1}$$

with the initial conditions,
$x(0) = x_0 > 0, v(0) = v_0 > 0$
Here x_0 is the initial plant population density and v_0 is the initial whitefly density.
For mathematical simplicity we consider the following transformation,
$x = aX, v = av, t = \frac{\tau}{r}$.
The transformed equations are,

$$\begin{aligned}\frac{dX}{d\tau} &= X(\alpha - X) - \frac{\beta XV}{1+X}\\ \frac{dV}{d\tau} &= V[\frac{\beta X}{1+X} - \gamma]\end{aligned} \tag{4.2}$$

where $\alpha = \frac{k}{a}$, $\beta = \frac{ck}{ra}$,$\gamma = \frac{(d+e+f)k}{ra}$.

4.3.1 Solution properties

lemma :

The solutions of (4.2) are positive.

Proof:
Since $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$, we have $X(0) = X_0 > 0$ and $V(0) = V_0 > 0$.Suppose $X(\tau)$ is not positive for all $\tau \geq 0$. Since $X_0 > 0$ then there exist τ_0 with $X(\tau_0) = 0$ and $X(\tau) > 0$ for $0 \leq \tau \leq \tau_0$. For $0 \leq \tau \leq \tau_0$
$\frac{X(\dot{\tau})}{X(\tau)} = \alpha - X - \frac{\beta V}{1+X} > -X - \frac{\beta V}{1+X}$
$X(\tau_0) > X_0 exp[-\frac{\tau_0^2}{2} - \int_0^{\tau_0} \beta V/(1+X)d\tau] > 0$

This is a contradiction and hence $X(\tau)$ is positive for all $\tau \geq 0$. Similarly it can be shown that $V(\tau)$ is also positive for all $\tau \geq 0$.

4.3.2 Equilibria

The equilibrium points are obtained by setting $\frac{dX}{d\tau} = 0$ and $\frac{dV}{d\tau} = 0$ and solving the equations
$X(\alpha - X) - \frac{\beta XV}{1+X} = 0$ and
$\frac{\beta XV}{1+X} - \gamma V = 0$.

We have seen that there are three equilibrium points i.e. $E_0(0,0)$,$E_1(\alpha,0)$ which is the whitefly free equilibrium, $E_2(X^*,V^*)$ which is the interior equilibrium. From the two equations we obtain
$X^* = \frac{\gamma}{\beta-\gamma}$
$V^* = \frac{\alpha\beta-\alpha\gamma-\gamma}{(\beta-\gamma)^2}$
Clearly $E_2(X^*,V^*)$ is feasible if $\alpha > \frac{\gamma}{\beta-\gamma} > 0$.

4.3.3 Stability

From the variational matrix we obtain the behavior of different equilibrium points of the system. The equilibrium $E_0(0,0)$ is saddle as its eigen values are α and$-\gamma$.

The eigen values of $E_1(\alpha, 0)$ are $-\alpha$, $\frac{\alpha\beta}{\alpha+1} - \gamma$

Therefore if $\frac{\alpha\beta}{1+\alpha} - \gamma > 0$ then $E_1(\alpha, 0)$ becomes saddle, in this case $E_2(X^*, V^*)$ exists. But if $\frac{\alpha\beta}{1+\alpha} - \gamma < 0$ then $E_1(\alpha, 0)$ becomes stable and in this case $E_2(X^*, V^*)$ does not exists. The characteristic equation for $E_2(X^*, V^*)$ is a quadratic equation which is as follows,
$\lambda^2 + \lambda(-\alpha + 2X^* + \frac{\beta V^*}{(1+X^*)^2}) + \frac{\beta^2 X^* V^*}{(1+X)^3} = 0$
Which can be written as
$\lambda^2 + A\lambda + B = 0$
Where,
$A = X^* - \frac{\beta X^* V^*}{(1+X^*)^2}$
and
$B = \frac{\beta^2 X^* V^*}{(1+X^*)^3} > 0$
A can be > 0 , equal to 0 or < 0.
Therefore if $A > 0$
then $1 - \frac{\beta V^*}{(1+X^*)^2} > 0$

i.e.$\frac{\beta+\gamma}{\beta-\gamma} > \alpha$
If $A = 0$, then $\frac{\beta+\gamma}{\beta-\gamma} = \alpha$
and if $A < 0$, then $\frac{\beta+\gamma}{\beta-\gamma} < \alpha$

Theorem 4.1

If $\frac{\beta+\gamma}{\beta-\gamma} > \alpha$ and $\alpha > \frac{\gamma}{\beta-\gamma}$ then the system (4.2) is globally asymptotically stable.

Proof :
If possible let Γ be any periodic orbit around $E_2(X^*, V^*)$ in the positive XV-plane. Then ,

$$\Delta = \int_\Gamma div(\dot{X}, \dot{V})\, d\tau$$
$$= \int_\Gamma [\alpha - 2X - \frac{\beta V}{(1+X)^2} + \frac{\beta X}{(1+X)} - \gamma]\, d\tau$$
$$= \int_\Gamma [\frac{\beta V}{(1+X)} - X - \frac{\beta V}{(1+X)^2}]\, d\tau$$
$$= \int_\Gamma [\frac{\gamma(\alpha\beta-\alpha\gamma-\gamma-\beta)}{\beta(\beta-\gamma)}]\, d\tau$$

Under the given assumption $E_2(X^*, V^*)$ is locally stable. Thus $\Delta < 0$. Then by Poincare criteria any periodic orbit Γ in the positive XV plane is stable , leads to a contradiction. Therefore, there is no periodic orbit around $E_2(X^*, V^*)$ in the positive XV plane and thus $E_2(X^*, V^*)$ is a global attractor (Konar et al., 1999) . This completes the proof of the theorem.

Theorem 4.2:

If $\frac{\beta+\gamma}{\beta-\gamma} = \alpha$ and $\alpha > \frac{\gamma}{\beta-\gamma}$ then the system (4.2) leads to small amplitude Hopf bifurcating periodic solutions near E_2.

Proof : To prove this theorem we have to satisfy all the conditions for hopf bifurcation. If $\frac{\beta+\gamma}{\beta-\gamma} = \alpha$ then the two roots of the characteristic equation $\lambda^2 + A\lambda + B = 0$ are purely imaginary namely $\pm i\eta$.
where $\eta^2 = \frac{\beta^2 XV}{4(1+X)^3}$.
The necessary and sufficient condition for hopf bifurcation to occur is that there exist a $\alpha = \alpha^*$ such that

i) $\frac{\beta+\gamma}{\beta-\gamma} = \alpha$ and

ii) $\frac{d(Real\lambda)}{d\alpha}\Big|_{\alpha=\alpha^*} \neq 0$

Now we have

$$\frac{d(-X+\frac{\beta VX}{(1+X)^2})}{d\alpha}\Big|_{\alpha=\frac{\beta+\gamma}{\beta-\gamma}} = \frac{\gamma}{\beta} \neq 0$$

Hence all the conditions for a Hopf bifurcation are satisfied.Then there exists small amplitude hopf bifurcating periodic solutions near E_2. This completes the proof of the theorem.

Theorem 4.3:

If $\frac{\beta+\gamma}{\beta-\gamma} < \alpha$ and $\alpha > \frac{\gamma}{\beta-\gamma}$ then there exists atleast one stable limit cycle around $E_2(X^*, V^*)$ in the positive XV plane.

Proof:
If possible let Γ be any periodic orbit around E_2 in the positive XV plane. Then

$$\begin{aligned}\Delta &= \int_\Gamma div(\dot{X}, \dot{V})\, d\tau \\ &= \int_\Gamma (\alpha - 2X - \frac{\beta V}{(1+X)^2} + \frac{\beta X}{(1+X)} - \gamma)\, d\tau \\ &= \int_\Gamma (\frac{\gamma(\alpha\beta-\alpha\gamma-\gamma-\beta)}{\beta(\beta-\gamma)})\, d\tau\end{aligned}$$

So, we can conclude that $\Delta > 0$ if $\frac{\beta+\gamma}{\beta-\gamma} < \alpha$. Hence by Poincare criteria any periodic orbit is stable. Therefore there exists atleast one stable limit cycle around $E_2(X^*, V^*)$ in the positive XV plane.

4.4 Persistence and Permanence of the system

The idea of persistence was first came to the light by Freedman and Waltman in 1984. From the biological point of view persistence implies that all the populations are present and none of them will become extinct (Konar et al., 1999).

Persistence and permanence are very useful to decide the questions of survival and extinction of n-species whose growth equations are governed by the differential equations

$$\dot{x}_i = x_i f_i(x_1, x_2, ..., x_n) \tag{4.3}$$

4.4.1 Some definitions

1) The system is said to be weakly persistent if $limsup x_i(t) > 0$ for all orbits in $int\mathbb{R}^n_+$ and strongly persistent if $liminf x_i(t) > 0$.

2)The system is said to be permanent if there exists a compact set $B \subset int\mathbb{R}^n_+$ such that all orbits in $int\mathbb{R}^n_+$ end up in B.

3) The system is uniformly persistence if there exist $\delta > 0$ such that for each compact set x_i , $liminf x_i(t) \geq \delta > 0$ for all $(x_1(t), x_2(t), ..., x_n(t)) = X(t) \in int\mathbb{R}^n_+$.

4) An equilibrium fixed point x^* is said to be saturated equilibrium if $x_i^* = 0$ then $f_i(x_1^*, x_2^*, ..., x_n^*) \leq 0$.

With the concept of saturated equilibria and by the method of average Lyapunov function we have the following theorem for permanent coexistence of both the species of the system (4.2).

4.4.2 Theorem 4.4

The system is permanent iff $\alpha > \frac{\gamma}{\beta - \gamma}$.

Proof:
The index theorem states that the system with dissipativeness assumption has atleast one saturated equilibrium. If all these saturated equilibria are regular, then the sum of their indices is +1. From the theorem 4.1 the system is dissipative and so there exists atleast one saturated equilibrium and the sum of their indices is +1 if they are regular. The permanence of the system implies that none of the boundary fixed points are saturated. Hence the interior fixed point exists and must be saturated. Hence all the eigen values are negative or have negative real parts.

We now construct the average Lyapunov function to prove the sufficient condition. In our model, we consider the average Lyapunov function as $\sigma(X) = X^{r_1}.V^{r_2}$ where $r_i > 0$ i=1,2.

Let,$\psi(X) = \frac{\dot{\sigma}(X)}{\sigma(X)}$
$= r_1 \frac{\dot{X}}{X} + r_2 \frac{\dot{V}}{V}$
$= r_1[\alpha - X - \frac{\beta V}{(1+X)}] + r_2[\frac{\beta X}{(1+X)} - \gamma]$

If $\psi(X) > 0$ for the ω -limit sets of trajectories initiated in $\mathbb{R}^3_+$, then the trajectories more away from the boundary and the system (4.2) is permanent. It is evident that there is no periodic trajectory. Hence if there exist $r_1 > 0$ such that $\Psi(E_1) > 0$,then (4.2) is permanent.

Therefore for $E_0(0,0)$, $\psi(X) = \alpha r_1 - \gamma r_2 > 0$

$$E_1(\alpha,0),\ \psi(X) = (\tfrac{\alpha\beta}{1+\alpha} - \gamma) r_2 > 0$$

The inequalities are evidently satisfied for atleast one positive $r = (r_1, r_2)$ if $\alpha > \frac{\gamma}{\beta-\gamma}$.

Hence the system is uniformly persistent(or permanent) if $\alpha > \frac{\gamma}{\beta-\gamma}$.
This completes the proof of the theorem.

4.5 The optimal control problem

We now reformulate the model as an optimal control problem to minimize the costs of insecticide spraying. The migration of infected whiteflies are not considered. Assuming that all the infected vectors are under the control of insecticide spraying (Chowdhury Jahangir, F.A.Basir et al.,2019; P.K.Roy, X.Z. Li et al,2015; Venturino et al., 2016). We now introduce the control variable u(t) such that $0 \leq u(t) \leq 1$ defined on $[t_0, t_f]$ where t_0 and t_f are the starting and finishing time of control respectively.
Now the model takes the form,

$$\begin{aligned} \frac{dX}{d\tau} &= X(\alpha - X) - (1 - u(t))\frac{\beta XV}{1+X} \\ \frac{dV}{d\tau} &= V[(1 - u(t))\frac{\beta X}{1+X} - \gamma] \end{aligned} \tag{4.4}$$

If we consider u(t)=0 then there is no reduction in the contact rate between the infected whiteflies and the plants.
If we consider u(t)=1 then there is no such contact rate between them. u(t) plays the key role to express the reduction of contact rate between them by the spraying of insecticide.

We define the objective functional to minimize the cost of insecticide spraying as follows :-

$J(u(t)) = \int t_0^{t_f} [Pu^2 - QX^2] d\tau$ Where $P \geq 0$ and $Q \geq 0$

Here the first term represents the costs of spraying insecticide and labor charge and the last term represents the extra revenues obtained by the larger population of healthy Jatropha Curcas plants.
Now we are going to find the optimal control.

4.5.1 Theorem 4.5:

The objective cost function $J(u^*)$ is minimum for the optimal control u^* corresponding to the interior equilibrium $E_2(X^*, V^*)$ and also there are adjoint variables ξ_1, ξ_2 satisfying the system of equations,

$$\begin{aligned} \frac{d\xi_1}{dt} &= 2QX - \xi_1[\alpha - 2X - (1-u(t))\frac{\beta V}{(1+X)^2]} - \xi_2[(1-u(t))\frac{\beta V}{(1+X)^2}] \\ \frac{d\xi_2}{dt} &= \xi_1(1-u(t))\frac{\beta X}{1+X} - \xi_2[(1-u(t))\frac{\beta X}{1+X} - \gamma] \end{aligned} \tag{4.5}$$

with the boundary condition $\xi_i(t_f) = 0$ (i=1,2). The optimal control can be given as,

$$u^*(t) = max\{0, min\{1, \frac{\beta XV(\xi_2 - \xi_1)}{2P(1+X)}\}\} \tag{4.6}$$

Proof:
Applying the Pontryagin Minimum Principle the optimal control variable $u^*(t)$ satisfies

$\frac{\partial H}{\partial u^*(t)} = 0$

Which implies

$2Pu^* + \xi_1\frac{\beta XV}{1+X} - \xi_2\frac{\beta XV}{1+X} = 0$

$\Rightarrow u^* = \frac{(\xi_2-\xi_1)\beta XV}{2P(1+X)}$

we first construct the Hamiltonian as follows:

$H = Pu^2 - QX^2 + \xi_1[X(\alpha - X) - (1-u(t))\frac{\beta XV}{1+X}] + \xi_2[(1-u(t))\frac{\beta XV}{1+X} - \gamma V]$

For the boundedness of the optimal control we have

$$u^*(t) = \begin{cases} 0 & \frac{\beta XV(\xi_2-\xi_1)}{2P(1+X)} \leq 0 \\ \frac{\beta XV(\xi_2-\xi_1)}{2P(1+X)} & 0 < \frac{\beta XV(\xi_2-\xi_1)}{2P(1+X)} < 1 \\ 1 & \frac{\beta XV(\xi_2-\xi_1)}{2P(1+X)} \geq 1 \end{cases}$$

According to Pontryagin Minimum Principle adjoint variables satisfy the following equations:

$$\frac{d\xi_i}{dt} = -\frac{\partial H}{\partial X_i} \tag{4.7}$$

where i=1,2 $X_i = X, V$.
i.e. $X_1 = X$, $X_2 = V$ and the equations can be determined by using (4.7). This completes the proof of the theorem.

4.6 Numerical simulation and discussions

To verify the theoretical results numerical simulations have been carried out using MATLAB-2016a. Here we have used MATLAB routine ODE23. Distinctive permissible estimations of the system parameters have been taken to ensure our theoritical results.
Keeping in mind the feasibility criteria we have chosen the value of α by using the following conditions :
1) $\frac{\beta+\gamma}{\beta-\gamma} > \alpha$

2) $\frac{\beta+\gamma}{\beta-\gamma} = \alpha$

3) $\frac{\beta+\gamma}{\beta-\gamma} < \alpha$

For the set of parameter values $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$ satisfying the condition $\frac{\beta+\gamma}{\beta-\gamma} > \alpha$ the equilibrium point becomes (1,10). The corresponding figure shows the local as well as global stability (see fig. 4.1,4.2). Phase portrait is also represented by the same parameter values and it also reflects the same results.

For the set of parameter values $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.1$ satisfying the condition $\frac{\beta+\gamma}{\beta-\gamma} = \alpha$ the equilibrium point becomes (1,20)). The corresponding figure shows the small amplitude hopf bifurcation (see fig. 4.3,4.4) around the equilibrium point. Phase portrait also justify the same results.

Similarly for the set of parameter values of $\alpha = 3.5$, $\beta = 0.2$, $\gamma = 0.1$ satisfying the condition $\frac{\beta+\gamma}{\beta-\gamma} < \alpha$ the equilibrium point becomes (1,25). The corresponding figure shows the large oscillation which leads to unstable condition (see fig. 4.5,4.6). Phase portrait is also represented by the same parameter values and it also reflects the same results.

We used control theory by using a control parameter u(t) in the basic model. It is observed that in the presence of control the growth of plants stabilized (see fig. 4.7,4.8,4.9) but the growth of infected whitefly declines.

In the realistic situation we observe the same phenomena.

