EIGENVALUE APPROACH TO STUDY SOME PROBLEMS OF THERMOELASTICITY

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BY

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled **"EIGENVALUE APPROACH TO STUDY SOME PROBLEMS OF THERMOELASTICITY''** submitted by **Sri Sumit Sovan Sardar** who got his name registered on 6th September, 2019 **(Index No.: 72/19/Maths./26)** for the award of Ph. D. (Science) Degree of Jadavpur University, is absolutely based upon his own work under the supervision of Dr. Abhijit Lahiri, Professor, Department of Mathematics, Jadavpur University, Kolkata and co-supervision of Dr. Bappa Das, Associate Professor, Department of Mathematics, Bankura University, Bankura and that neither this thesis nor any part of it has been submitted for either any degree / diploma or any other academic award anywhere before.

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Dedicated to

my parents

Sujit Kumar Sardar
and
Sagarika Sardar

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PREFACE

The present research work has been accomplished in the application oriented area of continuum mechanics regarding the thermoelastic and magnetothermoelastic properties of solids. The dissertation contains four chapters discussing six problems and their solutions in the context of generalized thermoelasticity and electromagneto thermoelasticy.

In the **first chapter** some fundamental concepts of thermoelasticity have been discussed in brief.

The **second chapter** is about the generalised thermoelasticity in context of multi-phase-lag model containing two different problems viz., Problem-1 and Problem-2.

Problem-1 is about a refined multi-phase lag model of generalized thermoelasticity in an anisotropic half-space medium. The normal mode analysis technique has been used to obtain vector matrix differential equation, which is then solved by the eigenvalue approach. Some earlier results, such as CTE, Lord Shulman (LS), Green–Nagdhi (GN)-II, and SPL, are deduced from the present investigation. Numerical computations for thermal strain and stress component, displacement components, and temperature distribution are calculated in a tabulated form and graphically to show the accuracy of the present model when mechanical and thermal loads are applied on the boundary.

Problem-2 deals with a two-dimensional multi-phase lag model in the context of generalized thermoelasticity for an isotropic half-space medium. A vector-matrix differential equation is obtained from the governing equations using normal mode analysis. The eigenvalue approach is applied to obtain the solutions. The temperature-dependent displacements, stresses, strains are calculated numerically and represented graphically to show the accuracy of the solution under mechanical and thermal loads.

The **third chapter** deals with non-local heat propagation for thermoelastic medium consisting of two different problems viz., Problem-3 and Problem-4.

Problem-3 concerns a generalized magnetothermoelastic problem for a homogeneous, isotropic and semiconducting medium under the non-local heat equation with dual-phase-lag(DPL) model. The boundary surface of the medium is subject to a prescribed time-dependent exponential order compression along with a prescribed temperature and carrier intensity gradient. Two integral transformations, Laplace transform for time variable and Fourier transform for space variable, are employed to equations of motion and heat conduction equation for formulation of a vector-matrix differential equation which is then solved by using eigenvalue approach. Inversion processes for the two integral transform are carried out numerically. Finally, the effects of physical field variables and stress components are analyzed and illustrated graphically under the variation of different physical parameters.

Problem-4 is about the new concept of non-local heat conduction equation to generalized magneto-thermoelastic problem of two dimensional isotropic and homogeneous half-space in presence of heat-flux at the boundary surface. By using the harmonic plane waves, the governing equations are transformed to the vector matrix differential equation which is then solved by eigenvalue method. The analytical closed form solutions for displacement component, temperature distribution and stress components have been made and comparisons are also illustrated graphically with the theory of non-local dual-phase-lag(NLDPL) and non-local Lord-Shulman(NLLS) theory for different values of physical parameters. The significant effects of non-local variables as well as phase lagging parameters on displacements, temperature distribution and stress components are studied graphically and concluding remarks are drawn.

In **fourth chapter** Thermoelastic behaviour in curvilinear co-ordinate system has been studied by solving two different problems viz., Problem-5 and Problem-6.

Problem-5 is a discussion about a refined multi-phase lag model for a homogeneous unbounded thermoelastic spherical cavity in a curvilinear coordinate system. The associated solutions are obtained by forming non-dimensional vector matrix equations and applying Eigen value approach in the transform domain. The results have been verified using the valid boundary conditions associated with physical and field variables.

Mathematica and MATLAB platform are used for numerical analysis and graphical representation.

Problem-6 concerns the longitudinal vibration of a circular cylinder in the context of generalized thermoelasticity where the fundamental equations for a homogeneous isotropic solid has been considered following Lord and Shulman, which has been reduced in cylindrical co-ordinate system. The equations have been expressed in the form of a vector matrix differential equation and solved using the eigenvalue method. At the end, the effects on physical field variables and stress components are analyzed graphically under the variation of different physical parameters.

List of Publications

A list of publications resulted from the work of this thesis is appended below.

(1) **S.S. Sardar, D. Ghosh, B. Das and A. Lahiri:** On a multi-phase lag model of three-dimensional coupled thermoelasticity in an anisotropic half-space. *Waves in Random and Complex Media*, published online, July 2022.
DOI: $https://doi.org/10.1080/17455030.2022.2094025$.

(2) **A. Lahiri, S.S. Sardar and D. Ghosh:** Modeling of a homogeneous isotropic half space in the context of multi-phase lag coupled thermoelasticity. *Mechanics of Time-Dependent Materials*, published online, December 2022.
DOI: $https://doi.org/10.1007/s11043-022-09584-7$.

(3) **B. Das, S.S. Sardar, D. Ghosh and A. Lahiri:** Wave propagation in a non-local magnetothermoelastic medium permeated by heat source. *International Journal for Computational Methods in Engineering Science and Mechanics*, published online, March 2023.
DOI: $https://doi.org/10.1080/15502287.2023.2186968$.

Contents

1
PRELIMINARIES

1.1 *Basic idea:*

1.1.1 Definitions, Relations and Theories :

Elastic Solid:

Most of the solids undergo deformation accompanying changes of shape and size under the action of a mechanical load. Such a deformation is measured by strain functions defined by changes in dimension per unit volume. The solid, if it is an ideal elastic solid, gets back to its original shape and volume under the influence of a system of reactionary forces which develop within the body due to application of loads. The intensity of these internal forces are termed as stress which is measured as force per unit area.

Thermoelasticity:

Thermoelascity is the study of an elastic body/material under the influence of non-uniform changes in it's temperature field. It is a generalisation of the theory of elasticity by accounting for thermal reactions as well as mechanical reactions. The theories of thermo-elasticity have been developed by a useful coupling of Fourier Law of heat conduction with the standard formulations developed in the theory of elasticity. During the second half of 20th century, the theory of thermo-elasticity and magneto-thermo-elasticity (interactions among strain, temperature and electromagnetic fields) has drawn the attention of many researchers because of it's extensive uses in diverse field, such as

a) Geophysics for understanding the effect of the Earth's magnetic field on seismic waves.
b) Damping of acoustic waves in a magnetic field.
c) Designing machine elements like heat exchangers, boiler's tubes where temperature induced elastic deformation occurs.
d) Biomedical engineering (problems involving thermal stress).
e) Emissions of electromagnetic radiations from nuclear devices.
f) Development of a highly sensitive super conducting magnetometer, electrical power engineering etc.

Strain-displacement relation:

If the displacement at a point within the body defined by cartesian coordinates x_i $(i = 1, 2, 3)$ is u_i, then the kinematic relation between the strain and displacement is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.1}$$

where $e_{ij} = e_{ji}$.

The stress component σ_{ij} is defined as the force along the x_j direction on unit area in the plane x_i =constant. So, there are nine such stress components. It can be shown that $\sigma_{ij} = \sigma_{ji}$. Then there are only six independent stress components needed to define the stress system in the body.

Constitutive relations:

The most general form of Hooke's law is represented by

$$\sigma_{ij} = C_{ijkl}e_{kl} \quad (i, j, k, l = 1, 2, 3), \tag{1.2}$$

where C_{ijkl} is the stiffness tensor which has 81 elements. Considering symmetry properties of both stress and strain components, these may be reduced to the form

$$\sigma_m = C_{mn}e_n, \qquad (m, n = 1, ..., 6). \tag{1.3}$$

In equation (1.3),

$$\begin{aligned} \sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3, \\ \sigma_{32} = \sigma_4,\ \sigma_{13} = \sigma_5,\ \sigma_{12} = \sigma_6, \end{aligned} \tag{1.4}$$

$$\begin{aligned} e_{11} = e_1,\ e_{22} = e_2,\ e_{33} = e_3, \\ e_{32} = e_4,\ e_{13} = e_5,\ e_{12} = e_6, \end{aligned} \tag{1.5}$$

and C_{mn} is a matrix of order 6 with $C_{mn} = C_{nm}$.

The coefficients C_{mn} in the generalized Hooke's law are symmetric due to the existence of strain energy density function. The number of independent elastic constants in the generalized Hooke's law (1.3) will include 6 different constants located along the diagonal and $\frac{36-6}{2} = 15$ among the remaining constants making altogether $15 + 6 = 21$ constants. So, the existence of strain energy density function reduces the number of coefficients from 36 to 21 in generalized Hooke's law. For isotropic material, the matrix is defined by only two independent parameters λ and μ called Lamè's constants. On the other hand, an orthotropic solid has 9, transversely isotropic has 5 and cubic crystalline has 3 independent elastic constants.

Thermal stresses:

It is known that if a body is heated, its change in dimension leads to deformation. So, even in absence of any external mechanical loads, the deformation and consequently stresses may develop in a body, called thermoelastic solid, when it is placed in an elevated temperature field. This temperature θ, excess over the ambient one, may develop due to external heating or due to internal heating arising out of straining.

Irrespective of the sources of heating, the thermal stresses are determined from the following constitutive relations

$$\sigma_m = C_{mn} e_n - \beta_m \theta \quad (m, n = 1, 2, 3, ..., 6), \tag{1.6}$$

where

$$\beta_{11} = \beta_1,\ \beta_{22} = \beta_2,\ \beta_{33} = \beta_3,\ \beta_{32} = \beta_4,\ \beta_{13} = \beta_5,\ \beta_{12} = \beta_6. \tag{1.7}$$

Equations (1.6) are called the Duhamel-Neumann relations.

For a transversely isotropic body, if the principal elastic axes coincide with the coordinate axes, the equations (1.6) may be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{32} \\ e_{31} \\ e_{12} \end{bmatrix} - \theta \begin{bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{1.8}$$

where

$$\beta_{11} = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3, \ \beta_{33} = 2c_{13}\alpha_1 + c_{33}\alpha_3, \tag{1.9}$$

and $\alpha_1, \ \alpha_3$ are coefficients of linear thermal expansion.

For isotropic bodies,

$$c_{12} = \lambda, \ c_{11} = \mu, \ c_{44} = \mu, \ \beta_{11} = (3\lambda + 2\mu)\alpha. \tag{1.10}$$

So, the constitutive relations may be written as

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \beta\theta, \tag{1.11}$$

where δ_{ij} is the Kronecker delta.

Strain-displacement relations in **cylindrical co-ordinate system** are given by

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r}, \ e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \ e_{zz} = \frac{\partial u_z}{\partial z}, \\ e_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \\ e_{rz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \\ e_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right). \end{aligned} \tag{1.12}$$

Equations of motion:

The equations of motion of an elastic body subjected to a system of stresses-mechanical, thermal or both are derived on the basis of conservation of linear momentum. For an arbitrary volume V of a body bounded by surface S, equating the inertial force with the total volume force along with the surface force

$$\rho\frac{\partial}{\partial t}\int_V v_i dv = \int_V X_i dv + \int_S p_i ds \ \ (i = 1, 2, 3), \tag{1.13}$$

where ρ is the mass density, v_i, X_i and p_i are respectively the components of velocity, body force per unit volume and surface traction acting on the surface S. Now, we also have

$$p_i = \sigma_{ij} n_j, \tag{1.14}$$

where n_i is the unit vector normal to the surface S. From equations (1.13) and (1.14) and using the Gauss divergence theorem

$$\int_S \sigma_{ij} n_j ds = \int_V \sigma_{ij,j} dv, \tag{1.15}$$

we can obtain,

$$\sigma_{ij,j} + X_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{1.16}$$

This together with appropriate constitutive relations help one to derive equations of motion in terms of displacement components as

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + X_i = \beta T_{,i} + \rho \ddot{u}_i \tag{1.17}$$

In vector form, the above equations can be written as

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}) + \mathbf{X} = \beta \nabla T + \rho \ddot{\mathbf{u}}. \tag{1.18}$$

Equations of motion in **cylindrical co-ordinate system** can be written in the following forms:

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + X_r = \rho \frac{\partial^2 u_r}{\partial t^2}, \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + X_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2}, \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + X_z = \rho \frac{\partial^2 u_z}{\partial t^2}.
\end{aligned} \tag{1.19}$$

Thermo-mechanical coupling:

During mechanical loading of an elastic body, some work is done due to straining. This energy dissipates as heat induces a temperature field within the material. So, in Fourier heat conduction equation this internal heat source should be appropriately included for accurately computing the temperature field. The coupling between the temperature and the strain fields also help in determining the temperature field due to time-varying forces and also accounts for the influence of temperature on the velocity of propagation of elastic waves. Only in stationary temperature fields, this coupling term may be neglected.

1.1.2 Various Theories OF Thermoelasticity :

Classical Coupled Thermoelasticity (CCTE):

Stress, strain and temperature relations in an isotropic and homogeneous thermoelastic solid body (Duhamel-Neumann relations) are given by

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda u_{i,i} - \beta\theta)\delta_{ij} \quad (i, j = 1, 2, 3), \tag{1.20}$$

where $\lambda,\ \mu$ are Lamè's constants, $\beta = (3\lambda + 2\mu)\alpha_t$, where α_t is the coefficient of linear thermal expansion of the material, σ_{ij} are the stress components, θ is the increase in temperature above the reference temperature T_0, $e = u_{i,i}$ is the dilatation, e_{ij}'s are given by equation (1.20). Later on these equations are modified by classical Fourier's law which connects heat flux vector $\mathbf{q}$ with temperature gradient $\nabla\theta$ as in the equation

$$\mathbf{q} = -k\nabla\theta \ \text{ or } \ q_i = -k\theta_{,i} \quad (i = 1, 2, 3) \tag{1.21}$$

where heat flux vector is the instantaneous result of a temperature gradient and k is the thermal conductivity.

While the coupling of strain and temperature field is being considered, the principal of conservation of local energy provides

$$\begin{aligned} -q_{i,i} + \rho Q &= \rho c_v\dot{\theta} + \beta T_0\dot{e} \\ \text{i.e.} \quad -\nabla\mathbf{q} + \rho Q &= \rho c_v\dot{\theta} + \beta T_0\dot{e}, \end{aligned} \tag{1.22}$$

where ρ is the mass density, c_v is the specific heat of the solid at constant volume, Q is the heat sources and t is the time. Thus, after eliminating q_i, coupled heat conduction equation is given by

$$k\nabla^2\theta + \rho Q = \rho c_v\dot{\theta} + \theta_0\beta\dot{u}_{k,k} \tag{1.23}$$

The term T_0 should be taken into account to establish a coupling between strain and temperature. Again, the principle of conservation of linear momentum yields the stress equations of motion in the following linearized form

$$\sigma_{ij,j} + \rho F_i = \rho\ddot{u}_i \ \ (i, j = 1, 2, 3), \tag{1.24}$$

where $F_i,\ i = 1, 2, 3$ are the components of external body force vector per unit mass.

Equations (1.1), (1.22) and (1.24) lead us to the displacement equations of motion

$$\begin{aligned} \mu\nabla^2 u_i + (\lambda + \mu)u_{k,k} + \rho F_i - \beta\theta_{,i} &= \rho\ddot{u}_i \\ \text{or,} \qquad \mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla.\mathbf{u}) + \rho F_i - \beta\nabla\theta &= \rho\ddot{\mathbf{u}}. \end{aligned} \tag{1.25}$$

The nature of equation (1.23) is a parabolic whereas equation (1.25) is of hyperbolic type. The equation (1.23), which was deduced by Biot [30], concerns the interaction of the thermal field and elastic deformation as these two are coupled. According to classical Fourier's law, equation (1.23) implies that if the material is subjected to a thermal disturbance, the effect on both temperature and displacement fields will take place at infinite distance from the heat source. Consequently the thermal waves propagate with infinite speed. This phenomenon is absurd and physically inappropriate, which diminishes the credibility of most of the classical thermodynamical theories.

Lord-Shulman Model [L-S Model] of linear thermoelasticity or Extended Thermoelasticity [ETE]:

To overcome the shortcomings of Biot's theory, Kaliski et. al. [68] generalised the classical Fourier's law (1.21) by introducing the following heat conduction law

$$\left(1+\tau\frac{\partial}{\partial t}\right)\mathbf{q} = -k\nabla\theta, \tag{1.26}$$

where τ is a non-negative constant called the relaxation time parameter. This law is a generalization of the classical Fourier's law (1.21). If we put $\tau = 0$ in equation(1.26), we get equation(1.21).

According to this generalized theory the energy equation for a homogenous and isotropic thermoelastic solid is given by

$$-\nabla\cdot\mathbf{q} + \rho Q = \rho c_E\dot{T} + \gamma T_0\dot{e}. \tag{1.27}$$

From equations (1.26) and (1.27), one can obtain the following generalized form of the heat conduction equation

$$k\nabla^2\theta = \left(1+\tau\frac{\partial}{\partial t}\right)\left[\rho c_e\dot{\theta} + \theta_0\beta\dot{u}_{k,k} - \rho Q\right] \tag{1.28}$$

As equation (1.26) is of hyperbolic nature, it is free from the paradox of infinite heat propagation speed and the thermal disturbances propagate with constant phase speed $\sqrt{\frac{k}{\tau}} < \infty$. The theory in which equation (1.28) is considered as the heat conduction equation is referred as Extended thermoelasticity (ETE) by Chandrasekharaiah [33].

The equation of motion of this model is given by

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \rho F_i + (\lambda+\mu)u_{j,ij} + \mu u_{i,jj} - \gamma T_{,i}. \tag{1.29}$$

The stress components are given by

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\left(T - T_0\right). \tag{1.30}$$

The generalized thermoelasticity theory introduced by Kaliski[80] was also established independently by Lord and Shulman [80] which is often refereed to as the L-S model.

Green-Lindsay Model [G-L Model] OF Linear Thermoelasticity or Temperature Rate Dependent Thermoelasticity (TRDTE):

Green and Lindsay [58] have established a theory of generalized thermoelasticity with certain special features that contrast with the L-S model having only one relaxation time parameter. In G-L model, Fourier's law of heat conduction is unchanged whereas the classical energy equation and the stress-strain temperature relations are modified. Two constitutive constants α and α_0 having the dimensions of time appear in the governing equations in place of one relaxation time τ in L-S model.

The field equations of generalized thermoelasticity model proposed by Green and Lindsay [58] are

$$-q_{i,i} + \rho Q = \rho c_v(\dot{\theta} + \alpha_0\ddot{\theta}) + \beta T_0\dot{e}, \tag{1.31}$$

which are modified energy equations. The stress-strain-temperature relations in this case are

$$\sigma_{ij} = \lambda u_{i,i}\delta_{ij} + 2\mu e_{ij} - \beta(\theta + \alpha\dot{\theta})\delta_{ij} \quad ; \quad i,j = 1,2,3 \tag{1.32}$$

which are modified constitutive equations with temperature rate term.

Elimination of q_i from equations (1.21) and (1.31) give the following heat conduction equation

$$k\nabla^2\theta + \rho Q = \rho c_e(\dot{\theta} + \alpha_0\ddot{\theta}) + \beta\theta_0\dot{u}_{k,k}. \tag{1.33}$$

The equation of motion in this case can be obtained as

$$\mu\nabla^2 u_i + (\lambda + \mu)u_{j,ij} - \beta(\theta + \alpha\dot{\theta})_{,i} + \rho F_i = \rho\ddot{u}_i, \tag{1.34}$$

where α and α_0 are two material constants satisfying the inequalities $\alpha \geq \alpha_0 \geq 0$ and are called the relaxation time parameters of this model.

Clearly, equation (1.33) is also hyperbolic type predicting finite speed for the propagation of thermal signals.

If we set $\alpha = \alpha_0 = 0$ in the equations (1.32), (1.33) and (1.34), then we recover equations (1.20), (1.23) and (1.25). Thus classical coupled thermoelasticity (CCTE) is a special case of the Temperature rate dependent thermoelasticity (TRDTE) with $\alpha = \alpha_0 = 0$.

Green-Naghdi Model:

I. Green-Naghdi model-III(1992) or Thermoelasticity With Energy Dissipation (TEWED)

The modified energy equation and the heat conduction law proposed by Green and Naghdi [60] are given by

$$-\nabla \mathbf{q} + \rho Q = \rho c_v \dot{\theta} + \beta T_0 \dot{e}, \tag{1.35}$$

and

$$\mathbf{q} = -(k\nabla\theta + k^*\nabla\nu). \tag{1.36}$$

Now,

$$\dot{\mathbf{q}} = -(k\nabla\dot{\theta} + k^*\nabla\dot{\nu}) \tag{1.37}$$

Using $\dot{\nu} = \theta$, we get

$$\dot{\mathbf{q}} = -(k\nabla\dot{\theta} + k^*\nabla\theta) \tag{1.38}$$

which gives

$$-\nabla\dot{\mathbf{q}} = k\nabla^2\dot{\theta} + k^*\nabla^2\theta \tag{1.39}$$

Eliminating $\mathbf{q}$ from equations (1.35) and (1.39), we get

$$k\nabla^2\dot{\theta} + k^*\nabla^2\theta + \rho\dot{Q} = \rho c_v \ddot{\theta} + \beta T_0 \ddot{e} \tag{1.40}$$

Equation (1.40) admits propagation of damped thermoelastic waves, damping due to the term $\dot{\theta}$ in the equation. If the heat source $Q = 0$, then equation (1.40) reduces to

$$k\nabla^2\dot{\theta} + k^*\nabla^2\theta = \rho c_v \ddot{\theta} + \beta T_0 \ddot{e}. \tag{1.41}$$

The equation of motion of this model is

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ij} - \gamma\theta_{,i} + \rho F_i = \rho\ddot{u}_i. \tag{1.42}$$

The constitutive relations are

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\theta. \tag{1.43}$$

II. Green-Naghdi Model II(1993) OR Thermoelasticity Without Energy Dissipation (TEWOED)

The governing equations of the generalized thermoelasticity proposed by Green and Naghdi [61] are

$$-\nabla\mathbf{q} + \rho Q = \rho c_v\dot{\theta} + \beta T_0\dot{e} \tag{1.44}$$

which modified the energy equation

$$-\mathbf{q} = -k^*\nabla\nu \tag{1.45}$$

which is also modified heat conduction law, where ∇v is the thermal displacement gradient such that $\dot{\nu} = \theta$.

Now,

$$-\dot{\mathbf{q}} = -k^*\nabla\dot{\nu} = -k^*\nabla\theta \tag{1.46}$$

which gives on taking divergence on both sides

$$-\nabla\dot{\mathbf{q}} = k^*\nabla^2\theta. \tag{1.47}$$

Eliminating $\mathbf{q}$ from equations (1.44) and (1.47), we get

$$k^*\nabla^2\theta + \rho\dot{Q} = \rho c_v\ddot{\theta} + \beta T_0\ddot{e}, \tag{1.48}$$

where $k^* > 0$ is a material constant. The finite thermal wave speed is equal to $\sqrt{\frac{k^*}{\rho c_v}}$.

The equation of motion of this case is

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ij} - \gamma\theta_{,i} + \rho F_i = \rho\ddot{u}_i. \tag{1.49}$$

The constitutive equations are

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\theta. \tag{1.50}$$

Equations (1.48), (1.49) and (1.50) are complete form of the Grenn-Nagdhi model without energy dissipation for an isotropic and homogeneous elastic solid medium.

Theory of Two-Temperature Linear Thermoelasticity :

A theory of heat conduction in deformable bodies which depends upon two distinct temperatures, the conductive temperature and the thermodynamic temperature has been established by Chen and Gurtin [37] and Chen et al. [36][38]. Youssef [123] has proposed a theory in the context of the generalized theory of thermoelasticity with two-temperature. By following this model, the constitutive relations and basic governing equations for thermoelastic interactions in a homogeneous isotropic elastic solid for CCTE, L-S and G-L theories may be written in a unified way as:

Equation of motion is

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ij} - \gamma\left(\theta + \nu\frac{\partial\theta}{\partial t}\right)_{,i} + \rho F_i = \rho\ddot{u}_i. \tag{1.51}$$

The heat conduction equation with two-temperature is given by

$$\begin{aligned} k\nabla^2\phi = \rho c_E\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\theta + \gamma T_0\left(\frac{\partial}{\partial t} + n_0\tau_0\frac{\partial^2}{\partial t^2}\right)u_{i,i} \\ - \left(1 + n_0\tau_0\frac{\partial}{\partial t}\right)Q, \end{aligned} \tag{1.52}$$

where n_0 is a constant.

The relation between the thermodynamic and conductive temperatures is given by

$$\phi - \theta = a\nabla^2\phi, \tag{1.53}$$

where $a > 0$ is the temperature discrepancy.

The constitutive relations are given in the form

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\left(\theta + \nu\frac{\partial\theta}{\partial t}\right). \tag{1.54}$$

Equations (1.51)-(1.54) constitute a complete system of **two-temperature generalized thermoelasticity**. This model can be applied to both classical generalizations, L-S model $(n0 = 0, \tau_0 > 0, \nu = 0, a = 0)$ and G-L model $(n0 = 0, \tau_0 > 0, \nu > 0, a = 0)$, as well as to CCTE model $(\tau_0 = \nu = a = 0)$. The main difference for this theorem from the other theorems is that this theorem differentiates between the wave propagation of the temperature that comes from the thermal process (heat conduction) and that which comes from the mechanical process (thermodynamics temperature).

Dual-Phase-Lag Model of Linear Thermoelasticity:

I. Parabolic thermoelasticity theory with two-phase-lag

The Tzou theory [114] is such a modified of CCTE in which the Fourier law $\mathbf{q}(P,t) = -k\nabla T(P,t)$ is replaced by an approximation of the equation

$$\mathbf{q}(P+\tau_q,t) = -[k\nabla T(P,t+\tau_T)], \tag{1.55}$$

in the following form

$$\left(1+\tau_q\frac{\partial}{\partial t}\right)\mathbf{q} = -k\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla T, \tag{1.56}$$

where the temperature gradient ∇T at a point P of the thermoelastic solid at time $t+\tau_q$ corresponds to the heat flux vector $\mathbf{q}$ at the same point at time $t+\tau_q$. The delay time τ_T is the *phase-lag of temperature gradient* that is interpreted as the delay time caused by the micro-structural interactions (a small scale effects of heat transport in space, such as phonon–electron interaction or phonon scattering) whereas the other delay time τ_q is interpreted as the relaxation time due to the fast transient effects of thermal inertia and is called the *phase-lag of the heat flux.* The model transmits thermoelastic disturbance in a wave-like manner if the approximation is linear with respect to τ_q and τ_T, and $0 \leq \tau_T < \tau_q$ or quadratic in τ_q and linear in τ_T, with $\tau_q > 0$ and $\tau_T > 0$. Eliminating $\mathbf{q}$ between Eqs. (1.55) and (1.56), the *parabolic type* heat conduction equation with *two-phase-lag* proposed by Tzou takes the form

$$k\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2 T = \left(1+\tau_q\frac{\partial}{\partial t}\right)\left(\rho c_E\dot{T}+\gamma T_0\dot{e}-\rho Q\right). \tag{1.57}$$

II. Hyperbolic thermoelasticity theory with two-phase-lag

Chandrasekharaiah [33] proposed a parabolic as well as a hyperbolic thermoelastic model with dual-phase-lag by expanding equation (1.57) in a Taylor series up to the first-order terms in τ_T and second order terms in τ_q in the following form

$$\left(\mathbf{q}+\tau_q\frac{\partial \mathbf{q}}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2\mathbf{q}}{\partial t^2}\right) = -k\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla T, \tag{1.58}$$

Eliminating $\mathbf{q}$ between Eqs. (1.56) and (1.58), the heat conduction equation in this case takes of the following form

$$k\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2 T=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_E\dot{T}+\gamma T_0\dot{e}\right)$$
$$-\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\rho Q. \tag{1.59}$$

Equations (1.59) is the *hyperbolic type* heat conduction equation with *two-phase-lag.*

Three-Phase-Lag Model Thermoelasticity:

Further generalization of coupled thermoelasticity theory was made by Roy Choudhury [102], where he discussed the concept of Three Phase Lag Model [3PHL] introducing three time parameters τ_q,τ_T,τ_ν where τ_q= heat flux time lag,τ_T= temperature gradient time lag and τ_v= thermal displacement gradient time lag satisfying the inequality $0\leq\tau_\nu\leq\tau_T\prec\tau_q$. To discuss the lagging behaviour, using $\vec{\nabla}v$, $\vec{q}$and $\vec{\nabla}T$ as thermal displacement gradient, heat flux vector and temperature gradient respectively, the constitutive equation of generalised heat conduction can be written as $\vec{q}(P,t+\tau_q)=-[\kappa^*\vec{\nabla}v(P,t+\tau_v)+\kappa\,\vec{\nabla}T(P,t+\tau_T)]$, where$P\,(\vec{r})$ is the point where material volume located at time $(t+\tau_v)$ and $(t+\tau_T)$ together with heat flux flow at different instant of τ_q for a finite time t>0.

Taking Taylor's series expansion from the above mentioned equation, we have $\vec{q}+\tau_q\frac{\partial\vec{q}}{\partial t}=-[\kappa^*\,\vec{\nabla}v+\kappa\,\tau_T\,\frac{\partial}{\partial t}\vec{\nabla}\theta+\tau_v{}^*\,\vec{\nabla}\theta]$

Where$\tau_v{}^*\ =\kappa+\kappa^*\tau_v\ \ and\ \dot{v}=\theta$

Now depending upon different values of τ_q,τ_T,τ_v and κ^*, different theory can be classified as below

1. *Classical Fourier's Law:* κ^**=0,* $\tau_q=\tau_T$
2. *Lord-Shulman (L-S) Theory:* κ^**=0,*$\tau_q=\tau$ *and* $\tau_T=0$,τ*is the relaxation time.*
3. *Green-Naghdi-III (G-N-III) theory:*$\tau_q=0,\tau_T=0$ *and* $\tau_v=0$

Neglecting the terms above the 2^{nd} order of τ_q in Taylor's expansion and then eliminating $div\,\vec{q}$, the generalized heat conduction equation reduced to $\kappa^*\nabla^2\,\dot{\theta}+k\,\tau_T\nabla^2\,\ddot{\theta}+\tau_v{}^*\nabla^2\,\dot{\theta}=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\partial^2}{\partial t^2}\frac{1}{2}\tau_q{}^2\right)F(x_1,x_2,x_3,t)$ where $F(x_1,x_2,x_3,t)=\left(\rho\,C_E\dot{\theta}+\gamma\,T_0\,\dot{e}\right)$ and ρ, C_E,γ, T_0 *and* e denote density, specific heat conduction, material constant, reference temperature and dilation respectively.