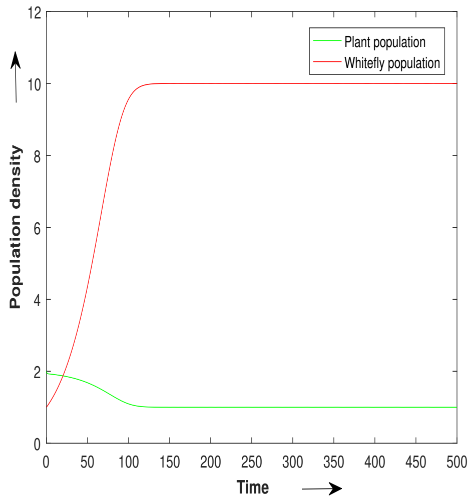

FIGURE 4.1: Variation of plant-herbivore densities with time for $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$. Here we observe local stability for the population with increasing time.

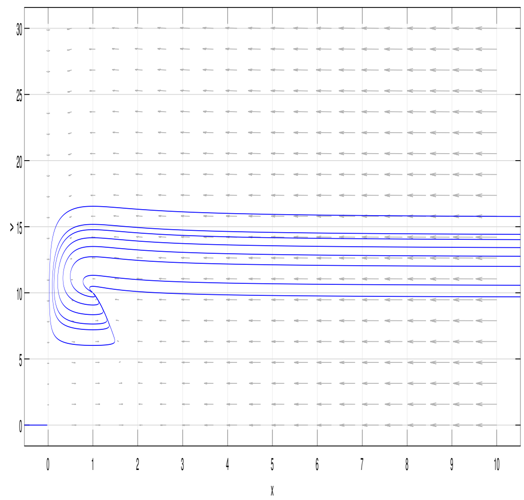

FIGURE 4.2: Variation of plant-herbivore densities $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$ .This shows the phase portrait in the XV plane which is globally asymptotically stable state of the model.

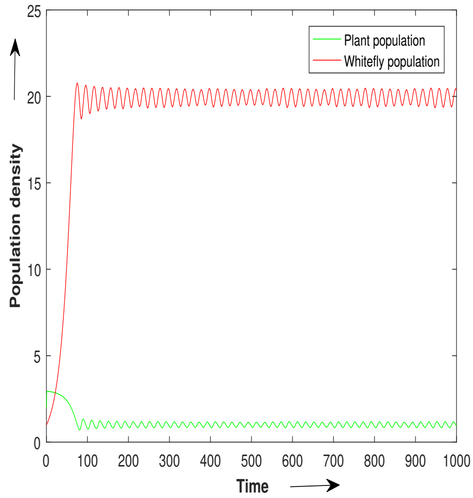

FIGURE 4.3: Small amplitude oscillation for both the population for the set of parameter values $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.1$.

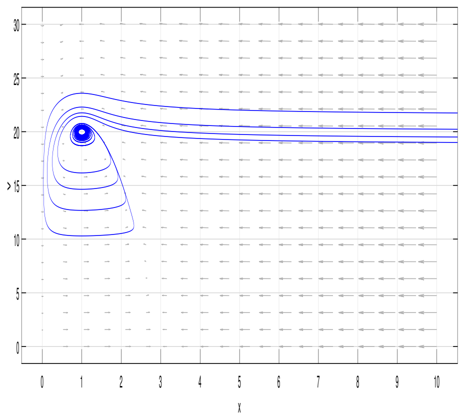

FIGURE 4.4: Variation of plant-herbivore densities in the model with $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.1$ .This shows the phase portrait in the XV plane which shows hopf bifurcation.

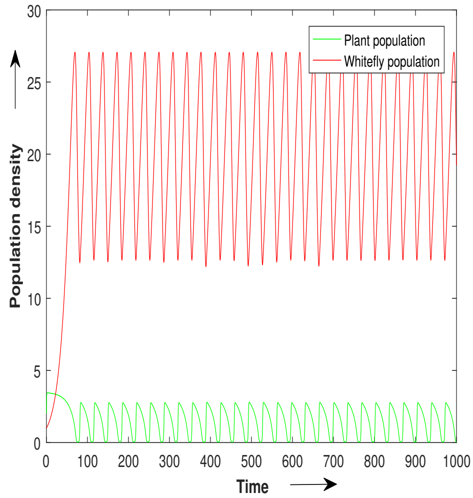

FIGURE 4.5: large oscillations of both the population for the set of parameter values $\alpha = 3.5$, $\beta = 0.2$, $\gamma = 0.1$ of the model .

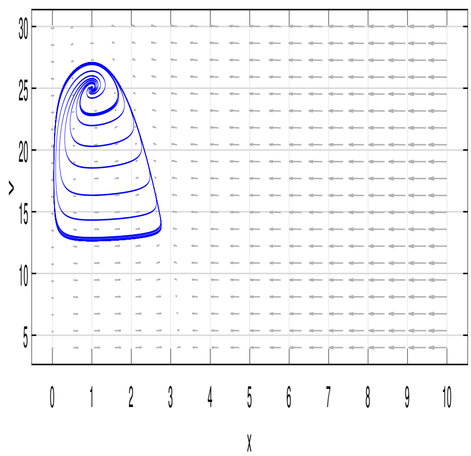

FIGURE 4.6: Limit cycle for the parameter values $\alpha = 3.5$, $\beta = 0.2$, $\gamma = 0.1$ of the model .

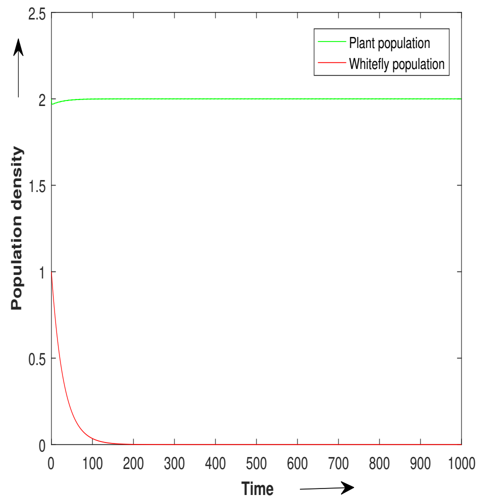

FIGURE 4.7: Effect of control on the stable state for the set of parameter values $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$ of the model.

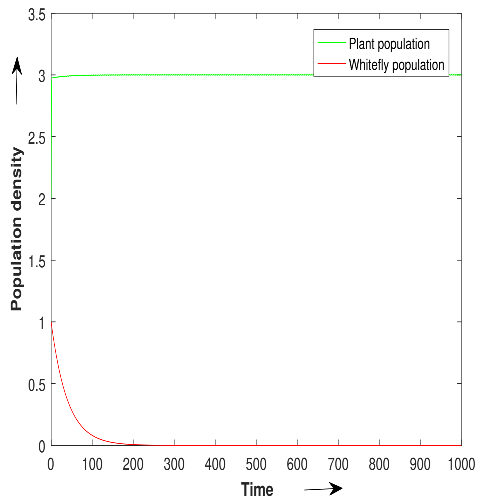

FIGURE 4.8: Effect of control on the hopf bifurcating parameter values $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.1$.

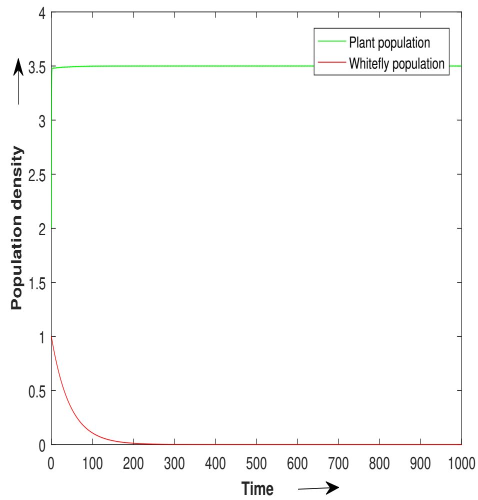

FIGURE 4.9: Effect of control on the limit cycle regarding parameter values $\alpha = 3.5$, $\beta = 0.2$, $\gamma = 0.1$.

4.7 Conclusions

This paper deals with the interaction between the Jatropha curcas plant and the whitefly. Here we observe that depending upon the parameter values of α we get stable, unstable and bifurcating nature of the system. We also discussed about the persistence and permanence of the system. We have tried to control the mosaic disease using the pesticide. So we introduced a control parameter u(t) on our basic model and observed that with the help of control the system becomes stabilized for all the pre-assumed parameter values. The results of insecticide spraying is also discussed in the numerical section. We have observed that spraying has a better effect on both the population.

5 Modelling Different Attack Patterns of Whitefly on Jatropha Curcas Plant and Control of the Mosaic Disease

[Chapter based on the paper published in Journal of the Calcutta Mathematical Society, Volume 18 (2), 2022, pp- 225-246]

5.1 Introduction

In this chapter we are interested to find whether the attack pattern of whitefly on plants affects the mosaic disease dynamics (Gao, Qu, Chua and Ye, 2010 and Guin, 2014). or not. So we have formulated a mathematical model regarding Jatropha curcas plant and whitefly interaction. Here we have taken the mosaic virus implicitly. There may be different theoretical possible results for mutual dependence of plant and the vector whitefly. This results mainly depends on both the population density, timing and the attack pattern of the vector.

Our objective is to show the effects on the plant populations due to different types of herbivore attack pattern. We here taken different types of herbivore attack function namely negative-binomial , poisson and binomial distribution. Besides this we also want to find out the way of controlling the mosaic disease (Guin, 2016, Narayana, Shankarappa, Govindappa, Prameela, Rao and Rangaswamy, 2006 and Sahoo, Kumar, Sharma and Naik, 2009) of the plant. This plant should be protected as it is one of the main solution of the future crisis of fossil fuel.

We here observed locally asymptotically stable state or hopf bifurcation or unstable condition depending upon the different parameter values regarding negative - binomial, poisson as well as binomial distribution.

Lastly we have discussed control approach to decrease the effect of the disease (Guin, 2015) by applying insecticide. This results stable condition of all the species which proves the efficiency of such spraying. A computer result using Matlab also supports the same.

For all the attack pattern we observe the different conditions of the system depending upon different parameter values though using the control theory the unstable state of the system can be stabilized. A computer results shows the behavior of the solution for different parameter values.

5.2 Statement of the model

In this paper we have consider the healthy as well as infected Jatropha Curcas plant and the infected whitefly while the mosaic virus is taken implicitly.

Let there are 'v' whiteflies which are distributed over 'x' number of healthy and 'y' number of infected Jatropha Curcas plant in such a way that some plants are whitefly free and others have 1,2,.........,i whiteflies per plant.

So we have

$$\sum_{i=1}^{x+y} i = v$$

In this paper we have assumed that number of whiteflies over (x+y) plants, follow a probability distribution, so that the proportion of plants with i whitefly is p(i). Now the number of plants with i whiteflies is p(i)(x+y). If the immanent plant loss-rate per whitefly is 'a' then the loss rate of plants with i whiteflies will be $aip(i)(x+y)$ (Roy P.K., Li X Z et al. 2015).
So the total loss-rate of plants with every possible number of whiteflies is:

$$a(x+y)\sum_{i=0}^{\infty} iP(i)$$

Here $\frac{v}{x+y}$ is the expectation of i. So the total loss-rate due to whitefly attack is axv, which is a Holling type-1 function.
Due to attack of whitefly a fraction of axv forms the infected plant class. On the other hand as the whiteflies consume both healthy as well as infected plant so there is a loss of infected plants due to such consumption. We here assume that the natural death rate of infected plant is same as the attack rate of whitefly i.e. 'a' .
Consumption of healthy plant contributes positive growth to whiteflies whereas consumption of infected plant contributes negative growth to whiteflies.
Here,
e = natural mortality of whitefly.
f = natural mortality of their host plant.
a = by their killing the host plant that they are on.
Self induced mortality occurs at a rate $ai^2p(i)(x+y)$.
Therefore for the whole plant population we have:

$$a(x+y)\sum_{i=0}^{\infty} i^2P(i)$$

where $\sum i^2P(i)$ is the expectation of i^2 the value of which depends upon different probability distribution. Ecologically binomial, poisson, negative binomial distributions represents the regular, random and aggregated attack pattern.

Firstly we have taken the binomial distribution. For this
$E(i^2) = \frac{v}{x+y} + \frac{(L-1)}{L}\frac{v^2}{(x+y)^2}$
Where 'L' is the binomial parameter.
Here $x(0) = x_0 > 0$ is the initial healthy plant population density.
$y(0) = y_0 > 0$ is the initial infected plant population density.
$v(0) = v_0 > 0$ is the initial infected whitefly population density.
Based on the above assumption our model becomes:

5.3 *Model -5.1: Regular attack pattern (Binomial distribution)*

$$\begin{aligned}
\frac{dx}{dt} &= rx - axv \\
\frac{dy}{dt} &= gxv - byv - ay \\
\frac{dv}{dt} &= v[(ax-by) - (e+f+a) - \frac{a(L-1)v}{L(x+y)}]
\end{aligned} \tag{5.1}$$

For mathematical simplicity the following transformation is considered,
$x = \frac{aX}{b}, y = \frac{aY}{b}, v = \frac{aV}{b}, t = \frac{\tau}{a}$.
Then system of equation (5.1) becomes,

$$\begin{aligned} \frac{dX}{d\tau} &= \alpha X - \beta XV \\ \frac{dY}{d\tau} &= \gamma XV - YV - Y \\ \frac{dV}{d\tau} &= V[\beta X - Y - \eta - \frac{\rho V}{X+Y}] \end{aligned} \tag{5.2}$$

Here $X(0) = X_0 > 0$, $Y(0) = Y_0 > 0$, $V(0) = V_0 > 0$ and $\alpha = \frac{r}{a}$, $\beta = \frac{a}{b}$, $\gamma = \frac{g}{b}$, $\eta = \frac{e+f+g}{a}$, $\rho = \frac{(L-1)}{L}$.

5.3.1 Solution properties

lemma 5.1:

The solutions of (5.2) are positive.

Proof:
Since $x(0) = x_0 > 0$, $y(0) = y_0 > 0$ and $v(0) = v_0 > 0$, we have $X(0) = X_0 > 0$, $Y(0) = Y_0 > 0$ and $V(0) = V_0 > 0$. Suppose $X(\tau)$ is not positive for all $\tau \geq 0$. Since $X_0 > 0$ then there exist τ_0 with $X(\tau_0) = 0$ and $X(\tau) > 0$ for $0 \leq \tau \leq \tau_0$. For $0 \leq \tau \leq \tau_0$

$\frac{\dot{X(\tau)}}{X(\tau)} = \alpha(1 - X - Y) - \beta Y - \gamma V > -\alpha X - \alpha Y - \beta Y - \gamma V$
$X(\tau_0) > X_0 exp[-\alpha \frac{\tau_0^2}{2} - \alpha \int_0^{\tau_0} Y d\tau - \beta \int_0^{\tau_0} Y d\tau - \gamma \int_0^{\tau_0} V d\tau] > 0$

This is a contradiction and hence $X(\tau)$ is positive for all $\tau \geq 0$. Similarly it can be shown that $Y(\tau)$, $V(\tau)$ are also positive for all $\tau \geq 0$.

5.3.2 Equilibria

The equilibrium points are obtained by setting $\frac{dX}{d\tau} = 0$, $\frac{dY}{d\tau} = 0$ and $\frac{dV}{d\tau} = 0$ and solving the equations

$\alpha X - \beta XV = 0$
$\gamma XV - YV - Y = 0$
$\beta XV - YV - \eta V - \frac{\rho V^2}{X+Y} = 0$

From these equations we obtained that the system has two equilibria namely

$E_1(\bar{X}, 0, \bar{V})$, $E_2(X^*, Y^*, V^*)$.
Here $E_1(\bar{X}, 0, \bar{V}) = E_1(\frac{\beta\eta \pm \sqrt{\beta^2\eta^2 + 4\beta^2\alpha\rho}}{2\beta^2}, 0, \frac{\alpha}{\beta})$ always exists and we have taken the positive sign.
For $E_2(X^*, Y^*, V^*)$, we have,

$V^* = \frac{\alpha}{\beta}$

$Y^* = \frac{\alpha\gamma X^*}{\alpha+\beta}$

Putting these values in the third equation of (5.2) we obtain , a quadratic equation of X^* which is given as,

$$X^{*2}(\alpha^2\beta^2\gamma + \alpha^2\beta^2 + 2\alpha\beta^3 + \alpha\beta^3\gamma + \beta^4 - \alpha^2\gamma^2\beta - \alpha^2\beta\gamma - \alpha\beta^2\gamma) - X^*(\alpha^2\beta\gamma\eta + \alpha\beta^2\gamma\eta + \alpha^2\beta\eta + 2\alpha\beta^2\eta + \beta^3\eta) - \alpha\rho(\alpha+\beta)^2 = 0$$

There is atleast one change of sign.Therefore by Descartes' rule of sign atleast one positive $E_2(X^*, Y^*, V^*)$ exists.

5.3.3 Dynamic behavior

The variational matrix is given by :

$$J(X,Y,V) = \begin{pmatrix} \alpha - \beta V & 0 & -\beta X \\ \gamma V & -1 - V & \gamma X - Y \\ \beta V + \frac{\rho V^2}{(X+Y)^2} & -V + \frac{\rho V^2}{(X+Y)^2} & \beta X - Y - \eta - \frac{2\rho V}{X+Y} \end{pmatrix}$$

For $E_1(\bar{X}, 0, \bar{V})$ the characteristic equation is given by ,
$$\lambda^3 + \lambda^2(\bar{V} + 1 + \frac{\rho\bar{V}}{\bar{X}}) + \lambda(\rho\frac{\bar{V}^2}{\bar{X}} + \frac{\rho\bar{V}}{\bar{X}} + \alpha\beta\bar{X} + \frac{\beta\rho\bar{V}^2}{\bar{X}} + \gamma\bar{X}\bar{V} - \frac{\rho\gamma\bar{V}^2}{\bar{X}}) - \beta\gamma\bar{X}\bar{V}^2 + \beta\rho\gamma\frac{\bar{V}^3}{\bar{X}} + \alpha\beta\bar{X}\bar{V} + \alpha\beta\bar{X} + \beta\rho\frac{\bar{V}^3}{\bar{X}} + \beta\rho\frac{\bar{V}^2}{\bar{X}} = 0$$

It can be written as,

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0$$

where
$A = \bar{V} + 1 + \frac{\rho\bar{V}}{\bar{X}} > 0$

$B = \rho\frac{\bar{V}^2}{\bar{X}} + \frac{\rho\bar{V}}{\bar{X}} + \alpha\beta\bar{X} + \frac{\beta\rho\bar{V}^2}{\bar{X}} + \gamma\bar{X}\bar{V} - \frac{\rho\gamma\bar{V}^2}{\bar{X}}$

$C = -\beta\gamma\bar{X}\bar{V}^2 + \beta\rho\gamma\frac{\bar{V}^3}{\bar{X}} + \alpha\beta\bar{X}\bar{V} + \alpha\beta\bar{X} + \beta\rho\frac{\bar{V}^3}{\bar{X}} + \beta\rho\frac{\bar{V}^2}{\bar{X}}$

$\quad = \alpha^2\bar{X}(1 - \frac{\gamma}{\beta}) + \frac{\rho\gamma\alpha^3}{\bar{X}\beta^2} + \alpha\beta\bar{X} + \frac{\rho\alpha^3}{\bar{X}\beta^2} + \frac{\rho\alpha^2}{\beta\bar{X}} > 0$

as $\beta > \gamma$

Now $AB - C$

$= \frac{\rho\alpha^3}{\beta^3\bar{X}} + 2\frac{\rho\alpha^2}{\beta^2\bar{X}} + \frac{\rho^2\alpha^3}{\beta^3\bar{X}^2} + \frac{\rho\alpha}{\beta\bar{X}} + \frac{\rho^2\alpha^2}{\beta^2\bar{X}^2} + \alpha^2\rho + \frac{\rho^2\alpha^3}{\beta^2\bar{X}^2} + \frac{\gamma\bar{X}\alpha^2}{\beta^2} + \frac{\gamma\bar{X}\alpha}{\beta} + \frac{\gamma\rho\alpha^2}{\beta^2} - \frac{\rho\gamma\alpha^3}{\beta^3\bar{X}} - \frac{\rho\gamma\alpha^2}{\beta^2\bar{X}} - \frac{\rho^2\gamma\alpha^3}{\beta^3\bar{X}^2} + \frac{\gamma\bar{X}\alpha^2}{\beta} - \frac{\rho\gamma\alpha^3}{\beta^2\bar{X}}$

The sign of which cannot be predictable. Hence $AB - C$ can be positive, negative, or equal to zero depending upon different parameter values which results stable, unstable or hopf bifurcation respectively. Here $\rho < 1$ and $\rho = 1$ if 'L' is very large.

For **interior equilibrium** $E_2(X^*, Y^*, V^*)$ the characteristic equation is given by:

$\lambda^3 + \lambda^2(V^* + 1 + \frac{\rho V^*}{X^*+Y^*}) + \lambda(\rho\frac{V^{*2}}{X^*+Y^*} + \frac{\rho V^*}{X^*+Y^*} + \gamma X^*V^* - \frac{\rho\gamma V^{*2}X^*}{(X^*+Y^*)^2} - Y^*V^* + \beta^2X^*V^* + \frac{\rho Y^*V^{*2}}{(X^*+Y^*)^2} + \frac{\rho\beta X^*V^{*2}}{(X^*+Y^*)^2}) - \beta\gamma X^*V^{*2} + \beta\rho\gamma\frac{X^*V^{*3}}{(X^*+Y^*)^2} + \beta\rho\frac{V^{*3}X^*}{(X^*+Y^*)^2} + \beta\rho\frac{V^{*2}X^*}{(X^*+Y^*)^2} + \beta^2X^*V^{*2} + \beta^2X^*V^* = 0$

It can be written as,

$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$

Where

$A_1 = V^* + 1 + \frac{\rho V^*}{X^*+Y^*} > 0$

$A_2 = \rho\frac{V^{*2}}{X^*+Y^*} + \frac{\rho V^*}{X^*+Y^*} + \gamma X^*V^* - \frac{\rho\gamma V^{*2}X^*}{(X^*+Y^*)^2} - Y^*V^* + \beta^2X^*V^* + \frac{\rho Y^*V^{*2}}{(X^*+Y^*)^2} + \frac{\rho\beta X^*V^{*2}}{(X^*+Y^*)^2}$

$A_3 = -\beta\gamma X^*V^{*2} + \beta\rho\gamma\frac{X^*V^{*3}}{(X^*+Y^*)^2} + \beta\rho\frac{V^{*3}X^*}{(X^*+Y^*)^2} + \beta\rho\frac{V^{*2}X^*}{(X^*+Y^*)^2} + \beta^2X^*V^{*2} + \beta^2X^*V^* > 0$

Now $A_1A_2 - A_3 =$

$\rho\frac{V^{*2}}{X^*+Y^*} + \rho\frac{V^{*3}}{X^*+Y^*} + \rho^2\frac{V^{*3}}{(X^*+Y^*)^2} + \rho\frac{V^{*2}}{X^*+Y^*} + \rho\frac{V^*}{X^*+Y^*} + \rho^2\frac{V^{*2}}{(X^*+Y^*)^2} - \gamma\rho X^*\frac{V^{*3}}{(X^*+Y^*)^2} - \gamma\rho X^*\frac{V^{*2}}{(X^*+Y^*)^2} - \gamma\rho^2X^*\frac{V^{*3}}{(X^*+Y^*)^3} + \rho Y\frac{V^{*3}}{(X^*+Y^*)^2} + \rho Y^*\frac{V^{*2}}{(X^*+Y^*)^2} + \rho^2Y^*\frac{V^{*3}}{(X^*+Y^*)^3} + \gamma X^*V^{*2} + \gamma X^*V^* + \rho\gamma\frac{X^*V^{*2}}{(X^*+Y^*)} - Y^*V^{*2} - Y^*V^* - \frac{\rho Y^*V^{*2}}{X^*+Y^*} + \frac{\beta^2\rho X^*V^{*2}}{X^*+Y^*} + \frac{\rho^2\beta X^*V^{*3}}{(X^*+Y^*)^3} + \beta\gamma X^*V^{*2} - \frac{\beta\gamma\rho X^*V^{*3}}{(X^*+Y^*)^2}$

5.4 Stability

The interior equilibrium point $E_2(X^*, Y^*, V^*)$ is stable if the following conditions are satisfied (Sarkar A.K., and Roy, 1989 and Venturino, Roy, Al Basir, et al., 2016) :

i)$A_1 > 0, A_3 > 0$

ii)$A_1A_2 - A_3 > 0$

Proof:

Using the Routh-Hurwitz criteria and from the above discussions we have

obtained that,
$A_1 = V^* + 1 + \frac{\rho V^*}{X^*+Y^*} > 0$
$A_3 > 0$ if
$-\beta\gamma X^* V^{*2} + \beta\rho\gamma \frac{X^* V^{*3}}{(X^*+Y^*)^2} + \beta\rho \frac{V^{*3} X^*}{(X^*+Y^*)^2} + \beta\rho \frac{V^{*2} X^*}{(X^*+Y^*)^2} + \beta^2 X^* V^{*2} + \beta^2 X^* V^* > 0$
and $A_1 A_2 - A_3 > 0$ if

$$\rho \frac{V^{*2}}{X^*+Y^*} + \rho \frac{V^{*3}}{X^*+Y^*} + \rho^2 \frac{V^{*3}}{(X^*+Y^*)^2} + \rho \frac{V^{*2}}{X^*+Y^*} + \rho \frac{V^*}{X^*+Y^*} + \rho^2 \frac{V^{*2}}{(X^*+Y^*)^2} - \gamma\rho X^* \frac{V^{*3}}{(X^*+Y^*)^2} - \gamma\rho X^* \frac{V^{*2}}{(X^*+Y^*)^2} - \gamma\rho^2 X^* \frac{V^{*3}}{(X^*+Y^*)^3} + \rho Y \frac{V^{*3}}{(X^*+Y^*)^2} + \rho Y^* \frac{V^{*2}}{(X^*+Y^*)^2} + \rho^2 Y^* \frac{V^{*3}}{(X^*+Y^*)^3} + \gamma X^* V^{*2} + \gamma X^* V^* + \rho\gamma \frac{X^* V^{*2}}{(X^*+Y^*)} - Y^* V^{*2} - Y^* V^* - \frac{\rho Y^* V^{*2}}{X^*+Y^*} + \frac{\beta^2 \rho X^* V^{*2}}{X^*+Y^*} + \frac{\rho^2 \beta X^* V^{*3}}{(X^*+Y^*)^3} + \beta\gamma X^* V^{*2} - \frac{\beta\gamma\rho X^* V^{*3}}{(X^*+Y^*)^2} > 0$$

So if these conditions hold then the system is stable around the interior equilibrium $E_2(X^*, Y^*, V^*)$.

5.5 Bifurcation Analysis

Now we will find the conditions of Hopf bifurcation for E^* by varying ρ over $\mathbb{R}$ (Basir, Venturino and Roy, 2016) .
Let $\psi : (0, \infty) \to \mathbb{R}$ be the following continuously differentiable function of ρ.
$\psi(\rho) = A_1 A_2 - A_3$
The conditions required for occurring the Hopf- bifurcation (Kar Tapan et al. 2019) at $\rho = \rho^*$ is that
i)$\psi(\rho^*) = 0$
ii)There exist $\rho^* \in (0, \infty)$ at which a pair of complex eigen values λ,$\bar{\lambda}$ are such that $Re\lambda(\rho^*) = 0$
and
iii) $\left.\frac{Re\lambda(\rho)}{\rho}\right|_{\rho=\rho^*} \neq 0$

Proof:

If $A_1 A_2 - A_3 = 0$ then the characteristic equation can be written as:
$(\lambda^2 + \frac{A_3}{A_1})(\lambda + A_1) = 0$
It has three roots with the pair of purely imaginary roots at $\rho = \rho^*$ so we have,
$\lambda_{1,2} = \pm i\sqrt{\frac{A_3}{A_1}}$ and $\lambda_3 = -A_1$
Since $\psi(\rho^*)$ is a continuous function of all its roots so there is an open interval $\rho \in (\rho^* - \epsilon, \rho^* + \epsilon)$ where $\lambda_{1,2}$ are complex conjugate for ρ.
Let their general form in this neighborhood are
$\lambda_1(\rho) = a(\rho) + ib(\rho)$
$\lambda_2(\rho) = a(\rho) - ib(\rho)$
Now we have to show that
$\left.\frac{Re\lambda_i(\rho)}{\rho}\right|_{\rho=\rho^*} \neq 0$

To prove this, substituting $\lambda_i(\rho) = a(\rho) \pm ib(\rho)$ into $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$ and then calculating the derivatives we get

$$\begin{aligned} P(\rho)a^{'}(\rho) - Q(\rho)b^{'}(\rho) + R(\rho) &= 0 \\ Q(\rho)a^{'}(\rho) + P(\rho)b^{'}(\rho) + S(\rho) &= 0 \end{aligned} \tag{5.3}$$

where
$P(\rho) = 3a^2 - 3b^2 + 2aA_1 + A_2$
$Q(\rho) = 6ab + 2bA_1$
$R(\rho) = a^2A_1^{'} - b^2A_1^{'} + aA_2^{'}$
$S(\rho) = A_1^{'}2ab + A_2^{'}b$

solving the equations (5.3) we obtain,
$\frac{Re\lambda_i(\rho)}{\rho}\Big|_{\rho=\rho^*} = a^{'}(\rho^*) = \frac{-Q(\rho)S(\rho)-P(\rho)R(\rho)}{P^2(\rho)+Q^2(\rho)} \neq 0$
Thus the transversality conditions hold. Therefore Hopf-bifurcation occurs at $\rho = \rho^*$.

5.6 Unstable condition

The interior equilibrium point $E_2(X^*, Y^*, V^*)$ is unstable if the following conditions are satisfied:
i)$A_1 > 0, A_3 > 0$
ii)$A_1A_2 - A_3 < 0$

Proof:
Using the Routh-Hurwitz criteria and from the above discussions we have obtained that,
$A_1 = V^* + 1 + \frac{\rho V^*}{X^*+Y^*} > 0$
$A_3 > 0$ if
$-\beta\gamma X^*V^{*2} + \beta\rho\gamma\frac{X^*V^{*3}}{(X^*+Y^*)^2} + \beta\rho\frac{V^{*3}X^*}{(X^*+Y^*)^2} + \beta\rho\frac{V^{*2}X^*}{(X^*+Y^*)^2} + \beta^2X^*V^{*2} + \beta^2X^*V^* > 0$
and $A_1A_2 - A_3 < 0$ if

$$\rho\frac{V^{*2}}{X^*+Y^*} + \rho\frac{V^{*3}}{X^*+Y^*} + \rho^2\frac{V^{*3}}{(X^*+Y^*)^2} + \rho\frac{V^{*2}}{X^*+Y^*} + \rho\frac{V^*}{X^*+Y^*} + \rho^2\frac{V^{*2}}{(X^*+Y^*)^2} - \gamma\rho X^*\frac{V^{*3}}{(X^*+Y^*)^2} - \gamma\rho X^*\frac{V^{*2}}{(X^*+Y^*)^2} - \gamma\rho^2X^*\frac{V^{*3}}{(X^*+Y^*)^3} + \rho Y\frac{V^{*3}}{(X^*+Y^*)^2} + \rho Y^*\frac{V^{*2}}{(X^*+Y^*)^2} + \rho^2Y^*\frac{V^{*3}}{(X^*+Y^*)^3} + \gamma X^*V^{*2} + \gamma X^*V^* + \rho\gamma\frac{X^*V^{*2}}{(X^*+Y^*)} - Y^*V^{*2} - Y^*V^* - \frac{\rho Y^*V^{*2}}{X^*+Y^*} + \frac{\beta^2\rho X^*V^{*2}}{X^*+Y^*} + \frac{\rho^2\beta X^*V^{*3}}{(X^*+Y^*)^3} + \beta\gamma X^*V^{*2} - \frac{\beta\gamma\rho X^*V^{*3}}{(X^*+Y^*)^2} < 0$$

So if these conditions hold then the system is unstable around the interior equilibrium $E_2(X^*, Y^*, V^*)$.

5.7 Model -5.2: Random attack pattern (Poisson distribution)

If we consider that the herbivore attack follows the Poisson distribution we have
$E(i^2) = \frac{v}{x+y} + \frac{v^2}{(x+y)^2}$

Now our model is

$$\begin{aligned}
\frac{dx}{dt} &= rx - axv \\
\frac{dy}{dt} &= gxv - byv - ay \\
\frac{dv}{dt} &= v[(ax - by) - (e + f + a) - \frac{av}{(x+y)}]
\end{aligned} \tag{5.4}$$

The transformations considered to dimensionless the system are
$x = \frac{aX}{b}, y = \frac{aY}{b}, v = \frac{aV}{b}, t = \frac{\tau}{a}$.
Then system of equation (5.4) becomes,

$$\begin{aligned}
\frac{dX}{d\tau} &= \alpha X - \beta XV \\
\frac{dY}{d\tau} &= \gamma XV - YV - Y \\
\frac{dV}{d\tau} &= V[\beta X - Y - \eta - \frac{\rho V}{X+Y}]
\end{aligned} \tag{5.5}$$

Here $X(0) = X_0 > 0$, $Y(0) = Y_0 > 0$, $V(0) = V_0 > 0$ and $\alpha = \frac{r}{a}$, $\beta = \frac{a}{b}$, $\gamma = \frac{g}{b}$, $\eta = \frac{e+f+g}{a}$.

5.7.1 Equilibria

The equilibrium points of the system (5.5) are $E_1(\bar{X}, 0, \bar{V})$ and $E_2(X^*, Y^*, V^*)$
Where $E_1(\bar{X}, 0, \bar{V}) = E_1(\frac{\beta\eta \pm \sqrt{\beta^2\eta^2 + 4\beta^2\alpha}}{2\beta^2}, 0, \frac{\alpha}{\beta})$.
For $E_2(X^*, Y^*, V^*)$
$V^* = \frac{\alpha}{\beta}$
$Y^* = \frac{\alpha\gamma X^*}{\alpha+\beta}$
Putting these values in the third equation of (5.5) we obtain , a quadratic equation of X^* which is given as,
$X^{*2}(\alpha^2\beta^2\gamma + \alpha^2\beta^2 + 2\alpha\beta^3 + \alpha\beta^3\gamma + \beta^4 - \alpha^2\gamma^2\beta - \alpha^2\beta\gamma - \alpha\beta^2\gamma) - X^*(\alpha^2\beta\gamma\eta + \alpha\beta^2\gamma\eta + \alpha^2\beta\eta + 2\alpha\beta^2\eta + \beta^3\eta) - \alpha(\alpha+\beta)^2 = 0$

Clearly interior equilibrium exists.