Theory of fractional order generalized thermoelasticity

Recently, fractional calculus has been introduced in the field of generalized thermoelasticity. Povstenko [98] constructed a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with a fractional order time derivative. He used the Caputo fractional derivative (Caputo [31]) and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. In 2010, Sherief et al. [106] was constructed a model in generalized thermoelasticity theory by using *fractional time-derivatives.*

The Duhamel–Neumann constitutive equations are

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e\delta_{ij} - \gamma(T - T_0)\delta_{ij}, \tag{1.60}$$

Equation of motion is

$$\sigma_{ij,j} + F_i = \rho\ddot{u}_i, \tag{1.61}$$

The heat conduction equation with fractional derivative heat transfer proposed by Sherief et al. [107] is

$$k\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right)(\rho c_E T + \gamma T_0 e - Q)\,, 0 < \alpha \leq 1, \tag{1.62}$$

where

$$\frac{\partial^\alpha}{\partial t^\alpha}f(x,t) = \begin{cases} f(x,t) - f(x,0), \quad \alpha \to 0, \\ I^{1-\alpha}\frac{\partial f(x,t)}{\partial t},\ 0 < \alpha < 1\ (Weak\ conductivity), \\ \frac{\partial f(x,t)}{\partial t},\ \alpha = 1\ (Normal\ conductivity). \end{cases}$$

In the above definition, the *Riemann–Liouville* fractional integral operator I^α is defined as-

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Equations (1.60)-(1.62) constitute a complete system of **fractional order generalized thermoelasticity**. This model can be applied to the classical generalization L-S model by setting $\alpha = 1$.

Basic relations and equations in magneto-elasticity:

Due to application of load, internal motion sets in a solid. In presence of an external strong magnetic field $\mathbf{H}$, the secondary electromagnetic fields due to their internal motion appear which interact again, in their turn, with the primary field. Having taken such interaction into account Maxwell's electrodynamic equations for an electromagnetically isotropic body, are given by Kaliski[80]. Maxwell's equations are as follows

$$\begin{aligned} \nabla \times \mathbf{h} &= \mathbf{J} + \varepsilon_0 \dot{\mathbf{D}}, \\ \nabla \times \mathbf{E} &= -\mu_0 \dot{\mathbf{h}}, \\ \nabla \cdot \mathbf{h} = 0,\ \nabla \cdot \mathbf{E} &= 0, \\ \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{h}),\ \mathbf{D} &= \varepsilon_0 \mathbf{E} \end{aligned} \tag{1.63}$$

Constitutive relations are

$$\begin{aligned} \mathbf{J} &= \eta_0[\mathbf{E} + \mu(\dot{\mathbf{u}} + \mathbf{H})] - \lambda' \nabla \theta + \rho_e \dot{\mathbf{u}}, \\ \mathbf{D} &= \varepsilon[\mathbf{E} + \frac{\varepsilon \mu c^2 - 1}{c^2}(\dot{\mathbf{u}} \times \mathbf{H})], \\ \mathbf{B} &= \mu \mathbf{H}, \end{aligned} \tag{1.64}$$

where $\mathbf{E}$ and $\mathbf{h}$ denote perturbations of the electric and magnetic fields respectively, $\mathbf{D}$ is the electric induction, $\mathbf{H}$ is the total magnetic field i.e., $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, $\mathbf{J}$ is the electric current, $\mathbf{u}$ is the displacement vector, ε & μ are electric and magnetic permeability of the medium respectively, η_0 is the electric conductivity, ρ_e is the change in density and λ' is the Thompson parameter.

The Lorentz force is given by

$$\mathbf{F} = \rho_e \mathbf{E} + \mu(\mathbf{J} \times \mathbf{H}). \tag{1.65}$$

1.2 *Different form of vector-matrix differential equations with solutions:*

Through our entire research work, we have solved some problems of generalized and magneto thermoelasticity. Using the Laplace transform, the governing equations of various generalized, magneto and fractional order thermoelasticity models are written in the form of a vector-matrix differential equation and then solved by eigenvalue approach. Some vector-matrix differential equations are given below:

- (i) $\frac{d\underline{v}}{dt} = \mathcal{A}\ \underline{v}$
- (ii) $\frac{d\underline{v}}{dx} = \underline{\underline{A}}\ \underline{v}\ +\ \underline{f}$
- (iii) $\mathcal{L}\underline{v} = \mathcal{A}\underline{v}$, $\mathcal{L} \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$.
- (iv) $L\tilde{x} = t^2\tilde{A}\tilde{x}$

Type-i:

Consider the vector-matrix differential equation of the form

$$\frac{d\underline{v}}{dt} = \mathcal{A}\underline{v} \tag{1.66}$$

with the initial condition

$$\underline{v}(0) = c, \tag{1.67}$$

where $\mathcal{A}$ is an $n \times n$ constant matrix and $\underline{v}$ is a column vector with n components which are functions of the variable t and c is a given constant n-vector.

Let us now assume that $\underline{v}(t) = \mathcal{X}\exp(\lambda^* t)$ where $\mathcal{X}$ is a non-zero n-vector independent of t and λ^* is a scalar. Then

$$\frac{d\underline{v}}{dt} - \mathcal{A}\underline{v} = \lambda^*\mathcal{X}\exp(\lambda^* t) - \mathcal{A}\mathcal{X}\exp(\lambda^* t) \tag{1.68}$$

$$= -(\mathcal{A}\mathcal{X} - \lambda^*\mathcal{X})\exp(\lambda^* t). \tag{1.69}$$

This shows that if λ^* is an eigenvalue of the matrix $\mathcal{A}$ and $\mathcal{X}$ is the corresponding eigenvector.

Hence $\underline{v}(t) = \mathcal{X}\exp(\lambda^* t)$ is a solution of the differential Eq. (1.66). Let $\lambda_1^*, \lambda_2^*, ..., \lambda_n^*$ be distinct eigenvalues of the matrix $\mathcal{A}$ and $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$ be the corresponding eigenvectors. Then clearly $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n$ form a basis of the space E^n, where E^n denotes the complex n-dimensional Euclidean space. So the vector c can be expressed in the form

$$c = c_1\mathcal{X}_1 + c_2\mathcal{X}_2 + ... + c_n\mathcal{X}_n, \tag{1.70}$$

where $c_1, c_2, ..., c_n$ are scalars.

Let

$$\underline{v}(t) = c_1\mathcal{X}_1\exp(\lambda_1^* t) + c_2\mathcal{X}_2\exp(\lambda_2^* t) + ... + c_n\mathcal{X}_n\exp(\lambda_n^* t). \tag{1.71}$$

Then $\underline{v}(t)$ is a solution of (1.66) and

$$\underline{v}(0) = c_1\mathcal{X}_1 + c_2\mathcal{X}_2 + ... + c_n\mathcal{X}_n = c. \tag{1.72}$$

Hence $\underline{v}(t)$ is the unique solution of (1.66) satisfying (1.67).

Type-ii:

Consider the vector-matrix differential equation

$$D\underline{v} = \mathcal{A}\underline{v} + f(x) \tag{1.73}$$

with the initial condition

$$\underline{v}(x_0) = c, \tag{1.74}$$

where $D \equiv \frac{d}{dx}$, $\mathcal{A}$ is a $n \times n$ constants real matrix, c is a real constant n-vector and $f(x)$ is real n-vector function. Let $f = (f_1, f_2, ..., f_n)^T$, $f_1, f_2, ..., f_n$ are scalar functions of x, $\underline{v} = (v_1, v_2, ..., v_n)^T$ and $\mathcal{A} = (a_{ij}),\ i, j = 1, 2, ..., n$.

Substituting $\mathcal{A} = \mathcal{V}\Lambda\mathcal{V}^{-1}$ in equation(1.78) and pre-multiplying the resulting equation by $\mathcal{V}^{-1}$, we obtain
$\mathcal{V}^{-1}D\underline{v} = \Lambda(\mathcal{V}^{-1}\underline{v}) + \mathcal{V}^{-1}f$
$\Rightarrow D(\mathcal{V}^{-1}\underline{v}) = \Lambda(\mathcal{V}^{-1}\underline{v}) + \mathcal{V}^{-1}f,$
where Λ is a diagonal matrix of order n whose elements are $\lambda_1, \lambda_2, ..., \lambda_n$, the distinct eigenvalues of $\mathcal{A}$. Let $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_n$ be the eigenvectors of $\mathcal{A}$ corresponding to $\lambda_1, \lambda_2, ..., \lambda_n$ respectively and
$\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_n) = (x_{ij}), \text{say}, i, j = 1, 2, ..., n.$

Substituting $\underline{y} = \mathcal{V}^{-1}\underline{v}$, we require to solve the system of equations,

$$D\underline{y} = \Lambda\underline{y} + \mathcal{V}^{-1}f. \tag{1.75}$$

Clearly, the above equation represents a set of n-decoupled ordinary differential equations. A typical r-th equation of this set may be taken as

$$Dy_r = \lambda_r y_r + Q_r,\ Q_r = V_r^{-1}f \tag{1.76}$$

Let $\mathcal{V}^{-1} = w_{ij},\ i, j = 1, 2, ..., n.$ Then we can obtain

$$Q_r = \sum_{j=1}^{n} w_{jr}f_j,\ r = 1, 2, ..., n. \tag{1.77}$$

The solution of the Eq. (1.76) may be written as-

$$y_r = e^{\lambda_r x}\left[y_r e^{-\lambda_r x}\right]_{x_0} + e^{\lambda_r x}\int_{x_0}^{x} Q_r e^{-\lambda_r x}dx. \tag{1.78}$$

Thus the complete solution of the Eq. (1.73) can now be written now as

$$\underline{v} = \sum_{r=1}^{n} \mathcal{V}_r y_r \tag{1.79}$$

Alternative process:

Considering a system of simultaneous differential equations in the form

$$\frac{d\vec{v}}{dx} = \vec{A}\vec{v} + \vec{f} \tag{1.80}$$

where $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T$, $\vec{A} = [a_{ij}]_{n\times n}$ and $\begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}^T$. Let us consider the coefficient matrix $\vec{A}$ can be written as

$$A = V\Lambda V^{-1} \tag{1.81}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ and $V = \begin{pmatrix} V_1 & V_2 & \cdots & V_n \end{pmatrix}$

Here $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigen values of the coefficient matrix A. $V_1, V_2, \cdots, V_n$ are the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

Now multiplying the equation (1.80) by V^{-1} we get $V^{-1}\frac{d\vec{v}}{dx} = V^{-1}(V\Lambda V^{-1})\vec{v} + V^{-1}\vec{f}$

$$\frac{d(V^{-1}\vec{v})}{dx} = \Lambda(V^{-1}\vec{v}) + V^{-1}\vec{f}$$

$$\frac{d\vec{y}}{dx} = \Lambda\vec{y} + V^{-1}\vec{f} \tag{1.82}$$

where $\vec{y} = V^{-1}\vec{v} \Rightarrow \vec{v} = V\vec{y}$

The r-th equation of (1.82) is

$$\frac{dy_r}{dx} = \lambda_r y_r + Q_r \tag{1.83}$$

where $Q_r = V_r^{-1}\vec{f}, \quad V_r^{-1} = (\omega_{ij})$

$$Q_r = \sum_{i=1}^{n} \omega_{ri} f_i$$

The solution of (1.83) is

$$y_r = c_r e^{\lambda_r x} + e^{\lambda_r x}\int Q_r e^{-\lambda_r x} dx \tag{1.84}$$

So, $\vec{v} = \sum_{r=1}^{n} V_r y_r$

Another process: Consider the Vector-matrix differential equation

$$\frac{dv}{dx} = A\ v + f(x) \tag{1.85}$$

with the condition

$$v(x_0) = C \tag{1.86}$$

where A is an $n \times n$ constants real matrix, C is given constant real n vector and f is real n vector function.
Let

$$v = X\ exp(\lambda x) \tag{1.87}$$

be the solution of the homogeneous equation.

$$\frac{dv}{dx} = A\ v \tag{1.88}$$

where λ is a scalar and X is an n vector independent of x. Substituting (1.87) in equation(1.88), we get,

$$(A\ X - \lambda\ X)e^{\lambda x} = 0 \Rightarrow A\ X = \lambda X \tag{1.89}$$

This may be interpreted that λ is an eigenvalue of the matrix A and X the corresponding right eigenvector. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be n distinct eigenvalues of the matrix A and $X_1, X_2, X_3, ..., X_n$ be the corresponding right eigenvector of the matrix A. Then the vectors $X_1, X_2, X_3, ..., X_n$ are linearly independent and so they form a basis of the space Γ^n ,where Γ denotes the field of complex numbers. We can find the scalers $b_1, b_2, b_3, ..., b_n$ such that
$C = b_1 X_1 + b_2 X_2 + b_3 X_3 + ... + b_n X_n$
Choose $c_i = b_i e^{\lambda_i x_0}$, $\quad (i = 1, 2, 3, ..., n)$
Let

$$u(x) = \sum_{i=1}^{n} c_i X_i\ e^{\lambda_i x} \tag{1.90}$$

Thus u(x) is the solution of the equation(1.88) and

$$u(x_0) = \sum_{i=1}^{n} c_i X_i\ e^{\lambda_i x_0} = \sum_{i}^{n} b_i X_i = C \tag{1.91}$$

Now,let

$$w(x) = \sum_{i=1}^{n} a_i(x) X_i \ e^{\lambda_i x} \tag{1.92}$$

be the solution of equation(1.85), $a_1(x), a_2(x), a_3(x), ..., a_n(x)$ are scalar function of x such that $a_i(x_0) = 0$, Differentiating equation(1.92) with respect to x, we get

$$w'(x) = \sum_{i=1}^{n} a_i'(x) X_i \ e^{\lambda_i x} + \sum_{i=1}^{n} a_i(x) \lambda_i X_i \ e^{\lambda_i x} \tag{1.93}$$

Substituting equations(1.92) and (1.93) in equation(1.85), we have

$$\begin{aligned} \sum_{i=1}^{n} a_i'(x) X_i \ e^{\lambda_i x} &+ \sum_{i=1}^{n} a_i(x) \lambda_i X_i \ e^{\lambda_i x} \\ &= \sum_{i=1}^{n} a_i(x) A X_i \ e^{\lambda_i x} + f(x) \end{aligned} \tag{1.94}$$

or,

$$\begin{aligned} \sum_{i=1}^{n} a_i'(x) X_i \ e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x) [A X_i - \lambda_i X_i] \ e^{\lambda_i x} \\ + f(x) = f(x) \end{aligned} \tag{1.95}$$

Multiplying equation(1.95) by $Y_j \ e^{-\lambda_j x}$ (where $Y_1, Y_2, Y_3, ..., Y_n$ are left eigenvector corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$), we get

$$\sum_{i=1}^{n} a_i'(x) Y_j \ e^{(\lambda_i - \lambda_j) x} = Y_j \ f(x) \ e^{-\lambda_j x} \tag{1.96}$$

or,

$$\begin{aligned} a_j'(x) Y_j X_j = Y_j \ f(x) \ e^{-\lambda_j x} \ , \ [Y_j X_j = 0 \quad for \ i \neq j] \\ a_j'(x) = \frac{1}{Y_j X_j} Y_j \ f(x) \ e^{-\lambda_j x} \\ a_j(x) = \int_{x_0}^{x} (Y_j X_j)^{-1} Y_j \ f(x) \ e^{-\lambda_j s} ds \end{aligned} \tag{1.97}$$

$[a_j(x_0) = 0, \quad for \quad j = 1, 2, 3, ..., n]$

Now take

$$v(x) = u(x) + w(x) \tag{1.98}$$

Differentiating we get,

$v'(x) = u'(x) + w'(x) = Au(x) + Aw(x) + f(x) = A[u(x) + v(x)] + f(x) = Av(x) + f(x)$

$v'(x_0) = u'(x_0) + w'(x_0) = C$
Hence, $v(x) = u(x) + w(x)$ is the unique solution of the differential equation(1.85), satisfying the condition (1.86).

Type-iii:
Consider the following vector-matrix differential equation

$$\mathcal{L}\underline{v} = \mathcal{A}\underline{v}, \tag{1.99}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} \equiv \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{x^2}. \tag{1.100}$$

This operator L is of frequent occurrence in problems on cylinders.
Let

$$A = V\Lambda V^{-1} \tag{1.101}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ is a diagonal matrix whose elements λ_1,λ_2, ... λ_n are the distinct eigenvalues of A. Let $\underset{\sim}{V_1}$, $\underset{\sim}{V_2}$, ..., $\underset{\sim}{V_n}$ be the eigenvectors of A corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, and
$V = \left[\underset{\sim}{V_1}, \;\; \underset{\sim}{V_2}, \;\; \ldots.., \;\; \underset{\sim}{V_n}\right]$ = (x_{ij}) (say); i, j = 1, 2, . . . , n.
Substituting (1.101) in (1.99) and pre-multiplying by V^{-1}, we get

$$\mathcal{L}\underset{\sim}{y} = \Lambda\underset{\sim}{y}, \qquad \text{where} \quad \underset{\sim}{y} = V^{-1}\underset{\sim}{v} \tag{1.102}$$

as a system of decoupled equations.
A typical r^{th} equation of (1.102) is
$\mathcal{L}y_r = \lambda_r \, y_r$
Or,

$$\frac{d^2y_r}{dx^2} + \frac{1}{x}\frac{dy_r}{dx} - \left(\lambda_r + \frac{n^2}{x^2}\right)y_r = 0 \tag{1.103}$$

<u>Case (i)</u>
When$\lambda_r = \alpha_r^2$, the solution of equation (1.103) can be written as,
$y_r = A_rK_n(\alpha_r x) + B_rI_n(\alpha_r x)$,

n is an integer and A_r, B_r are constants. K_n , I_n are modified Bessel functions of the second kind of order n.

__Case (ii)__

When $\lambda_r = -\, \alpha_r^2$, the solution can be written as

$y_r = A_r\, J_n\, (\alpha_r x) + B_r\, y_n(\alpha_r x)$, n is integral

J_n, y_n are Bessel functions of the first kind of order n.

Hence the complete solution in this case can be written as $\underset{\sim}{v} = \sum_{r=1}^{n} V_r\, \mathrm{y_r}$

Type-iv:

Let the Vector-matrix differential equation be of the form

$$L\tilde{x} = t^2 \tilde{A}\tilde{x} \tag{1.104}$$

where L is the linear second order differential operator and

$$L = t^2 \frac{d^2}{dt^2} + tp(t)\frac{d}{dt} + q(t) \tag{1.105}$$

$\tilde{A}$ is an $n \times n$ constant matrix and $p(t)$,$q(t)$ are two real valued continuous on [0,1]. The initial conditions are assumed as

$$\tilde{x}(1) = \tilde{a} \quad and \quad \tilde{x'}(1) = \tilde{b} \tag{1.106}$$

where $\tilde{x}$,$\tilde{a}$and $\tilde{b}$ are n-vectors.

Assume that $\tilde{x}(t) = \tilde{X}(\lambda)\omega(t,\lambda)$ be a solution of the equation (1.104), where λ is scalar, $\tilde{X}$is an n-vector independent of t and $\omega(t,\lambda)$ is a non-trivial solution of scalar differential equation

$$t^2 \frac{d^2y}{dt^2} + tp(t)\frac{dy}{dt} + q(t) = t^2\lambda y$$
$$i.e. \quad Ly = \lambda t^2 y \tag{1.107}$$

Applying the operator L on x, we get

$$\begin{aligned} L\tilde{x} &= L(\tilde{X},\omega) = \tilde{X}L\omega \\ &= \tilde{X}(t^2\lambda\omega) = \lambda t^2 \tilde{X}\omega \end{aligned} \tag{1.108}$$

Thus the equation (1.104) becomes

$$\begin{aligned} \lambda t^2 \tilde{X}\omega &= t^2 \tilde{A}(\tilde{X}\omega) = t^2(\tilde{A}\tilde{X})\omega \\ or \quad t^2(\lambda\tilde{X} &- \tilde{A}\tilde{X})\omega = 0 \end{aligned} \tag{1.109}$$

Since $\omega(t,\lambda)$ is non-trivial and $(\lambda\tilde{X} - \tilde{A}\tilde{X})$ is independent of t, it follows that

$$\lambda\tilde{X} = \tilde{A}\tilde{X} \tag{1.110}$$

This gives rise to the algebraic eigenvalue problem where λ is the eigenvalue of the matrix $\tilde{A}$ and $\tilde{X}$ is the corresponding eigenvector.
Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the distinct eigenvalues of the matrix $\tilde{A}$ and let $\underbrace{X_1}, \underbrace{X_2}, ..., \underbrace{X_n}$ be the corresponding eigenvectors. Then $\underbrace{X_1}, \underbrace{X_2}, ..., \underbrace{X_n}$ are linearly independent and so they form a basis of the space Γ^n ,where Γ is the field of the complex numbers.
We can find the scalars $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ such that
$\underbrace{a} = a_1X_1 + a_2X_2 + ... + a_nX_n \quad and \quad \underbrace{b} = b_1X_1 + b_2X_2 + ... + b_nX_n$
Let $u(t,\lambda_i)$ and $v(t,\lambda_i)$ denote two linearly independent solutions of the differential equations

$$Ly = \lambda_i t^2 y$$

With the initial conditions $\quad u(1,\lambda_i) = 1, u^{'}(1,\lambda_i) = 0 \quad and \quad v(1,\lambda_i) = 1, v^{'}(1,\lambda_i) = 1$
Now let

$$\underbrace{x(t)} = \sum_1^n \underbrace{X_i}[a_i u(t,\lambda_i) + b_i v(t,\lambda_i)] \tag{1.111}$$

Clearly $\underbrace{x(t)}$ is the solution of the differential equation (23) and

$$\begin{aligned} \underbrace{x(1)} &= \sum_1^n \underbrace{X_i}[a_i u(1,\lambda_i) + b_i v(1,\lambda_i)] \\ &= \sum_1^n a_i X_i = \underbrace{a} \end{aligned} \tag{1.112}$$

$$\begin{aligned} \underbrace{x^{'}(1)} &= \sum_1^n \underbrace{X_i}[a_i u^{'}(1,\lambda_i) + b_i v^{'}(1,\lambda_i)] \\ &= \sum_1^n b_i X_i = \underbrace{b} \end{aligned} \tag{1.113}$$

Therefore $\underbrace{x(t)}$ is given by (1.111), is the unique solution of the system(1.104) and (1.106).

1.3 ***Numerical Inversion of Laplace Transform:***

The inversion of laplace transform in numerical computation usually performed by two methods-
a. Bellman method [29] and
b. Zakian Method [127]

a. Bellman method:

Through our research work, the numerical inversion of Laplace transform is carried out by *Bellman Method.*

Laplace transform $F(p)$ (say) of a function $u(t)$ is defined as

$$F(p) = \int_0^\infty u(t)\exp(-pt)dt \tag{1.114}$$

We assume that $u(t)$ is sufficiently smooth to permit the approximate method so that we employ to put $x = e^{-t}$ in equation(1.114) which gives

$$F(p) = \int_0^\infty x^{p-1}g(x)dx \tag{1.115}$$

where $u(-\log x) = g(x)$.

Applying the Gaussian quadrature formula in Eq. (1.115), we get

$$\sum_{i=1}^{n} W_i x_i^{p-1} g(x_i) = F(p) \tag{1.116}$$

where x_i are the roots of the shifted Legendre polynomial $P_N(x) = 0$ and W_i are the corresponding coefficients. Thus x_i and W_i are known.

The equation (1.115) can be written as

$$W_1 x_1^{p-1} g(x_1) + W_2 x_2^{p-1} g(x_2) + ... + W_N x_N^{p-1} g(x_N) = F(p) \tag{1.117}$$

We now put $p = 1, 2, 3, ..., N$ in (1.116), then the resulting equations become

$$\begin{aligned} W_1 g(x_1) + W_2 g(x_2) + .. + W_N g(x_N) &= F(1) \\ W_1 x_1 g(x_1) + W_2 x_2 g(x_2) + .. + W_N x_N g(x_N) &= F(2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ W_1 x_1^{N-1} g(x_1) + W_2 x_2^{N-1} g(x_2) + .. + W_N x_N^{N-1} g(x_N) &= F(N) \end{aligned}$$

Thus, we can write the above system of equations in the matrix form as

$$\begin{bmatrix} g(x_1) \\ g(x_2) \\ .... \\ g(x_N) \end{bmatrix} = \begin{bmatrix} W_1 & W_2 & .... & W_N \\ W_1 x_1 & W_2 x_2 & .... & W_N x_N \\ .... & .... & .... & .... \\ W_1 x_1^{N-1} & W_2 x_2^{N-1} & .... & W_N x_N^{N-1} \end{bmatrix}^{-1} * \begin{bmatrix} F(1) \\ F(2) \\ .... \\ F(N) \end{bmatrix}$$

Hence $g(x_1), g(x_2), ..., g(x_N)$ can be evaluated by solving the above system.

Now, $u(-logx_1) = g(x_1), u(-logx_2) = g(x_2), ..., u(-logx_N) = g(x_N)$.
For N=9,

Roots of the Legendre Polynomial x_i'	$u(-logx_i) = g(x_i)$
$x_1' = -0.968160239$	4.140186636
$x_2' = -0.836031107$	2.501225729
$x_3' = -0.613371432$	1.643438002
$x_4' = -0.324253423$	1.085084341
$x_5' = 0.000000000$	0.693147180
$x_6' = 0.324253423$	0.412298334
$x_7' = 0.613371432$	0.214821133
$x_8' = 0.836031107$	0.085540945
$x_9' = 0.968160239$	0.016047962

b. Zakian method:

Let $F(p)$ be the Laplace transform of $f(t)$ be given by

$$F(p) = \int_0^{\infty} f(t)exp(-pt)dt \quad ; \quad Re(p) > \sigma \tag{1.118}$$

Thus it assumed that $f(t)$ is integrable and of exponential order σ.
Let $\delta(\frac{\lambda}{t} - 1)$ denote the scaled delta function defined by

$$\int_0^T \delta(\frac{\lambda}{t} - 1)d\lambda = t \quad ; \quad 0 < t < T \tag{1.119}$$

$$\delta(\frac{\lambda}{t} - 1) = 0 \quad ; \quad t \neq \lambda \tag{1.120}$$

$$I = \frac{1}{t}\int_0^T f(\lambda)\delta(\frac{\lambda}{t} - 1)d\lambda \quad ; \quad 0 < t < T \tag{1.121}$$

Making use of the property of the delta function given in equation (1.120), whenever t is a point of continuity of f, we can replace the integrand of equation (1.121) by $f(t)\delta(\frac{\lambda}{t} - 1)$ and therefore

$$I = \frac{f(t)}{t}\int_0^T \delta(\frac{\lambda}{t} - 1)d\lambda \quad ; \quad 0 < t < T \tag{1.122}$$

Hence, using equations (1.119) and (1.121), we obtain the shifting integral associated with $\delta(\frac{\lambda}{t}-1)$;

$$f(t)=\frac{1}{t}\int_0^T f(\lambda)\delta(\frac{\lambda}{t}-1)d\lambda \quad ; \quad 0<t<T \tag{1.123}$$

At those points t where the function f jumps discontinuously from $f(t-)$ to $f(t+)$, then left hand side of equation (1.123) should be replaced by $\frac{1}{2}\{k_1 f(t-)+k_2 f(t+)\}$, where k_1 and k_2 are real non negative constants such that $k_1+k_2=2$. In particular, $k_1=k_2$ if $\delta(\frac{\lambda}{t}-1)$ is defined as the 'limit' of a sequence of functions which are symmetrical about the vertical line $\lambda=t$.
It can be proved that the scaled delta function $\delta(\frac{\lambda}{t}-1)$ can be expanded into the series

$$\delta(\frac{\lambda}{t}-1)=\sum_{i=1}^{\infty}k_i exp(-\alpha_i\frac{\lambda}{t}) \tag{1.124}$$

More precisely, it can be shown that a sequence of functions $\{\delta_N(\frac{\lambda}{t}-1)\}$ exists, so that at every continuity point t of f,

$$f(t)=\lim_{N\to\infty}f_N(t)\ ; \qquad 0<t<T \tag{1.125}$$

where

$$f_N(t)=\frac{1}{t}\int_0^T f(\lambda)\delta_N(\frac{\lambda}{t}-1)d\lambda \tag{1.126}$$

$$\delta_N(\frac{\lambda}{t}-1)=\sum_{i=1}^{N}k_i\ exp(-\alpha_i\frac{\lambda}{t}) \tag{1.127}$$

(a) the constants α_i and k_i are either real or occur in complex conjugate pairs, e.g., $\alpha_i=\alpha_2^*$, and hence $k_1=k_2^*$.
(b) α_i and k_i depend on N.
(c) as $N\to\infty$, so also $Re(\alpha_i)\to\infty$ and $\mid k_i\mid\to\infty$
(d) $Re(\alpha_i)>0$.
(e) the α_i are distinct, i.e., $\alpha_i=\alpha_j$ if and only if $i=j$.

From equations (1.126) and (1.127), we get

$$f_N(t)=\frac{1}{t}\int_0^T f(\lambda)\sum_{i=1}^{N}k_i exp(-\alpha_i\frac{\lambda}{t})d\lambda \tag{1.128}$$

Hence

$$f_N(t) = \frac{1}{t}\sum_{i=1}^{N} k_i \int_0^T f(\lambda)exp(-\alpha_i \frac{\lambda}{t})d\lambda \tag{1.129}$$

Allowing $T \to \infty$ and using equation (1.118), we obtain

$$f_N(t) = \frac{1}{t}\sum_{i=1}^{N} k_i F(\frac{\alpha_i}{t}) \;\; ; \;\; 0 < t < t_c \tag{1.130}$$

where

$$t_c = \min_{i=1,2,\ldots,N}\{Re(\frac{\alpha_i}{\sigma})\} \;\; ; \;\; \sigma > 0 \tag{1.131}$$

As $N \to \infty$, $Re(\alpha_i) \to \infty$, and hence $t_c \to \infty$.
Therefore, using equation (1.149), we obtain the explicit inversion formula

$$f(t) = \lim_{N\to\infty} \frac{1}{t}\sum_{i=1}^{N} k_i F(\frac{\alpha_i}{t}) \;\; ; \;\; 0 < t < \infty \tag{1.132}$$

A number of methods for obtaining optimal sets of constants α_i and k_i are being investigated and we get the following results when $A_i = \frac{k_i}{\alpha_i}$.