5.8 Stability

The variational matrix is given by :

$$J(X,Y,V) = \begin{pmatrix} \alpha - \beta V & 0 & -\beta X \\ \gamma V & -1-V & \gamma X - Y \\ \beta V + \frac{V^2}{(X+Y)^2} & -V + \frac{V^2}{(X+Y)^2} & \beta X - Y - \eta - \frac{2V}{X+Y} \end{pmatrix}$$

The characteristic equation for the interior equilibrium is
$\lambda^3 + \lambda^2(V^* + 1 + \frac{V^*}{X^*+Y^*}) + \lambda(\frac{V^{*2}}{X^*+Y^*} + \frac{V^*}{X^*+Y^*} + \gamma X^* V^* - \frac{\gamma V^{*2} X^*}{(X^*+Y^*)^2} - Y^* V^* + \beta^2 X^* V^* + \frac{Y^* V^{*2}}{(X^*+Y^*)^2} + \frac{\beta X^* V^{*2}}{(X^*+Y^*)^2}) - \beta\gamma X^* V^{*2} + \beta\gamma \frac{X^* V^{*3}}{(X^*+Y^*)^2} + \beta \frac{V^{*3} X^*}{(X^*+Y^*)^2} + \beta \frac{V^{*2} X^*}{(X^*+Y^*)^2} + \beta^2 X^* V^{*2} + \beta^2 X^* V^* = 0$
This can be written as,
$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$
where
$B_1 = V^* + 1 + \frac{V^*}{X^*+Y^*} > 0$
$B_2 = \frac{V^{*2}}{X^*+Y^*} + \frac{V^*}{X^*+Y^*} + \gamma X^* V^* - \frac{\gamma V^{*2} X^*}{(X^*+Y^*)^2} - Y^* V^* + \beta^2 X^* V^* + \frac{Y^* V^{*2}}{(X^*+Y^*)^2} + \frac{\beta X^* V^{*2}}{(X^*+Y^*)^2}$
$B_3 = -\beta\gamma X^* V^{*2} + \beta\gamma \frac{X^* V^{*3}}{(X^*+Y^*)^2} + \beta \frac{V^{*3} X^*}{(X^*+Y^*)^2} + \beta \frac{V^{*2} X^*}{(X^*+Y^*)^2} + \beta^2 X^* V^{*2} + \beta^2 X^* V^*$
$Now B_1 B_2 - B_3 =$
$\frac{V^{*2}}{X^*+Y^*} + \frac{V^{*3}}{X^*+Y^*} + \frac{V^{*3}}{(X^*+Y^*)^2} + \frac{V^{*2}}{X^*+Y^*} + \frac{V^*}{X^*+Y^*} + \frac{V^{*2}}{(X^*+Y^*)^2} - \gamma X^* \frac{V^{*3}}{(X^*+Y^*)^2} - \gamma X^* \frac{V^{*2}}{(X^*+Y^*)^2} - \gamma X^* \frac{V^{*3}}{(X^*+Y^*)^3} + Y \frac{V^{*3}}{(X^*+Y^*)^2} + Y^* \frac{V^{*2}}{(X^*+Y^*)^2} + Y^* \frac{V^{*3}}{(X^*+Y^*)^3} + \gamma X^* V^{*2} + \gamma X^* V^* + \gamma \frac{X^* V^{*2}}{(X^*+Y^*)} - Y^* V^{*2} - Y^* V^* - \frac{Y^* V^{*2}}{X^*+Y^*} + \frac{\beta^2 X^* V^{*2}}{X^*+Y^*} + \frac{\beta X^* V^{*3}}{(X^*+Y^*)^3} + \beta\gamma X^* V^{*2} - \frac{\beta\gamma X^* V^{*3}}{(X^*+Y^*)^2}$

We have shown numerically that the value of $B_1 B_2 - B_3$ is positive which results stable state of the interior equilibrium .

5.9 *Model-5.3: Aggregated attack pattern (Negative-Binomial distribution)*

For Negative-Binomial distribution
$E(i^2) = \frac{v}{x+y} + \frac{(k+1)}{k} \frac{v^2}{(x+y)^2}$
where 'k' is the negative-binomial parameter.
Thus our model transformed into the form as

$$\begin{aligned}\frac{dx}{dt} &= rx - axv \\ \frac{dy}{dt} &= gxv - byv - ay \\ \frac{dv}{dt} &= v[(ax - by) - (e + f + a) - \frac{a(k+1)v}{k(x+y)}]\end{aligned} \tag{5.6}$$

For mathematical simplicity the following transformation is considered, $x = \frac{aX}{b}, y = \frac{aY}{b}, v = \frac{aV}{b}, t = \frac{\tau}{a}$.
Then system of equation (5.6) becomes,

$$\begin{aligned}\frac{dX}{d\tau} &= \alpha X - \beta XV \\ \frac{dY}{d\tau} &= \gamma XV - YV - Y \\ \frac{dV}{d\tau} &= V[\beta X - Y - \eta - \frac{\xi V}{X+Y}]\end{aligned} \tag{5.7}$$

Here $X(0) = X_0 > 0$, $Y(0) = Y_0 > 0$, $V(0) = V_0 > 0$ and $\alpha = \frac{r}{a}$, $\beta = \frac{a}{b}$, $\gamma = \frac{g}{b}$, $\eta = \frac{e+f+g}{a}$, $\xi = \frac{(k+1)}{k}$.

The dynamics of this model is same as that of Binomial distribution. But here $\xi > 1$ and $\xi = 1$ if k is very large. This creates a difference between the dynamics of two distribution. We here numerically shown that for negative binomial distribution it always gives stable state. Here $A_1A_2 - A_3 > 0$ for the interior equilibrium if $\xi \geq 1$.

5.10 The optimal control problem

To decrease the effect of the disease we use insecticide spraying.
Mathematically we here use the control theory (Chowdhury, Basir, Takeuchi, Ghosh and Roy, 2019; Holt, Jeger, Thresh and Otim-Na, 1997 and Roy, Li, Al Basir, Datta, Chowdhury, 2015). Assuming that all the infected whiteflies of a particular region fall possibly under the control of insecticide spraying. We choose the control parameter u(t) . Now our reformulated model becomes:

$$\begin{aligned}
\frac{dX}{d\tau} &= \alpha X - (1-u(t))\beta XV \\
\frac{dY}{d\tau} &= (1-u(t))\gamma XV - YV - Y \\
\frac{dV}{d\tau} &= V[(1-u(t))\beta X - \eta - Y - \frac{\rho V}{X+Y}]
\end{aligned} \tag{5.8}$$

If we consider $u(t) = 0$ then there is no effect of control and if $u(t) = 1$ then there is no such contact rate between the whitefly and the plant. So the control parameter u(t) lies between 0 and 1 that is $0 \leq u(t) \leq 1$ defined on $[t_0, t_f]$ where t_0 and t_f are the starting and ending time of control respectively. The objective functional to minimize the cost of insecticide spraying is thus given as :-

$J(u(t)) = \int t_0^{t_f} [P_1 u^2(t) + Q_1 Y^2 - R_1 X^2] d\tau$

Where
$P_1 u^2(t)$=Cost regarding insecticide spraying and labor.
$Q_1 Y^2$=Loss of crop due to infection.
$R_1 X^2$=Extra revenue obtained by a larger population of healthy plant.

5.10.1 Theorem :

The objective functional $J(\gamma^*(t))$ is minimum for the optimal control variable $u^*(t)$ corresponding to the interior equilibrium $E_2(X^*, Y^*, V^*)$ and the adjoint variables ξ_1, ξ_2, ξ_3 satisfy the system of equations,

$$\begin{aligned}
\frac{d\xi_1}{dt} &= 2R_1 X - \xi_1[\alpha - (1-u(t))\beta V] - \xi_2(1-u(t))\gamma V - \xi_3[(1-u(t))\beta V + \frac{\rho V^2}{(X+Y)^2}] \\
\frac{d\xi_2}{dt} &= -2Q_1 Y + \xi_2(1+V) + \xi_3[V - \frac{\rho V^2}{(X+Y)^2}] \\
\frac{d\xi_3}{dt} &= \xi_1 \beta X(1-u(t)) - \xi_2[\gamma X(1-u(t)+Y] - \xi_3[\beta X(1-u(t)) - \eta - Y - \frac{2\rho V}{X+Y}]
\end{aligned} \tag{5.9}$$

with the boundary condition $\xi_i(t_f) = 0$ (i=1,2,3). The optimal control can be given as,

$$u^*(t) = max\{0, min\{1, \frac{(\xi_3 - \xi_1)\beta XV + \xi_2 XV}{2P_1}\}\} \tag{5.10}$$

Proof:
The Hamiltonian is constructed as:

$H = P_1 u^2(t) + Q_1 Y^2 - R_1 X^2 + \xi_1[\alpha X - (1-u(t))\beta XV] + \xi_2[(1-u(t))\gamma XV - YV - Y] + \xi_3[(1-u(t))\beta XV - \eta V - YV - \frac{\rho V^2}{X+Y}]$

According to Pontryagin Minimum Principle the optimal control variable $u^*(t)$ satisfies

$\frac{\partial H}{\partial u^*(t)} = 0$

Which implies

$2P_1 u^*(t) + \xi_1 \beta XV - \xi_2 \gamma XV - \xi_3 \beta XV = 0$
$u^*(t) = \frac{\beta XV(\xi_3 - \xi_1) + \gamma XV \xi_2}{2P_1}$

For the boundedness of the optimal control we have

$$u^*(t) = \begin{cases} 0 & \frac{\beta XV(\xi_3-\xi_1)+\gamma XV\xi_2}{2P_1} \leq 0 \\ \frac{\beta XV(\xi_3-\xi_1)+\gamma XV\xi_2}{2P_1} & 0 < \frac{\beta XV(\xi_3-\xi_1)+\gamma XV\xi_2}{2P_1} < 1 \\ 1 & \frac{\beta XV(\xi_3-\xi_1)+\gamma XV\xi_2}{2P_1} \geq 1 \end{cases}$$

According to Pontryagin Minimum Principle adjoint variables satisfy the following equations:

$$\frac{d\xi_i}{dt} = -\frac{\partial H}{\partial X_i} \tag{5.11}$$

where $i = 1, 2, 3$, $X_i = X, Y, V$ and $\xi_i = \xi_1, \xi_2, \xi_3$
and the equations (5.9) can be determined by using (5.11). This completes the proof of the theorem.

5.11 Numerical simulations

Numerical simulations are performed to verify the theoretical outcomes obtained in the previous sections by ode23 solver using MATLAB 2017a. In this paper our goal is to find out the role of different probability distribution on the disease dynamics. We have taken the initial values as $X(0) = 0.30$, $Y(0) = 0.15$, $V(0) = 0.20$.
Keeping in mind all the feasibility criteria the parameter values are chosen as

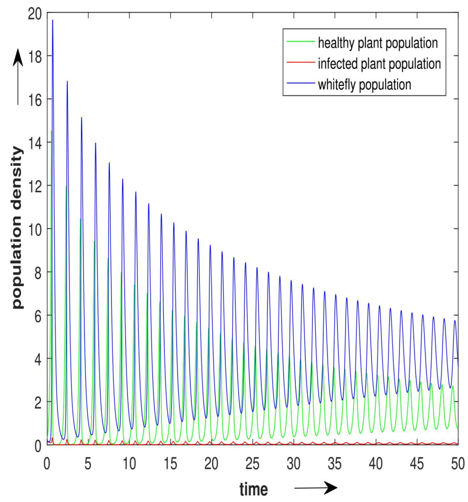

FIGURE 5.1: Variation of plant-herbivore densities for $\alpha = 8$, $\beta = 2$, $\gamma = 0.05$, $\eta = 3$, $\rho = 0.5$. Here we observe unstable condition for all the populations with increasing time for binomial distribution.

$\alpha = 8, \beta = 2, \gamma = 0.05, \eta = 3.$

Firstly, for the Binomial distribution we observed that $\rho = \frac{L-1}{L}$ that implies $\rho < 1$ (see fig. 5.3) always and $\rho = 1$ (see fig. 5.6) if L is very large. So while performing the time-series plotting we have taken $\rho \leq 1$. Taking ρ as the bifurcating parameter we here observed that when $\rho = 0.0439$ then it possesses hopf bifurcation (see fig. 5.2) while $\rho = 0.5$ leads to stable state and $\rho = 0.008$ leads to unstable condition (see fig. 5.1) of the interior equilibrium which ecologically reflects unstable condition of all the species.

Secondly, when we take the Poisson distribution (see fig. 5.4) then it results stable state of the system as well as all the species.

Thirdly, for the negative-binomial distribution we observed that $\xi = \frac{k+1}{k}$ that implies $\xi > 1$ and $\xi = 1$ if k is very large. So when plotting for this distribution we have taken $\xi \geq 1$ and observed that the interior equilibrium is stable (see fig. 5.5) for all $\xi \geq 1$. It also gives the steady state of all the species.

Lastly we have illustrated the analytical method using optimal control theory on the model (5.2) to reduce the effect of the disease by controlling whitefly population. We have seen that with the effect of control all the population goes to steady state (see fig. 5.7) .

So it is evident from our study that different distribution left an influence on the plant-whitefly interaction. Also controlling by spraying insecticide gives a better results for all the population.

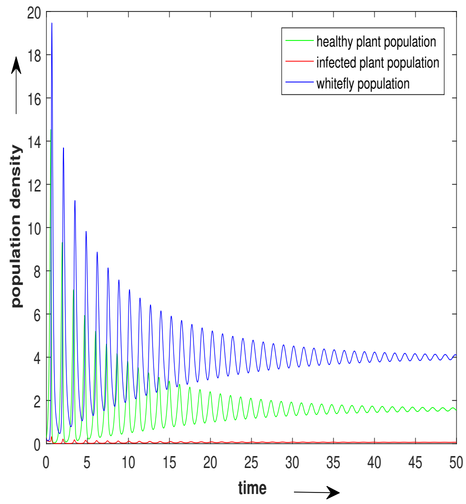

FIGURE 5.2: Variation of plant-herbivore densities for the system with $\alpha = 8$, $\beta = 2$, $\gamma = 0.05$, $\eta = 3$, $\rho = 0.0439$.This shows small oscillating hopf bifurcation for binomial distribution.

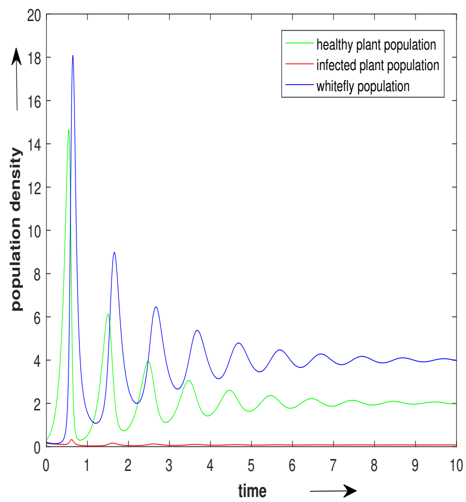

FIGURE 5.3: Variation of plant whitefly densities with $\alpha = 8$, $\beta = 2$, $\gamma = 0.05$, $\eta = 3$,$\rho = 0.005$ which shows the stable state for binomial distribution.

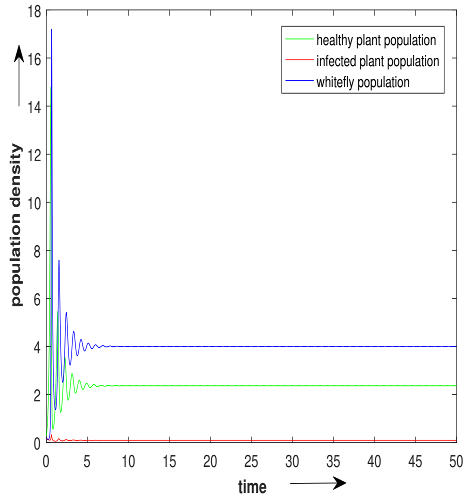

FIGURE 5.4: Variation of plant whitefly densities with poisson distribution with $\alpha = 8$, $\beta = 2$, $\gamma = 0.05$, $\eta = 3$.

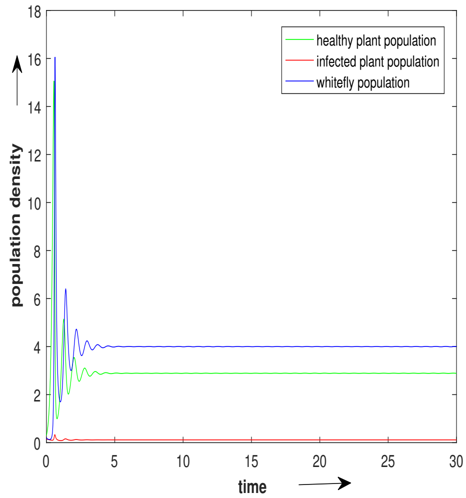

FIGURE 5.5: Time-series plotting for the system for $\alpha = 8$, $\beta = 2$, $\gamma = 0.05$, $\eta = 3$, $\rho = 2$ with negative-binomial distribution.

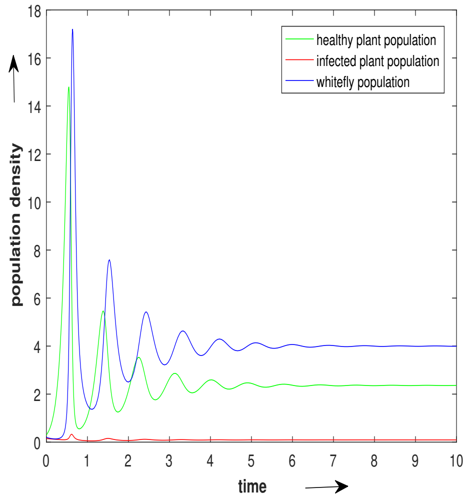

FIGURE 5.6: Time series plotting for the system for $\alpha = 8$, $\beta = 2$, $\gamma = 0.05$, $\eta = 3$,$\rho = 1$ with binomial distribution by taking $\rho = 1$.