For N=10

i	α_i	A_i
1	$5.2038 - 15.7212i$	$-10.15471 - 4.260437i$
2	$5.2038 + 15.7212i$	$-10.15471 + 4.260437i$
3	$8.7980 - 11.9391i$	$189.2250 + 250.7353i$
4	$8.7980 + 11.9391i$	$189.2250 - 250.7353i$
5	$10.9343 - 8.4096i$	$-866.2283 - 2313.588i$
6	$10.9343 + 8.4096i$	$-866.2283 + 2313.588i$
7	$12.2261 - 5.0127i$	$1560.540 + 8422.502i$
8	$12.2261 + 5.0127i$	$1560.540 - 8422.502i$
9	$12.8376 - 1.666i$	$-872.8822 - 15431.37i$
10	$12.8376 + 1.666i$	$-872.8822 + 15431.37i$

2

ANALYSIS OF THERMOELASTIC MEDIUM IN THE CONTEXT OF MULTI PHASE LAG MODEL

PROBLEMS :

- PROBLEM -1: **On a multi-phase lag model of three-dimensional coupled thermoelasticity in an anisotropic halfspace.**

  Published in *Waves in Random and Complex Media*, Published online: 06 Jul 2022., DOI: $https://doi.org/10.1080/17455030.2022.2094025$.

- PROBLEM -2: **Modeling of a homogeneous isotropic half space in the context of multi-phase lag coupled thermoelasticity.**

  Published in *Mechanics of Time-Dependent Materials* , 21st January, 2022 , $DOI : https://doi.org/10.1007/s11043-022-09584-7$.

2.1 On a multi-phase lag model of three-dimensional coupled thermoelasticity in an anisotropic halfspace

2.1.1 Introduction

For last several decades, it has been seen that non-isothermal problems in the theory of elasticity has become popular due to its effective application in different diverse filed such as aircraft engineering, nuclear reactors etc. Generally, the thermoelasticity theory deals with the direct and inverse impact of heat on the elastic medium due to deformation. When the structural characteristics in a medium is compared with either the time rate of variation of thermal boundary conditions or the time rate of variation of a heat source, the thermal stress is generated. Using this concept, the solutions to the problems associated with physical variables like- stress components, temperature with respect to space variables are obtained using coupled thermoelasticity equations.

The uncoupled theory of thermoelasticity, known as classical thermoelasticity theory predicts two non-compatible phenomena about heat conduction equation. They are a) heat conduction not containing any elastic term and b) heat equation predicting infinite speed of heat wave propagation (parabolic type). To overcome the paradox of infinite propagation of heat wave in classical thermoelasticity, Biot [30] proposed the theory of coupled thermoelasticity (CTE) using Fourier's Law. However, the resulting coupled equation is a mixed parabolic-hyperbolic type as equation of motions are of hyperbolic type and the thermal equation is of diffusion (parabolic) type. Wang and Melnick [120] proposed a general procedure to reform coupled problems of dynamical thermoelasticity to differential- algebraic systems.To overcome the ambiguity in CTE, various researcher proposed a modified dynamical thermoelasticity, known as generalized thermoelasticity.

Both of coupled and uncoupled thermoelasticity problems had been solved using potential functions. But this method is not always applicable as studied by Sherief and Anwar [106] and Dhaliwal and Sheriff [44]. The reason behind it is the initial and boundary conditions to a problem are associated to physical quantities but not the potential function. In this regard, the alternative methods are (a) Eigen value approach: it is the method where a vector-matrix differential equation is obtained from the governing equations and the solution of the corresponding field variables are derived applying eigen value approach and (b) State- space approach: in this process, the coefficient matrix of field variables expanded in a series and the solutions are obtained using Caley-Hamilton method.

Different engineering model proposed to determine the stress, strain, displacement and temperature are three dimensional. Also, the three-dimensional anisotropic wave propagation of heat leads to study of three-dimensional anisotropic elasticity to analyze the mechanical behavior of the materials like microcline, rhodonite, Vosges sandstone, turquoise etc.

The first generalized thermoelasticity theory was proposed by Lord and Shulman [80] known as L-S model. In L-S model, the Fourier law of heat conduction was replaced by Maxwell- Cattanew Law by involving one relaxation time parameter. Green-Lindsay [58] proposed the second generalized thermoelasticity introducing two relaxation time parameters to modify both motion equations and thermal equation. It has been seen that Fourier law and Green-Lindsay theory can be obtained as a particular case of the other though they are structurally different. Later on, Hetnarski- Ignaczak [63] proposed third generalization which is known as theory of low-temperature thermoelasticity. By considering thermal displacement gradient in the constitutive equations, Green- Naghdi introduced a new theory of thermoelasticity as fourth generalized thermoelasticity. The proposed models are known as G-N model of type-I, type-II and type-III. Type-I model [59] can be considered as identical as classical heat equation based on Fourier's Law. Type II [60] and type-III [61] theories are comprising of finite heat wave propagation. Type-II theory does not accommodate energy dissipation whereas type-III theory is compatible with energy dissipation of thermal energy. Several studies have been followed to investigate energy dissipation in G-N theory by Sharma and Chouhan [105], Chandrashekhariah and Srinath [35], Mukhopadhay and Kumar [92] etc. In fifth generalization, Dual Phase Lag (DPL) theory was introduced by Tzou[113][117] and Chandrashekhariah[33].

In DPL, Tzou considered both of the heat flux vector and the temperature gradient in the delayed response in time to inspect the micro-structural effects due to photo-electron interaction in the macroscopic level. According to the DPL model, the constitutive equation in classical Fourier law,$q = -K\nabla T$ is replaced by $q(P, t + \tau_q) = -K\nabla T(P, t + \tau_T)$ where K is the thermal conductivity with temperature gradient ∇T

, at a point P in the medium at a time $t + \tau_T$ for the heat flux q at the same point at the time $t + \tau_q$ where τ_q is the phase lag variable for the heat flux which is the relaxation time due to the transient effects of thermal inertia. Recently Roychowdhury [102] introduced the theory of three phase lag modifying the Fourier Law as $q(P, t + \tau_q) = -K\nabla T(P, t + \tau_T) + K^*\nabla V(P, t + \tau_v)$, where τ_v is the another phase lag variable here which is associated to thermal displacement gradient. Using DPL, 3PHL models several studies have been done by Ghosh et al.[49], Kar and Kanoria [71]. Also, Abbas [3], Ghosh et al. [52] discussed the 3 PHL model considering fiber-reinforced medium in their respective studies. Abd-Alla et. al[5] investigated the surface wave propagation in fiber-reinforce anisotropic half space in context of fractional order thermoelasticity Abouelregal and Abo-Dahab [9] proposed a model of Rayleigh waves on granular medium under influence of initial stress in presence of gravitational field. Ghazanfarian and Somali [48] and Ghosh and Lahiri[50] studied transient temperature in a two-dimensional DPL model of heat conduction . Nayfeh and Nasser [94] studied the plane wave propagation in a solid considering the generalized magneto-thermoelastic field. Sarkar et. al[104] used generalized thermoelasticity to solve a boundary value problem for a isotropic medium to obtain the solution for displacement, stress, strain, temperature using Laplace transformation. A general theoretical analysis of thermoelastic (TE) mechanism due to photothermal generation and electronic deformation (ED) effects in a semiconductor medium during photothermal process is studied by Das et. al.[41]

Recently, Riha[101]investigated the heat conduction in microtemperatures for a thermoelastic medium. The theory of microtemperatures deals with the Nano materials which has a great impact in engineering field as the microelements of a thermoelastic solid have different temperatures. Aouadi [23] and Iesan and Quintanilla [64] studied some theories of microstretch thermoelastic linear solids with microtemperatures and inner structure. Zenkour [128] investigated a refined multiphase lag (RPL) theory in thermoelasticity and proposed a model to reduce the previous cases of phase lag models to special cases.

There are various vital approaches for determining the solutions of the governing equations. Akbar M.A et. al.[18] studied the mechanics of closed-form wave solutions using Kudryashov method . Again The nonlinear interaction between Langmuir waves and electrons produce non-linear parabolic law which is known as cubic-quintic nonlinearity. To solve this type of problem Akinyemi L. et. al.[19] have taken help of generalized auxiliary equation . Ahmad H. et. al.[15] approached for fractional iteration algorithm to solve non-linear fractional order differential equations. Also Ahmad H. et. al.[17] solved three different types of Burgers' equations with the help of variational iteration algorithm-II which is a modification of variational iteration algorithm-I whereas Mohammed and Abbo [84] studied variational iteration algorithm-I to solve Learning Fuzzy Neural Networks problem using Conjugate Gradient Method. As in Bazighifan, Ahmad and Yao [28], since previous decade, Riccati transformation is being used to study the oscillation theory consisting of higher order advanced differential equations. Using meshless method with radial basis functions, Wang et. al.[119] proposed a direct meshless scheme on Gaussian radial basis function to solve both linear and non-linear convection-diffusion problem. Also Ahmad, Alam and Omri [16] studied the modified (G'/G)-expansion method to obtain the numerical solutions for a non-linear biological model.

In our recent studies we have investigated a thermoelastic anisotropic medium in the context of RPL theory. Here we tried to make generalization of phase lag variables and also we have shown special cases drawn from the RPL .

2.1.2 Formulation of The Problem

In the orthogonal co-ordinate system $x_i (i = 1, 2, 3)$, we consider three dimensional anisotropic half space defined in the region $\Omega = \{0 \leq x_1 < \infty, -\infty < x_2 < \infty, -\infty < x_3 < \infty\}$ subject to fraction free boundary $x_1 = 0$, x_1-axis is vertically downwards and $x_2 x_3$ along the free surface of the anisotropic half space.

We may assume, without any loss of generality, that the propagation of waves along x_1 direction , and as such field variables are functions of x_1, x_2, x_3 and t only i.e. $u = [u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t)]$ and temperature field $\theta = \theta(x_1, x_2, x_3, t)]$. Assuming that a body having mass density ρ is in unstressed and undeformed state at the constant reference temperature θ_0. The body undergoes deformation $u_i, i = 1, 2, 3$ and respective temperature increment acquired $(\theta - \theta_0)$, the absolute temperature θ is chosen such that $|\frac{\theta - \theta_0}{\theta_0}| \ll 1$, under the influence of thermal and mechanical stresses act on the free surface.

In the absence of body force and heat source, the equations of motion and the heat

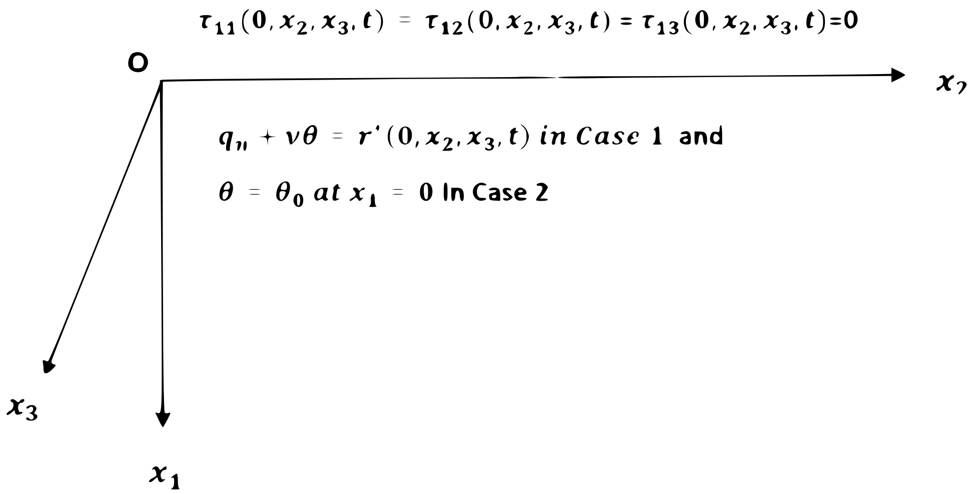


Figure 1: Schematic diagram

conduction equation in multiphase lag model are as follows:

$$\begin{aligned}
\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} &= \rho \frac{\partial^2 u_1}{\partial t^2} \\
\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} &= \rho \frac{\partial^2 u_2}{\partial t^2} \\
\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2}
\end{aligned} \tag{2.1}$$

$$\left(1 + \sum_{n=1}^{N} \frac{\tau_\theta^n}{n!} \frac{\partial^n}{\partial t^n}\right) \left[K^* \left(\frac{\partial^2 \theta}{\partial x_1^2} + \frac{\partial^2 \theta}{\partial x_2^2} + \frac{\partial^2 \theta}{\partial x_3^2}\right)\right]$$

$$= \left[\overline{R} + \tau_0 \frac{\partial}{\partial t} + \sum_{n=1}^{N} \frac{\tau_q^n + 1}{(n+1)!} \frac{\partial^{n+1}}{\partial t^(n+1)}\right] \left[\rho c_E \frac{\partial^2 \theta}{\partial t^2} + \gamma \theta_0 \frac{\partial^2}{\partial t^2} (\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3})\right] \tag{2.2}$$

The stress-strain-temperature relations are given by

$$\tau_{ij} = c_{ijkl} e_{kl} - \beta ij(\theta - \theta_0 + \tau_1 \dot{\theta}) \tag{2.3}$$

where $\beta_{ij} = c_{ijkl} \alpha_{kl}$

Using Hooke's law, stress-strain-temperature relations are given as

$$\begin{aligned}
\tau_{11} &= c_{11} e_{11} + c_{12} e_{22} + c_{13} e_{33} + 2(c_{14} e_{23} + c_{15} e_{13} + c_{16} e_{12}) \\
\tau_{22} &= c_{21} e_{11} + c_{22} e_{22} + c_{23} e_{33} + 2(c_{24} e_{23} + c_{25} e_{13} + c_{26} e_{12}) \\
\tau_{33} &= c_{31} e_{11} + c_{32} e_{22} + c_{33} e_{33} + 2(c_{34} e_{23} + c_{35} e_{13} + c_{36} e_{12}) \\
\tau_{12} &= c_{41} e_{11} + c_{42} e_{22} + c_{43} e_{33} + 2(c_{44} e_{23} + c_{45} e_{13} + c_{46} e_{12}) \\
\tau_{13} &= c_{51} e_{11} + c_{52} e_{22} + c_{53} e_{33} + 2(c_{54} e_{23} + c_{55} e_{13} + c_{56} e_{12}) \\
\tau_{23} &= c_{61} e_{11} + c_{62} e_{22} + c_{63} e_{33} + 2(c_{64} e_{23} + c_{65} e_{13} + c_{66} e_{12})
\end{aligned} \tag{2.4}$$

Using (2.4) in (2.1), we obtain

$$\left\{c_{11}\frac{\partial^2 u_1}{\partial x_1^2}+c_{66}\frac{\partial^2 u_1}{\partial x_2^2}+c_{55}\frac{\partial^2 u_1}{\partial x_3^2}+2\left(c_{16}\frac{\partial^2 u_1}{\partial x_1\partial x_2}+c_{15}\frac{\partial^2 u_1}{\partial x_1\partial x_3}+c_{56}\frac{\partial^2 u_1}{\partial x_2\partial x_3}\right)\right\}$$
$$+\left\{c_{16}\frac{\partial^2 u_2}{\partial x_1^2}+c_{26}\frac{\partial^2 u_2}{\partial x_2^2}+c_{45}\frac{\partial^2 u_2}{\partial x_3^2}+(c_{12}+c_{16})\frac{\partial^2 u_2}{\partial x_1\partial x_2}+(c_{14}+c_{56})\frac{\partial^2 u_2}{\partial x_1\partial x_3}+(c_{46}+c_{25})\frac{\partial^2 u_2}{\partial x_2\partial x_3}\right\}$$
$$+\left\{c_{15}\frac{\partial^2 u_3}{\partial x_1^2}+c_{46}\frac{\partial^2 u_3}{\partial x_2^2}+c_{35}\frac{\partial^2 u_3}{\partial x_3^2}+(c_{14}+c_{56})\frac{\partial^2 u_3}{\partial x_1\partial x_2}+(c_{13}+c_{55})\frac{\partial^2 u_3}{\partial x_1\partial x_3}+(c_{36}+c_{45})\frac{\partial^2 u_3}{\partial x_2\partial x_3}\right\}$$
$$-\beta_{11}\frac{\partial\theta}{\partial x_1}=\rho\frac{\partial^2 u_1}{\partial t^2} \quad (2.5)$$

$$\left\{c_{16}\frac{\partial^2 u_1}{\partial x_1^2}+c_{26}\frac{\partial^2 u_1}{\partial x_2^2}+c_{45}\frac{\partial^2 u_1}{\partial x_3^2}+(c_{12}+c_{66})\frac{\partial^2 u_1}{\partial x_1\partial x_2}+(c_{14}+c_{56})\frac{\partial^2 u_1}{\partial x_1\partial x_3}+(c_{46}+c_{25})\frac{\partial^2 u_1}{\partial x_2\partial x_3}\right\}$$
$$+\left\{c_{66}\frac{\partial^2 u_2}{\partial x_1^2}+c_{22}\frac{\partial^2 u_2}{\partial x_2^2}+c_{44}\frac{\partial^2 u_2}{\partial x_3^2}+2\left(c_{26}\frac{\partial^2 u_2}{\partial x_1\partial x_2}+c_{46}\frac{\partial^2 u_2}{\partial x_1\partial x_3}+c_{24}\frac{\partial^2 u_2}{\partial x_2\partial x_3}\right)\right\}$$
$$+\left\{c_{56}\frac{\partial^2 u_3}{\partial x_1^2}+c_{24}\frac{\partial^2 u_3}{\partial x_2^2}+c_{34}\frac{\partial^2 u_3}{\partial x_3^2}+(c_{46}+c_{25})\frac{\partial^2 u_3}{\partial x_1\partial x_2}+(c_{36}+c_{45})\frac{\partial^2 u_3}{\partial x_1\partial x_3}+(c_{23}+c_{44})\frac{\partial^2 u_3}{\partial x_2\partial x_3}\right\}$$
$$-\beta_{22}\frac{\partial\theta}{\partial x_2}=\rho\frac{\partial^2 u_2}{\partial t^2} \quad (2.6)$$

$$\left\{c_{15}\frac{\partial^2 u_1}{\partial x_1^2}+c_{46}\frac{\partial^2 u_1}{\partial x_2^2}+c_{35}\frac{\partial^2 u_1}{\partial x_3^2}+(c_{56}+c_{14})\frac{\partial^2 u_1}{\partial x_1\partial x_2}+(c_{55}+c_{13})\frac{\partial^2 u_1}{\partial x_1\partial x_3}+(c_{45}+c_{36})\frac{\partial^2 u_1}{\partial x_2\partial x_3}\right\}$$
$$+\left\{c_{56}\frac{\partial^2 u_2}{\partial x_1^2}+c_{24}\frac{\partial^2 u_2}{\partial x_2^2}+c_{34}\frac{\partial^2 u_2}{\partial x_3^2}+(c_{25}+c_{46})\frac{\partial^2 u_2}{\partial x_1\partial x_2}+(c_{45}+c_{36})\frac{\partial^2 u_2}{\partial x_1\partial x_3}+(c_{44}+c_{23})\frac{\partial^2 u_2}{\partial x_2\partial x_3}\right\}$$
$$+\left\{c_{55}\frac{\partial^2 u_3}{\partial x_1^2}+c_{44}\frac{\partial^2 u_3}{\partial x_2^2}+c_{33}\frac{\partial^2 u_3}{\partial x_3^2}+2\left(c_{45}\frac{\partial^2 u_3}{\partial x_1\partial x_2}+c_{35}\frac{\partial^2 u_3}{\partial x_1\partial x_3}+c_{34}\frac{\partial^2 u_3}{\partial x_2\partial x_3}\right)\right\}$$
$$-\beta_{33}\frac{\partial\theta}{\partial x_3}=\rho\frac{\partial^2 u_3}{\partial t^2} \quad (2.7)$$

and heat conduction equation in multiple phase lag model is given by

$$\left(1+\sum_{n=1}^{N}\frac{\tau_\theta^n}{n!}\frac{\partial^n}{\partial t^n}\right)\left[K^*\left(\frac{\partial^2\theta}{\partial x_1^2}+\frac{\partial^2\theta}{\partial x_2^2}+\frac{\partial^2\theta}{\partial x_3^2}\right)\right]$$
$$=\left[\overline{R}+\tau_0\frac{\partial}{\partial t}+\sum_{n=1}^{N}\frac{\tau_q^n+1}{(n+1)!}\frac{\partial^{n+1}}{\partial t^(n+1)}\right]\left[\rho c_E\frac{\partial^2\theta}{\partial t^2}+\gamma\theta_0\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\right] \quad (2.8)$$

To transform the equations (2.5)-(2.8) in the non-dimensional form, we use the following non-dimensional variables:

$$\begin{gathered}(x_i', u_i') = \frac{1}{l}(x_i, \frac{\rho c_1^2}{\beta\theta_0}u_i),\ i = 1, 2, 3;\ t' = \frac{c_1 t}{l}\theta;\ \tau_{ij}' = \frac{\tau_{ij}}{\beta\theta_0} \\ e_{ij}' = \beta\theta_0 e_{ij};\ \tau_q' = \frac{c_1}{l}\tau_q;\ \tau_\theta' = \frac{c_1}{l}\tau_\theta; \tau_0' = \frac{c_1}{l}\tau_0 \\ l = \frac{K^*}{\rho c_E c_1};\ c_1^2 = \frac{c_{11}}{\rho}\end{gathered} \tag{2.9}$$

Using the non-dimensional variables (2.9) and omitting the primes for convenience, equations (2.5)-(2.8) becomes

$$\begin{aligned}&\left\{\frac{\partial^2 u_1}{\partial x_1^2} + \frac{c_{66}}{c_{11}}\frac{\partial^2 u_1}{\partial x_2^2} + \frac{c_{55}}{c_{11}}\frac{\partial^2 u_1}{\partial x_3^2} + 2\left(\frac{c_{16}}{c_{11}}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{c_{15}}{c_{11}}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{c_{56}}{c_{11}}\frac{\partial^2 u_1}{\partial x\_2 \partial x_3}\right)\right\} \\ &+ \left\{\frac{c_{16}}{c_{11}}\frac{\partial^2 u_2}{\partial x_1^2} + \frac{c_{26}}{c_{11}}\frac{\partial^2 u_2}{\partial x_2^2} + \frac{c_{45}}{c_{11}}\frac{\partial^2 u_2}{\partial x_3^2} + \frac{(c_{12} + c_{16})}{c_{11}}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{(c_{14} + c_{56})}{c_{11}}\frac{\partial^2 u_2}{\partial x_1 \partial x_3}\right. \\ &\left. + \frac{(c_{46} + c_{25})}{c_{11}}\frac{\partial^2 u_2}{\partial x_2 \partial x_3}\right\} \\ &+ \left\{\frac{c_{15}}{c_{11}}\frac{\partial^2 u_3}{\partial x_1^2} + \frac{c_{46}}{c_{11}}\frac{\partial^2 u_3}{\partial x_2^2} + \frac{c_{35}}{c_{11}}\frac{\partial^2 u_3}{\partial x_3^2} + \frac{(c_{14} + c_{56})}{c_{11}}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} + \frac{(c_{13} + c_{55})}{c_{11}}\frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right. \\ &\left. + \frac{(c_{36} + c_{45})}{c_{11}}\frac{\partial^2 u_3}{\partial x_2 \partial x_3}\right\} - \frac{\partial \theta}{\partial x_1} = \frac{\partial^2 u_1}{\partial t^2}\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\left\{\frac{c_{16}}{c_{11}}\frac{\partial^2 u_1}{\partial x_1^2} + \frac{c_{26}}{c_{11}}\frac{\partial^2 u_1}{\partial x_2^2} + \frac{c_{45}}{c_{11}}\frac{\partial^2 u_1}{\partial x_3^2} + \frac{(c_{12} + c_{66})}{c_{11}}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{(c_{14} + c_{56})}{c_{11}}\frac{\partial^2 u_1}{\partial x_1 \partial x_3}\right. \\ &\left. + \frac{(c_{46} + c_{25})}{c_{11}}\frac{\partial^2 u_1}{\partial x_2 \partial x_3}\right\} \\ &+ \left\{\frac{c_{66}}{c_{11}}\frac{\partial^2 u_2}{\partial x_1^2} + \frac{c_{22}}{c_{11}}\frac{\partial^2 u_2}{\partial x_2^2} + \frac{c_{44}}{c_{11}}\frac{\partial^2 u_2}{\partial x_3^2} + 2\left(\frac{c_{26}}{c_{11}}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{c_{46}}{c_{11}}\frac{\partial^2 u_2}{\partial x_1 \partial x_3} + \frac{c_{24}}{c_{11}}\frac{\partial^2 u_2}{\partial x_2 \partial x_3}\right)\right\} \\ &+ \left\{\frac{c_{56}}{c_{11}}\frac{\partial^2 u_3}{\partial x_1^2} + \frac{c_{24}}{c_{11}}\frac{\partial^2 u_3}{\partial x_2^2} + \frac{c_{34}}{c_{11}}\frac{\partial^2 u_3}{\partial x_3^2} + \frac{(c_{46} + c_{25})}{c_{11}}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} + \frac{(c_{36} + c_{45})}{c_{11}}\frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right. \\ &\left. + \frac{(c_{23} + c_{44})}{c_{11}}\frac{\partial^2 u_3}{\partial x_2 \partial x_3}\right\} - \beta_2\frac{\partial \theta}{\partial x_2} = \frac{\partial^2 u_2}{\partial t^2}\end{aligned} \tag{2.11}$$

$$\left\{\frac{c_{15}}{c_{11}}\frac{\partial^2 u_1}{\partial x_1^2}+\frac{c_{46}}{c_{11}}\frac{\partial^2 u_1}{\partial x_2^2}+\frac{c_{35}}{c_{11}}\frac{\partial^2 u_1}{\partial x_3^2}+\frac{(c_{56}+c_{14})}{c_{11}}\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\frac{(c_{55}+c_{13})}{c_{11}}\frac{\partial^2 u_1}{\partial x_1\partial x_3}+\frac{(c_{45}+c_{36})}{c_{11}}\frac{\partial^2 u_1}{\partial x_2\partial x_3}\right\}$$
$$+\left\{\frac{c_{56}}{c_{11}}\frac{\partial^2 u_2}{\partial x_1^2}+\frac{c_{24}}{c_{11}}\frac{\partial^2 u_2}{\partial x_2^2}+\frac{c_{34}}{c_{11}}\frac{\partial^2 u_2}{\partial x_3^2}+\frac{(c_{25}+c_{46})}{c_{11}}\frac{\partial^2 u_2}{\partial x_1\partial x_2}+\frac{(c_{45}+c_{36})}{c_{11}}\frac{\partial^2 u_2}{\partial x_1\partial x_3}+\frac{(c_{44}+c_{23})}{c_{11}}\frac{\partial^2 u_2}{\partial x_2\partial x_3}\right\}$$
$$+\left\{\frac{c_{55}}{c_{11}}\frac{\partial^2 u_3}{\partial x_1^2}+\frac{c_{44}}{c_{11}}\frac{\partial^2 u_3}{\partial x_2^2}+\frac{c_{33}}{c_{11}}\frac{\partial^2 u_3}{\partial x_3^2}+2\left(c_{45}\frac{\partial^2 u_3}{\partial x_1\partial x_2}+\frac{c_{35}}{c_{11}}\frac{\partial^2 u_3}{\partial x_1\partial x_3}+\frac{c_{34}}{c_{11}}\frac{\partial^2 u_3}{\partial x_2\partial x_3}\right)\right\}$$
$$-\beta_3\frac{\partial\theta}{\partial x_3}=\frac{\partial^2 u_3}{\partial t^2} \quad (2.12)$$

$$\left(1+\sum_{n=1}^{N}\frac{\tau_\theta^n}{n!}\frac{\partial^n}{\partial t^n}\right)\left[K_{11}^*\frac{\partial^2\theta}{\partial x_1^2}+K_{22}^*\frac{\partial^2\theta}{\partial x_2^2}+K_{33}^*\frac{\partial^2\theta}{\partial x_3^2}\right]$$
$$=\left[\overline{R}+\tau_0\frac{\partial}{\partial t}+\sum_{n=1}^{N}\frac{\tau_q^n+1}{(n+1)!}\frac{\partial^{n+1}}{\partial t^(n+1)}\right]\left[c_E c_{11}\frac{\partial^2\theta}{\partial t^2}+T_0\beta_{11}^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_1}+\beta_2\frac{\partial u_2}{\partial x_2}+\beta_3\frac{\partial u_3}{\partial x_3}\right)\right] \quad (2.13)$$

where $\beta_2=\frac{\beta_{22}}{\beta_{11}};\ \beta_3=\frac{\beta_{33}}{\beta_{11}}$

Non-dimensional stress components are given by

$$\begin{aligned}
\tau_{11}&=\frac{1}{c_{11}}[c_{11}e_{11}+c_{12}e_{22}+c_{13}e_{33}+2(c_{14}e_{23}+c_{15}e_{13}+c_{16}e_{12})]-\theta\\
\tau_{22}&=\frac{1}{c_{11}}[c_{21}e_{11}+c_{22}e_{22}+c_{23}e_{33}+2(c_{24}e_{23}+c_{25}e_{13}+c_{26}e_{12})]-\beta_2\theta\\
\tau_{33}&=\frac{1}{c_{11}}[c_{31}e_{11}+c_{32}e_{22}+c_{33}e_{33}+2(c_{34}e_{23}+c_{35}e_{13}+c_{36}e_{12})]-\beta_3\theta\\
\tau_{12}&=\frac{1}{c_{11}}[c_{41}e_{11}+c_{42}e_{22}+c_{43}e_{33}+2(c_{44}e_{23}+c_{45}e_{13}+c_{46}e_{12})]\\
\tau_{13}&=\frac{1}{c_{11}}[c_{51}e_{11}+c_{52}e_{22}+c_{53}e_{33}+2(c_{54}e_{23}+c_{55}e_{13}+c_{56}e_{12})]\\
\tau_{23}&=\frac{1}{c_{11}}[c_{61}e_{11}+c_{62}e_{22}+c_{63}e_{33}+2(c_{64}e_{23}+c_{65}e_{13}+c_{66}e_{12})]
\end{aligned} \quad (2.14)$$

2.1.3 Method of Solution: Formulation of a Vector-Matrix Differential Equation

For the solution of the equations (2.10)-(2.13), the physical quantities can be decomposed in terms of normal modes in the following form

$$[u_1, u_2, u_3, \tau_{ij}, \theta](x_1, x_2, x_3, t) = [u_1^*, u_2^*, u_3^*, \tau_{ij}^*, \theta](x_1, x_2, x_3, t)e^{\omega t + i(ax_2 + bx_3)} \tag{2.15}$$

where $i = \sqrt{-1}$, ω is the angular frequency and a,b are the wave numbers along x_2 and x_3 direction respectively.