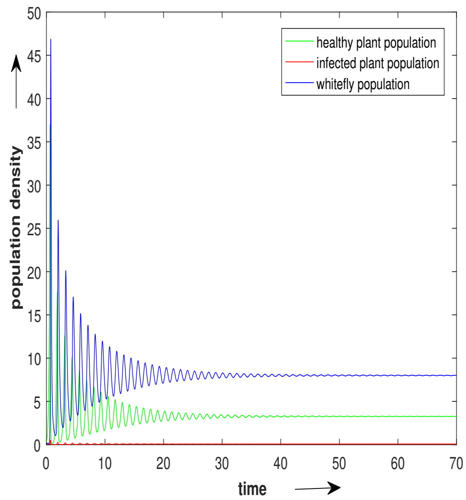

FIGURE 5.7: Variation of plant-whitefly densities with control for $\alpha = 8$, $\beta = 2$, $\gamma = 0.05$, $\eta = 3$,$\rho = 0.05$,$u = 0.5$ of the model which shows the stability of the system .

5.12 Conclusions

This study highlights on the effect of different attack pattern of whitefly on the Jatropha Curcas plant. Here the different distribution parameters are playing a very important role for the dynamics. It is clear from the study that the whiteflies which kill the plants will also kill themselves. It is also observed that when k and L are very large such that $\frac{1}{k} = 0 = \frac{1}{L}$ then the dynamics for regular, random and aggregated attack patterns give similar dynamic behavior. From the present study it is observed that the application of control will help to minimize the application of spraying insecticide. It also helps to minimize the cost for the marginal farmers in the real system. Using time-delay these models can be renovated in near future.

6 Role of Insecticide Spraying in Reduction of Mosaic Disease of Jatropha Curcas Plant

[Chapter based on the paper submitted in the Journal of the Calcutta Mathematical Society]

6.1 Introduction

At first we here discuss again about the Jatropha curcas plant and the vector of the mosaic disease, the whitefly. The demand of alternative fuel is increasing proportionally with the increasing consumption of fossil fuel world-wide. Biodiesel is a significant name as an alternative fuel for diesel engine as it possesses environment friendly characteristics. The seeds of the Jatrophsa curcas plant contains approximately 37% oil that can be used to obtain a better quality of biodiesel fuel (Narayana D.S.A. et al., 2006 ; Venturino et al., 2016) . Jatropha Curcas which is also known as physic nut, is a species of flowering plant in the spurge family Euphorbiaceae. This plant is originated at Africa and Asia but now its popularity spreads world-wide. More than 40 species of insects affect Jatropha curcas plant, among them Mosaic virus is specifically mentionable. In this paper mosaic virus is taken implicitly.
The vector of this virus is whitefly. The population of whitefly is controlled by the temperature and rainfall. Heavy rainfall (Roy, Li, Al Basir, Datta, Chowdhury, 2015) is an obstruction for the growth of whiteflies. The spread of the whitefly is highly dependent on the plant density. Whiteflies are tremendously productive (Gao et al., 2010 ; Holt et al., 1997). Normally whitefly needs three hours feeding time to procure the virus and a latent phase of eight hours. It requires ten minutes time to contaminate the young leaves. Symptoms seem to be appeared after a latent period of three to five weeks. The symptoms occurs for mosaic disease are severe mosaic, mottling, blistering of leaves, yellowing of leaves , reduced leaf size, stunting of diseased plants. Besides these the fruits production reduces and quality of seeds becomes low.
We here tried to control the population of whiteflies by spraying the insecticide. It breaks the gatherings from the laying more amounts of eggs. It also prevents adult whiteflies to migrate to the neighbour plants (Narayana D.S.A. et al., 2006 ; Venturino et al., 2016).
Based on these a mathematical model is formulated to analyze the disease dynamics of Jatropha Curcas plant. Persistence and permanence is also performed to ensure the permanent coexistence of all the species. The theoretical outcomes are supported numerically by Matlab.

6.2 Statement of the model

We here divide the plant population into two classes in presence of infection that is susceptible plant 'x' and infected plant 'y' population. The disease is spread by the whitefly to a susceptible plant. It is assumed that only susceptible plant is capable of reproduction and the infected prey does not recover and is removed by death. Although the infected plant 'y' contributes with the susceptible plant 'x' to population growth towards the carrying capacity as both of them are still in the environment. Growth of plant population is assumed as logistic. Consumption of susceptible plant contributes positive growth while consumption of infected plants contribute negative growth to the whitefly. Biologically all parameters are assumed to be positive. The description of parameters are as follows:

r=Growth rate of susceptible plant.
k= Carrying capacity.
a= Force of infection.
b=Rate of attack of whitefly to the susceptible plant.
c=Rate of consumption of infected plant of whitefly.
m=Death rate of infected plant for all causes except the consumption of whitefly.
n=Natural death rate of whitefly.
Based on the above assumptions our model is formed as:

6.3 Model

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x+y}{k}) - axy - bxv \\ \frac{dy}{dt} &= axy - cyv - my \\ \frac{dv}{dt} &= bxv - cyv - nv \end{aligned} \tag{6.1}$$

with the initial conditions,
$x(0) = x_0 > 0, y(0) = y_0 > 0, v(0) = v_0 > 0$
Here x_0 is the initial susceptible plant population density, y_0 is the initial infected plant population density and v_0 is the initial whitefly density.
For mathematical simplicity we consider the following transformation,
$x = kX, y = kY, v = kV, t = \frac{\tau}{n}$.
Based on the above mentioned transformation the model (6.1) becomes,

$$\begin{aligned} \frac{dX}{d\tau} &= \alpha X(1 - X - Y) - \beta XY - \gamma XV \\ \frac{dY}{d\tau} &= \beta XY - \eta YV - \rho Y \\ \frac{dV}{d\tau} &= \gamma XV - \eta YV - V \end{aligned} \tag{6.2}$$

where $\alpha = \frac{r}{n}, \beta = \frac{ak}{n}, \gamma = \frac{bk}{n}, \eta = \frac{ck}{n}, \rho = \frac{m}{n}$.

6.3.1 Solution properties

lemma :

The solutions of (6.2) are positive.

Proof:
Since $x(0) = x_0 > 0, y(0) = y_0 > 0$ and $v(0) = v_0 > 0$,
we have $X(0) = X_0 > 0$, $Y(0) = Y_0 > 0$ and $V(0) = V_0 > 0$. Suppose $X(\tau)$ is not positive for all $\tau \geq 0$. Since $X_0 > 0$ then there exist τ_0 with $X(\tau_0) = 0$ and $X(\tau) > 0$ for $0 \leq \tau \leq \tau_0$. For $0 \leq \tau \leq \tau_0$
$\frac{\dot{X(\tau)}}{X(\tau)} = \alpha(1 - X - Y) - \beta Y - \gamma V > -\alpha X - \alpha Y - \beta Y - \gamma V$
$X(\tau_0) > X_0 exp[-\alpha\frac{\tau_0^2}{2} - \alpha\int_0^{\tau_0} Y d\tau - \beta\int_0^{\tau_0} Y d\tau - \gamma\int_0^{\tau_0} V d\tau] > 0$
This is a contradiction and hence $X(\tau)$ is positive for all $\tau \geq 0$. Similarly it can be shown that $Y(\tau)$, $V(\tau)$ are also positive for all $\tau \geq 0$.

6.3.2 Equilibria

The equilibrium points are obtained by setting $\frac{dX}{d\tau} = 0$, $\frac{dY}{d\tau} = 0$ and $\frac{dV}{d\tau} = 0$ and solving the equations
$\alpha X(1 - X - Y) - \beta XY - \gamma XV = 0$
$\beta XY - \eta YV - \rho Y = 0$
$\gamma XV - \eta YV - V = 0$
We have observed that there are five equilibrium points i.e. $E_0(0,0,0)$, $E_1(1,0,0)$,$E_2(X,Y,0)$, $E_3(X,0,V)$, $E_*(X^*,Y^*,V^*)$ which is the interior equilibrium. Where
$(X,Y,0) = (\frac{\rho}{\beta}, \frac{\alpha(\beta-\rho)}{\beta(\alpha+\beta)}, 0)$ exists if $\rho < \beta$
$(X,0,V) = (\frac{1}{\gamma}, 0, \frac{\alpha(\gamma-1)}{\gamma^2})$ exists if $\gamma > 1$
$(X^*,Y^*,V^*) = (\frac{\alpha+\beta+\alpha\eta+\gamma\rho}{\alpha\eta+\alpha\gamma+2\beta\gamma}, \frac{\gamma X^*-1}{\eta}, \frac{\beta X^*-\rho}{\eta})$ exists if $\rho < \beta, \gamma > 1$ and $max(\frac{\rho}{\beta}, \frac{1}{\gamma}) < X^*$

6.3.3 Dynamic behavior

The variational matrix is given by :

$$J(X,Y,V) = \begin{pmatrix} \alpha - 2\alpha X - \alpha Y - \beta Y - \gamma V & -\alpha X - \beta X & -\gamma X \\ \beta Y & \beta X - \eta V - \rho & -\eta Y \\ \gamma V & -\eta V & \gamma X - \eta Y - 1 \end{pmatrix}$$

From the variational matrix we show that $E_0(0,0,0)$ is always unstable as its eigen values are $\alpha, -\rho, -1$
The equilibrium $E_1(1,0,0)$ is locally asymptotically stable if $\rho > \beta, \gamma < 1$ as its eigen values are $-\alpha, \beta - \rho, \gamma - 1$. But in this case $E_2(X,Y,0)$, $E_3(X,0,V)$ does not exist.

$E_2(X,Y,0)$ is locally asymptotically stable if $\beta > \rho$ and $\gamma < \frac{\beta}{\rho} + \frac{\alpha\eta(\beta-\rho)}{\rho(\alpha+\beta)}$

$E_3(X,0,V)$ is locally asymptotically stable if $\rho\gamma^2 + \gamma(\eta\alpha - \beta) - \eta\alpha > 0$ and $\gamma > 1$

$$J(X^*,Y^*,V^*) = \begin{pmatrix} -\alpha X^* & -\alpha X^* - \beta X^* & -\gamma X^* \\ \beta Y^* & 0 & -\eta Y^* \\ \gamma V^* & -\eta V^* & 0 \end{pmatrix}$$

The characteristic equation is given by:

$$\begin{pmatrix} -\alpha X^* - \lambda & -\alpha X^* - \beta X^* & -\gamma X^* \\ \beta Y^* & -\lambda & -\eta Y^* \\ \gamma V^* & -\eta V^* & -\lambda \end{pmatrix} = 0$$

The characteristic equation for $E_*(X^*, Y^*, V^*)$ is a cubic equation which is as follows,

$\lambda^3 + \lambda^2 \alpha X^* + \lambda(\gamma^2 X^* V^* + \beta^2 X^* Y^* + \alpha\beta X^* Y^* - \eta^2 V^* Y^*) - \alpha\eta^2 X^* Y^* V^* - \alpha\eta\gamma X^* Y^* V^* - 2\beta\gamma\eta X^* Y^* V^* = 0$

Which can be written as

$\lambda^3 + A\lambda^2 + B\lambda + C = 0$
Where,
$A = \alpha X^* > 0$,$B = \gamma^2 X^* V^* + \beta^2 X^* Y^* + \alpha\beta X^* Y^* - \eta^2 V^* Y^*$, $C = -\alpha\eta^2 X^* Y^* V^* - \alpha\eta\gamma X^* Y^* V^* - 2\beta\gamma\eta X^* Y^* V^* < 0$ and $AB - C > 0$
But since $C < 0$ so $E_*(X^*, Y^*, V^*)$ is always unstable.

6.4 Persistence and permanence

Freedman and Waltman first implemented the idea of persistence and permanence. It decides the questions of survival and extinction of n-species whose growth equations are represented as (Konar et al., 1999) ,

$$\dot{x}_i = x_i f_i(x_1, x_2, ..., x_n) \quad (6.3)$$

6.4.1 Theorem 6.1

The system is permanent iff $\beta > \rho$, $\gamma > 1$, $\gamma > \frac{\beta}{\rho} + \frac{\alpha\eta(\beta-\rho)}{\rho(\alpha+\beta)}$. *and*
$\rho\gamma^2 + \gamma(\eta\alpha - \beta) - \eta\alpha < 0$

Proof:
The permanence of the system implies that none of the boundary fixed points are saturated. Hence the interior fixed point exists and must be saturated. Hence all the eigen values are negative or have negative real parts.

Firstly,we construct the average Lyapunov function to prove the sufficient condition. In our model, we consider the average Lyapunov function as $\sigma(X) = X^{r_1}.Y^{r_2}.V^{r_3}$ where $r_i > 0$ i=1,2,3.

Let,$\psi(X) = \frac{\dot{\sigma}(X)}{\sigma(X)}$
$= r_1\frac{\dot{X}}{X} + r_2\frac{\dot{Y}}{Y} + r_3\frac{\dot{V}}{V}$
$= r_1(\alpha(1 - X - Y) - \beta Y - \gamma V) + r_2(\beta X - \eta V - \rho)$
$+ r_3(\gamma X - \eta Y - 1)$
If $\psi(X) > 0$ then the trajectories more away from the boundary and the system (6.2) becomes permanent. It is evident that there is no periodic trajectory. Hence if there exist $r_i > 0$ such that $\Psi(E_1) > 0$, then (6.2) is permanent.

Therefore for $E_0(0,0,0)$, $\psi(X) = \alpha r_1 - \rho r_2 - r_3 > 0$

For $E_1(1,0,0)$, $\psi(X) = r_2(\beta - \rho) + r_3(\gamma - 1) > 0$

$E_1(\frac{\rho}{\beta}, \frac{\alpha(\beta-\rho)}{\beta(\alpha+\beta)}, 0)$, $\psi(X) = r_3(\frac{\gamma\rho}{\beta} - \frac{\alpha\eta(\beta-\rho)}{\beta(\alpha+\beta)} - 1) > 0$

$E_3(\frac{1}{\gamma}, 0, \frac{\alpha(\gamma-1)}{\gamma^2})$, $\psi(X) = r_2\frac{(-\rho\gamma^2-\gamma(\eta\alpha-\beta)+\eta\alpha)}{\gamma^2} > 0$

The inequalities are evidently satisfied for atleast one positive $r = (r_1, r_2, r_3)$ iff $\beta > \rho$, $\gamma > 1$, $\gamma > \frac{\beta}{\rho} + \frac{\alpha\eta(\beta-\rho)}{\rho(\alpha+\beta)}$ and $\rho\gamma^2 + \gamma(\eta\alpha - \beta) - \eta\alpha < 0$
Hence the system is uniformly persistent(or permanent).This completes the proof of the theorem.

6.5 The optimal control problem

We now introduce control theory (Sahoo et al., 2009 ;Pandey et al., 2012 ; Chowdhury, Basir, Takeuchi, Ghosh and Roy, 2019). to the model (6.2) to minimize the cost of insecticide spraying. Assuming that all the infected whiteflies of a particular region fall possibly under the control of insecticide spraying. We introduce the control parameter $\gamma(t)$ such that $0 \leq \gamma(t) \leq 1$. Now our reformulated model becomes:

$$\begin{aligned}
\frac{dX}{d\tau} &= \alpha X(1 - X - Y) - (1 - \gamma(t))(\beta XY + \gamma XV) \\
\frac{dY}{d\tau} &= (1 - \gamma(t))\beta XY - \eta YV - \rho Y \\
\frac{dV}{d\tau} &= (1 - \gamma(t))\gamma XV - \eta YV - V
\end{aligned} \tag{6.4}$$

If we consider $\gamma(t) = 0$ then there is no effect of control between the infected whiteflies and the plants.
If $\gamma(t) = 1$ then it disproves the interaction between them. Here $\gamma(t)$ is the control parameter defined on $[t_0, t_f]$ where t_0 and t_f are the starting and

ending time of control respectively. The objective functional for minimizing the cost of insecticide spraying is thus formed as :-

$J(u(t)) = \int t_0^{t_f} [P\gamma^2(t) + QY^2 - RX^2] d\tau$ Where $P \geq 0$ and $Q \geq 0$

Here the first term represents the costs of spraying insecticide and labor charge, second term represents the loss of crop due to infection which should be minimized and the last term implies the extra revenues obtained by the larger population of healthy Jatropha Curcas plants.
Now we are going to find the optimal control.

6.5.1 Theorem 6.2:

The objective functional $J(\gamma^*(t))$ is minimum for the optimal control γ^* corresponding to the interior equilibrium $E_*(X^*, V^*)$ and also there are adjoint variables ξ_1, ξ_2, ξ_3 satisfying the system of equations,

$$\begin{aligned}
\frac{d\xi_1}{dt} &= 2RX - \xi_1[\alpha - 2\alpha X - \alpha Y - (1-\gamma(t))(\beta Y + \gamma V)] - \xi_2[(1-\gamma(t))\beta Y - \xi_3(1-\gamma(t))\gamma V] \\
\frac{d\xi_2}{dt} &= -2QY - \xi_1[-\alpha XY - (1-\gamma(t))\beta X] - \xi_2[(1-\gamma(t))\beta X - \eta V - \rho] + \xi_3 \eta V \\
\frac{d\xi_3}{dt} &= -\xi_1[(1-\gamma(t))\gamma X] + \xi_2 \eta Y - \xi_3[\gamma X(1-\gamma(t)) - \eta Y - 1]
\end{aligned} \tag{6.5}$$

with the boundary condition $\xi_i(t_f) = 0$ (i=1,2,3). The optimal control can be given as,

$$\gamma^*(t) = max\{0, min\{1, \frac{\gamma XV(\xi_3 - \xi_1) + \beta XY(\xi_2 - \xi_1)}{2P}\}\} \tag{6.6}$$

Proof:
we first construct the Hamiltonian as follows:

$H = P\gamma^2(t) + QY^2 - RX^2 + \xi_1[\alpha X(1 - X - Y) - (1-\gamma(t))(\beta XY + \gamma XV] + \xi_2[(1-\gamma(t))\beta XY - \eta YV - \rho Y] + \xi_3[(1-\gamma(t))\gamma XV - \eta YV - V]$

Applying the Pontryagin Minimum Principle the optimal control variable $\gamma^*(t)$ satisfies

$\frac{\partial H}{\partial \gamma^*(t)} = 0$

Which implies

$2P\gamma(t) + \xi_1(\beta XY + \gamma XV) - \xi 2\beta XY - \xi_3 \gamma XV = 0$
$\gamma(t) = \frac{\gamma XV(\xi_3 - \xi_1) + \beta XY(\xi_2 - \xi_1)}{2P}$

For the boundedness of the optimal control we have

$$\gamma^*(t) = \begin{cases} 0 & \frac{\gamma XV(\xi_3-\xi_1)+\beta XY(\xi_2-\xi_1)}{2P} \leq 0 \\ \frac{\gamma XV(\xi_3-\xi_1)+\beta XY(\xi_2-\xi_1)}{2P} & 0 < \frac{\gamma XV(\xi_3-\xi_1)+\beta XY(\xi_2-\xi_1)}{2P} < 1 \\ 1 & \frac{\gamma XV(\xi_3-\xi_1)+\beta XY(\xi_2-\xi_1)}{2P} \geq 1 \end{cases}$$

According to Pontryagin Minimum Principle adjoint variables satisfy the following equations:

$$\frac{d\xi_i}{dt} = -\frac{\partial H}{\partial X_i} \tag{6.7}$$

where $i = 1, 2, 3$, $X_i = X, Y, V$.
i.e. $X_1 = X$, $X_2 = Y$,$X_3 = V$ and the equations (6.5) can be determined by using (6.7). This completes the proof of the theorem.