After imposing (2.15), equations (2.10)-(2.13) are as follows (omitting the asterisks for convenience):

$$\frac{d^2u_1}{dx_1^2}+a_{11}\frac{du_1}{dx_1}+a_{12}u_1+a_{13}\frac{d^2u_2}{dx_1^2}+a_{21}\frac{du_2}{dx_1}+a_{22}u_2+a_{23}\frac{d^2u_3}{dx_1^2}+a_{31}\frac{du_3}{dx_1}+a_{32}u_3-a_{33}\frac{d\theta}{dx_1}=0 \tag{2.16}$$

$$b_{11}\frac{d^2u_1}{dx_1^2}+b_{12}\frac{du_1}{dx_1}+b_{13}u_1+\frac{d^2u_2}{dx_1^2}+b_{21}\frac{du_2}{dx_1}+b_{22}u_2+b_{23}\frac{d^2u_3}{dx_1^2}+b_{31}\frac{du_3}{dx_1}+b_{32}u_3-b_{33}\theta=0 \tag{2.17}$$

$$m_{11}\frac{d^2u_1}{dx_1^2}+m_{12}\frac{du_1}{dx_1}+m_{13}u_1+m_{21}\frac{d^2u_2}{dx_1^2}+m_{22}\frac{du_2}{dx_1}+m_{23}u_2+\frac{d^2u_3}{dx_1^2}+m_{31}\frac{du_3}{dx_1}+m_{32}u_3-m_{33}\theta = 0 \tag{2.18}$$

and

$$\theta'' = d_{41}\frac{du_1}{dx_1} + d_{44}u_2 + d_{46}u_3 + d_{48}\theta \tag{2.19}$$

and stress components are

$$\begin{aligned}
\tau_{11} &= h_{11}\frac{du_1}{dx_1} + h_{12}\frac{du_2}{dx_1} + h_{13}\frac{du_3}{dx_1} + h_{14}u_1 + h_{15}u_2 + h_{16}u_3 - \theta \\
\tau_{22} &= h_{21}\frac{du_1}{dx_1} + h_{22}\frac{du_2}{dx_1} + h_{23}\frac{du_3}{dx_1} + h_{24}u_1 + h_{25}u_2 + h_{26}u_3 - \beta_2\theta \\
\tau_{33} &= h_{31}\frac{du_1}{dx_1} + h_{32}\frac{du_2}{dx_1} + h_{33}\frac{du_3}{dx_1} + h_{34}u_1 + h_{35}u_2 + h_{36}u_3 - \beta_3\theta \\
\tau_{23} &= h_{41}\frac{du_1}{dx_1} + h_{42}\frac{du_2}{dx_1} + h_{43}\frac{du_3}{dx_1} + h_{44}u_1 + h_{45}u_2 + h_{46}u_3 \\
\tau_{13} &= h_{51}\frac{du_1}{dx_1} + h_{52}\frac{du_2}{dx_1} + h_{53}\frac{du_3}{dx_1} + h_{54}u_1 + h_{55}u_2 + h_{56}u_3 \\
\tau_{12} &= h_{61}\frac{du_1}{dx_1} + h_{62}\frac{du_2}{dx_1} + h_{63}\frac{du_3}{dx_1} + h_{64}u_1 + h_{65}u_2 + h_{66}u_3
\end{aligned} \tag{2.20}$$

where a_{ij}, m_{ij} and h_{ij} $(i, j = 1, 2, 3)$ are given in the appendix A.

Equations (2.16)-(2.19) can be written as vector-matrix differential equation as

$$\frac{d\underset{\sim}{v}}{dx_1} = A\underset{\sim}{v} \tag{2.21}$$

where $\tilde{v} = \begin{bmatrix} u_1 & u_2 & u_3 & \theta & u_1' & u_2' & u_3' & \theta' \end{bmatrix}^T$ and $A = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{21} \end{bmatrix}$
where L_{11} and L_{12} are respectively the null matrix and the identity matrix of order 4×4 and L_{21} and L_{22} are given in the Appendix I.

2.1.4 Solution of the Vector-Matrix Differential Equation

The characteristic equation of the matrix A is given by

$$|A - \lambda I| = 0 \tag{2.22}$$

The roots (eigenvalues) of the characteristic equation (22) and $\lambda = \lambda_i$ $(i = 1, 2, 3, 4)$ and are of the form

$$\lambda = \pm\lambda_1, \lambda = \pm\lambda_2, \lambda = \pm\lambda_3, \lambda = \pm\lambda_4.$$

The eigen vector $\underset{\sim}{X}$ corresponding to the eigen value λ calculated as

$$\underset{\sim}{X} = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \lambda\delta_1 & \lambda\delta_2 & \lambda\delta_3 & \lambda\delta_4 \end{bmatrix}^T \tag{2.23}$$

where

$$\begin{aligned}
\delta_1 =& (f_{24}f_{13} - f_{14}f_{23})(f_{22}f_{33} - f_{32}f_{23}) - (f_{34}f_{23} - f_{24}f_{33})(f_{12}f_{23} - f_{22}f_{13}) \\
\delta_2 =& (f_{34}f_{23} - f_{24}f_{33})(f_{11}f_{23} - f_{21}f_{13}) - (f_{24}f_{13} - f_{14}f_{23})(f_{21}f_{33} - f_{31}f_{23}) \\
\delta_3 =& (f_{12}f_{21} - f_{11}f_{22})(f_{21}f_{34} - f_{31}f_{24}) - (f_{22}f_{31} - f_{21}f_{32})(f_{11}f_{24} - f_{14}f_{21}) \\
\delta_4 =& (f_{11}f_{23} - f_{21}f_{13})(f_{22}f_{33} - f_{32}f_{23}) - (f_{12}f_{23} - f_{22}f_{13})(f_{21}f_{33} - f_{31}f_{23})
\end{aligned} \tag{2.24}$$

where f_{ij} $(i, j = 1, 2, 3)$ are given in Appendix-I.
The solution of the equation (2.21) is given by

$$\vec{v} = \sum_{i=1}^{4} X_i\ y_i \tag{2.25}$$

Where X_i is the eigen vector corresponding to the eigen value λ_i .

Thus we obtain the displacement components and temperature as follows:

$$u_1 = \sum_{j=1}^{4} A_j\, x_{1j}\, e^{\lambda_j x_1}, u_2 = \sum_{j=1}^{4} A_j\, x_{2j}\, e^{\lambda_j x_1}, u_3 = \sum_{j=1}^{4} A_j\, x_{3j}\, e^{\lambda_j x_1}\, and\, \theta = \sum_{j=1}^{4} A_j\, x_{4j}\, e^{\lambda_j x_1} \tag{2.26}$$

Using equation(2.26) in the equations given at (2.20), we obtain the stress components as below

$$\tau_{11} = \textstyle\sum_{j=1}^{4}\ A_i\, R_{1i}(x_1)\ , \tau_{22} = \sum_{j=1}^{4}\ A_i\, R_{2i}(x_1)\ , \tau_{33} = \sum_{j=1}^{4}\ A_i\, R_{3i}(x_1)$$

$$\tau_{12} = \textstyle\sum_{j=1}^{4}\ A_i\, R_{4i}(x_1)\ , \tau_{13} = \sum_{j=1}^{4}\ A_i\, R_{5i}(x_1)\ , \tau_{23} = \sum_{j=1}^{4}\ A_i\, R_{6i}(x_1) \tag{2.27}$$

$$\theta = \textstyle\sum_{j=1}^{4}\ A_i\, R_{7i}(x_1)$$

Where R_{ji}(j=1,2,..,7 and i=1,2,..4) are given in the Appendix-II. A_r are constants, which are to be determined by using boundary conditions.

2.1.5 BOUNDARY CONDITIONS

As we consider the problem of a half-space in the region
$R = \{(x_1, x_2, x_3) :\ 0 \leq x_1 \leq \infty, -\infty \leq x_2 \leq \infty, -\infty \leq x_3 \leq \infty\}$.
To determine arbitrary constants ($A_i,\ i = 1, 2, 3, 4$) we consider the following two boundary conditions are prescribed as the follows

CASE 1:

Mechanical Boundary Condition:

We consider the traction free boundary surface of the half-space x_1=0 by means of

$$\tau_{11}(0, x_2, x_3, t) = \tau_{12}(0, x_2, x_3, t) = \tau_{13}(0, x_2, x_3, t) = 0 \tag{2.28}$$

Thermal Boundary Condition:

$$q_n + \nu\theta = r^*(0, x_2, x_3, t)\ at\, x_1 = 0; \tag{2.29}$$

CASE 2:

We consider the traction free boundary surface of the half-space x_1=0 by means of

$$\tau_{11}(0, x_2, x_3, t) = \sigma_0, \quad \tau_{12}(0, x_2, x_3, t) = \tau_{13}(0, x_2, x_3, t) = 0 \tag{2.30}$$

Thermal Boundary Condition:

$$\theta = \theta_0; \tag{2.31}$$

2.1.6 Numerical analysis

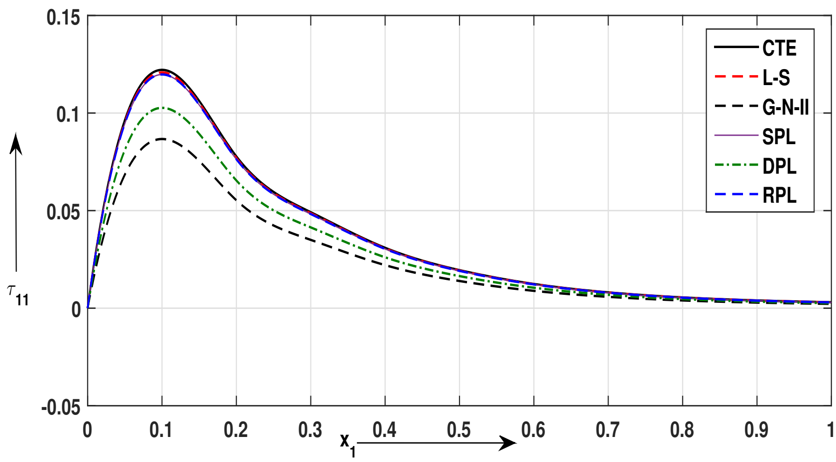
Numerical computations were performed by considering two different sets of boundary conditions. To study the nature of characteristic curves,we consider the following physical constants in triclinic half space as in Chattopadhaya and Rogerson .

$$
\begin{aligned}
&C_{11} = 16.248; C_{22} = 11.88; C_{33} = 12.216; C_{44} = 5.64;\\
&C_{55} = 5.88; C_{66} = 9.91; C_{12} = 1.48 = C_{21};\\
&C_{13} = 2.4 = C_{31}; C_{14} = -1.152 = C_{41}; C_{15} = 0.0 = C_{51};\\
&C_{16} = -0.561 = C_{61}; C_{23} = 1.032 = C_{32};\\
&C_{24} = 0.912 = C_{42}; C_{25} = 1.608 = C_{52};\\
&C_{26} = 1.248 = C_{62}; C_{33} = 12.216; C_{34} = -0.672 = C_{43};\\
&C_{35} = 0.216 = C_{53}; C_{36} = -0.216 = C_{63};\\
&C_{45} = 2.16 = C_{54}; C_{46} = 0 = C_{64}; C_{56} = 0.0 = C_{65};\\
&\rho = 2.40; g = 9.8; \nu = 2.0; a = 1.2; b = 1.3; z = 200.0;
\end{aligned}
\tag{2.32}
$$

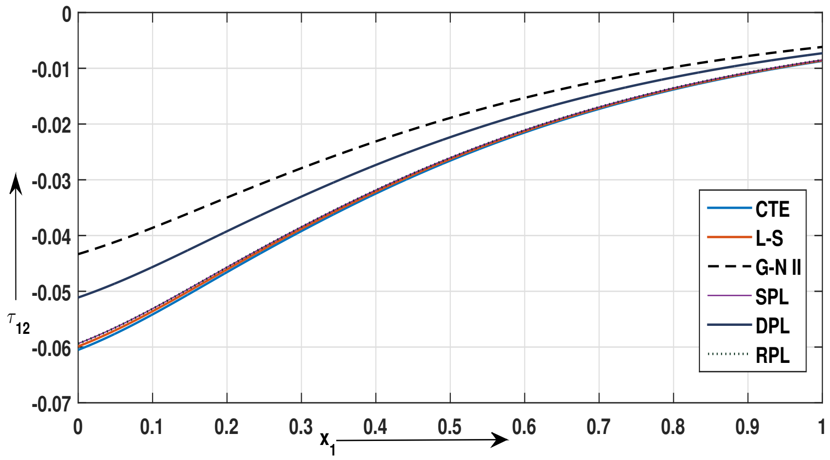
2.1.7 Geometrical Representation and analysis

Depending upon the boundary conditions and using above mentioned numerical values, the geometrical representation of different physical variables are provided in two separate cases as follows

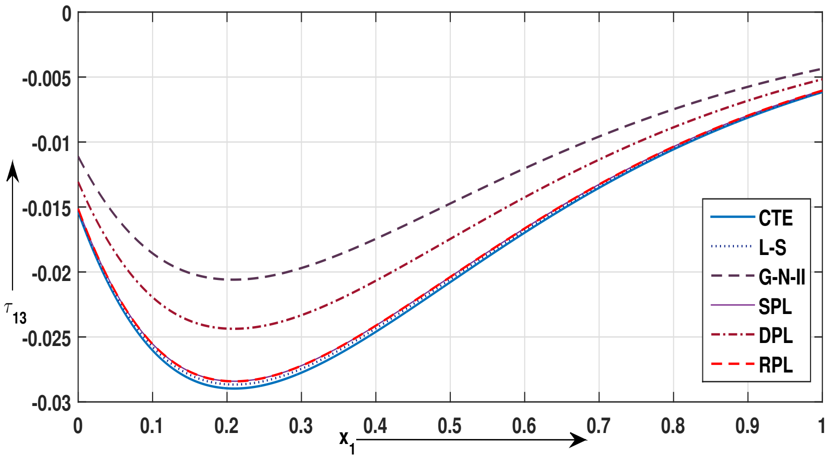
CASE 1:



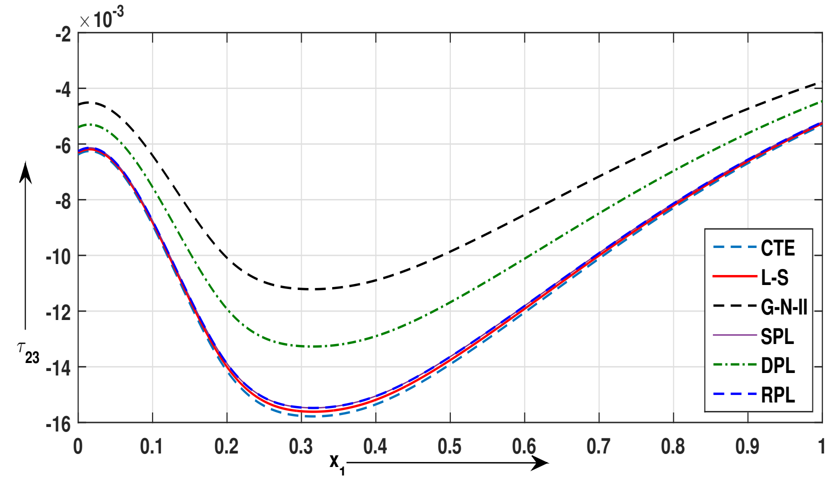
Case1: Figure 2.



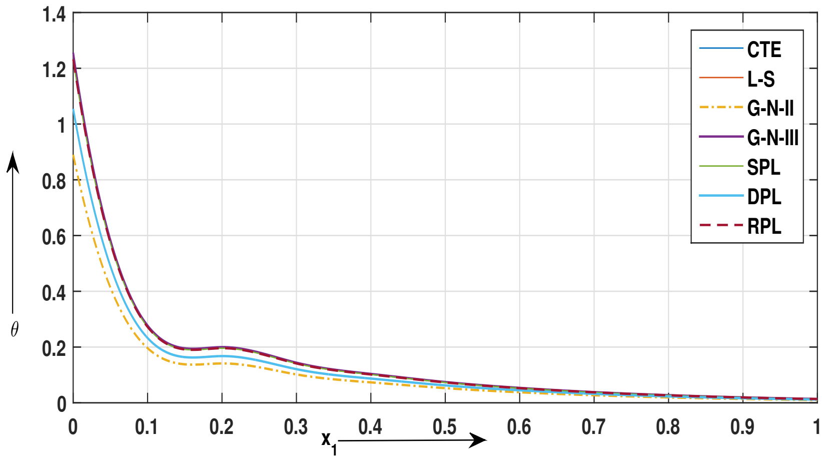
Case1: Figure 3.



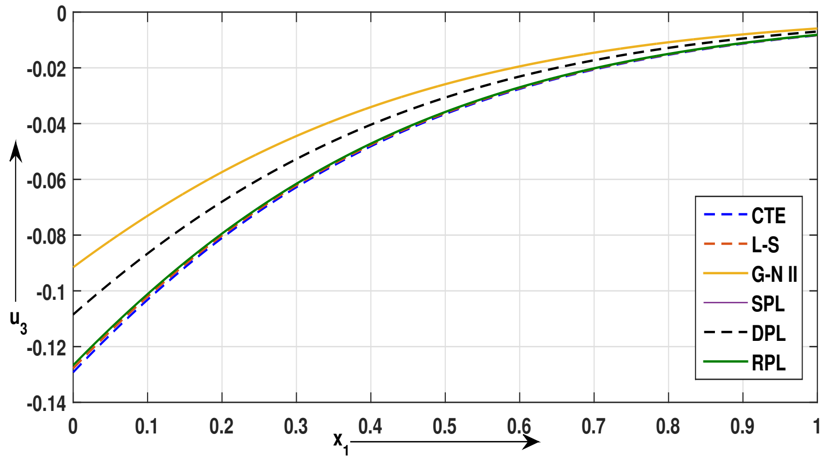
Case1: Figure 4.



Case1: Figure 5.

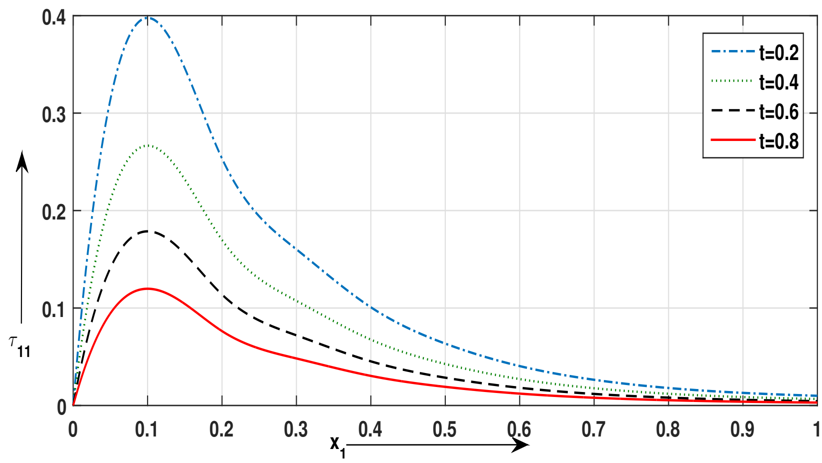


Case1: Figure 6.

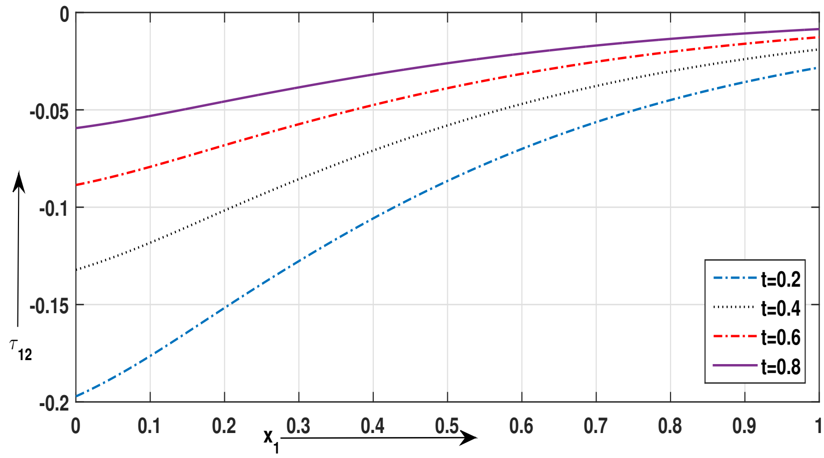


Case1: Figure 7.

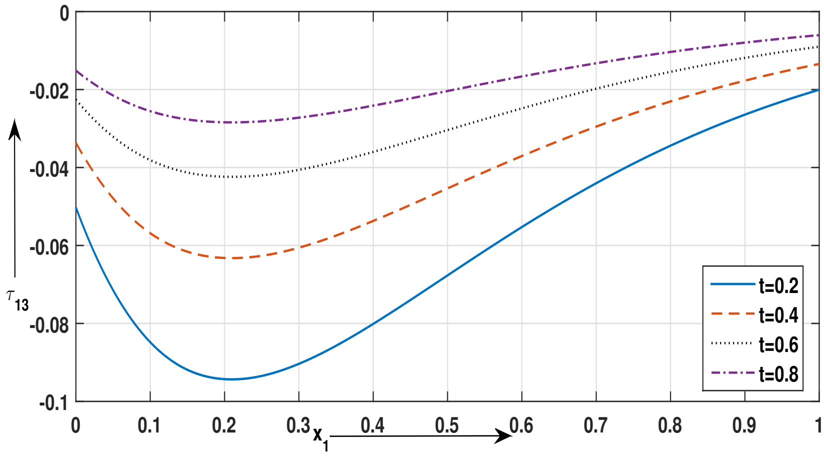
Figure 2-7 represents the characteristic curves of different stress components, temperature and displacement components with respect to x_1 for different models. The characteristic curves are obtained for fixed values of $R = 0, x_2 = 0.4$ and $x_3 = 0.5$.



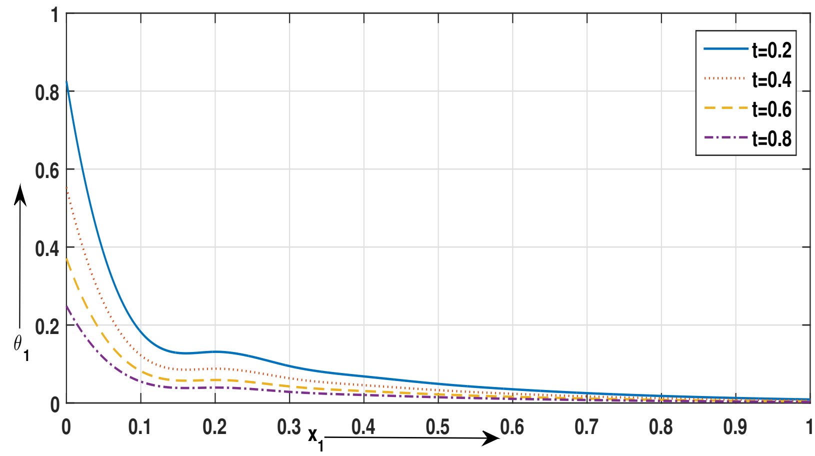
Case1: Figure 8.



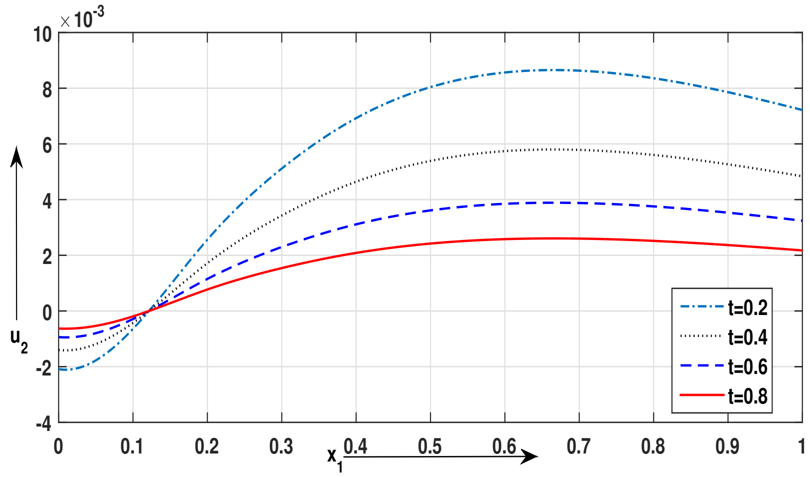
Case1: Figure 9.



Case1: Figure 10.

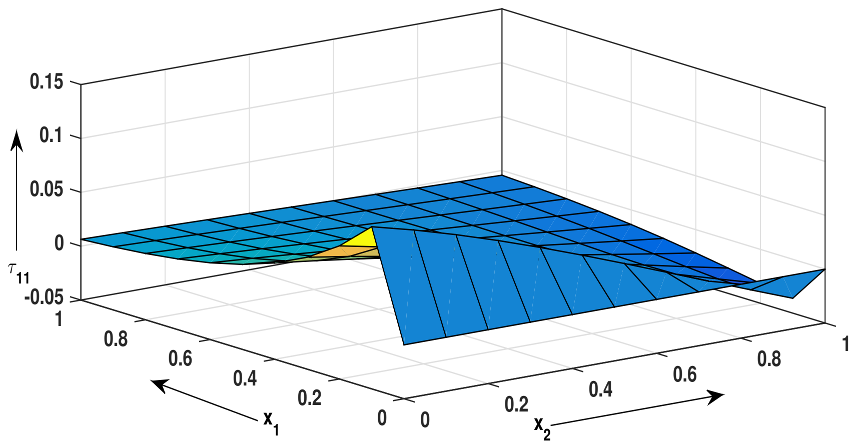


Case1: Figure 11.

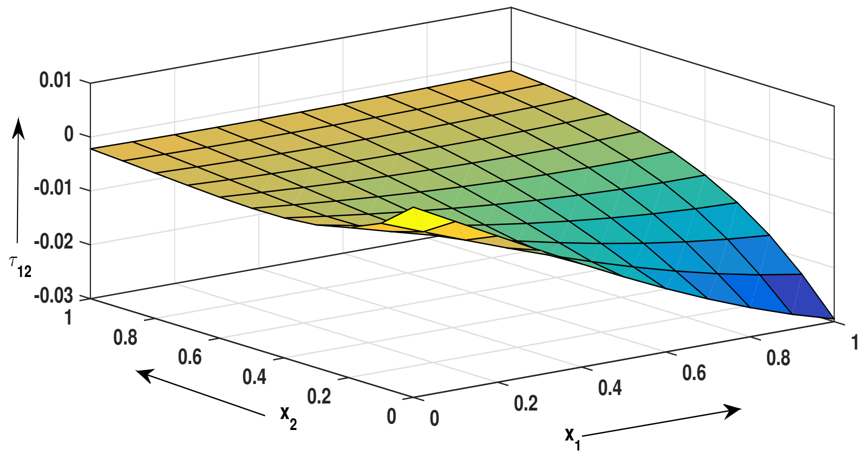


Case1: Figure 12.

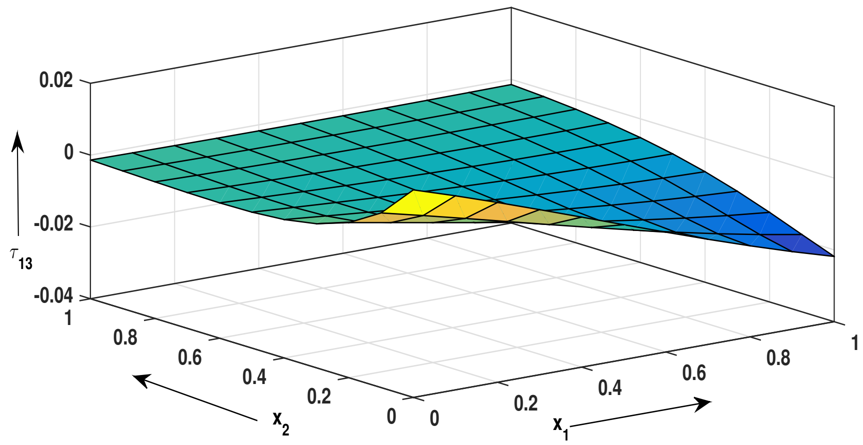
Figure 8-12 represents the characteristic curves of different components of stress, temperature na displacement components for different values of $t(0.1, 0.3, 0.5, 0.7)$ and for fixed values of $\tau_q = 0.01$, $\tau_\theta = 0.0001$, $\tau_0 = 0.01$ and R=1.



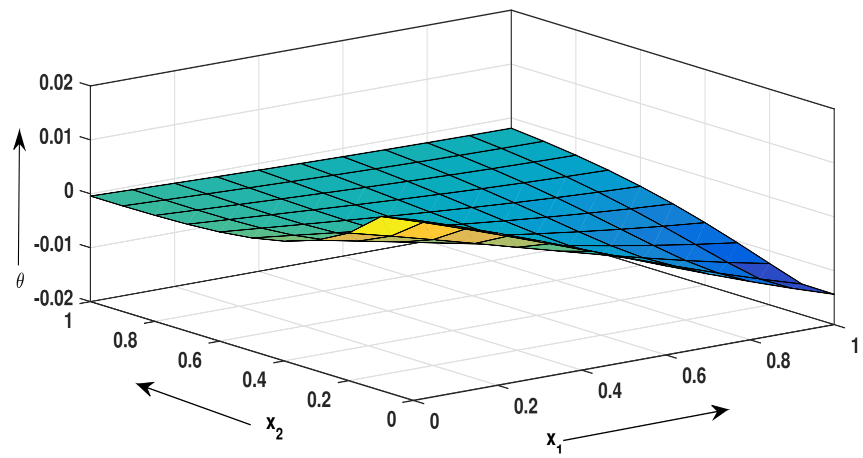
Case1: Figure 13.



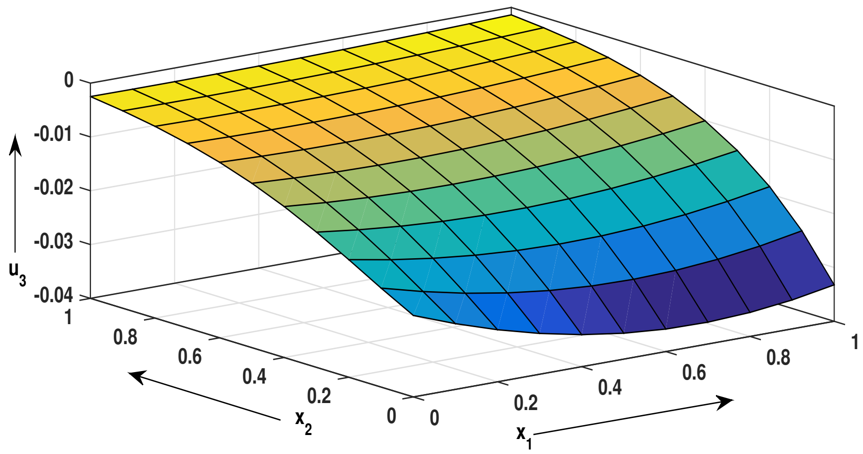
Case1: Figure 14.



Case1: Figure 15.