6.6 Numerical simulation and discussions

In this section, numerical simulations are performed to validate the theoretical results. Keeping in mind all the feasibility criteria the parameter values and the initial values of the healthy, infected plants and the whiteflies are chosen. Firstly we have assumed different values for the parameters of the system (6.1). Then the corresponding transformed parameter values are calculated for the system (6.2).
To plot the system (6.2) we have used Matlab ODE 23. Assuming that the initial values $I_1 = [0.030, 0.010, 0.015]$ we here observed that if $\alpha = 0.01$, $\beta = 15$, $\gamma = 20$, $\eta = 0.7$, $\rho = 0.7$ then all the populations become unstable. Also we have illustrated the analytical method using control theory for the qualitative analysis of the dynamical system to control the whitefly population. Numerically We have observed that β plays an important role for the dynamics of the system (6.2). As depending upon the values of β the system changes from unstable to stable condition.
The figure (6.2) represents the control effect on the dynamics of the system. Here we observe that due to effect of control the healthy plant population slowly increases from its initial value and then become stable. The infected plants as well as the infected whiteflies go to extinction very shortly. Hence spraying of insecticide plays a key role to make the system stable and maintaining the stability in the remaining portion of time span.

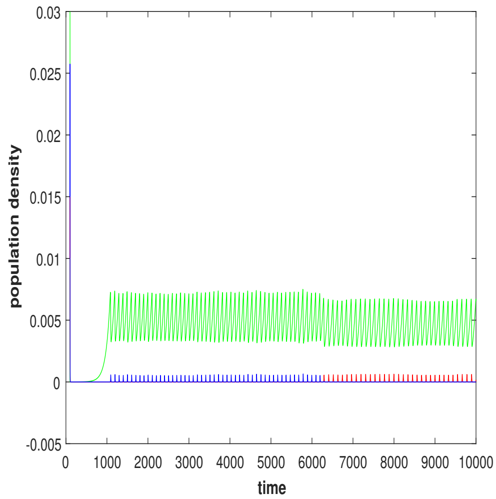

FIGURE 6.1: Variation of plant-herbivore densities for $\alpha = 0.01$, $\beta = 15$, $\gamma = 20$, $\eta = 0.7$, $\rho = 0.7$. Here we observe unstable condition for all the populations with increasing time.

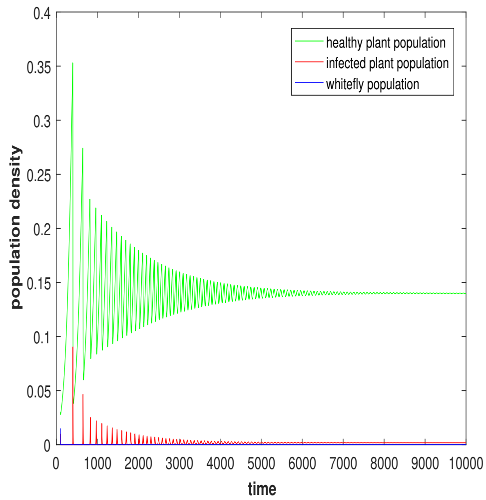

FIGURE 6.2: Variation of plant-herbivore densities for the system with $\alpha = 0.01$, $\beta = 5$, $\gamma = 2$, $\eta = 0.7$, $\rho = 0.7$, $u = 0.00001$ .This shows the stable condition in the presence of control.

6.7 Conclusions

Here we have proposed a mathematical model to study the dynamics of mosaic disease of Jatropha Curcas plant which is spread by the vector whitefly. We here taken healthy as well as infected Jatropha curcas plant and the

infected whitefly into consideration. We observed that the system possesses unstable condition for some parameter values. By spraying the insecticide the unstable nature of the system can be stabilized that biologically reflects the stable state of the healthy Jatropha plants. The numerical results shows the stability of healthy Jatropha curcas plants and extinction of both the infected Jatropha curcas plant as well as the infected vector in the presence of control. Hence we can apply insecticide on the host plant which would be liberally developed ourselves for advanced socio-economical insight in the upcoming days.

7 Discretization of a mathematical model regarding Jatropha Curcas plant and whitefly interaction with aggregated attack pattern of whitefly

[Chapter based on the paper submitted in Journal Ganita, Bharata Ganita parisad]

7.1 Introduction

In the present chapter we consider a mathematical model of the growth of Jatropha curcas plant population under aggregated attack pattern of whitefly, the vector of mosaic disease.Assuming that the distribution of whiteflies on plants(Sarkar and Roy , 1989) follows negative-binomial distribution, we studied the dynamics of the model analytically. Also we have discretized the continuous-time model by Mickens non-standard finite difference (NSFD) scheme (R.E.Mickens,1989) as well as standard Euler forward scheme, Since it gives better approximation of the solution as well as the dynamics of the disease (R.E.Mickens,2010). We have compared all the cases (M.Biswas N.Bairagi ,2017) and obtained global stability in the cases of continuous time model and Euler model. But NSFD model results unstable condition. Persistence and permanence of the model are also discussed. Numerical simulations are also performed to validate the theoretical results.

7.2 Statement of the model

In this model we assume that v whiteflies are distributed on x plants by following negative binomial distribution. It is considered that there are 1,2,.....,i herbivores per plant and some of the plants are herbivore-free.
Therefore we have,

$$\sum_{i=1}^{x} i = v$$

The loss rate of plants by i herbivores is $aiP(i)x$
where,
a= intrinsic plant loss-rate per herbivore.
P(i)= Proportion of plants with i herbivores.
Therefore the total plant loss rate with every possible number of herbivores is

$$ax\sum_{i=0}^{\infty} iP(i)$$

where $\sum iP(i)$ is the mean number of herbivores per plant. For negative-binomial distribution the expectation is $E(i) = \frac{v}{x}$
The loss-rate due to whitefly consumption is av. The loss of whiteflies occur in the following ways.
e=natural mortality of whitefly.
b=natural mortality of the host plant.
a=by their killing the host plant.
The self-induced mortality occurs at a rate $ai^2P(i)x$. Hence for the whole plant population it is

$$ax\sum_{i=0}^{\infty} i^2P(i)$$

where $\sum i^2P(i)$ is the expectation of i^2.

In this model we have chosen the negative binomial distribution which ecologically reflects aggregated attack pattern of whitefly. Here whitefly-inflicted losses through the plant death are $axE(i^2)$ where $E(i^2) = \frac{v}{x} + \frac{(k+1)}{k}\frac{v^2}{x^2}$ for negative binomial distribution.
We have chosen here logistic growth for the plant population and the attack pattern of whitefly on the plant is taken as holling type-I function. 'r'is the growth rate of the plant, 'k_1'is the carrying capacity. 'k'is the negative binomial parameter. Based on the above assumptions the continuous-time system takes the form,

7.3 Model

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k_1}) - axv \\
\frac{dv}{dt} &= v[ax - (e+b+a) - a\frac{(k+1)}{k}\frac{v}{x})]
\end{aligned} \tag{7.1}$$

with the initial conditions,
$x(0) = x_0 > 0, v(0) = v_0 > 0$
Here x_0 is the initial plant population density and v_0 is the initial whitefly density.
For mathematical simplicity we consider the following transformation,
$x = k_1X, v = k_1v, t = \frac{\tau}{r}$.
Based on the above mentioned transformation the model becomes,

$$\begin{aligned}
\frac{dX}{d\tau} &= X(1-X) - \alpha XV \\
\frac{dV}{d\tau} &= V[\alpha X - \beta - \gamma\frac{V}{X}]
\end{aligned} \tag{7.2}$$

where $\alpha = \frac{ak_1}{r}, \beta = \frac{a+b+e}{r}, \gamma = \frac{a}{r}\frac{(k+1)}{k}$.

7.3.1 Solution properties

lemma 7.1 :

The solutions of (7.2) are positive.

Proof:
Since $x(0) = x_0 > 0$ and $v(0) = v_0 > 0$, we have $X(0) = X_0 > 0$ and $V(0) = V_0 > 0$. Suppose $X(\tau)$ is not positive for all $\tau \geq 0$. Since $X_0 > 0$ then there exist τ_0 with $X(\tau_0) = 0$ and $X(\tau) > 0$ for $0 \leq \tau \leq \tau_0$. For $0 \leq \tau \leq \tau_0$

$\frac{\dot{X(\tau)}}{X(\tau)} = 1 - X - \alpha V > -X - \alpha V$
$X(\tau_0) > X_0 exp[-\frac{\tau_0^2}{2} - \alpha \int_0^{\tau_0} V d\tau] > 0$
This is a contradiction and hence $X(\tau)$ is positive for all $\tau \geq 0$. Similarly it can be shown that $V(\tau)$ is also positive for all $\tau \geq 0$.

7.3.2 Equilibria

The equilibrium points are obtained by setting $\frac{dX}{d\tau} = 0$ and $\frac{dV}{d\tau} = 0$ and solving the equations

$X(1 - X) - \alpha XV = 0$ and
$V(\alpha X - \beta - \gamma \frac{V}{X}) = 0$.
We have observed that the continuous time system has two equilibrium points i.e. $E_1(1, 0)$ which is the whitefly free equilibrium, $E_2(X^*, V^*)$ which is the interior equilibrium. From the two equations we obtain
$X^* = \frac{(\alpha\beta - \gamma) \pm \sqrt{(\alpha\beta - \gamma)^2 + 4\alpha^2\gamma}}{2\alpha^2}$
$V^* = \frac{1 - X^*}{\alpha}$
Therefore $E_2(X^*, V^*)$ always exists.

7.3.3 Dynamic behavior

From the variational matrix we have found the dynamics of the equilibrium points of the system. The equilibrium $E_1(1, 0)$ is saddle as its eigen values are -1 and $\alpha - \beta$ and $\alpha > \beta$.
The characteristic equation for $E_2(X^*, V^*)$ is a quadratic equation which is as follows,
$\lambda^2 + \lambda(X^* + \gamma \frac{V^*}{X^*}) + \gamma V^* + \alpha^2 X^* V^* + \alpha\gamma \frac{V^{*2}}{X^*} = 0$
Which can be written as
$\lambda^2 + A\lambda + B = 0$
Where,
$A = X^* + \gamma \frac{V^*}{X^*} > 0$
and
$B = \gamma V^* + \alpha^2 X^* V^* + \alpha\gamma \frac{V^{*2}}{X^*} > 0$
Therefore $E_2(X^*, V^*)$ is locally asymptotically stable.

7.4 Global Stability

lemma 7.2 :

The XV subsystem is globally asymptotically stable.

Proof: $H_1(X, V) = \frac{1}{XV}$ then $H_1 > 0$ if $X > 0, V > 0$.
$h_1(X, V) = X(1 - X) - \alpha XV$
$h_2(X, V) = V[\alpha X - \beta - \gamma \frac{V}{X}]$
Therefore $\nabla(X, V) = \frac{\partial(h_1 H_1)}{\partial X} + \frac{\partial(h_2 H_1)}{\partial V}$

$=\frac{\partial(\frac{1-X}{V}-\alpha)}{\partial X}+\frac{\partial(\alpha-\frac{\beta}{X}-\frac{\gamma V}{X^2})}{\partial V}$
$=-\frac{1}{V}-\frac{\gamma}{X^2}$
Hence by Bendixon-Dulac criteria $E_2(X^*, V^*)$ is globally asymptotically stable in the positive XV plane. This completes the proof of the lemma.

7.5 Euler's discrete time system

Using Euler's forward scheme the continuous time system can be discretized as (M.Biswas N.Bairagi, 2017),

$$\begin{aligned} X_{n+1} &= X_n + h[X_n(1-X_n) - \alpha X_n V_n] \\ V_{n+1} &= V_n + h[\alpha X_n V_n - \beta V_n - \gamma \frac{{V_n}^2}{X_n}] \end{aligned} \tag{7.3}$$

Where $h > 0$ is the step-size. Since there is many negative signs in the R.H.S. of (7.3) then the solutions for all step-size h may not be positive although the initial values are assumed to be positive. So there is a possibility of numerical instability.

7.5.1 Equilibria

The equilibrium points are obtained by setting
$X_{n+1} = X_n = X, V_{n+1} = V_n = V$ in (7.3)
Then (7.3) becomes

$$\begin{aligned} X &= X + h[X(1-X) - \alpha XV] \\ V &= V + h[\alpha XV - \beta V - \gamma \frac{V^2}{X}] \end{aligned} \tag{7.4}$$

We have observed that the model (7.3) has the same equilibrium points with that of the continuous time system i.e. $E_1(1,0)$ which is the whitefly free equilibrium and $E_2(X^*, V^*)$ which is the interior equilibrium. From the two equations we obtain
$X^* = \frac{(\alpha\beta-\gamma)\pm\sqrt{(\alpha\beta-\gamma)^2+4\alpha^2\gamma}}{2\alpha^2}$
$V^* = \frac{1-X^*}{\alpha}$
Therefore $E_2(X^*, V^*)$ always exists.

7.5.2 Dynamic behavior

The variational matrix is given by

$$J(X,V) = \begin{pmatrix} 1+h[1-2X-\alpha V] & -h\alpha X \\ h\alpha V + h\gamma\frac{V^2}{X^2} & 1+h[\alpha X-\beta-2\gamma\frac{V}{X}] \end{pmatrix}$$

$$J(1,0) = \begin{pmatrix} 1-h & -h\alpha \\ 0 & 1+h\alpha-h\beta] \end{pmatrix}$$

$E_1(1,0)$ is unstable if $h < 1$ otherwise saddle, since $\alpha > \beta$

$$J(X^*,V^*) = \begin{pmatrix} 1-hX & -h\alpha X \\ h\alpha V + h\gamma\frac{V^2}{X^2} & 1-h\gamma\frac{V}{X}] \end{pmatrix}$$

Now the characteristic equation can be written as:
$P(\lambda) = \lambda^2 + A\lambda + B = 0$
Where,
$A = -2 + hX^* + h\gamma\frac{V^*}{X^*}$
and
$B = 1 - hX^* - h\gamma\frac{V^*}{X^*} + h^2\gamma V^* + h^2\alpha^2 X^* V^* + h^2\alpha\gamma\frac{V^{*2}}{X^*}$
After some algebraic manipulation we have,

$$P(1) = 1 + A + B = h^2\gamma V^* + h^2\alpha^2 X^* V^* + h^2\alpha\gamma\frac{V^{*2}}{X^*} > 0$$

It is observed that P(1) is always positive. Therefore (7.3) is always stable around $E_2(X^*,V^*)$ for all parameter values So it possesses dynamical consistency with its continuous counter-part (R E. Mickens,1989) .

7.6 Global Stability

lemma 7.3 :

The system (7.3) is globally asymptotically stable irrespective of step-size h.

Proof: $H_2(X,V) = \frac{1}{X_n V_n}$ then $H_2 > 0$ if $X > 0$, $V > 0$.

$h_1(X,V) = X_n + h[X_n(1-X_n) - \alpha X_n V_n]$

$h_2(X,V) = V_n + h[V_n\alpha X_n - V_n\beta - \gamma\frac{V_n^2}{X_n}]$

Therefore $\nabla(X,V) = \frac{\partial(h_1 H_2)}{\partial X} + \frac{\partial(h_2 H_2)}{\partial V}$
$= -\frac{h}{V_n} - \frac{h\gamma}{X_n^2}$
Hence by Bendixon-Dulac criteria $E_2(X^*,V^*)$ is globally asymptotically stable in the positive XV plane. This completes the proof of the lemma.

7.7 Non-standard finite difference (NSFD) scheme

Here we have used NSFD scheme (M.Biswas N. Bairagi ,2017) to discretize the continuous time model.
For convenience we first write the continuous time model.

$$\begin{aligned}\frac{dX}{d\tau} &= X - X^2 - \alpha XV \\ \frac{dV}{d\tau} &= \alpha XV - \beta V - \gamma\frac{V^2}{X}]\end{aligned} \tag{7.5}$$

We now apply the following non-local approximations term-wise.