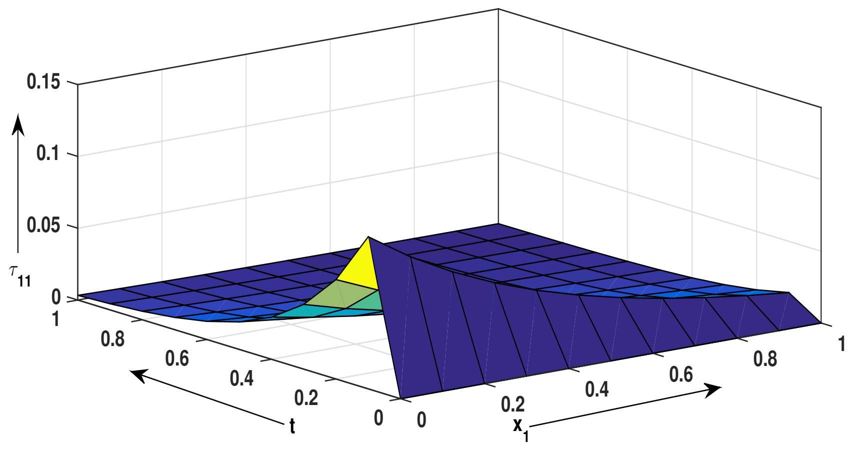


Case1: Figure 16.

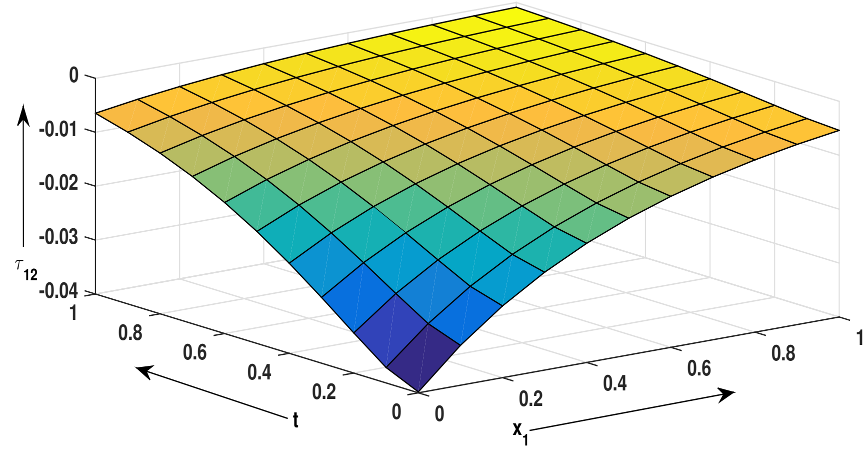


Case1: Figure 17.

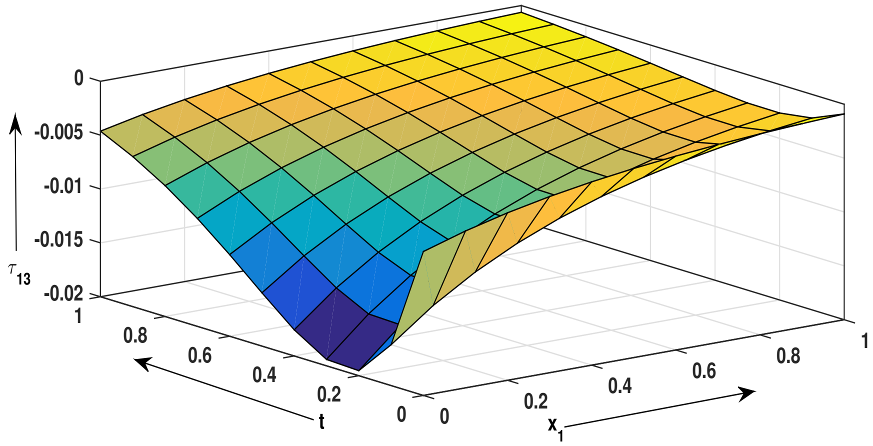
Figure 13-17 are showing the three dimensional characteristics behaviour of temperature and different components of stresses and displacements with respect to x_2 and x_2 for fixed values of $x_3 = 0.5$, $\tau_q = 0.02$, $\tau_\theta = 0.04$, $\tau_0 = 0.1$, t=0.5 and R=0.



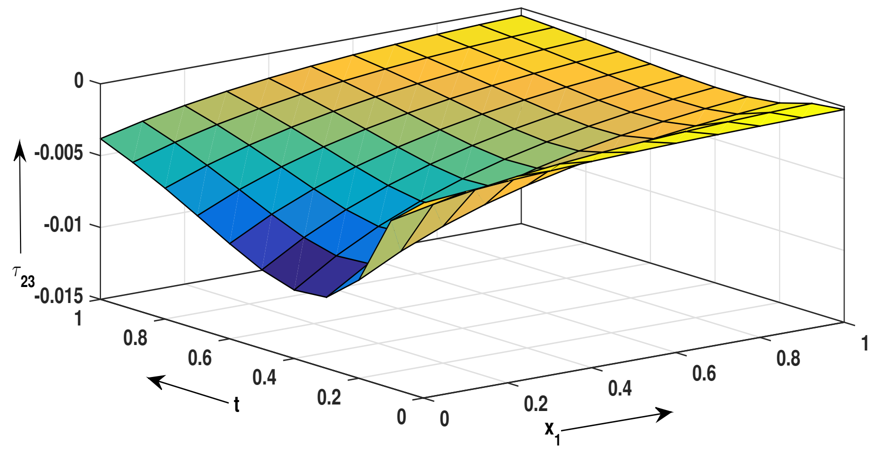
Case1: Figure 18.



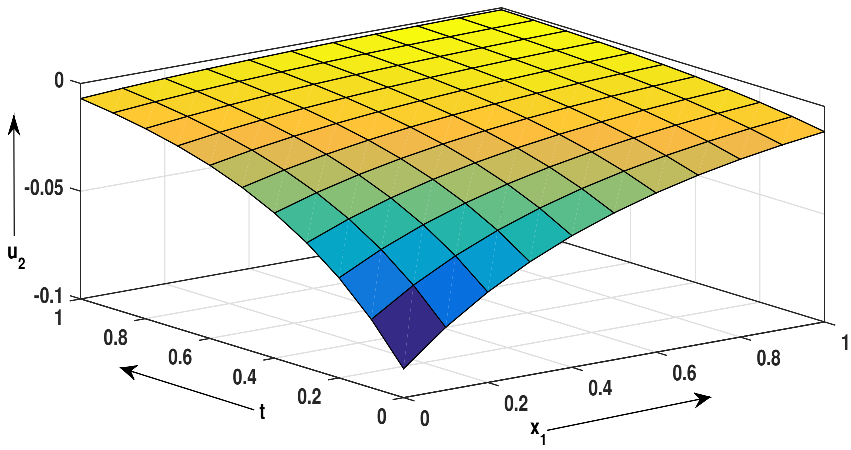
Case1: Figure 19.



Case1: Figure 20.



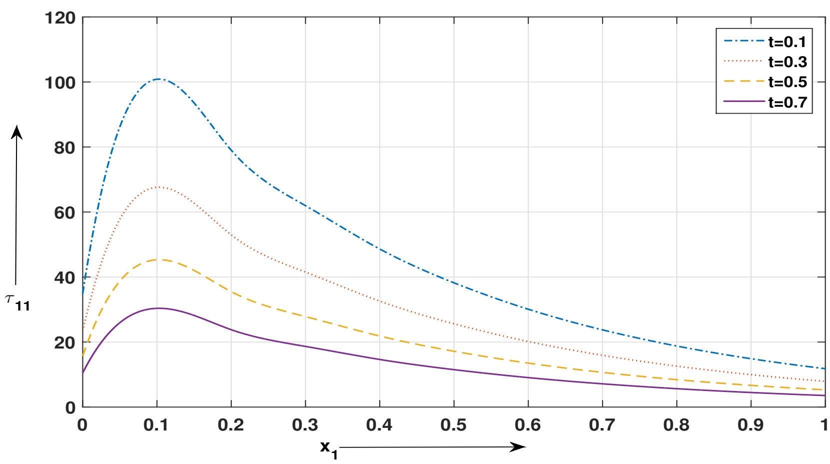
Case1: Figure 21.



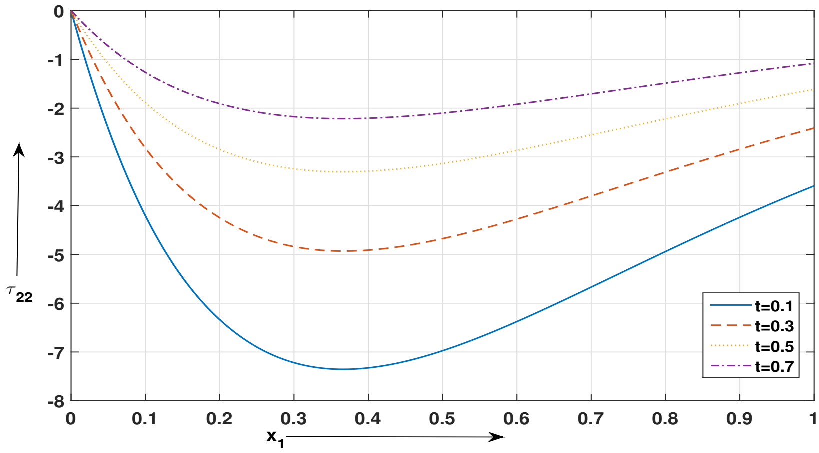
Case1: Figure 22.

Figure 18-22 represents the three-dimensional behaviours of different components of stresses, displacements and temperature with respect to x_1 and t with respect to $x_2 = 0.5$ and $x_3 = 0.5$ for fixed values of $\tau_q = 0.02$, $\tau_\theta = 0.04$, $\tau_0 = 0.1$ and R=0.

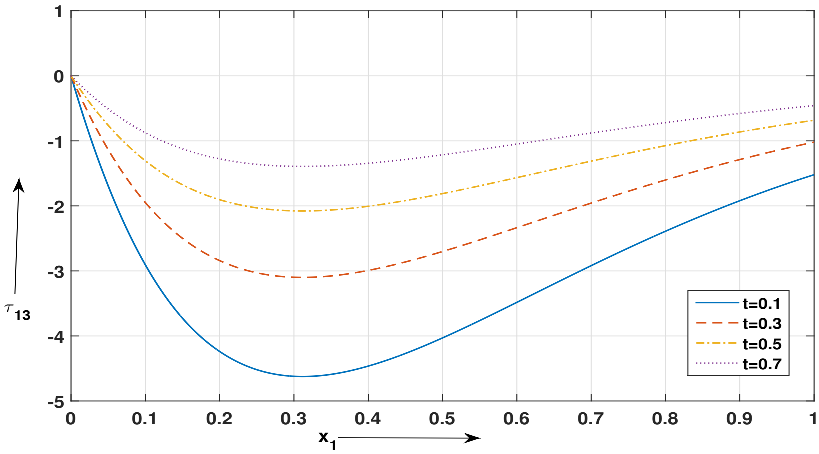
CASE 2:



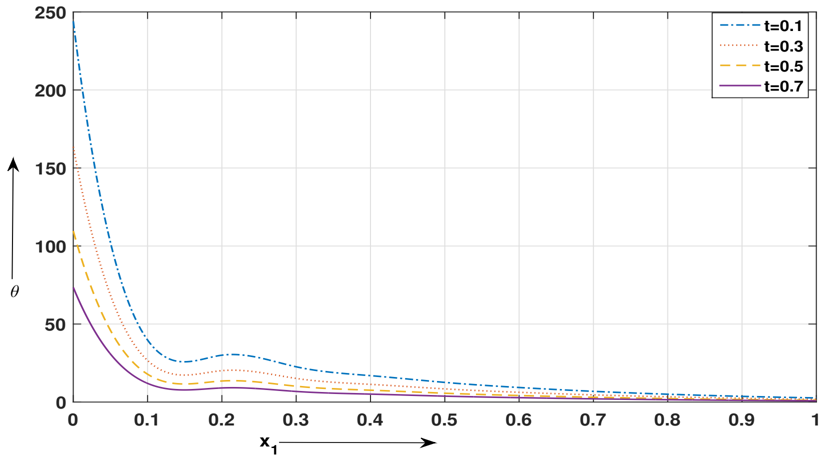
Case2: Figure 23.



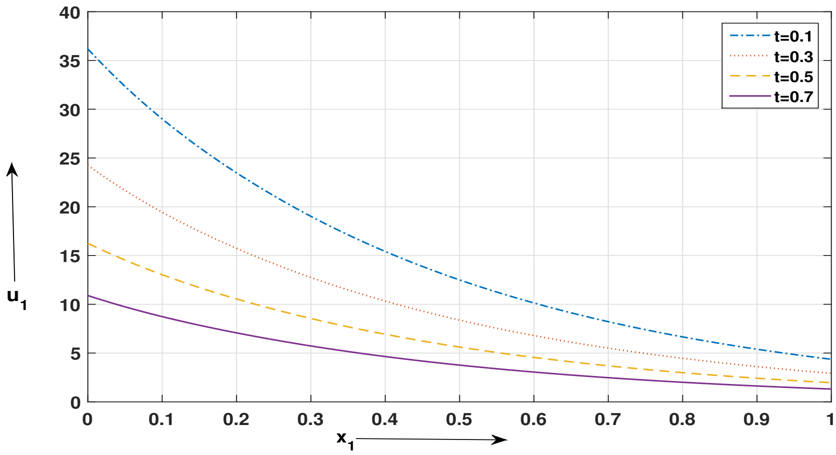
Case2: Figure 24.



Case2: Figure 25.



Case2: Figure 26.



Case2: Figure 27.

Figure 23-27 represents the characteristic curves different physical variables with respect to x_1 with $x_2 = 0.4$ and $x_3 = 0.5$ for different values of t for fixed values of $\tau_q = 0.01$, $\tau_\theta = 0.0001$, $\tau_0 = 0.01$ and R=1 satisfying boundary conditions.

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Physical Variables	CTE	L-S	G-N II	G-N III	SPL	DPL	RPL	N=1	N=2	N=3	N=4
τ_{11}	1.2982998110	1.2855458900	0.9190918256	1.2982998110	1.2726913170	1.0896629870	1.2730626970	1.2731854430	1.2730635140	1.2730627010	1.2730626970
τ_{22}	0.5275016524	0.5223242031	0.3735251250	0.5275016524	0.5171060242	0.4427994880	0.5172565719	0.5173064020	0.5172569035	0.5172565735	0.5172565719
τ_{33}	0.5222683007	0.5171363796	0.3696941698	0.5222683007	0.5119640114	0.4383209990	0.5121134196	0.5121628100	0.5121137483	0.5121134213	0.5121134196
τ_{12}	0.0777237510	0.0769357947	0.0545040287	0.0777237510	0.0761425915	0.0649056245	0.0761650379	0.0761726144	0.0761650883	0.0761650381	0.0761650379
τ_{13}	0.1304906540	0.1291877804	0.0919312982	0.1304906540	0.1278748069	0.1092246771	0.1279129682	0.1279255014	0.1279130516	0.1279129686	0.1279129682
τ_{23}	0.1915670394	0.1896842939	0.1355957839	0.1915670394	0.1877865969	0.1607677414	0.1878415314	0.1878596511	0.1878416520	0.1878415320	0.1878415314
u_1	1.2804261560	1.2676927090	0.9031484675	1.2804261560	1.2548560300	1.0723847800	1.2552325570	1.2553550620	1.2552333720	1.2552325610	1.2552325570
u_2	1.4301869650	1.4162445530	1.0147361300	1.4301869650	1.4021893280	1.2018277970	1.4025960450	1.4027302600	1.4025969380	1.4025960500	1.4025960450
u_3	1.2261773990	1.2141247110	0.8678764274	1.2261773990	1.2019750540	1.0289850900	1.2023279920	1.2024439880	1.2023287640	1.2023279960	1.2023279920
θ	0.1947619169	0.1928404798	0.1377023483	0.1947619169	0.1909050725	0.1633728846	0.1909599880	0.1909784780	0.1909601111	0.1909599887	0.1909599880

Case1: Figure. 28:

Physical Variables	CTE	L-S	G-N II	G-N III	SPL	DPL	RPL	N=1	N=2	N=3	N=4
τ_{11}	144.3238084000	144.3221134000	144.2727274000	144.3238084000	144.3205449000	144.2994920000	144.3204528000	144.3204691000	144.3204529000	144.3204528000	144.3204528000
τ_{22}	43.1622178800	43.1617184900	43.1471587900	43.1622178800	43.1612399900	43.1546326900	43.1612292200	43.1612340400	43.1612292600	43.1612292200	43.1612292200
τ_{33}	49.6914479600	49.6914711900	49.6918560100	49.6914479600	49.6915033200	49.6919844000	49.6914932800	49.6914930600	49.6914932700	49.6914932800	49.6914932800
τ_{12}	-9.0675977360	-9.0673977860	-9.0616187240	-9.0675977360	-9.0671814760	-9.0639407390	-9.0672020020	-9.0672039280	-9.0672020150	-9.0672020020	-9.0672020020
τ_{13}	-0.1834701574	-0.1830308874	-0.1703936970	-0.1834701574	-0.1825965304	-0.1764940243	-0.1826009071	-0.1826051352	-0.1826009352	-0.1826009072	-0.1826009071
τ_{23}	10.7626468100	10.7630063500	10.7732553800	10.7626468100	10.7633303700	10.7675293200	10.7633580800	10.7633546200	10.7633580600	10.7633580800	10.7633580800
u_1	100.7220924000	100.7100991000	100.3660651000	100.7220924000	100.6978023000	100.5206903000	100.6983617000	100.6984771000	100.6983625000	100.6983617000	100.6983617000
u_2	136.9013211000	136.9026363000	136.9393944000	136.9013211000	136.9035957000	136.9132386000	136.9039211000	136.9039085000	136.9039211000	136.9039211000	136.9039211000
u_3	120.6248041000	120.6315237000	120.8234582000	120.6248041000	120.6378979000	120.7242853000	120.6380980000	120.6380334000	120.6380976000	120.6380980000	120.6380980000
θ	27.6133708800	27.6132716500	27.6103270000	27.6133708800	27.6134927600	27.6199307100	27.6131743400	27.6131753000	27.6131743500	27.6131743400	27.6131743400

Case2: Figure. 29

The above two tables depicts the variations of values of the different components of shearing and normal stress,temperature and the displacement components for different models and for different values of N to clarify the effects of multi phase lag variables depending upon the boundary conditions provided in case 1 and case 2 respectively.

2.1.8 Conclusion

The thermomechanical analysis of the anisotropic half space has been investigated applying two-temperature parameter with modified coupled stress . As the extension of the study by Tzou, the concept of multi-phase lag thermoelasticity has been established and verified here successfully. The characteristic behaviours of different components of stress, displacement and temperature with respect to several thermoelasticity model(CTE, L-S, G-N, RPL etc)has been represented analytically and graphically. The validity and accuracy of this study has been assured through the boundary conditions provided in two different cases.

2.1.9 Appendix:

APPENDIX-I

$$a_{11} = 2ia\frac{c_{16}}{c_{11}} + 2ib\frac{c_{15}}{c_{11}}, a_{12} = -a^2\frac{c_{66}}{c_{11}} - b^2\frac{c_{55}}{c_{11}} - 2ab\frac{c_{55}}{c_{11}} - \omega^2, a_{21} = \frac{c_{16}}{c_{11}},$$

$$a_{22} = ia\frac{c_{12}+c_{66}}{c_{11}} + ib\frac{c_{14}+c_{56}}{c_{11}},$$

$$a_{23} = -a^2\frac{c_{26}}{c_{11}} - b^2\frac{c_{45}}{c_{11}} - ab\frac{c_{46}+c_{25}}{c_{11}}, a_{31} = \frac{c_{15}}{c_{11}},$$

$$a_{32} = ia\frac{c_{14}+c_{56}}{c_{11}} + ib\frac{c_{13}+c_{55}}{c_{11}}, a_{34} = -(1+T_1\omega)\frac{d\theta}{dx_1},$$

$$a_{33} = -a^2\frac{c_{46}}{c_{11}} - b^2\frac{c_{35}}{c_{11}} - ab\frac{c_{36}+c_{45}}{c_{11}}, b_{11} = \frac{c_{16}}{c_{66}}, b_{12} = ia\frac{c_{12}+c_{66}}{c_{66}} + ib\frac{c_{14}+c_{56}}{c_{66}},$$

$$b_{13} = -a^2\frac{c_{26}}{c_{66}} - b^2\frac{c_{45}}{c_{66}} - ab\frac{c_{46}+c_{25}}{c_{66}}, b_{21} = 2ia\frac{c_{26}}{c_{66}} + 2ib\frac{c_{46}}{c_{66}},$$

$$b_{22} = -a^2\frac{c_{22}}{c_{66}} - b^2\frac{c_{44}}{c_{66}} - 2ab\frac{c_{24}}{c_{66}} - \frac{c_{11}}{c_{66}}\omega^2,$$

$$b_{31} = \frac{c_{56}}{c_{66}}, b_{32} = ia\frac{c_{46}+c_{25}}{c_{66}} + ib\frac{c_{36}+c_{45}}{c_{66}},$$

$$b_{33} = -a^2\frac{c_{24}}{c_{66}} - b^2\frac{c_{34}}{c_{66}} - ab\frac{c_{23}+c_{44}}{c_{66}}, b_{34} = \frac{ia\beta_2 c_{11}(1+T_1\omega)}{c_{66}},$$

$$m_{11} = \frac{c_{15}}{c_{55}}, m_{12} = ia\frac{c_{56}+c_{14}}{c_{55}} + ib\frac{c_{55}+c_{13}}{c_{55}},$$

$$m_{13} = -a^2\frac{c_{46}}{c_{55}} - b^2\frac{c_{35}}{c_{55}} - ab\frac{c_{45}+c_{36}}{c_{55}}, m_{21} = \frac{c_{56}}{c_{55}},$$

$$m_{22} = ia\frac{c_{25}+c_{46}}{c_{55}} + ib\frac{c_{45}+c_{36}}{c_{55}},$$

$$m_{23} = -a^2\frac{c_{24}}{c_{55}} - b^2\frac{c_{34}}{c_{55}} - ab\frac{c_{44}+c_{23}}{c_{55}}, m_{31} = 2ia\frac{c_{45}}{c_{55}} + 2ib\frac{c_{35}}{c_{55}},$$

$$m_{32} = -a^2\frac{c_{44}}{c_{55}} - b^2\frac{c_{33}}{c_{55}} - 2ab\frac{c_{34}}{c_{55}} - \frac{c_{11}}{c_{55}}\omega^2, m_{33} = \frac{ib\beta_3 c_{11}}{c_{55}},$$

$$h_{12} = \frac{c_{16}}{c_{11}}, h_{13} = \frac{c_{15}}{c_{11}}, h_{14} = \frac{i(bc_{15}+ac_{16})}{c_{11}}, h_{15} = \frac{i(bc_{14}+ac_{12})}{c_{11}},$$

$$h_{16} = \frac{i(bc_{13}+ac_{14})}{c_{11}}, h_{21} = \frac{c_{21}}{c_{11}},$$

$$h_{22} = \frac{c_{26}}{c_{11}}, h_{23} = \frac{c_{25}}{c_{11}}, h_{24} = \frac{i(bc_{25}+ac_{26})}{c_{11}}, h_{25} = \frac{i(bc_{24}+ac_{22})}{c_{11}},$$

$$h_{26} = \frac{i(bc_{23}+ac_{24})}{c_{11}}, h_{31} = \frac{c_{31}}{c_{11}},$$

$$h_{32}=\frac{c_{36}}{c_{11}},h_{33}=\frac{c_{35}}{c_{11}},h_{34}=\frac{i(bc_{35}+ac_{26})}{c_{11}},h_{35}=\frac{i(bc_{34}+ac_{32})}{c_{11}},$$
$$h_{36}=\frac{i(bc_{33}+ac_{34})}{c_{11}},h_{41}=\frac{c_{41}}{c_{11}},$$
$$h_{42}=\frac{c_{46}}{c_{11}},h_{43}=\frac{c_{45}}{c_{11}},h_{44}=\frac{i(bc_{45}+ac_{46})}{c_{11}},h_{45}=\frac{i(bc_{44}+ac_{42})}{c_{11}},$$
$$h_{46}=\frac{i(bc_{43}+ac_{44})}{c_{11}},h_{51}=\frac{c_{51}}{c_{11}},$$
$$h_{52}=\frac{c_{56}}{c_{11}},h_{53}=\frac{c_{55}}{c_{11}},h_{54}=\frac{i(bc_{55}+ac_{56})}{c_{11}},h_{55}=\frac{i(bc_{54}+ac_{52})}{c_{11}},$$
$$h_{56}=\frac{i(bc_{53}+ac_{54})}{c_{11}},h_{61}=\frac{c_{61}}{c_{11}},$$
$$h_{62}=\frac{c_{66}}{c_{11}},h_{63}=\frac{c_{65}}{c_{11}},h_{64}=\frac{i(bc_{65}+ac_{66})}{c_{11}},$$
$$h_{65}=\frac{i(bc_{64}+ac_{62})}{c_{11}},h_{66}=\frac{i(bc_{63}+ac_{64})}{c_{11}},$$
$$d_{11}=1+\frac{a_{21}(b_{31}m_{11}-b_{11})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{21}b_{11}-m_{11})}{1-b_{31}m_{21}},$$
$$d_{12}=a_{11}+\frac{a_{21}(b_{31}m_{12}-b_{12})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{21}b_{12}-m_{12})}{1-b_{31}m_{21}},$$
$$d_{13}=a_{12}+\frac{a_{21}(b_{31}m_{13}-b_{13})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{21}b_{13}-m_{13})}{1-b_{31}m_{21}},$$
$$d_{14}=a_{22}+\frac{a_{21}(b_{31}m_{22}-b_{21})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{21}b_{21}-m_{22})}{1-b_{31}m_{21}},$$
$$d_{15}=a_{23}+\frac{a_{21}(b_{31}m_{23}-b_{22})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{21}b_{22}-m_{23})}{1-b_{31}m_{21}},$$
$$d_{16}=a_{32}+\frac{a_{21}(b_{31}m_{31}-b_{32})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{21}b_{32}-m_{31})}{1-b_{31}m_{21}},$$
$$d_{17}=a_{33}+\frac{a_{21}(b_{31}m_{32}-b_{33})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{21}b_{33}-m_{31})}{1-b_{31}m_{21}},$$
$$d_{18}=-\frac{a_{21}(b_{31}m_{33}+b_{34})}{1-b_{31}m_{21}}+\frac{a_{31}(m_{33}-m_{21}b_{34})}{1-b_{31}m_{21}},$$

$$d_{21}=(-b_{12}-\frac{b_{31}(m_{21}b_{12}-m_{12})}{1-b_{31}m_{21}})-\frac{d_{12}}{d_{11}}(-b_{11}-\frac{b_{31}(m_{21}b_{11}-m_{11})}{1-b_{31}m_{21}}),$$
$$d_{22}=(-b_{13}-\frac{b_{31}(m_{21}b_{13}-m_{13})}{1-b_{31}m_{21}})-\frac{d_{13}}{d_{11}}(-b_{11}-\frac{b_{31}(m_{21}b_{11}-m_{11})}{1-b_{31}m_{21}}),$$
$$d_{23}=(-b_{21}-\frac{b_{31}(m_{21}b_{21}-m_{22})}{1-b_{31}m_{21}})-\frac{d_{14}}{d_{11}}(-b_{11}-\frac{b_{31}(m_{21}b_{11}-m_{11})}{1-b_{31}m_{21}}),$$

$$d_{24} = (-b_{22} - \frac{b_{31}(m_{21}b_{22} - m_{23})}{1 - b_{31}m_{21}}) - \frac{d_{15}}{d_{11}}(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}),$$

$$d_{25} = (-b_{32} - \frac{b_{31}(m_{21}b_{32} - m_{31})}{1 - b_{31}m_{21}}) - \frac{d_{16}}{d_{11}}(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}),$$

$$d_{26} = (-b_{33} - \frac{b_{31}(m_{21}b_{33} - m_{32})}{1 - b_{31}m_{21}}) - \frac{d_{17}}{d_{11}}(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}),$$

$$d_{27} = \frac{1}{d_{11}}(-b_{11} - \frac{b_{31}(m_{21}b_{11} - m_{11})}{1 - b_{31}m_{21}}), d_{31} = \frac{m_{11}d_{12}}{d_{11}} - m_{12} - m_{21}d_{21},$$

$$d_{32} = \frac{m_{11}d_{13}}{d_{11}} - m_{13} - m_{21}d_{22},$$

$$d_{33} = \frac{m_{11}d_{14}}{d_{11}} - m_{22} - m_{21}d_{23}, d_{34} = \frac{m_{11}d_{15}}{d_{11}} - m_{23} - m_{21}d_{24},$$

$$d_{35} = \frac{m_{11}d_{16}}{d_{11}} - m_{31} - m_{21}d_{25},$$

$$d_{36} = \frac{m_{11}d_{17}}{d_{11}} - m_{32} - m_{21}d_{26}, d_{37} = -(\frac{m_{11}\beta_1}{d_{11}} + m_{21}d_{27}),$$

$$d_{38} = \frac{m_{11}d_{18}}{d_{11}} + m_{33} - m_{21}d_{28},$$

$$d_{41} = \frac{\varepsilon_1\omega^2}{C_T^2}, d_{42} = 0, d_{43} = 0, d_{44} = \frac{ia\varepsilon_2}{C_T^2}, d_{45} = 0, d_{46} = \frac{ib\varepsilon\beta_3}{C_T^2},$$

$$d_{47} = 0, d_{48} = \frac{C_T^2(a^2 + b^2) + \omega^2}{C_T^2},$$

$$g_{51} = -\frac{d_{13}}{d_{11}}, g_{52} = -\frac{d_{15}}{d_{11}}, g_{53} = -\frac{d_{17}}{d_{11}}, g_{54} = -\frac{d_{18}}{d_{11}},$$

$$g_{55} = -\frac{d_{12}}{d_{11}}, g_{56} = -\frac{d_{14}}{d_{11}}, g_{57} = -\frac{d_{16}}{d_{11}}, g_{58} = \frac{1}{d_{11}},$$

$$g_{61} = d_{22}, g_{62} = d_{24}, g_{63} = d_{26}, g_{64} = d_{28},$$

$$g_{65} = d_{21}, g_{66} = d_{23}, g_{67} = d_{25}, g_{68} = d_{27},$$

$$g_{71} = d_{32}, g_{72} = d_{34}, g_{73} = d_{36}, g_{74} = d_{38},$$

$$g_{75} = d_{31}, g_{76} = d_{33}, g_{77} = d_{35}, g_{78} = d_{37},$$

$$g_{81} = 0, g_{82} = d_{44}, g_{83} = d_{46}, g_{84} = d_{48},$$

$$g_{85} = d_{41}, g_{86} = g_{87} = g_{88} = 0,$$

$$f_{11} = g_{51} + \lambda g_{55} - \lambda^2, f_{12} = g_{52} + \lambda g_{56},$$

$$f_{13} = g_{53} + \lambda g_{57}, f_{14} = g_{54} + \lambda g_{58},$$

$$f_{21} = g_{61} + \lambda g_{65}, f_{22} = g_{62} + \lambda g_{66} - \lambda^2, f_{23} = g_{63} + \lambda g_{67},$$

$$f_{24} = g_{64} + \lambda g_{68}, f_{31} = g_{71} + \lambda g_{65},$$

$$f_{32} = g_{72} + \lambda g_{76}, f_{33} = g_{73} + \lambda g_{77} - \lambda^2, f_{34} = g_{74} + \lambda g_{78},$$

$$f_{41} = \lambda g_{85}, f_{42} = g_{82}, f_{43} = g_{83}, f_{44} = g_{84},$$

$$L_{21} = \begin{bmatrix} g_{51} & g_{52} & g_{53} & g_{54} \\ g_{61} & g_{62} & g_{63} & g_{64} \\ g_{71} & g_{72} & g_{73} & g_{74} \\ g_{81} & g_{82} & g_{83} & g_{84} \end{bmatrix} L_{22} = \begin{bmatrix} g_{55} & g_{56} & g_{57} & g_{58} \\ g_{65} & g_{66} & g_{67} & g_{68} \\ g_{75} & g_{76} & g_{77} & g_{78} \\ g_{85} & g_{86} & g_{87} & g_{88} \end{bmatrix}$$