$\frac{dX}{d\tau} \to \frac{X_{n+1}-X_n}{\phi_1(h)}$, $\frac{dV}{d\tau} \to \frac{V_{n+1}-V_n}{\phi_2(h)}$,
$X \to X_n$, $XV \to X_nV_n$,
$X^2 \to X_nX_{n+1}$, $V \to V_{n+1}$,
$XV \to X_{n+1}V_n$, $\frac{V^2}{X} \to \frac{V_nV_{n+1}}{X_n}$

where $h(>)0$ is the step-size. We have chosen here $\phi_1(h) = h$, $\phi_2(h) = \frac{1-e^{-\beta h}}{\beta}$

where $\phi_1(h), \phi_2(h) > 0$ for all $h > 0$
Based on the above transformation (7.5) becomes:

$$\begin{aligned}\frac{X_{n+1} - X_n}{\phi_1(h)} &= X_n - X_nX_{n+1} - \alpha X_{n+1}V_n \\ \frac{V_{n+1} - V_n}{\phi_2(h)} &= \alpha X_nV_n - \beta V_{n+1} - \gamma\frac{V_nV_{n+1}}{X_n}\end{aligned} \tag{7.6}$$

From (7.6)

$$\begin{aligned}X_{n+1} &= \frac{X_n(1+\phi_1(h))}{1+\phi_1(h)X_n + \alpha\phi_1(h)V_n} \\ V_{n+1} &= \frac{V_n(1+\alpha\phi_2(h)X_n)}{1+\beta\phi_2(h)+\gamma\frac{V_n\phi_2(h)}{X_n}}\end{aligned} \tag{7.7}$$

The R.H.S. of (7.7) is always positive for all step-size h.

7.7.1 Equilibria

The fixed points are obtained by putting $X_{n+1} = X_n = X, V_{n+1} = V_n = V$ in (7.6)
Clearly the equilibrium points for the NSFD scheme are same as that of the continuous-time system.

7.7.2 Dynamic behavior

The variational matrix is given by,

$J(X, V) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
Where
$a_{11} = \frac{1+\alpha\phi_1(h)V_n+\phi_1(h)+\alpha\phi_1^2(h)V_n}{(1+\phi_1(h)X_n+\alpha\phi_1(h)V_n)^2}$

$a_{12} = \frac{-X_n(1+\phi_1(h))\alpha\phi_1(h)}{(1+\phi_1(h)X_n+\alpha\phi_1(h)V_n)^2}$

$a_{21} = \frac{\alpha\phi_2(h)V_n+\alpha\beta\phi_2^2(h)V_n+\alpha\gamma\phi_2^2(h)\frac{V_n^2}{X_n}+\gamma\phi_2^2(h)\frac{V_n^2}{X_n^2}}{(1+\beta\phi_2(h)+\phi_2(h)\gamma\frac{V_n}{X_n})^2}$

$a_{22} = \frac{1+\phi_2(h)\beta+\alpha\phi_2(h)X_n+\alpha\beta\phi_2^2(h)X_n}{(1+\phi_2(h)\beta+\phi_2(h)\gamma\frac{V_n}{X_n})^2}$

$J(1, 0) = \begin{pmatrix} \frac{1}{1+\phi_1(h)} & \frac{-\alpha\phi_1(h)}{1+\phi_1(h)} \\ 0 & \frac{1+\alpha\phi_2(h)}{1+\beta\phi_2(h)} \end{pmatrix}$

The corresponding eigen values are $\lambda_1 = \frac{1}{1+\phi_1(h)}$ and $\lambda_2 = \frac{1+\alpha\phi_2(h)}{1+\beta\phi_2(h)}$. It is clear that $|\lambda_1| < 1$ if $\phi_1(h) > 0$ which is obvious and $|\lambda_2| < 1$ if $\beta > \alpha$ but it is not possible according to our model. Therefore $E_1(1, 0)$ is saddle.

Characteristic equation for $E_2(X^*, V^*)$ is $P(\lambda) = \lambda^2 + A\lambda + B = 0$
Where,
$A = -a_{11} - a_{22} < 0$
and
$B = a_{11}a_{22} - a_{12}a_{21} > 0$
Thus (7.7) is always unstable.

7.8 Persistence and Permanence of the system

The idea of persistence was first came to the light by Freedman and Waltman. From the biological point of view persistence means that all the populations are present and none of them will become extinct. Persistence and permanence are very useful to decide the questions of survival and extinction of n-species whose growth equations are governed by the differential equations

$$\dot{x}_i = x_i f_i(x_1, x_2, ..., x_n) \tag{7.8}$$

With the concept of saturated equilibria and by the method of average Lyapunov function we have the following theorem for permanent coexistence of both the species of the system (Konar et al., 1999).

7.8.1 Theorem 7.1

The system is permanent.

Proof:
The index theorem states that the system with dissipativeness assumption has atleast one saturated equilibrium. If all these saturated equilibria are regular, then the sum of their indices is +1. From the theorem the system is dissipative and so there exists atleast one saturated equilibrium and the sum of their indices is +1 if they are regular. The permanence of the system implies that none of the boundary fixed points are saturated. Hence the interior fixed point exists and must be saturated. Hence all the eigen values are negative or have negative real parts.

We now construct the average Lyapunov function to prove the sufficient condition. In our model, we consider the average Lyapunov function as $\sigma(X) = X^{r_1}.V^{r_2}$ where $r_i > 0$ i=1,2.

$$\text{Let,}\psi(X) = \frac{\dot{\sigma}(X)}{\sigma(X)} = r_1\frac{\dot{X}}{X} + r_2\frac{\dot{V}}{V} = r_1(1 - X - \alpha V) + r_2(\alpha X - \beta - \gamma\frac{V}{X})$$

If $\psi(X) > 0$ for the ω -limit sets of trajectories initiated in $\mathbb{R}^3_+$, then the trajectories more away from the boundary and the system (7.2) is permanent. It is evident that there is no periodic trajectory. Hence if there exist $r_1 > 0$ such that $\Psi(E_1) > 0$, then (7.2) is permanent.

$$\text{For } E_1(1,0),\ \psi(X) = (\alpha - \beta)r_2 > 0$$

The inequalities are evidently satisfied for atleast one positive $r = (r_1, r_2)$. Hence the system is uniformly persistent(or permanent).This completes the proof of the theorem.

7.9 Numerical simulation and discussions

To verify the theoretical results numerical simulations are performed using MATLAB-2016a. Here we have used MATLAB routine ODE23. Distinctive permissible estimations of the system parameters have been taken to ensure our theoritical results for continuous -time model.We here shown that if $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$ then the continuous-time system is locally as well as globally stable.

For Euler's discrete time system , taking the same parameter values as of continuous time system and taking the step-size $h = 0.01$ we have seen that the plant population declines and the whitefly population becomes stable. But if we keep all the parameter values fixed and increase the value of the step-size h by taking $h = 1$ then both the population becomes stable.

Again for NSFD scheme, keeping all parameter values same and taking $h = 1$

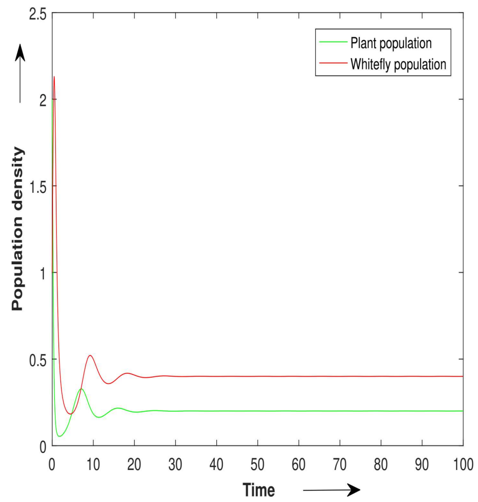

FIGURE 7.1: Variation of plant-herbivore densities with continuous time for $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$. Here we observe local stability for the population with increasing time.

we have seen that the plant population declines and whitefly population increases. But if we decrease h and take $h = 0.01$ then the plant population increases exponentially and the whitefly population is less than that of the plant population. If we take $h = 0.0001$ then the whitefly population declines and the plant population still grows exponentially.
So h is a key factor for the dynamics.

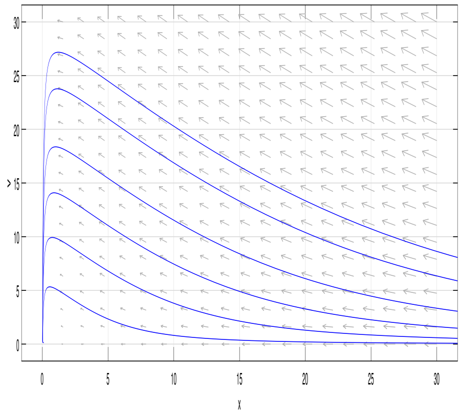

FIGURE 7.2: Variation of plant-herbivore densities for continuous time system $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$ .This shows the phase portrait in the XV plane which is globally asymptotically stable state of the model.

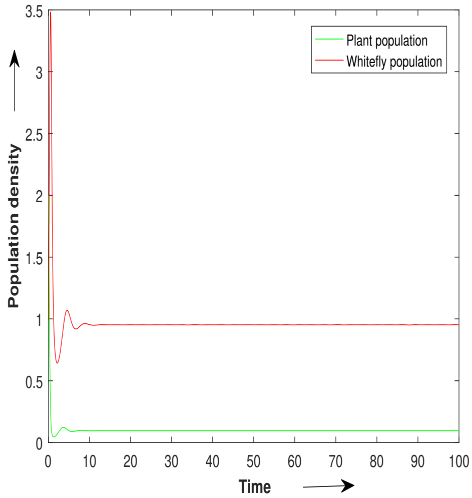

FIGURE 7.3: Variation of plant whitefly densities with Euler's discrete time model with $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$, $h = 1$.

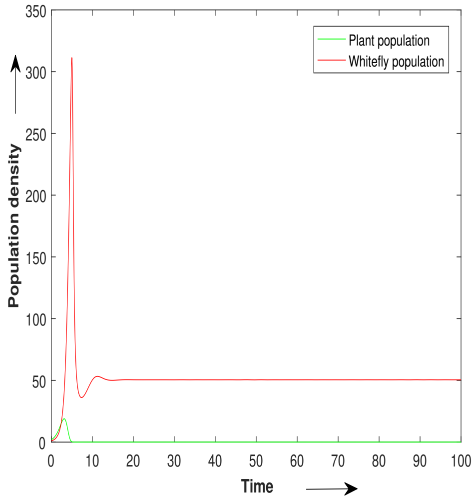

FIGURE 7.4: Variation of plant whitefly densities with Euler's discrete time model with $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$, $h = 0.01$.

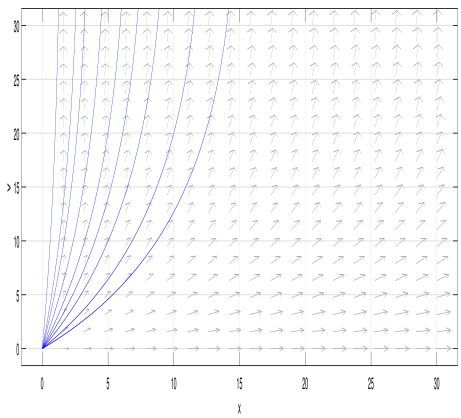

FIGURE 7.5: Phase portrait of Euler's discrete time system for $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$, $h = 1$.

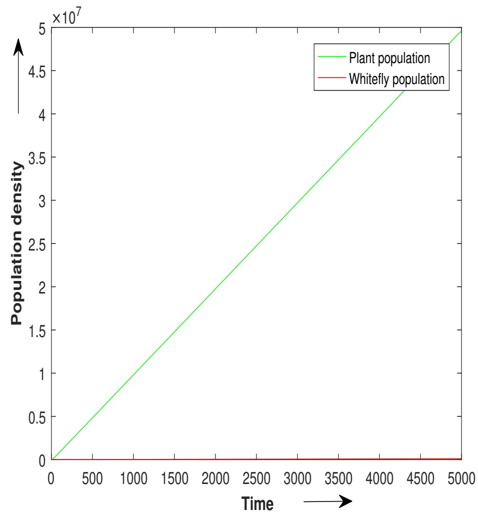

FIGURE 7.6: Variation of plant-whitefly densities with NSFD scheme for $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$, $h = 0.0001$ of the model .

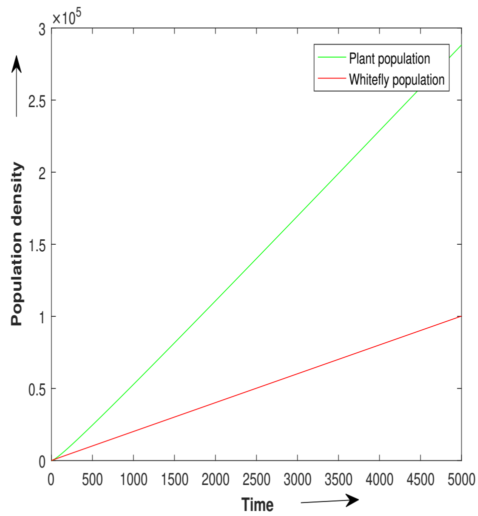

FIGURE 7.7: Plant-Whitefly densities for NSFD scheme for $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$, $h = 0.01$.

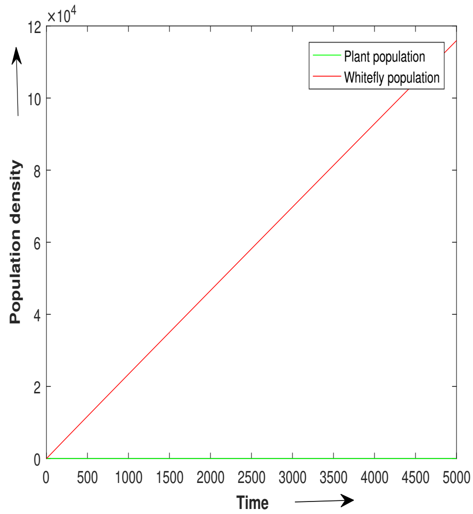

FIGURE 7.8: Plant-Whitefly densities for NSFD scheme for $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$, $h = 1$.

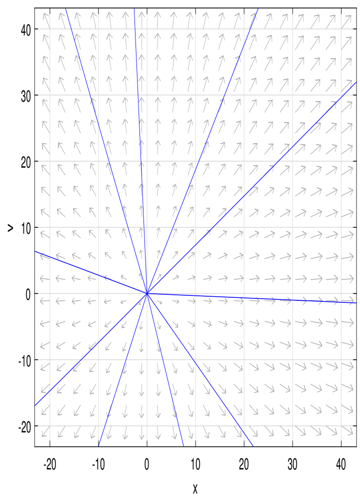

FIGURE 7.9: Phase portrait of Plant-Whitefly densities for NSFD scheme for $\alpha = 2$, $\beta = 0.2$, $\gamma = 0.1$, $h = 0.0001$.

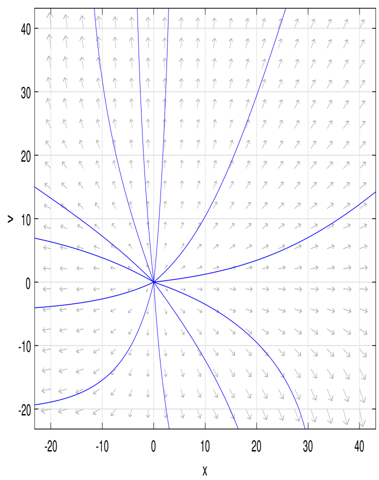

FIGURE 7.10: Phase portrait of Plant-Whitefly densities for NSFD scheme for $\alpha = 0.75$, $\beta = 2$, $\gamma = 0.2$, $h = 0.01$.

7.10 Conclusions

This paper is divided into three sections such as continuous time system, Euler discrete time system and Non-standard finite difference scheme to show the effect of discretization on the continuous time system and to make a comparison among them. We have seen that although the equilibrium points are same for all the cases , the dynamics are not same for all the cases. The dynamics is mainly depends on the step size h. For continuous time model the system is locally as well as globally stable. For Euler's system we have seen the same phenomena. But NSFD scheme results unstable condition.Persistence and permanence is also performed to verify the permanent coexistence of all the species.

Conclusion and future work

We will discuss here about the results obtained from different chapters of this thesis. In this thesis we have seven chapters including the Introduction.

1) In the first chapter we discussed about some topics which we have used in the next chapters. Also the outline of the thesis and scope are also discussed here.

2) In the second chapter two different growth functions namely logistic and exponential are compared taking the attack function of whitefly as Holling type-I function with random attack pattern of whitefly. Persistence and permanence of the system is also discussed here. Comparing these two we can conclude that growth function plays an important role to the mosaic disease dynamics.

3) We have taken in the third chapter all the same as second chapter except the attack function of whitefly as Holling type-II function and made a comparison study between them. Persistence and permanence of the system is also verified.

4) In the fourth chapter we have taken the logistic growth of the Jatropha Curcas plant with attack function of whitefly as Holling type-II function. We here observed that the system is uniformly persistent or permanent. We here also introduced control theory and observed that by spraying the insecticide the effect of mosaic disease can be reduced.

5) In the fifth chapter we have used different probability distribution function namely Poisson, Negative-binomial and Binomial distribution to express the random, aggregated and regular attack pattern of whitefly. It is revealed that different attack pattern gives different disease dynamics. From the present study it is observed that the application of control will help to minimize the application of spraying insecticide as well as the cost for the marginal farmers in the real system.

6) In the sixth chapter we have considered healthy as well as infected Jatropha Curcas plant and infected whitefly population which results unstable condition of the system but with the effect of control the system can be stabilised.

7) In the seventh chapter we have compared the continuous and discrete time system. We have introduced here Euler's discrete time system and Mickens Non-standard finite difference scheme. We can conclude that discrete time system gives more accurate results than the continuous counter part and the system is uniformly persistent or permanent.