APPENDIX II:

$R_{11}(x_1) = [(h_{14} - \lambda_1)\{(\delta_1)_{\lambda=-\lambda_1} + (h_{15} - \lambda_1 h_{12})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{16} - \lambda_1 h_{13})\{(\delta_3)_{\lambda=-\lambda_1}\}$
$-\{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}$
$R_{12}(x_1) = [(h_{14} - \lambda_2)\{(\delta_1)_{\lambda=-\lambda_2} + (h_{15} - \lambda_2 h_{12})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{16} - \lambda_2 h_{13})\{(\delta_3)_{\lambda=-\lambda_2}\}$
$-\{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}$
$R_{13}(x_1) = [(h_{14} - \lambda_3)\{(\delta_1)_{\lambda=-\lambda_3} + (h_{15} - \lambda_3 h_{12})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{16} - \lambda_3 h_{13})\{(\delta_3)_{\lambda=-\lambda_3}\}$
$-\{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}$
$R_{14}(x_1) = [(h_{14} - \lambda_4)\{(\delta_1)_{\lambda=-\lambda_4} + (h_{15} - \lambda_4 h_{12})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{16} - \lambda_4 h_{13})\{(\delta_3)_{\lambda=-\lambda_4}\}$
$-\{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}$
$R_{21}(x_1) = [(h_{24} - \lambda_1 h_{21})\{(\delta_1)_{\lambda=-\lambda_1} + (h_{25} - \lambda_1 h_{22})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{26} - \lambda_1 h_{23})\{(\delta_3)_{\lambda=-\lambda_1}\}$
$-\beta_2\{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}$
$R_{22}(x_1) = [(h_{24} - \lambda_2 h_{21})\{(\delta_1)_{\lambda=-\lambda_2} + (h_{25} - \lambda_2 h_{22})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{26} - \lambda_2 h_{23})\{(\delta_3)_{\lambda=-\lambda_2}\}$
$-\beta_2\{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}$
$R_{23}(x_1) = [(h_{24} - \lambda_3 h_{21})\{(\delta_1)_{\lambda=-\lambda_3} + (h_{25} - \lambda_3 h_{22})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{26} - \lambda_3 h_{23})\{(\delta_3)_{\lambda=-\lambda_3}\}$
$-\beta_2\{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}$
$R_{24}(x_1) = [(h_{24} - \lambda_4 h_{21})\{(\delta_1)_{\lambda=-\lambda_4} + (h_{25} - \lambda_4 h_{22})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{26} - \lambda_4 h_{23})\{(\delta_3)_{\lambda=-\lambda_4}\}$
$-\beta_2\{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}$
$R_{31}(x_1) = [(h_{34} - \lambda_1 h_{31})\{(\delta_1)_{\lambda=-\lambda_1} + (h_{35} - \lambda_1 h_{32})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{36} - \lambda_1 h_{33})\{(\delta_3)_{\lambda=-\lambda_1}\}$
$-\beta_3\{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}$
$R_{32}(x_1) = [(h_{34} - \lambda_2 h_{31})\{(\delta_1)_{\lambda=-\lambda_2} + (h_{35} - \lambda_2 h_{32})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{36} - \lambda_2 h_{33})\{(\delta_3)_{\lambda=-\lambda_2}\}$
$-\beta_3\{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}$
$R_{33}(x_1) = [(h_{34} - \lambda_3 h_{31})\{(\delta_1)_{\lambda=-\lambda_3} + (h_{35} - \lambda_3 h_{32})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{36} - \lambda_3 h_{33})\{(\delta_3)_{\lambda=-\lambda_3}\}$
$-\beta_3\{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}$
$R_{34}(x_1) = [(h_{34} - \lambda_4 h_{31})\{(\delta_1)_{\lambda=-\lambda_4} + (h_{35} - \lambda_4 h_{32})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{36} - \lambda_4 h_{33})\{(\delta_3)_{\lambda=-\lambda_4}\}$
$-\beta_3\{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}$

$R_{41}(x_1) =$
$[(h_{44} - \lambda_1 h_{41})\{(\delta_1)_{\lambda=-\lambda_1}\} + (h_{45} - \lambda_1 h_{42})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{46} - \lambda_1 h_{43})\{(\delta_3)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}$

$R_{42}(x_1) =$
$[(h_{44} - \lambda_2 h_{41})\{(\delta_1)_{\lambda=-\lambda_2}\} + (h_{45} - \lambda_2 h_{42})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{46} - \lambda_2 h_{43})\{(\delta_3)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}$

$R_{43}(x_1) =$
$[(h_{44} - \lambda_3 h_{41})\{(\delta_1)_{\lambda=-\lambda_3}\} + (h_{45} - \lambda_3 h_{42})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{46} - \lambda_3 h_{43})\{(\delta_3)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}$

$R_{44}(x_1) =$
$[(h_{44} - \lambda_4 h_{41})\{(\delta_1)_{\lambda=-\lambda_4}\} + (h_{45} - \lambda_4 h_{42})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{46} - \lambda_4 h_{43})\{(\delta_3)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}$

$R_{51}(x_1) =$
$[(h_{54} - \lambda_1 h_{51})\{(\delta_1)_{\lambda=-\lambda_1}\} + (h_{55} - \lambda_1 h_{52})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{56} - \lambda_1 h_{53})\{(\delta_3)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}$
$R_{52}(x_1) =$
$[(h_{54} - \lambda_2 h_{51})\{(\delta_1)_{\lambda=-\lambda_2} + (h_{55} - \lambda_2 h_{52})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{56} - \lambda_2 h_{53})\{(\delta_3)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}$

$R_{53}(x_1) =$
$[(h_{54} - \lambda_3 h_{51})\{(\delta_1)_{\lambda=-\lambda_3}\} + (h_{55} - \lambda_3 h_{52})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{56} - \lambda_3 h_{53})\{(\delta_3)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}$

$R_{54}(x_1) =$
$[(h_{54} - \lambda_4 h_{51})\{(\delta_1)_{\lambda=-\lambda_4}\} + (h_{55} - \lambda_4 h_{52})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{56} - \lambda_4 h_{53})\{(\delta_3)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}$

$R_{61}(x_1) =$
$[(h_{64} - \lambda_1 h_{61})\{(\delta_1)_{\lambda=-\lambda_1}\} + (h_{65} - \lambda_1 h_{62})\{(\delta_2)_{\lambda=-\lambda_1}\} + (h_{66} - \lambda_1 h_{63})\{(\delta_3)_{\lambda=-\lambda_1}\}]e^{-\lambda_1 x_1}$

$R_{62}(x_1) =$
$[(h_{64} - \lambda_2 h_{61})\{(\delta_1)_{\lambda=-\lambda_2}\} + (h_{65} - \lambda_2 h_{62})\{(\delta_2)_{\lambda=-\lambda_2}\} + (h_{66} - \lambda_2 h_{63})\{(\delta_3)_{\lambda=-\lambda_2}\}]e^{-\lambda_2 x_1}$

$R_{63}(x_1) =$
$[(h_{64} - \lambda_3 h_{61})\{(\delta_1)_{\lambda=-\lambda_3}\} + (h_{65} - \lambda_3 h_{62})\{(\delta_2)_{\lambda=-\lambda_3}\} + (h_{66} - \lambda_3 h_{63})\{(\delta_3)_{\lambda=-\lambda_3}\}]e^{-\lambda_3 x_1}$

$R_{64}(x_1) =$
$[(h_{64} - \lambda_4 h_{61})\{(\delta_1)_{\lambda=-\lambda_4}\} + (h_{65} - \lambda_4 h_{62})\{(\delta_2)_{\lambda=-\lambda_4}\} + (h_{66} - \lambda_4 h_{63})\{(\delta_3)_{\lambda=-\lambda_4}\}]e^{-\lambda_4 x_1}$

$$R_{71}(0) = (\nu - \lambda_1)\{(\delta_4)_{\lambda=-\lambda_1}\}]e^{-\lambda_1}$$
$$R_{72}(0) = (\nu - \lambda_2)\{(\delta_4)_{\lambda=-\lambda_2}\}]e^{-\lambda_2}$$
$$R_{73}(0) = (\nu - \lambda_3)\{(\delta_4)_{\lambda=-\lambda_3}\}]e^{-\lambda_3}$$
$$R_{74}(0) = (\nu - \lambda_4)\{(\delta_4)_{\lambda=-\lambda_4}\}]e^{-\lambda_4}$$

-

$$D_1 = \begin{vmatrix} 0 & R_{12}(0) & R_{13}(0) & R_{14}(0) \\ 0 & R_{22}(0) & R_{23}(0) & R_{24}(0) \\ 0 & R_{32}(0) & R_{33}(0) & R_{34}(0) \\ r^* & R_{72}(0) & R_{73}(0) & R_{74}(0) \end{vmatrix} D_{2=} \begin{vmatrix} R_{11}(0) & 0 & R_{13}(0) & R_{14}(0) \\ R_{21}(0) & 0 & R_{23}(0) & R_{24}(0) \\ R_{31}(0) & 0 & R_{33}(0) & R_{34}(0) \\ R_{71}(0) & r^* & R_{73}(0) & R_{74}(0) \end{vmatrix}$$

$$D_{3=} \begin{vmatrix} R_{11}(0) & R_{12}(0) & 0 & R_{14}(0) \\ R_{21}(0) & R_{22}(0) & 0 & R_{24}(0) \\ R_{31}(0) & R_{32}(0) & 0 & R_{34}(0) \\ R_{71}(0) & R_{72}(0) & r^* & R_{74}(0) \end{vmatrix} D_{4=} \begin{vmatrix} R_{11}(0) & R_{12}(0) & R_{13}(0) & 0 \\ R_{21}(0) & R_{22}(0) & R_{23}(0) & 0 \\ R_{31}(0) & R_{32}(0) & R_{33}(0) & 0 \\ R_{71}(0) & R_{72}(0) & R_{73}(0) & r^* \end{vmatrix}$$

$$\Delta = \begin{vmatrix} R_{11}(0) & R_{12}(0) & R_{13}(0) & R_{14}(0) \\ R_{21}(0) & R_{22}(0) & R_{23}(0) & R_{24}(0) \\ R_{31}(0) & R_{32}(0) & R_{33}(0) & R_{34}(0) \\ R_{71}(0) & R_{72}(0) & R_{73}(0) & R_{74}(0) \end{vmatrix}$$

$$N_{1i} = \frac{1}{\lambda_i}\sum_{i=1}^{8}(h_{14}x_{1i} + h_{15}x_{2i} + h_{16}x_{3i} - x_{4i})$$

$$N_{2i} = \frac{1}{\lambda_i}\sum_{i=1}^{8}(h_{24}x_{1i} + h_{25}x_{2i} + h_{26}x_{3i} - \beta_2 x_{4i})$$

$$N_{3i} = \frac{1}{\lambda_i}\sum_{i=1}^{8}(h_{34}x_{1i} + h_{35}x_{2i} + h_{36}x_{3i} - \beta_3 x_{4i})$$

2.2 Modeling of a homogeneous isotropic half space in the context of multi-phase lag coupled thermoelasticity

2.2.1 Introduction

The modified coupled stress-strain theory has become popular in Nano-systems to study strain, displacement, vibration, buckling, bending etc. in the engineering structures like beams, plates and shells. The modified theory of coupled stress was introduced by Yang et al.[122]. The theory of generalised thermoelasticity was introduced to remove the finiteness of heat equation in the conventional classical thermoelasticity (CTE) theory. The generalised thermoelasticity theory has become acceptable to the different engineering fields as well as to the researchers as it is capable of avoiding the finiteness of heat propagation.

In the history of generalised thermoelasticity, Lord-Shulman [80] introduced one-relaxation time parameter in the conventional heat conduction equation to modify classical Fourier law (CFL) which is also known as first generalisation theory of thermoelasticity. In the second generalisation theory, Green-Lindsay [58] proposed temperature rate dependent theory (TRDTE) by introducing two relaxation time parameters in the coupled theory of thermoelasticity. In third generalisation, Hetnarski and Ignaczak [63] proposed the non-linear model introducing the concept of coupled thermoelasticity with low temperature. Green and Nagdhi ([59], [60], and [61]) proposed three thermoelastic models known as Green-Naghdi type-I, type-II and type-III respectively. Type-I is considered as similar as classical theory of thermoelasticity. Type-II model provides the idea of non-dissipation of energy associated with zero production rate of entropy. Type-III Green-Naghdi model is associated with type-I and type-II together with the study of energy dissipation and damped thermoelastic waves. Later on Tzou [113], [117] and Chandrasekhariah [33] discussed the Dual Phase Lag (DPL) model to inspect the lagging behaviour in thermoelastic medium. Again, Roy Choudhury [102] discussed the concept of Three Phase Lag Model [TPL] introducing three time parameters in conventional heat conduction equation. Ghosh et al. [49] [52], Quintanilla and Racke [99] found the solutions of the heat equation in the theory of TPL heat conduction in their research work. Also Zenkour [128] proposed a refined two-temperature multi-phase-lag (RPL) model for a generalized thermoelastic medium consisting of both the heat flux vector and the temperature gradient. In their work, Sardar et al.[103], studied a three-dimensional coupled thermoelasticity for an anisotropic half-space us-

ing multi-phase lag gradients. Zenkour [129] also studied thermomechanical effects of mocrobeams using refined multiphase-lag theory. In association, Alharabi et al. [20] studied a multi-phase-lag model to investigate the influence of variable thermal conductivity with initial stress on a fibre-reinforced thermoelastic material in magnetic field.
In this study we have investigated the effect of multi-phase lag variables on a two dimensional thermoelastic isotropic medium.

2.2.2 Formulation of the problem

In the orthogonal co-ordinate system XOY, we consider a two dimensional isotropic half space defined in the region $W = \{0 \le x < \infty, -\infty < y < \infty\}$ (as in Fig. 2.1) subject to traction free boundary $x = 0$. Also y-axis is considered as vertically downwards and the xy-plane is along the free surface of the half space.

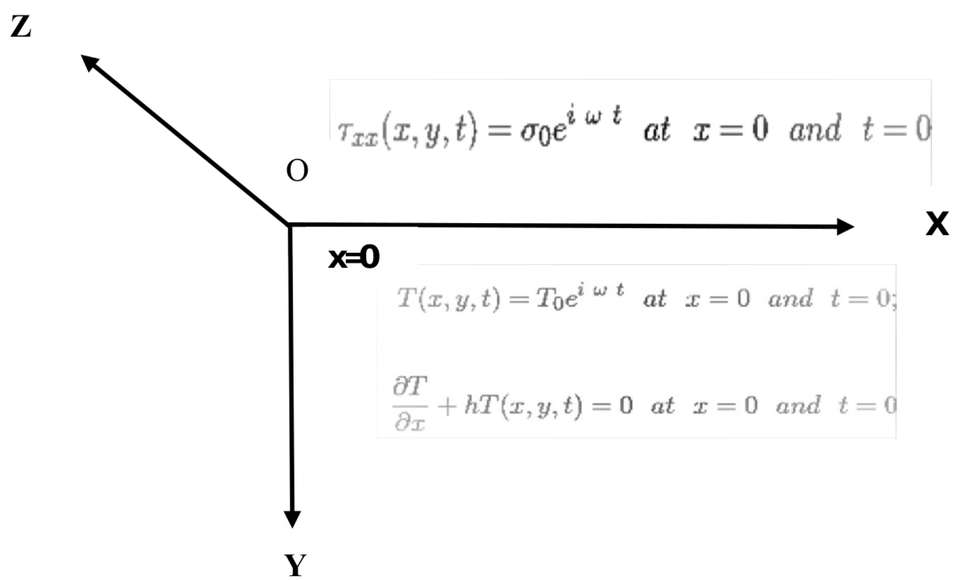

Figure 2.1: Schematic diagram of the problem.

2.2.3 Basic Equations

Equation of motion:

For a homogeneous isotropic half space, as in Ghosh et al.[53], Zenkour [128] and Eringen [47] respectively, we consider the following governing equations .

$$(\lambda+\mu)\frac{\partial e}{\partial x}+\mu\nabla^2 u-\frac{P}{2}\frac{\partial^2 u}{\partial y^2}+\frac{P}{2}\frac{\partial^2 v}{\partial x\partial y}-\gamma\frac{\partial T}{\partial x}=\rho\left[\frac{\partial^2 u}{\partial t^2}-\Omega^2 u\right] \tag{2.33}$$

$$(\lambda+\mu)\frac{\partial e}{\partial y}+\mu\nabla^2 v+\frac{P}{2}\frac{\partial^2 u}{\partial x\partial y}-\frac{P}{2}\frac{\partial^2 v}{\partial x^2}+\gamma\frac{\partial T}{\partial y}=\rho\left[\frac{\partial^2 v}{\partial t^2}-\Omega^2 v\right] \tag{2.34}$$

The stress-displacement relation:

$$\tau_{xx}=(\lambda+2\mu+P)\frac{\partial u}{\partial x}+(\lambda+P)\frac{\partial v}{\partial y}-\gamma T \tag{2.35}$$

$$\tau_{yy}=(\lambda+2\mu+P)\frac{\partial v}{\partial y}+(\lambda+P)\frac{\partial u}{\partial x}-\gamma T \tag{2.36}$$

$$\tau_{xy}=\left(\mu-\frac{P}{2}\right)\frac{\partial v}{\partial x}+\left(\mu+\frac{P}{2}\right)\frac{\partial u}{\partial y} \tag{2.37}$$

Heat conduction equation*(In the context of multi-phase Lag)***:**

$$\left(1+\sum_{n=1}^{N}\frac{\tau_\theta^n}{n!}\frac{\partial^n}{\partial t^n}\right)\left(K_{11}\frac{\partial^2 T}{\partial x^2}+K_{22}\frac{\partial^2 T}{\partial y^2}\right)$$
$$=\left[\overline{R}+\tau_0\frac{\partial}{\partial t}+\sum_{n=1}^{N}\frac{\tau_q^{n+1}}{(n+1)!}\frac{\partial^{n+1}}{\partial t^{n+1}}\right]\left[\rho C_E\frac{\partial^2 T}{\partial t^2}+(3\lambda+2\mu)\alpha_0 T_0\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right] \tag{2.38}$$

where $\gamma=(3\lambda+2\mu)\alpha_t$, $\lambda+2\mu=\rho {c_1}^2$ and $e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$

2.2.4 Nomenclature

Column 1	Column 2
u, v: Displacement Components	e: Dilatation
T: Absolute thermodynamic temperature.	t: -Time variable
C_E: -Specific heat at constant strain	λ, μ: Lame's Constant
T_0: Reference temperature	τ: Relaxation Time
τ_q , τ_θ: Dual-phase-lag, τ_0:thermal relaxation time	ρ : - Density of the material
K_{11}, K_{22} : Thermal conductivity	Ω : angular velocity in the domain W
α_t : Coefficient of linear thermal expansion	P: Initial stress

2.2.5 Method of Solution: Formulation of a Vector-Matrix Differential Equation

For the solution of the equations(2.38)-(2.43), the physical quantities can be decomposed in the following form

$$\begin{array}{cc} t' = \frac{C_1^2}{K_1}t, \quad (\ x',\ y') = \frac{C_1}{K_1}(x,y), & (u',v') = \frac{c_1^3\rho}{K_1(3\lambda+2\mu)\alpha_t T_0}(u,v), \\ \tau_0' = \frac{C_1^2}{K_1}\tau_0, & T' = \frac{T}{T_0} \\ (\tau_{xx}',\tau_{yy}',\tau_{xx}') = \frac{1}{(3\lambda+2\mu)\alpha_t T_0}(\tau_{xx},\tau_{yy},\tau_{xy}) & \Omega' = \frac{K_1}{C_1^2}\Omega, K_1 = \frac{K_{11}}{\rho C_E}, \\ \rho = \frac{\rho'}{\alpha T_0}, c_1^2 = \frac{\lambda+2\mu}{\rho} & (\tau_q',\tau_\theta') = \frac{C_1^2}{K_1}(\tau_q,\tau_\theta) \end{array} \tag{2.39}$$

Introducing non-dimensional variables we obtain from equations (2.38), (2.39) and (2.43) (omitting primes for convenience),

$$\frac{\partial^2 u}{\partial x^2} + C_{11}\frac{\partial^2 v}{\partial x\partial y} + C_{12}\frac{\partial^2 u}{\partial y^2} - \frac{\partial T}{\partial x} = \frac{\partial^2 u}{\partial t^2} - \Omega^2 u \tag{2.40}$$

$$\frac{\partial^2 v}{\partial y^2} + C_{21}\frac{\partial^2 v}{\partial x^2} + C_{22}\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial T}{\partial y} = \frac{\partial^2 v}{\partial t^2} - \Omega^2 v \tag{2.41}$$

$$\begin{aligned} &\frac{K_{11}}{\rho C_E c_1^2}\left(1 + \sum_{n=1}^{N}\frac{\tau_\theta^n}{n!}\frac{\partial^n}{\partial t^n}\right)\left(\frac{\partial^2 T}{\partial x^2} + \frac{K_{22}}{K_{11}}\frac{\partial^2 T}{\partial y^2}\right) \\ &= \left(\overline{R} + \tau_0\frac{\partial}{\partial t} + \sum_{n=1}^{N}\frac{\tau_q^{n+1}}{(n+1)!}\frac{\partial^{n+1}}{\partial t^{n+1}}\right)\left(\frac{\partial^2 T}{\partial t^2} + \frac{\gamma^2 T_0}{\rho^2 c_1^2 c_E}\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) \end{aligned} \tag{2.42}$$

After introducing non-dimensional variables, the stress-displacement relations (equations (2.40)-(2.42)) reduce to (omitting primes for convenience),

$$\tau_{xx} = C_{41}\frac{\partial u}{\partial x} + C_{42}\frac{\partial v}{\partial y} - T \tag{2.43}$$

$$\tau_{yy} = C_{41}\frac{\partial v}{\partial y} + C_{42}\frac{\partial u}{\partial x} - T \tag{2.44}$$

$$\tau_{yy} = C_{51}\frac{\partial v}{\partial x} + C_{52}\frac{\partial u}{\partial y} \tag{2.45}$$

2.2.6 Normal Mode Analysis:

To decompose the physical variables in terms of normal modes, as in Ghosh et al.[54] we consider the following normal mode analysis

$$(u,v,\tau_{xx},\tau_{yy},\tau_{xy},T)(x,y,t) = (u^*,v^*,\tau_{xx}^*,\tau_{yy}^*,\tau_{xy}^*,T^*)(x)e^{\omega t+iay} \tag{2.46}$$

where ω is the angular frequency , a is the wave number along x-axis and $i = \sqrt{-1}$.

Introducing normal mode analysis to equation (2.45), (2.46) and (2.47), we obtain (omitting asterisks for convenience):

$$\frac{d^2u}{dx^2} = M_{11}u + M_{12}v + M_{13}T + M_{14}\frac{du}{dx} + M_{15}\frac{dv}{dx} + M_{16}\frac{dT}{dx} \tag{2.47}$$

$$\frac{d^2v}{dx^2} = M_{21}u + M_{22}v + M_{23}T + M_{24}\frac{du}{dx} + M_{25}\frac{dv}{dx} + M_{26}\frac{dT}{dx} \tag{2.48}$$

$$\frac{d^2T}{dx^2} = M_{31}u + M_{32}v + M_{33}T + M_{34}\frac{du}{dx} + M_{35}\frac{dv}{dx} + M_{36}\frac{dT}{dx} \tag{2.49}$$

Introducing normal mode analysis to equations (2.48)-(2.50), we obtain the stress components as (omitting asterisks for convenience)

$$\tau_{xx} = C_{41}\frac{du}{dx} + C_{42}\;\; iav - T \tag{2.50}$$

$$\tau_{yy} = C_{41}\;\; iav + C_{42}\frac{du}{dx} - T \tag{2.51}$$

$$\tau_{xy} = C_{51}\frac{dv}{dx} + C_{52}\;\; iau \tag{2.52}$$

where $M_{ij}(i = 1, 2, 3 \; and \; j = 1, 2, .., 6)$ and $C_{ij}(i = 1, 2, 3, 4, 5 \; and \; j = 1, 2)$ are mentioned in the appendix.

2.2.7 Solution of the Vector-Matrix Differential Equation

The equations (2.52)-(2.54) reduce to the compact form of vector-matrix differential equation as follows

$$\frac{d}{dx}(\vec{v}) = A\vec{v} \tag{2.53}$$

where $\vec{v} = \begin{pmatrix} u & v & T & \frac{du}{dx} & \frac{dv}{dx} & \frac{dT}{dx} \end{pmatrix}$ and A is given in the appendix.

For the solution of the vector-matrix differential equation (2.58), we apply the method of eigenvalue approach as in Ghosh et al. [54]. The characteristic equation of matrix A is given by

$$\left| A - \lambda I \right| = 0 \tag{2.54}$$

The roots of the characteristic equation (2.58) are $\lambda = \lambda_i$(i=1(1)6) and the corresponding eigen vector **X** is given below-

$$\mathbf{X} = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \lambda\delta_1 \quad \lambda\delta_2 \;\; \lambda\delta_3 \;\;]^T \tag{2.55}$$

where

$$\begin{aligned} \delta_1 &= f_{11}f_{23} - f_{13}f_{22}, \\ \delta_2 &= f_{13}f_{21} - f_{11}f_{23}, \\ \delta_3 &= f_{11}f_{22} - f_{12}f_{21} \end{aligned} \tag{2.56}$$

and f_{ij} (i, j = 1,2,3) are given in the appendix.

The solution of the vector-matrix equation is given by

$$\begin{aligned} u &= \textstyle\sum_{i=1}^{3} A_i(\delta_1)_{\lambda=-\lambda_i} e^{-\lambda_i x} \\ v &= \textstyle\sum_{i=1}^{3} A_i(\delta_2)_{\lambda=-\lambda_i} e^{-\lambda_i x} \\ T &= \textstyle\sum_{i=1}^{3} A_i(\delta_3)_{\lambda=-\lambda_i} e^{-\lambda_i x} \end{aligned} \tag{2.57}$$

Thus the stress components are as follows-

$$\begin{aligned} \tau_{xx} &= \textstyle\sum_{j=1}^{3} A_j R_{1j}(x), \\ \tau_{yy} &= \textstyle\sum_{j=1}^{3} A_j R_{2j}(x), \\ \tau_{xy} &= \textstyle\sum_{j=1}^{3} A_j R_{3j}(x), \end{aligned} \tag{2.58}$$

where R_{ij} $i, j = 1, 2, 3$ are given in the appendix and $A_j, j = 1, 2, 3$ are to be obtained using the boundary conditions..

2.2.8 Boundary Conditions

Due to the regularity condition of the solution at infinity, there are three terms containing exponentials of growing in nature in the space variables x has been discarded and the remaining arbitrary constants A_i, (i=1,2,...4) are to be determined from the following boundary conditions.

2.2.9 Mechanical Boundary:

The boundary of the half-space x=0 has no traction elsewhere i.e.,

$$\tau_{xx}(x, y, t) = \sigma_0 e^{i\omega t} \quad at \quad x = 0 \quad and \quad t = 0; \tag{2.59}$$

2.2.10 The Thermal Boundary Condition:

$$T(x,y,t) = T_0 e^{i\omega t} \quad at \quad x = 0 \quad and \quad t = 0; \tag{2.60}$$

$$\frac{\partial T}{\partial x} + hT(x,y,t) = 0 \quad at \quad x = 0 \quad and \quad t = 0; \tag{2.61}$$

Applying above boundary conditions in equation (2.62) and (2.63) we get the following simultaneous equations:-

$$\begin{aligned} A_1 S_{11} + A_2 S_{12} + A_3 S_{13} &= z_1 \\ A_1 S_{41} + A_2 S_{42} + A_3 S_{43} &= z_2 \\ A_1 S_{51} + A_2 S_{52} + A_3 S_{53} &= 0 \end{aligned} \tag{2.62}$$

The arbitrary constants can be obtained by solving above simultaneous equations where, $A_i = \frac{D_i}{D}, i = 1,2,3, D, D_i : i = 1,2,3$, $S_{ij} : \quad i,j = 1,2,3$ and $z_i : i = 1,2$ which are given in the Appendix.

2.2.11 Numerical analysis

Numerical analysis and computation have been done using the mechanical and thermal conditions as mentioned in equations (2.64)-(2.66) to study the characteristic behaviours of the physical constants with respect to space variables in triclinic half space. The numerical values (in SI unit) of constants are taken as in Eringen [47] , Zenkour [128]:

$$\begin{array}{ccc} \lambda = 9.4 \times 10^{10}\, N/m^2, & \mu = 4.0 \times 10^{10} kg/ms^2, & \rho = 1.7 \times 10^3 kg/m^3, \\ a = 2.0, b = 0.5, T_0 = 293, & K, \alpha_T = 7.4033 \times 10^{-7} & K^{-1}, t = 0.3s, \sigma_0 = 200.0, \\ K_{11} = 113 \times 10^{-4}\, N/m^2, & K_{22} = 117 \times 10^{-4}\, N/m^2, & C_E = 1.4 \times 10^3 J/(kgk), \\ \gamma = 210 \times 10^4, \Omega = 0.5, & k = 348, \Omega = 0.5, & E_T = 0.0016, \omega = 2.0 \end{array}$$

2.2.12 Geometrical Representation and analysis

The expressions for displacements, stress, and temperature are very complex and we prefer to develop an efficient computer programme for the numerical computations.

We now depict some graphs to illustrate the problem.