All the results of chapter 2-7 are numerically verified and the supporting pictures using Matlab are provided.

In future we will try to extend these research works using time delay and also suitable modifications.

Bibliography

Ali N, Chakravarty S. , 'Stability analysis of a food chain model consisting of two competitive preys and one predator.' *Nonlinear dynamics* **82**(3): 1303-1306 (2015)

Basir F.A., Venturino E. , Roy P.K. 'Effects of awareness program for controlling mosaic disease in Jatropha curcas plantations'. *Mathematical methods in the Applied sciences* **40**(7) : 2441-2453 (2016) .

Cheng K. , H Su S, Lin S 'Some results on global stability of a predator- prey system'. *Journal of Mathematical Biology* **12**(1): 115-126 (1982).

Chowdhury Jahangir , Basir F.A. , Takeuchi Yasuhiro , Ghosh Mini , Roy P.K. 'A mathematical model for pest management in Jatropha curcas with integrated pesticides- An optimal control approach'.*Ecological complexity* **37** : 24-31 (2019).

Das Dhiraj kumar and Das Kunal and Kar Tapan. 'Dynamical Behaviour of infected predator-prey Eco-epidemics with Harvesting effort. *International journal of Applied and Computational Mathematics***7** (2022).

Desharnais R. 'Population Dynamics and Laboratory Ecology'. *Academic Press* (2005).

Freedman H. 'Stability analysis of a predator-prey system with mutual interference and density-dependent death rates'.*Bulletin of Mathematical Biology*.**41**(1): 67-78 (1979). Gao S.Q. , Qu J., Chua N.H., Ye J. 'A new strain of Indian cassava mosaic virus causes a mosaic disease in the biodiesel crop Jatropha curcas'. *Archives of virology* **155** : 607-612 (2010).

Gazi Huda Nurul, Das Kalyan. 'Control of parameters of a delayed diffusive autotroph herbivore system.'*Journal of Biological systems***18**(2) : 509-529 (2010).

Guin L.N. 'Existence of spatial patterns in a predator-prey model with self- and cross-diffusion'. *Applied Mathematics and Computation* **226** : 320-335 (2014).

Guin L.N. 'Spatial patterns through Turing instability in a reaction-diffusion predator-prey model'. *Mathematics and Computers in simulation* **109** : 174-185 (2015).

Guin L.N. , Mondal Benukar , chakravarty santabrata 'Existence of spatiotemporal patterns in the reaction-diffusion predator-prey model incorporating prey refuge'. *International Journal of Biomathematics* **9**(6) : pp. 1650085(1-25) (2016).

Haque M., Venturino E, 'An Ecoepidemiological model with disease in predator, the ratio dependent case'. *Mathematical Methods in the Applied sciences***30**(14) : 1791-1809 (2007)

Hassel M , Varley G. 'New Inductive population model for insect parasites and its bearing on biological control'.*Nature***223(5211)**: 1133-1137 (1969)

Holling C. 'The functional response of predators to prey density and its role in mimicry and population regulation. *Memoirs of the Entomological Society of Canada***97(45)** : 1-60 (1965).

Holt J.Jeger M.J., Thresh J.M., Otim-Nape G.W. 'An epidemiological model incorporating vector population dynamics applied to African cassava mosaic virus disease'. *Appl Ecol* **34**(3) : 793-806 (1997).

Hsu SB, Huang TW 'Global stability for a class of predator-prey systems'. *SIAM Journal on Applied Mathematics***55**(3) :763-783 (1995).

Huang J, Gong Y, Ruan S 'Bifurcation analysis in a predator-prey model with constant-yield predator harvesting'. *Discrete and continuous Dynamical Systems-Series B***18**(8) : 2101-2121 (2013).

Jana Soovoojeet and Guria Srabani and Das Uttam and Kar Tapan and Ghorai Abhijit. ' Effect of harvesting and infection on predator in a prey - predator system. *Nonlinear dynamics***81** (2015).

Jana Soovoojeet Khatua Anupam Mandal Manotosh Kar Tapan. 'Dynamics of a prey-predator type ecological model in relevance to pest control'. *International Journal of Modelling and Simulation***42** : 1-14 (2022).

Kar Tapan and Nandi Swapan and Jana Soovoojeet and Mandal Manotosh. 'Stability and bifurcation analysis of an epidemic model with the effect of media'.*Chaos, Solitons & Fractals***120** : 188-199 (2019).

Khan M.A., Ali K. Bonyah, E.et al. 'Mathematical modeling and stability analysis of pine wilt Disease with optimal control'. *Scientific Reports.***7** : 3115 (2017).

Khan Muhammad and Ahmed L and Mandal Prashanta and Smith Robert and Haque Maniul. 'Modelling the dynamics of pine wilt Disease with Asymptomatic carriers and optimal control'.*Reports.***10** :11412 (2020).

Khatua Anupam and Kar Tapan. 'Dynamical behavior and control strategy of a dengue epidemic model'. *The European physical Journal Plus***135** (2020).

Konar Debajyoti and Das Kalyan and Sarkar A.K... 'Global stability and persistence of a three species donar-controlled model with extensive omnivory'.*The Bulletin of the Hong Kong Mathematical Society* **2** (1999).

May R.M. 'Stability and complexity in model ecosystems'. *Princeton University Press* (2001).

M.Biswas and N. Bairagi. 'Discretization of an eco-epidemiological model and its dynamic consistency'. *Journal of Difference Equations and Applications* **1**: 23:5,860-877 (2017).

Mondal Prasanta Kumar , Jana Soovoojeet and Kar T. K. 'A Theoretical Approach On Controlling Agricultural Pest By Biological Controls'. *Acta Biotheoretica* **62**(1):47-67 (2013).

Mondal Prasanta and Jana Soovoojeet and Guria Srabani and Kar Tapan.' A Mathematical approach of a pest control management policy through viral infection 3504 and natural enemy.*inbook* (2014).

Murray JD. 'Mathematical Biology'.*Springer-Verlag, Berlin* (1993).

Narayana D.S.A., Shankarappa K.S., Govindappa M.R., Prameela H.A., Rao M.R.G., Rangaswamy K.T.. 'Natural occurence of Jatropha mosaic virus disease in India'. *Current Science***91**(5):584-586 (2006).

Pal, Pallav and Sarwardi, Sahabuddin and Saha , Tapan and Mandal, Prashanta. 'Mean Square Stability in a modified Leslie-Gower and Holling- type II predator-prey model'.*Journal of Applied Mathematics and Informatics***29**:781-802 (2011) .

Pal, Pallav and Mandal , Prashanta .'Dynamics of a predator-prey model with disease in the predator'.*Mathematical Methods in the Applied Sciences***10**:2988 (2014) .

Pandey V.C., Singh K., Singh J.S., Kumar A., Singh B., Singh R.P. 'Jatropha curcas: A potential biofuel plant for sustainable environmental development'. *Renew sustain energy* Rev**16** : 2870-2883 (2012).

Raj, S K Snehi, Dr. Sunil Kumar Kumar, Susheel Khan, M.S. Pathre, U. 'First molecular identification of a distinct begomovirus causing mosaic disease on Jatropha curcas L. in India which is closely related to Indian cassava mosaic virus'. *Australasian Plant Disease Notes* **3**: 69-72 (2008).

R.E. Mickens , 'Exact solutions to a finite-difference model of a non-linear reaction-advection equation:Implications for numerical analysis'.*Numerical Methods for Partial Differential Equations***54** :313-325 (1989).

R.E. Mickens, 'A SIR-model with square-root dynamics: An NSFD scheme'. *Journal of Difference Equations and Applications***16**2-3 :209-216 (2010).

Roy P.K., Li XZ, Al Basir F., Datta A., Chowdhury J. 'Effect of insecticide spraying on Jatropha curcas plant to control mosaic virus: a mathematical study'. Commun. *Math Biol Neurosci* **36** : 1-21 (2015).

Sahoo N.K., Kumar A., Sharma S., Naik S.N. 'Interaction of Jatropha Curcas plantation with ecosystem'. Proceedings of international conference on *Energy and environment*, chandigarh, India : 19-21 (2009).

Sawardi, Sahabuddin and Haque, Mainul and Mandal, Prashanta. 'Persistence and global stability of Bazykin predator-prey model with Bedding - De Angelis response function'.*Communications in Non-linear Science and Numerical simulation***19** : 189-209 (2014).

Sarkar A.K., and Roy A.B.. 'Role of herbivore attack pattern in growth of plant populations'.*Ecol. Modelling* .**45** : 307-316 (1989) .

Venturino, E., Roy, P.K., Al Basir, F. et al. 'A model for the control of the mosaic virus disease in Jatropha curcas plantations'. *Energ. Ecol. Environ*. **1**6 : 360-369.(2016).

Ye Meng, Li Caiyan, Francis George, P.S Makkar Harinder. Current situatio and Prospects of Jatropha Curcas as a multipurpose tree in China. *Agroforest Syst*.**76**:487-497. (2009)

LIST OF PUBLISHED/ COMMUNICATED PAPERS

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2. Roshmi Das and Ashis Kumar Sarkar. Comparison between Different Growth Functions of the Jatropha Curcas plant with Random Attack Pattern of Whitefly. Global Journal of Engineering Science and Researches. Vol-7, Issue-9, pp. 16-26 September (2020).

3. Roshmi Das and Ashis Kumar Sarkar. An Effort For Controlling The Mosaic Disease of Jatropha Curcas Plant. Advances and Applications in Mathematical Sciences. Vol-21, Issue-11, pp. 6437-6454, September (2022).

4. Roshmi Das and Ashis Kumar Sarkar. Modeling Different Attack Patterns of Whitefly on Jatropha Curcas Plant and Control of The Mosaic Disease. Journal of the Calcutta Mathematical Society. Vol-18(2). pp. 225-246 (2022).

5. Roshmi Das and Ashis Kumar Sarkar. Role of Insecticide Spraying in Reduction of Mosaic Disease of Jatropha Cucas Plant. Paper submitted for publication in Bulletin of Calcutta Mathematical Society.

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Effect of Growth Functions on Jatropha Curcas Plant with Random Attack Pattern of Whitefly: A mathematical study

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Abstract

Jatropha curcas is an important plant which can provide an affordable solution of shrinkage of fossil fuel by producing alternative fuel (biodiesel). The seeds of this plant contain a high amount of oil that can be used to obtain a better quality of biodiesel . So the economic value of the plant is very high , but this plant is affected by the mosaic virus (*Begomovirus*) through whitefly (Bemisia tabaci) which causes mosaic disease. In this paper we consider two mathematical models of different growth functions of the Jatropha curcas plant with random attack pattern of the whitefly. These two nonlinear deterministic models of jatropha curcas plant and whitefly is studied analytically where the distribution of whitefly on plants follows poisson distribution. The result shows that the first system possesses a fragile behavior and the other shows a steady state which is globally asymptotically stable.

Keywords: Jatropha Curcas plant ; Mosaic virus (Begomovirus) ; Whitefly (Bemisia tabaci); Mosaic disease ; Poisson distribution ; Random attack ; Fragile behaviour ; Global stability .

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COMPARISON BETWEEN DIFFERENT GROWTH FUNCTIONS OF THE JATROPHA CURCAS PLANT WITH RANDOM ATTACK PATTERN OF WHITEFLY

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ABSTRACT

We have proposed here two deterministic models of Jatropha Curcas plant and Whitefly that recreate the dynamics of cooperation between them where the conveyance of Whitefly on plant follows Poisson distribution.In the first model growth rate of the plant is thought to be in logistic form whereas in the second model it is taken as exponential form. The attack pattern and the growth of the whitefly are assumed as Holling type II function.The first model outcomes a globally stable state and in the second one we discover a globally attracting steady state for some parameter values,and a stable limit cycle for some other parameter esteems. It is likewise demonstrated that there exist Hopf bifurcation regarding some parameter values. The paper additionally examine the inquiry regarding persistence and permanence of the model. It is discovered that the particular growth rate of both the population and attack pattern of the whitefly administers the dynamics of both the models.

Keywords: *Jatropha curcas plant, Whitefly (Bemisia tabaci), Random attack , Global stability, Limit cycle and Hopf bifurcation investigation*.

I. INTRODUCTION

With the upgradation of human civilisation the interest for the elective vitality sources is also likewise expanding. Among the potential methods for creating vitality in a situation invitingway,the creation of biofuels is getting generally well known. Jatropha curcas is such a significant plant the seeds of which plant contains 37% oil that can be utilissed to obtain a superior nature of biodiesel [8]. The beginning of this plant is tropical zone at first from Mexico and central part of the USA and is currently developed overall [6]. The tree is of critical financial significance for its various mechanical and medicinal use.

Jatropha curcas is a semi-evergreen little bush with huge green to light green leaves. Normally it develops between $(3-5)$meter in tallness yet accomplishes a stature upto $(8-10)$meter under good condition. It is normally known as physic or purging nut. It is a multipurpose and drought resistant crop which is developed in marginal grounds with lesser input. The tree can be developed in dry and barren conditions and can be developed likewise in rough,sandy and salty soils. It has low plantation cost. It developes rapidly and lives delivering seeds for 50 years. Yet such a significant plant is influenced by the mosaic virus (begomovirus).

Mosaic virus is one kind of plant virus that causes the leaves of plants with a spotted and speckled look. They move oftentimes in nature. The indications are serious mosaic, mottling, blistering of leaves, yellowing of leaves, decreased leaf size, hindering of infected plants. It basically attacks its fruits extensively decreasing the creation and nature of seeds. The mosaic virus spreading chiefly relies upon the vector whitefly [2][3] . The number of inhabitants in whitefly is constrained by temperature and rainfall. Heavy rainfall makes an obstacle for the develoment of whiteflies [1]. In this ailment the mosaic virus passes from an infected whitefly to a susceptible plant and the other way around. The spread of the virus is profoundly depends on the plant thickness. A solitary whitefly is satisfactory to contaminate the host plants however transmission of the ailment spread when various infected whiteflies feed on the host plants through massive flux of saliva. Accordingly have host plant (Jatropha Curcas) faces leaf harm and sap seepage because of such feeding. Whiteflies reproduces very quickly [7] , if once they get traditional on any aspect of the plants they will willfully roam and try to attempt to assault some other immediate vegetation [5][10]. Ordinarily they need 3 hours feeding time to secure the infection and a latent phase of 8 hours. It requires 10 minutes time to taint the youthful

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AN EFFORT FOR CONTROLLING THE MOSAIC DISEASE OF JATROPHA CURCAS PLANT

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Abstract

Jatropha Curcas is very much essential plant for ecological as well as environmental purpose. The seeds of the plant contain 37% of oil that can be used to obtain a better quality of biodiesel which is very useful as an alternative fuel. But such an important plant is affected by the Mosaic virus (Begomovirus) through the vector whitefly (Bemisia tabaci) which causes mosaic disease. In this paper we propose a model for the dynamics of this disease and its possible control via insecticide spraying. The result shows that the system possesses a steady state for some parameter values, Hopf bifurcation for some other parameter values and unstable condition for some other parameter values. Pontryagin minimum principle is applied to minimize the cost of spraying.

1. Introduction

The genus Jatropha of family Euphorbiaceae has more than 400 species distributed worldwide and among them Jatropha Curcas is recorded from India. It is commonly known as physic or purging nut. The seeds of this plant produce biodiesel which is an efficient substitute fuel for diesel engine. It is also an essential ingredient in various soap, dye and wood industries [6]. Jatropha Curcas is semi evergreen shrub or small tree with large green to pale green leaves.

It grows between (3-5) meter in height but grows upto (8-10) meter under favourable conditions. It is a multipurpose and drought resistant crop which is grown in marginal lands with lesser input. The tree can be grown in dry

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Keywords: Jatropha curcas plant, Whitefly (Bemisia tabaci), Mosaic virus (Begomovirus), Mosaic disease, Optimal control.

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MODELING DIFFERENT ATTACK PATTERNS OF WHITEFLY ON JATROPHA CURCAS PLANT AND CONTROL OF THE MOSAIC DISEASE

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Abstract. A mathematical model has been formulated on Jatropha curcas plant and whitefly interaction. In this model we have chosen different probability distribution functions on whitefly attack pattern such as Binomial, Poisson and Negative-binomial distribution which biologically express the regular, random and aggregated attack pattern of whitefly. Our goal is to find out the effects on the disease dynamics due to different attack pattern of whitefly. Analytically we have shown that due to aggregated attack pattern of whitefly the system possesses stable, unstable or small amplitude hopf bifurcating oscillations for different values of the system parameters. Due to regular and random attack pattern of whitefly the system possesses stable steady state irrespective of all system parameter values. Using control theory a possible way of controlling mosaic disease is shown. The theoretical outcomes are justified using numerical simulations.

Keywords: Jatropha curcas plant, Mosaic disease, Whitefly (Bemisia tabaci), Poisson distribution, Negative binomial distribution, Binomial distribution, Control theory.

1. Introduction. Jatropha Curcas plant is such a plant which has utmost importance in economy as well as medicinal purpose. Bio-diesel which is most demanding alternative fuel is obtained from the seeds of the Jatropha Curcas plant. This important plant is easily effected by the mosaic virus by the vector whitefly (Guin, 2015). The symptoms which occurs in this mosaic disease is that severe mosaic, mottling, blistering of leaves, yellowing of leaves, reduced leaf size, stunting of diseased plants (Gao, Qu, Chua and Ye, 2010 and Guin, 2014). Not only the leaves but also the fruits and the seeds become affected. Fruit production reduction, low quality seed production are also the symptoms of this disease (Guin, 2016, Narayana, Shankarappa, Govindappa, Prameela, Rao and Rangaswamy, 2006 and Sahoo, Kumar, Sharma and Naik, 2009).