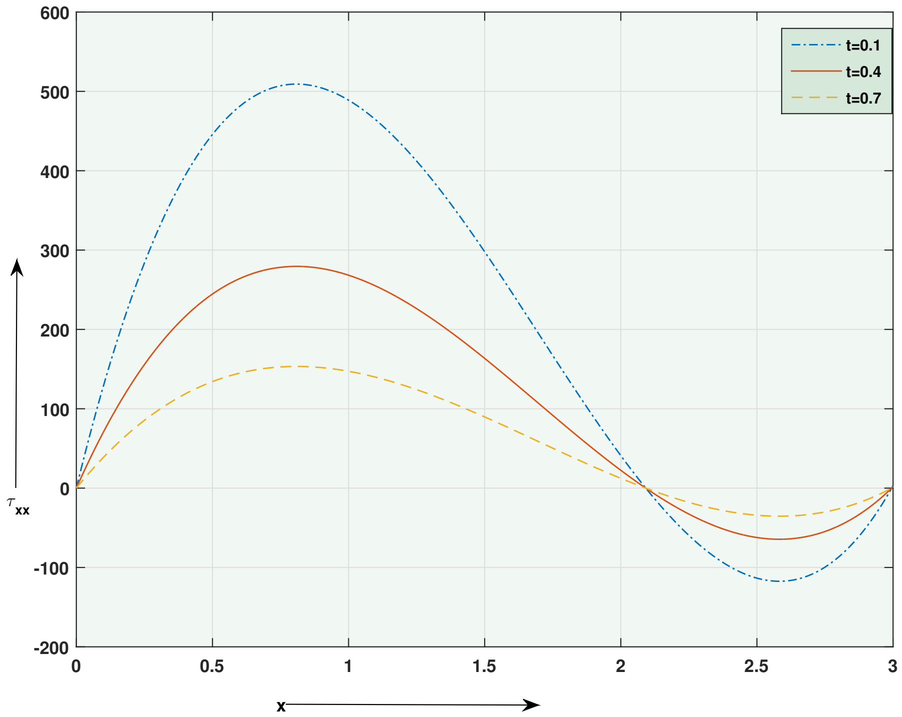

Fig. 2.2: Distribution of τ_{xx} vs. x for different t at y =0.7

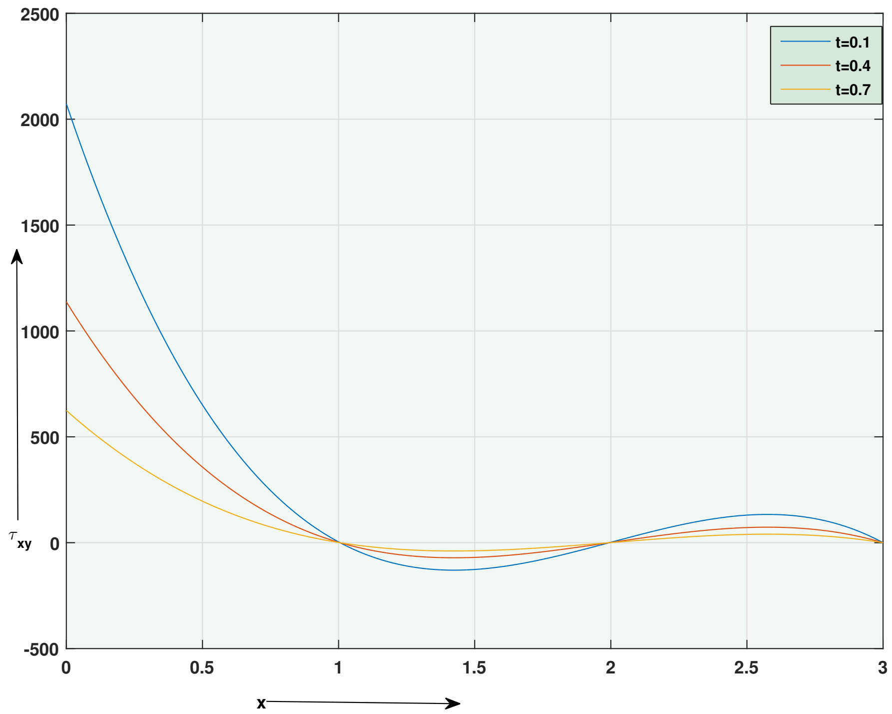

Fig. 2.3: Distribution of τ_{xy} vs. x for different t at y =0.7

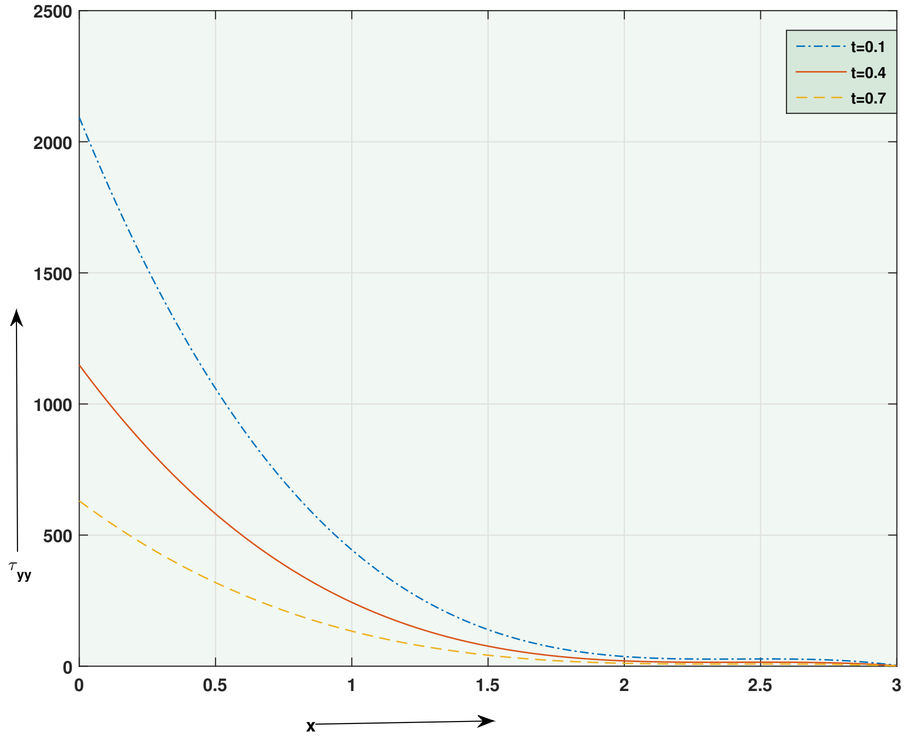

Fig. 2.4: Distribution of τ_{yy} vs. x for different t at y =0.7

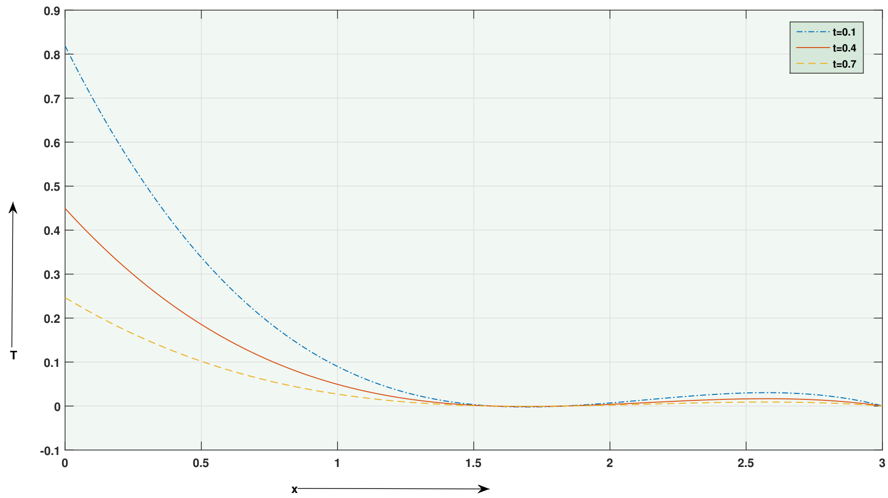

Fig. 2.5: Distribution of temperature (T) vs. x for different t at y =0.7

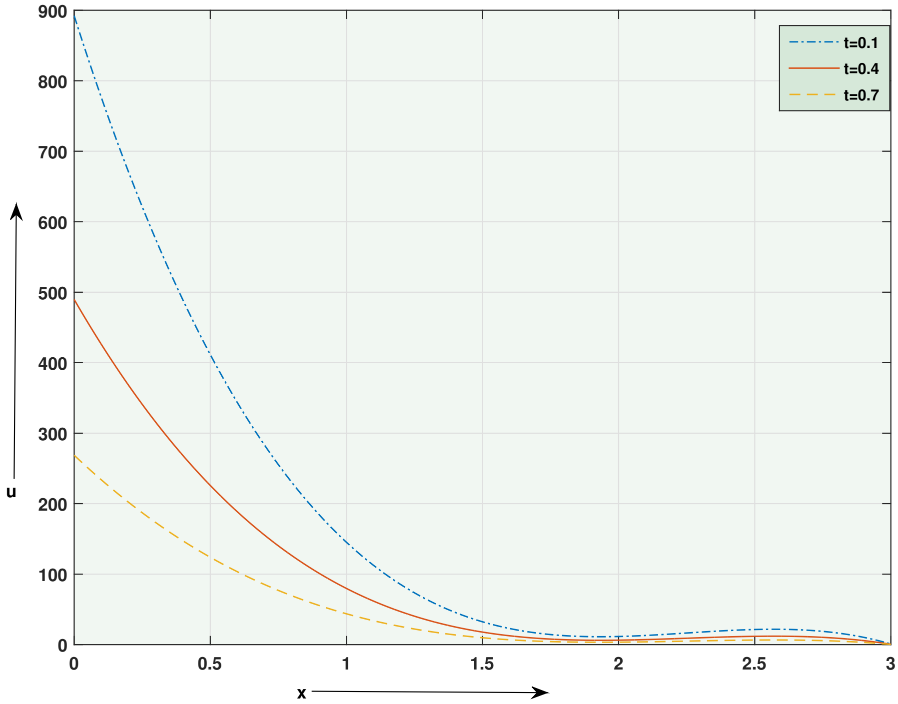

Fig. 2.6: Distribution of displacement component (u) vs. x for different t at y =0.7

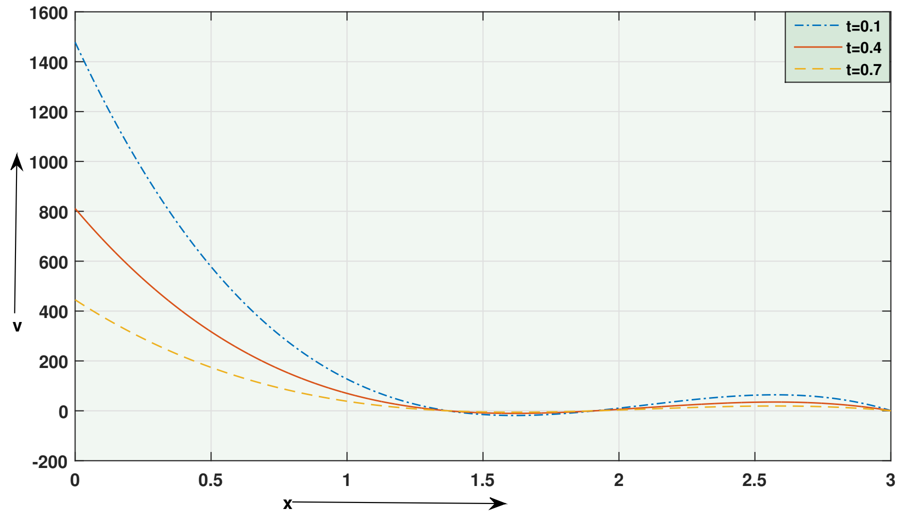

Fig. 2.7: Distribution of displacement component (v) vs. x for different t at y =0.7

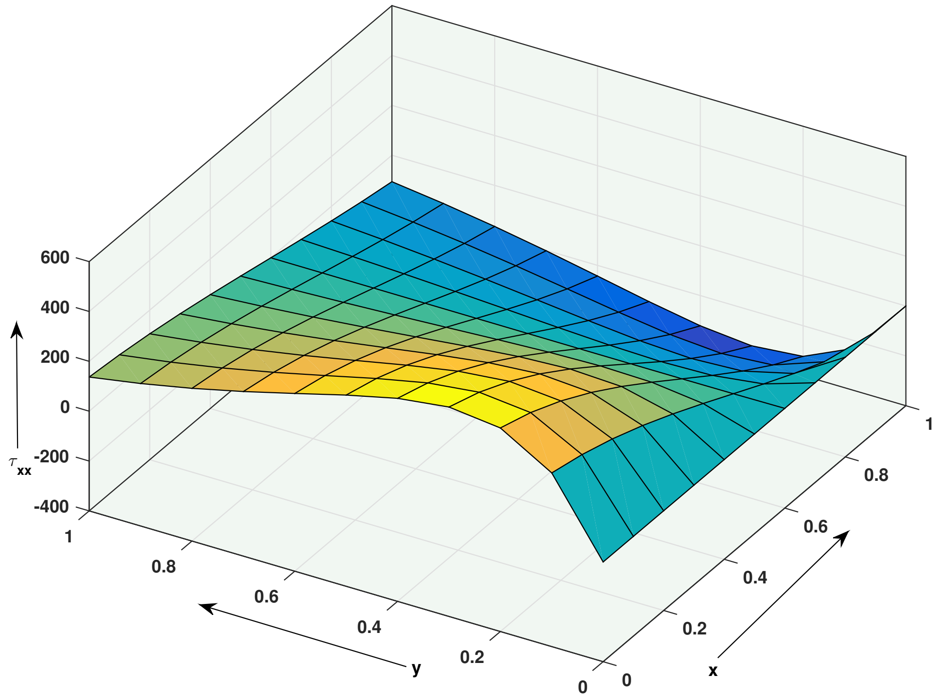

Fig. 2.8: Three dimensional representation of τ_{xx} vs. x and y for fixed t=0.3

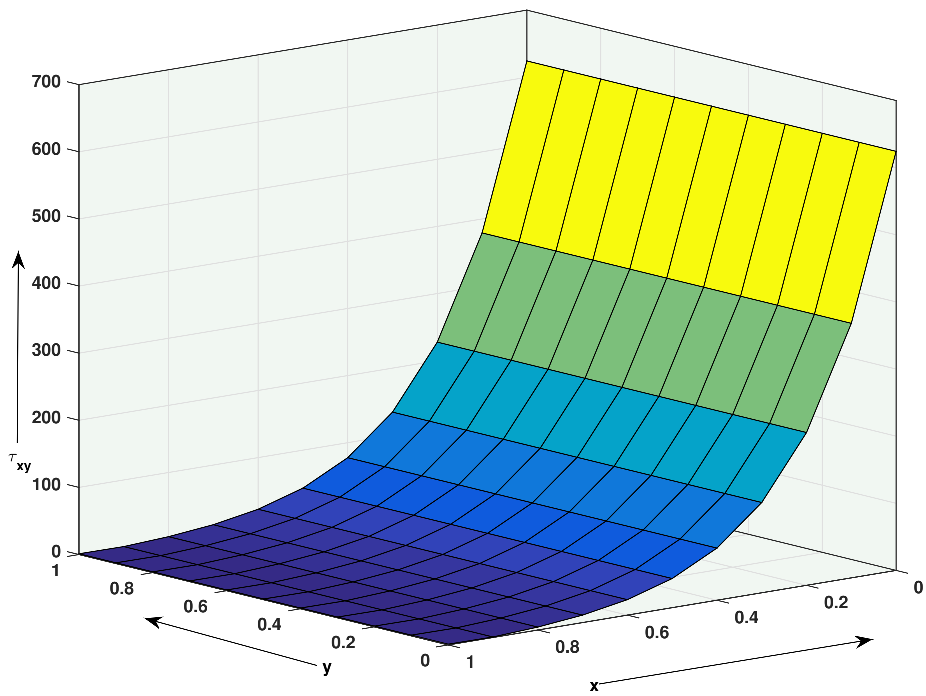

Fig. 2.9: Three dimensional representation of τ_{xy} vs. x and y for fixed t=0.3

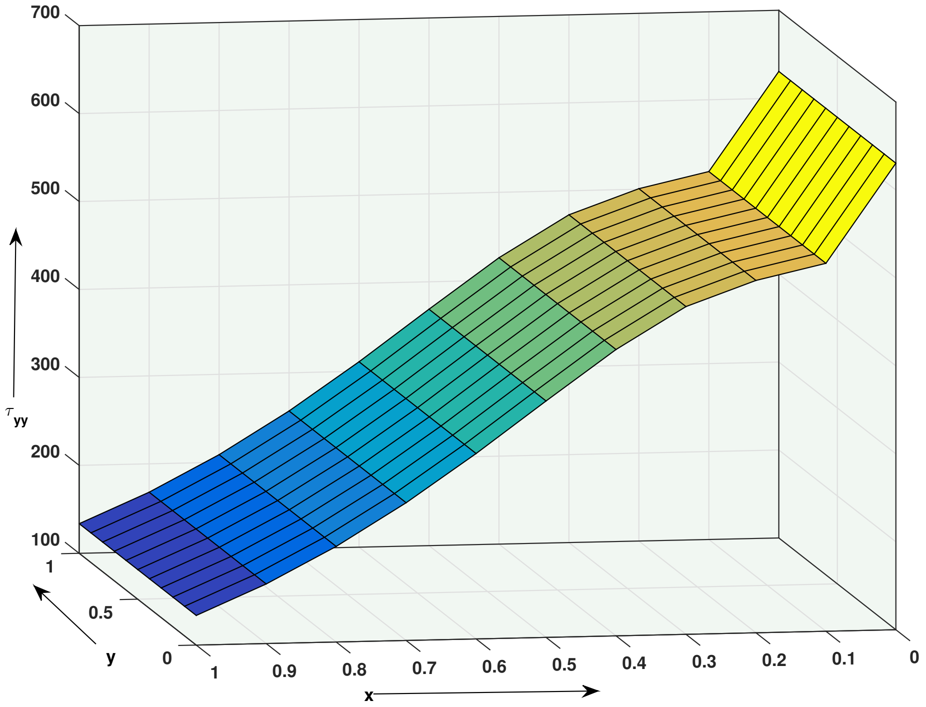

Fig. 2.10: Three dimensional representation of τ_{yy} vs. x and y for fixed t=0.3

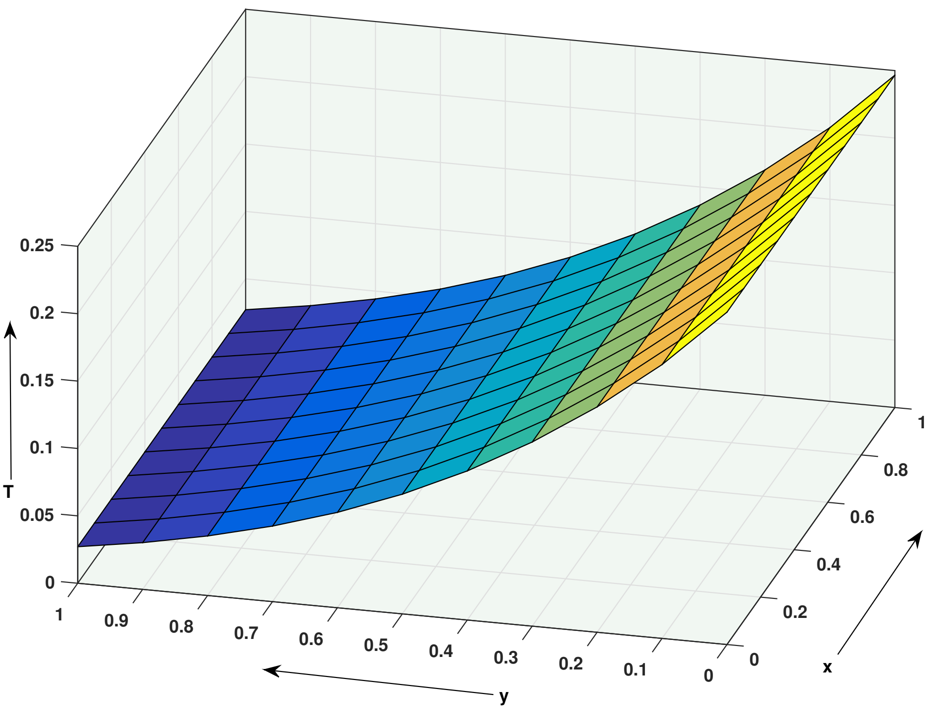

Fig. 2.11: Three dimensional representation of T vs. x and y for fixed t=0.3

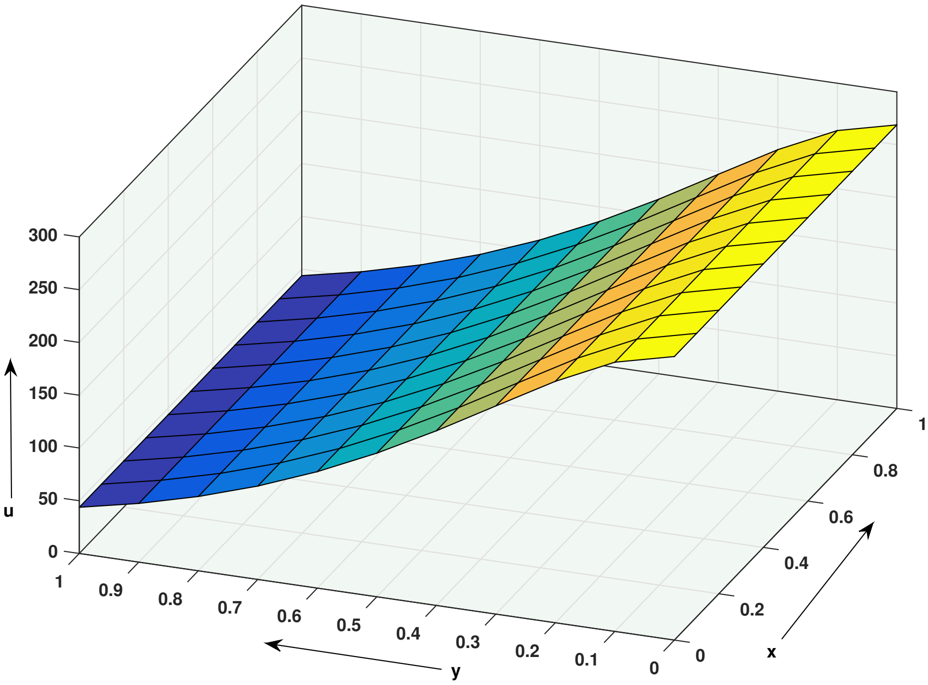

Fig. 2.12: Three dimensional representation of u vs. x and y for fixed t=0.3

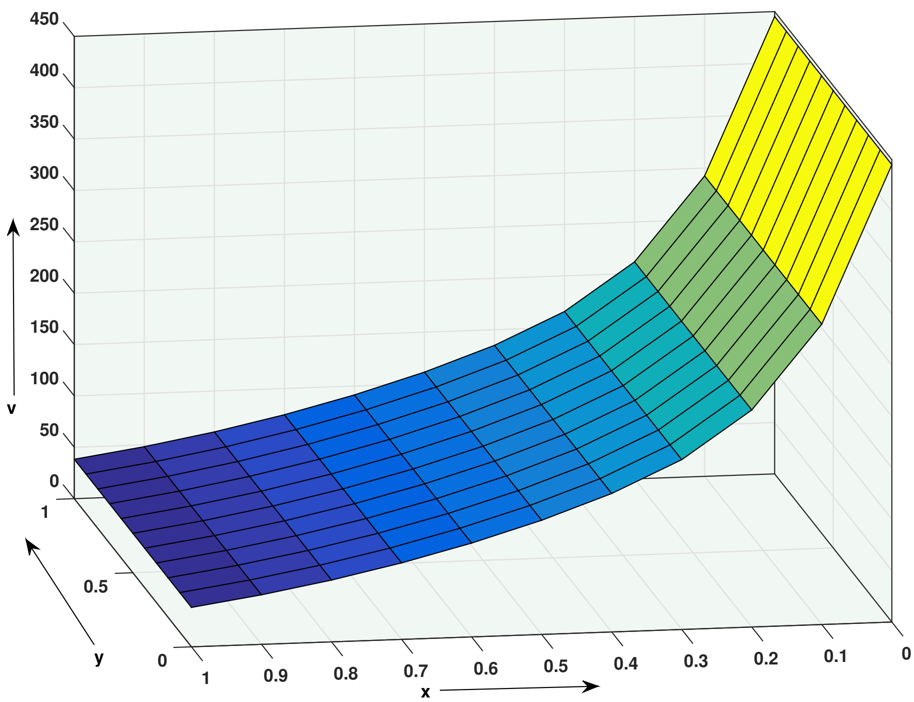

Fig. 2.13: Three dimensional representation of v vs. x and y for fixed t=0.3

2.2.13 Concluding Remarks:

Fig. 2.2, 2.3 and 2.4 depict the characteristic behaviour of the different stress components τ_{xx}, τ_{xy}, and τ_{yy} respectively along $x-$axis respect to space variable(x) in different time (t=0.1, t=0.4, t=0.7). Also Fig. 2.5, 2.6 and 2.7 represent the space variation of non-dimensional displacement components (u and v) and temperature (T) along x-direction for different time mentioned in legend.

Fig. 2.8, 2.9, 2.10 and 2.11 are pointing out the three dimentional variations of different stress components τ_{xx}, τ_{xy}, τ_{yy} and temperature (T) respectively with respect to space variable(x and y) in a particular time span(t=0.3).

Also Fig. 2.12, 2.13 are about the three dimentional depiction of the two elementary displacement components (u and v) w.r.to x and y for fixed time (t=0.3).

2.2.14 Significance and applications:

The Dual Phase Lag (DPL) model by Tzou, Chandrasekhariah and Three Phase Lag (TPL) by Roy Choudhury has been extended here using the refined techniques known as multiphase lag model. In our work, the multiphase lag concept is studied and verified successfully using the prominent mechanical and thermal boundary conditions associated to governing equations. The two and three dimensional variations of the different stress components, strain components and temperature curves have been represented graphically.

PHYSICAL VARIABLES	CTE	L-S	G-N -II	G-N -III	SPL	DPL	RPL				
							N=1	N=2	N=3	N=4	N=5
τ_{xx}	473.1505319	473.1502386	473.1494963	473.1505319	473.1505222	473.1502103	473.1502346	473.1502103	473.150038	473.1501995	473.1502717
τ_{yy}	472.6831795	472.6828864	472.6821447	472.6831795	472.6831697	472.6828581	472.6828824	472.68296	472.682686	472.6828473	472.6829194
τ_{xy}	134.462778	134.4626947	134.4624838	134.462778	134.4627753	134.4626866	134.4626936	134.4627156	134.4626377	134.4626836	134.4627041
u	197.2866122	197.28649	197.2861806	197.2866122	197.2866083	197.2864782	197.2864883	197.2865207	197.2864064	197.2864737	197.2865038
v	165.3005608	165.3004583	165.300199	165.3005608	165.3005573	165.3004484	165.3004569	165.300484	165.3003882	165.3004446	165.3004698
T	0.13659991	0.13659992	0.1365999	0.13659991	0.13659999	0.13659991	0.13659992	0.13659992	0.136599908	0.136599909	0.136599924

Fig. 2.14: Data table

The tabular data in Fig. 2.14 represents the compact variations of the numerical value of different stress components, temperature and displacement components in the context of different thermoelastic models compared to multiphase lag model. From the data table it is possible to differentiate the effect of different phase lag models and multi phase lags on different physical variables.

2.2.15 Appendix:

$$M_{11} = aC_{12} + \omega^2 - \Omega^2,\ M_{12} = M_{13} = M_{14} = 0,\ M_{15} = -iaC_{11},\ M_{16} = 1,$$

$$M_{21} = M_{25} = M_{26} = 0,\ M_{22} = \frac{a^2 + \omega^2 + \Omega^2}{C_{21}},\ M_{23} = -\frac{ia}{C_{21}},\ M_{24} = -\frac{iaC_{22}}{C_{21}},$$

$$M_{31} = M_{35} = M_{36} = 0,\ M_{32} = ia\frac{\gamma^2 T_0}{\rho^2 C_1^2 C_E},$$

$$M_{33} = a^2\frac{K_{22}}{K_{11}} + \frac{\omega^2\left(\overline{R} + \tau_0\omega + \sum_{n=1}^{N}\frac{\tau_q^{n+1}}{(n+1)!}\omega^{n+1}\right)}{\frac{K_{11}}{\rho C_E C_1^2}\left(1 + \sum_{n=1}^{N}\frac{\tau_\theta^n}{n!}\omega^n\right)},$$

$$M_{34} = \frac{\gamma^2 T_0}{\rho^2 C_1^2 C_E}\frac{\omega^2\left(\overline{R} + \tau_0\omega + \sum_{n=1}^{N}\frac{\tau_q^{n+1}}{(n+1)!}\omega^{n+1}\right)}{\frac{K_{11}}{\rho C_E C_1^2}\left(1 + \sum_{n=1}^{N}\frac{\tau_\theta^n}{n!}\omega^n\right)}$$

$$C_{11} = C_{22} = \tfrac{\lambda+\mu+\frac{P}{2}}{\lambda+2\mu},\ C_{12} = C_{21} = \tfrac{\mu-\frac{P}{2}}{\lambda+2\mu}$$

$$C_{41} = \frac{\lambda + 2\mu + \rho}{\rho c_1^2},\ C_{42} = \frac{\lambda + P}{\rho c_1^2},\ C_{51} = \frac{\mu - \frac{P}{2}}{\rho c_1^2},\ c_{52} = \frac{\mu + \frac{P}{2}}{\rho c_1^2}$$

$$A = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\ L_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ L_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_{21} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\ L_{22} = \begin{pmatrix} M_{14} & M_{15} & M_{16} \\ M_{24} & M_{25} & M_{26} \\ M_{34} & M_{35} & M_{36} \end{pmatrix}$$

$$\begin{array}{lll} f_{11} = M_{11} + \lambda M_{14} - \lambda^2 & f_{21} = M_{21} + \lambda M_{24} & f_{31} = M_{31} + \lambda M_{34} \\ f_{12} = M_{12} + \lambda M_{15} & f_{22} = M_{22} + \lambda M_{25} - \lambda^2 & f_{32} = M_{32} + \lambda M_{35} \\ f_{13} = M_{13} + \lambda M_{16} & f_{23} = M_{23} + \lambda M_{26} & f_{33} = M_{33} + \lambda M_{36} - \lambda^2 \end{array}$$

$$R_{1i}(x) = [-C_{41}\lambda_i(delta_1)_{\lambda=\lambda_i} + iaC_{42}(\delta_2)_{\lambda=-\lambda_i} - (\delta_3)_{\lambda=\lambda_i}]e^{-\lambda_i x}, i = 1,2,3$$
$$R_{2i}(x) = [iaC_{41}(\delta_2)_{\lambda=-\lambda_i} - C_{42}\lambda_1(\delta_1)_{\lambda=-\lambda_i} - (\delta_3)_{\lambda=-\lambda_i}]e^{-\lambda_i x}, i = 1,2,3$$
$$R_{3i}(x) = [-C_{51}\lambda_i(\delta_2)_{\lambda=-\lambda_i} + iaC_{52}(\delta_1)_{\lambda=-\lambda_i}]e^{-\lambda_i x}, i = 1,2,3$$

$$\begin{aligned} z_1 &= \sigma_0 e^{i\omega t} \\ z_2 &= T_0 e^{i\omega t} \end{aligned}$$

$$\begin{aligned} S_{ij} &= R_{ij}(0),\ i = 1,4,\ j = 1,2,3 \\ S_{5k} &= R_{5k}(x),\ k = 1,2,3 \end{aligned}$$

$$D_1 = \begin{vmatrix} z_1 & S_{12} & S_{13} \\ z_2 & S_{42} & S_{43} \\ 0 & S_{52} & S_{53} \end{vmatrix} D_2 = \begin{vmatrix} S_{11} & z_1 & S_{13} \\ S_{41} & z_2 & S_{43} \\ S_{51} & 0 & S_{53} \end{vmatrix}$$
$$D_3 = \begin{vmatrix} S_{11} & S_{12} & z_1 \\ S_{41} & S_{42} & z_2 \\ S_{51} & S_{52} & 0 \end{vmatrix} D = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{41} & S_{42} & S_{43} \\ S_{51} & S_{52} & S_{53} \end{vmatrix}$$

3

ANALYSIS OF NON-LOCAL HEAT PROPAGATION FOR THERMOELASTIC MEDIUM

PROBLEMS :

- PROBLEM -3 **Photothermal Effects of Semiconducting Medium with Non-local Theory.**

  This paper has been communicated.

- PROBLEM -4 **Wave Propagation in a Non-local Magneto-thermoelastic Medium Permeated by Heat Source.**

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  DOI: *https://doi.org*/10.1080/15502287.2023.2186968.

3.1 Photothermal Effects of Semiconducting Medium with Non-local Theory

3.1.1 Introduction

Biot [30] introduced the classical coupled theory of thermoelasticity which contains conduction theory with conventional Fourier's law. In this theory, it obviously proves that the energy equation is of parabolic type and does not include any elastic term. This implies that the parabolic equation predicts infinite speed of deformative wave propagation. Actually, this theory contradicts the fact of thermoelastic wave propagation which gives the finite speed and elastic changes.

Eventually, this theory regarding infinite speed of wave propagation was swiped out due to introduction of generalized theory of thermoelasticity developed by Lord-Shulman [80] which is also known as extended thermo-elasticity(ETE). Lord-Shulman [80] modified the classical coupled theory without violating the conventional Fourier's law of heat conduction theory. One thermal relaxation time parameter was introduced to the energy equation which automatically transformed to the hyperbolic type of equation. First type of paradox of the infinite speed of propagation was eliminated due to introduction of generalized thermoelasticity. The heat conduction equation does not contain any elastic term so it does not cause any elastic changes which is known as second paradox. This type of paradox was removed by the introduction of Green-Lindsay (G-L) [58] theory just remodeling the energy equation and employing another relaxation time parameter to the equation of motion. This G-L [58] theory admits two relaxation time parameters: one in equation of motion and another in energy equation and overcomes the two well known paradoxes. This theory is also known as temperature rate dependent thermoelasticity(TRDTE). Green and Naghdi [59]-[61] developed three theories-I,II,III relating to the theory of generalized thermoelasticity with and/or without energy dissipation for isotropic and homogeneous thermoelastic medium. The theory concerned with energy dissipation is known as thermoelasticity with energy dissipation theory(TEWEDT) and similarly the theory as regards without energy dissipation is called thermoelasticity without energy dissipation theory (TEWOEDT).

Increasing attention has been devoted to the field of thermoelasticity in presence of electromagnetic field because of its widely spread application of material structure. Electro-magneto-thermoelasticity deals with the material structure related to the interaction between continuum deformations along with heat propagation. A significant thermoelastic fact is characterized due to the combination of the "electronic deformation" in the semiconducting medium which is based on the photogeneration theory in the crystal lattice and the "thermoelastic mechanism" due to integral photothermal process. Recently, Das et al. [41] focussed on the analysis of the electromagnetothermoelastic theory for a semiconductor.

According to Eringen [46], the hypothesis of the continuum physics is the measurement of strain components which are combination of inner products of non-local deformations and their gradients compared to the inner products of the non-local and local member of the continuum. The non-local thermoelastic theory proposed the long-gap in between the classical continuum hypothesis and the construction theory of lattice. The constitutive relations regarding the atomic structure of lattices as well as the dispersion of phonon can be treated according to the classical continuum hypothesis. Later, Tzou [113], [115] introduced the dual-phase-lag heat conduction theory by introductions of two phase-lags associated with the heat flux and the temperature gradient. Dual-phase-lag(DPL) thermoelastic theory admits two-phase lagging parameters - (i) τ_q illustrates the fast-transient effects of thermal inertia and (ii) τ_θ depicts the micro-structural interactions. Recently, Gupta and Mukhopadhyay [62] discussed generalized thermoelastic theory by using non-local theory of heat conduction.

Later, Roychoudhuri [102] extended this idea of DPL on G-N model-III by introducing the third-phase-lag and modified another generalized thermoelastic theory which is known as three phase-lag(TPL) theory. Tzou and Guo [118] developed a new theory on energy equation which is known as non-local behavior with thermal lagging. It also captures the lagging response along with the time variable. Phase-lagging behavior is responsible to capture the ultrafast response in fem-to-second domain whereas the non-local theory enlightens the physical mechanism at nano scale.

Generally, thermoelastic or magnetothermoelastic (coupled or generalized) problems are solved in four different processes viz., (i) Potential function method where the analytical solutions are represented in terms of potential function which is not always convergent where the initial and boundary conditions of the given problems are related to the physical considerations not to the potential function approach. Das and Lahiri [42] solved a generalized problem with functionally graded spherical cavity by using potential function methodology. (ii) State-space method where the physical problems are transformed in terms of coefficient matrix which is expanded by Cayley-Hamilton theorem. (iii) Eigen value method where the considered problem is transformed to the vector-matrix differential equation which is then solved by eigenvalue methodology. Lahiri, Das and Sarkar [74] solved a problem for an unbounded body with a spherical cavity by using eigenvalue approach methodology. (iv) Finite element and difference method where the basic equations of considered problem are discretised in terms of algebraic equations considering the stability and/or convergence condition of the system. Recently, Patra et al. [97] analyzed a magnetothermoelastic problem for a rotating cylinder by using finite difference method.

Generalized magneto-thermoelastic problem with internal heat source was studied by Othman et al. [96], Mahdy et al. [81] described the electromagnetic effect for photo-excited semiconducting medium in presence of fractional order heat equation. Abo-Dahab and Lotfy [8] predicted the interactions of fibre-reinforced thermoelasticity for a rotating medium. Magnetic photo-thermal diffusion process for nano-composite semiconducting medium was studied by Lotfy [79], [76]. Lotfy et al. [77]-[78] and Khamis et al. [72] also analyzed the photothermal and thermochemical responses of semiconductor under different conditions. Refined multi-phase-lag model for an infinite medium has been introduced by Kutubi and Zenkour [73]. Zenkour [130]-[132] modeled a generalized magneto-shock problem for isotropic and anisotropic medium respectively. Sobhy and Zenkour [110] illustrated the modified phase-lagging models for axisymmetric annular disc. Zenkour [134], [136]-[138] described the refined multi-phase-lag photo-thermoelastic theory for different types of medium. Coupled thermoelastic model with phase-lag theory is also analyzed by Zenkour and El-Mekawy [139]. Zenkour [135] studied the diffusion problem under refined dual-phase lagging with Green-Naghdi's generalized thermoelasticity. Recently, Abouelregal et al.[10]-[14] solved different problems of thermoelastic and visco-elastic by using Moore-Gibson-Thompson and two-phase lag theory.

In this problem, we have studied a model which interacts with the generalized thermoelasticity and plasma transportation under the non-local heat conduction theory in presence of electromagnetic field components. Two integral transformations, Laplace transform for time variable and Fourier transform for space variable, are employed to the equations of motion and the heat conduction equation for formulation of a vector-matrix differential equation which is then solved by using eigenvalue approach. Variations of the physical field variables are presented by the several graphs with fixed values of physical parameters.

3.1.2 Basic equations for theoretical model

As in Todorvic [112] and Das et. al. [41], the plasma equation is of the form

$$\frac{\partial N}{\partial t} = D_E \nabla^2 N - \frac{N}{\tau} + k_1 \theta \tag{3.1}$$

As in Gupta and Mukhopadhyay[62], the modified heat transportation law for non-local theory is as

$$K(1+\tau_\theta \frac{\partial}{\partial t})\theta_{,ii} + \frac{E_g}{\tau}N = (1+(\lambda_q)_k \frac{\partial}{\partial x_k} + \tau_q \frac{\partial}{\partial t})(\rho c_e \dot{\theta} + \alpha \theta_0 \dot{e}) \tag{3.2}$$

Due to the presence of electro-magnetic field, it is necessary to incorporate the Lorentz force components F_i's, where, $\mathbf{F}_i = \mu_0(\mathbf{J} \times \mathbf{H})_\mathbf{i}$, into the transportation equation of motion which is as of the following form

$$\rho \ddot{\mathbf{u}}_i + \mathbf{F}_i = (\lambda + 2\mu)\nabla^2 u - \alpha \theta_{,i} - \delta_n N_{,i} \tag{3.3}$$

The constitutive stress components are

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e - \alpha \theta - \delta_n N)\delta_{ij} \tag{3.4}$$

According to Baltz [27], when a semiconducting medium experiences an electro-magnetic field, then it follows Maxwell's equations, which are described as

$$\begin{aligned} \text{curl } \mathbf{h} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\ , \quad & \text{curl } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t} \\ \text{div } \mathbf{h} = 0, \qquad & \mathbf{E} = -\mu_0(\dot{\mathbf{u}} \times \mathbf{H}) \\ \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{h}), \qquad & \mathbf{D} = \varepsilon_0 \mathbf{E} \end{aligned} \tag{3.5}$$

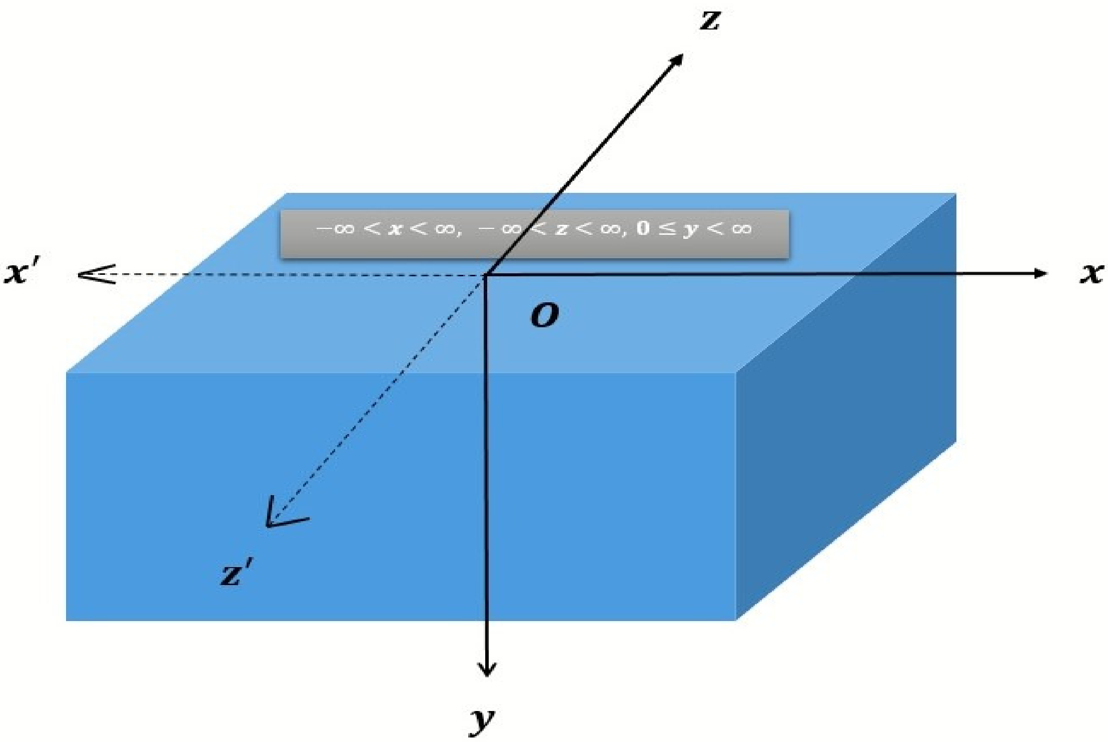
Three theories such as the non-local classical dynamical coupled (NLCDC) theory, the first generalized thermoelastic theory i.e., Lord and Shulman's (L-S) [80] theory and the dual-phase-lag(DPL) theory proposed by Tzou [116], [114] are derived from the modified heat transportation equation (3.2) by assigning the values of $\tau_\theta = \tau_q = 0$; $\tau_\theta = 0$, $\tau_q = \tau_0 > 0$ and $\tau_q \geq \tau_\theta > 0$ respectively.
where, $k_1 = \frac{\partial N_0}{\partial T}\frac{T}{\tau}$, N_0 and E_g are equilibrium carrier concentration at temperature T and the energy gap of semiconductor respectively, N is the phase of the carrier intensity and τ is the photogenerated carrier lifetime, μ_0 and ϵ_0 are the magnetic and electric permeability, respectively, θ_0 is uniform reference temperature, θ is the increase in temperature from θ_0 such that $\left|\frac{\theta-\theta_0}{\theta_0}\right| << 1$, $\mathbf{u}$ displacement vector, c_e is specific heat at constant strain, ρ is constant mass density, λ, μ are *Lamé* elastic constants,

$\delta_n = (3\lambda + 2\mu)d_n$, K is the thermal conductivity, α is thermoelasticity constant with $\alpha = (3\lambda + 2\mu)\alpha_0$, α_0 is the coefficient of linear thermal expansion of the material, τ_θ is the phase-lag of the temperature gradient, $c_0 = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$ is the speed of propagation of isothermal elastic waves, $\eta_0 = \frac{\rho c_e}{K}$, τ_q is the phase-lag of the heat flux, e=Cubical dialatation= $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and $(\lambda_q)_k$ is component of non-local length vector.
The subscripts i, j and k take the values 1, 2, 3 and the subscripted comma notation is used to represent the partial derivative with respect to the respective space variable.

Proper Validation of the Proposed Formulation
(a) If we consider the following numerical values of the parameter for copper $\lambda = 7.76\times 10^{10}\ kgm^{-1}s^{-2}$, $\mu = 3.86 \times 10^{10}\ kgm^{-1}s^{-2}$, $\kappa = 386\ Wm^{-1}K^{-1}$, $\alpha_t = 1.78 \times 10^{-5}\ K^{-1}$, $\rho = 8954Kgm^{-3}$, $C_E = 383.1\ JKg^{-1}K^{-1}$, $d_n = 0.0$, $D_E = 0.0$, $E_g = 0.0$, $\tau = 5\times10^{-5}\ s$, $\theta_0 = 293\ K$, $H_0 = 0.0$, $N = 0.0$, $\lambda_{q1} = \lambda_{q2} = 0.0$, all the numerical results and discussions are similar in nature with Gupta and Mukhopadhyay [62] .
(b) In equation (2), if we consider $(\lambda_q)_k = 0$, the basic equations are identical with the equation (3.2) (Cylindrical form) of Das et al.[7].
In absence of the carrier intensity parameter, energy gap(E_g) and electromagnetic field components as well as considering only one non-local parameter λ_q, equations (3.2) and (3.3) coincide with the equations (13) and (14) of Gupta and Mukhopadhyay [62].

3.1.3 Formulation and solution of the Problem

Now, we consider a two dimensional homogeneous, isotropic and thermoelastic half-space with cartesian co-ordinates (x, y, z) under the electromagnetic fields. The magnetic field is given by $\mathbf{H} = \mathbf{H_0} + \mathbf{h}$, $\mathbf{H_0} = (0, 0, H_0)$ is the initial magnetic field acting along z-direction. The Maxwell's equation (3.5) gives the expressions of the electromagnetic fields such as



Schematic Representation of the Physical Problem

$$\mathbf{h} = -H_0 e, \quad \mathbf{F_1} = -\mu_0^2 {H_0}^2 \epsilon_0 \frac{\partial^2 u}{\partial t^2} - \mu_0 {H_0}^2 \frac{\partial^2 u}{\partial x^2} - \mu_0 {H_0}^2 \frac{\partial^2 v}{\partial x \partial y},$$
$$\mathbf{F_2} = -\mu_0^2 {H_0}^2 \epsilon_0 \frac{\partial^2 v}{\partial t^2} + \mu_0 {H_0}^2 \frac{\partial^2 u}{\partial x \partial y} + \mu_0 {H_0}^2 \frac{\partial^2 v}{\partial y^2} \tag{3.6}$$

Using the equation (3.6), the equations (3.1)-(3.4) are obtained as

$$\frac{\partial N}{\partial t} = D_E \left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} \right) - \frac{N}{\tau} + k_1 \theta \tag{3.7}$$

$$K \left(1 + \tau_\theta \frac{\partial}{\partial t} \right) \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + \frac{E_g}{\tau} N = \left(1 + \lambda_{q_1} \frac{\partial}{\partial x} + \lambda_{q_2} \frac{\partial}{\partial y} + \tau_q \frac{\partial}{\partial t} \right)$$
$$\left\{ \rho c_e \frac{\partial \theta}{\partial t} + \alpha \theta_0 \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right\} \tag{3.8}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y} - \alpha \frac{\partial \theta}{\partial x} - \delta_n \frac{\partial N}{\partial x}$$
$$+ \mu_0^2 H_0^2 \epsilon_0 \frac{\partial^2 u}{\partial t^2} + \mu_0 H_0^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) \tag{3.9}$$

$$\rho\frac{\partial^2 v}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x^2} + (\lambda+\mu)\frac{\partial^2 u}{\partial x\partial y} - \alpha\frac{\partial\theta}{\partial y} - \delta_n\frac{\partial N}{\partial y} + \mu_0^2 H_0^2\epsilon_0\frac{\partial^2 v}{\partial t^2} - \mu_0 H_0^2\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial y^2}\right) \quad (3.10)$$

$$\sigma_{xx} = (\lambda+2\mu)\tfrac{\partial u}{\partial x} + \lambda\tfrac{\partial v}{\partial y} - \alpha\theta - (3\lambda+2\mu)\delta_n N \quad (3.11)$$

$$\sigma_{yy} = (\lambda+2\mu)\tfrac{\partial v}{\partial y} + \lambda\tfrac{\partial u}{\partial x} - \alpha\theta - (3\lambda+2\mu)\delta_n N \quad (3.12)$$

$$\sigma_{zz} = \lambda\left(\tfrac{\partial u}{\partial x} + \tfrac{\partial v}{\partial y}\right) - \alpha\theta - (3\lambda+2\mu)\delta_n N \quad (3.13)$$

$$\sigma_{xy} = \mu\left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right) \quad (3.14)$$

The non-dimensional parameters are as

$$(x', y', u', v', \lambda'_{qi}) = c_0\eta_0(x, y, u, v, \lambda_{qi}), \quad i = 1,2\ ,\ (t', \tau'_q, \tau'_\theta) = c_0^2\eta_0(t, \tau_q, \tau'_\theta), \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\mu},\ \theta' = \frac{\alpha\theta}{\rho c_0^2},\ N' = \frac{\delta_n N}{\lambda+2\mu} \quad (3.15)$$

By using the above non-dimensional parameters given in equation (3.15), the non-dimensional form of the equations (3.7)-(3.14), ignoring the prime notation conventionally, are obtained as

$$\frac{\partial N}{\partial t} = \epsilon_3\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right) - \epsilon_4 N + \epsilon_5\theta \quad (3.16)$$

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \epsilon_2 N = \left(1 + \lambda_{q_1}\frac{\partial}{\partial x} + \lambda_{q_2}\frac{\partial}{\partial y} + \tau_q\frac{\partial}{\partial t}\right)\left(\frac{\partial\theta}{\partial t} + \epsilon_1\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) \quad (3.17)$$

$$m_2\frac{\partial^2 u}{\partial t^2} = m_1\frac{\partial^2 u}{\partial x^2} + \beta_0\frac{\partial^2 u}{\partial y^2} + a_3\frac{\partial^2 v}{\partial x\partial y} - \frac{\partial\theta}{\partial x} - \frac{\partial N}{\partial x} \quad (3.18)$$

$$m_2\frac{\partial^2 v}{\partial t^2} = m_1\frac{\partial^2 v}{\partial y^2} + \beta_0\frac{\partial^2 v}{\partial x^2} + a_3\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial\theta}{\partial y} - \frac{\partial N}{\partial y} \quad (3.19)$$

and the expression for cubical dilatation is

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{3.20}$$

The non-dimensional stress components are

$$\sigma_{xx} = \frac{1}{\beta_0}\left[\frac{\partial u}{\partial x} - (1-2\beta_0)\frac{\partial v}{\partial y} - \theta - N\right] \tag{3.21}$$

$$\sigma_{yy} = \frac{1}{\beta_0}\left[\frac{\partial v}{\partial y} - (1-2\beta_0)\frac{\partial u}{\partial x} - \theta - N\right] \tag{3.22}$$

$$\sigma_{zz} = \frac{1}{\beta_0}\left[(1-2\beta_0)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \theta - N\right] \tag{3.23}$$

$$\sigma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{3.24}$$

where, the constants are given by
$a_2 = \frac{\mu_0^2 H_0^2 \varepsilon_0^2}{\rho}$, $a_1 = \frac{\mu_0 H_0^2}{\rho c_0^2}$, $\beta_0 = \frac{\mu}{\rho c_0^2}$, $a_3 = m_1 - \beta_0$, $\beta = \frac{\lambda}{\lambda+2\mu}$, $m_2 = 1 - a_2$, $m_1 = 1 + a_1$, $\varepsilon_1 = \frac{\alpha^2}{\gamma c_e}$, $\varepsilon_2 = \frac{E_g \alpha c^2}{\tau \delta_n k_1}$, $\varepsilon_3 = \frac{D_E}{c c_0}$, $\varepsilon_4 = \frac{c}{c_0 \tau}$, $\varepsilon_5 = \frac{k_1 \delta_n c}{\alpha c_0}$.

We now apply the Laplace integral transform for time variable and Fourier integral transform for space variable x which are defined by

$$\overline{T}(x,y,p) = \int_0^\infty T(x,y,t)e^{-pt}dt$$
$$\overline{T}_1(\xi,y,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \overline{T}(x,y,p)e^{i\xi x}dx \tag{3.25}$$

to the non-dimensional form of the basic equations. As in Das and Lahiri [42], the transformed basic equations are written compactly in the form of vector-matrix differential equation

$$D\underline{V}(\xi,y,p) = \underline{A}(\xi,p)\ \underline{V}(\xi,y,p) \quad ; \quad D \equiv \frac{d}{dy} \tag{3.26}$$

where, $\underline{V}(\xi,y,p) = \left[\bar{N}_1 \ \ \bar{e}_1 \ \ \bar{\theta}_1 \ \ \frac{d\bar{N}_1}{dy} \ \ \frac{d\bar{e}_1}{dy} \ \ \frac{d\bar{\theta}_1}{dy}\right]^T$ and $\underline{A}(\xi,p) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$, L_{11} and L_{12} are null and identity matrix of order 3 respectively and L_{21} and L_{22} are given in the Appendix.

Initially i.e., at time$(t) = 0$, we now consider the cubical dilatation and carrier intensity are zero and the time derivatives of these field variables and electromagnetic fields, temperature distribution are also zero whereas the system maintains a reference temperature θ_0.

$$e(x,y,0)=\frac{\partial e(x,y,0)}{\partial t}=0\ ;\quad \frac{\partial\theta(x,y,0)}{\partial t}=0\ ;\quad N(x,y,0)=\frac{\partial N(x,y,0)}{\partial t}=0\ ;$$
$$\frac{\partial \mathbf{h}(x,y,0)}{\partial t}=\frac{\partial \mathbf{E}(x,y,0)}{\partial t}=0 \tag{3.27}$$

As discussed in Das and Lahiri [14], we apply eigenvalue approach methodology to obtain the solution of vector-matrix differential equation (3.26). The characteristic equation of the coefficient matrix $\underline{A}$ is derived from the equation

$$\underline{A}\,\underline{Y}=\lambda\,\underline{Y} \tag{3.28}$$

where, $\underline{Y}$=$[y_i]^T$ are the eigenvectors corresponding to the eigenvalues $\lambda=\lambda_i$ which are obtained from the characteristic equation (3.28) and $y_i=[Y]_{\lambda=\lambda_i}, i=1(1)6$ also given in Appendix.

By using the expression of the cubical dilatation, the general solution of the equation (3.26)(omitting prime symbol, conventionally) is

$$(\bar{N}_1,\bar{u}_1,\bar{v}_1,\bar{\theta}_1)=\sum_{i=1}^{4}P_i(G_{1i},G_{2i},G_{3i},G_{4i}) \tag{3.29}$$

where, P_i's are the arbitrary parameters. With the help of equations (3.25), the analytical expression of the stress components are derived from the equation (3.21)-(3.24) which are of the form $[(\bar{\sigma}_{xx})_1,(\bar{\sigma}_{yy})_1,(\bar{\sigma}_{zz})_1,(\bar{\sigma}_{xy})_1]=\sum_{i=1}^{4}P_i[G_{5i},G_{6i},G_{7i},G_{8i}]$, where, the values of G_{ij}'s are given in the Appendix. Equation (3.6) also gives the values of the electromagnetic field components and the expressions of Lorentz force components.

3.1.4 Boundary Conditions

To determine the values of the arbitrary parameters P_i's, we have prescribed the following boundary conditions:

We now consider a homogeneous, isotropic and thermoelastic semiconducting medium which is extended along both the directions of x-axis and occupying the specific region $\Omega:\ \Omega=\{(x,y,z): -\infty<x<\infty, y\geq 0, -\infty<z<\infty\}$. The arbitrary parameters P_i's have to be chosen such that the boundary conditions on the surface at $y=\pm 0$ (adjacent to vacuum) satisfies the following thermal and mechanical boundary conditions:

1. Mechanical Boundary Condition:

a) The boundary surface $y = 0$ is experienced a prescribed time-dependent exponential compression i.e. $\sigma_{xx}(x, 0, t) = -p_1^*$, where, $p_1^* = p_1\ exp(\omega t + ibx)$, p_1 is the absolute value of the mechanical force, ω is the complex circular frequency and b is a wave number in the x-direction.

b) The boundary surface $y = 0$ of the half-space semiconducting medium is traction free i.e. $\sigma_{xy}(x, 0, t) = 0$.

2. Thermal Boundary Condition:

The temperature gradient is zero at the thermally insulated boundary surface $y = 0$ i.e. $\frac{\partial\theta(x,0,t)}{\partial y} = 0$

3. Sample Carrier Intensity Restriction:

During the diffusion process, the carrier intensity of the semiconducting half-space medium reaches at the boundary surface $y = 0$. So, the gradient of the carrier density of the sample size with a finite probability of recombinational value i.e. $\frac{\partial N(x,0,t)}{\partial x} = \frac{S}{D_e}N$, where, S and D_e is pre-assigned positive number.

Applying the equations (3.15) and (3.25) to the above three boundary conditions, we can obtain a system of simultaneous equations satisfied by the four arbitrary parameters.

3.1.5 Numerical Results and Discussions

As discussed in Das and Lahiri [42], the Laplace and Fourier inversion of the solutions in equations (3.29) and the expressions for the stress components in space-time domain are carried out by using an efficient programming language (MATLAB R2016a). For the numerical inversion of the Laplace transform for time variable, we follow the Zakian [127] method and for the numerical inversion of Fourier transform, we carry out by seven-point Gaussian quadrature formula for different values of y.

For analyzing the numerical results, we illustrate the physical field variables and the stress components graphically and comparison is made for different values of physical parameters. As in Das, Ghosh and Lahiri [41], the values of the material constants for silicon(in SI units) are: $\lambda = 3.64 \times 10^{10}\ Nm^{-2}$, $\mu = 5.46 \times 10^{10}\ Nm^{-2}$, $\kappa = 150\ Wm^{-1}K^{-1}$, $\alpha_t = 3 \times 10^{-6}\ K^{-1}$, $\rho = 2.33 \times 10^3\ Kgm^{-3}$, $C_E = 695\ JKg^{-1}K^{-1}$, $d_n = -9 \times 10^{-31}\ m^3$, $D_E = 2.5 \times 10^{-3}\ m^2s^{-1}$, $E_g = 1.12\ ev$, $\tau = 5 \times 10^{-5}\ s$, and the values of the associated constants used in this problem are: $\theta_0 = 293\ K$, $H_0 = \frac{10^7}{4\pi}$, $S = 2.0$.

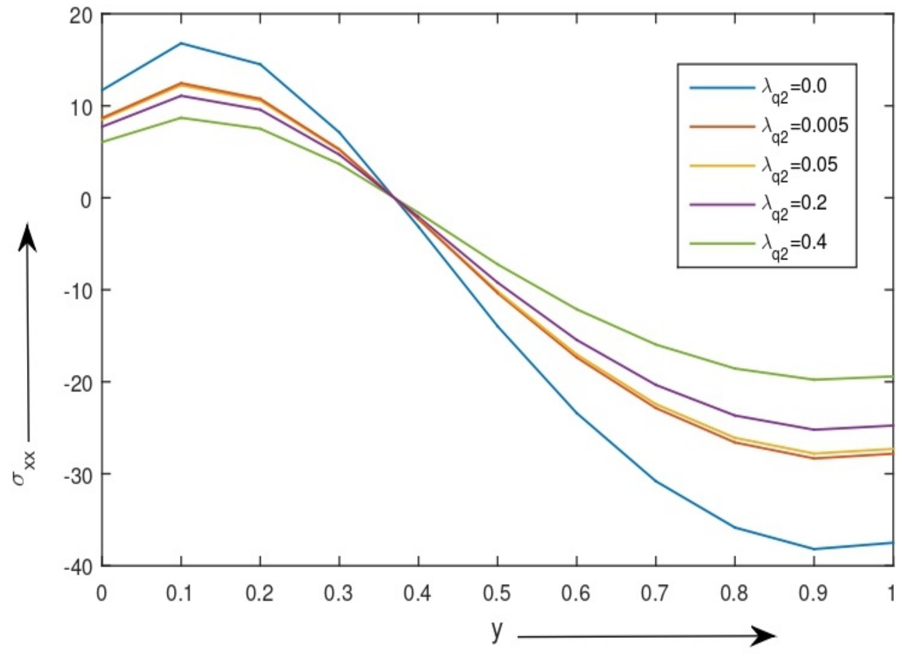


Fig. 1: Distribution of stress component (σ_{xx})

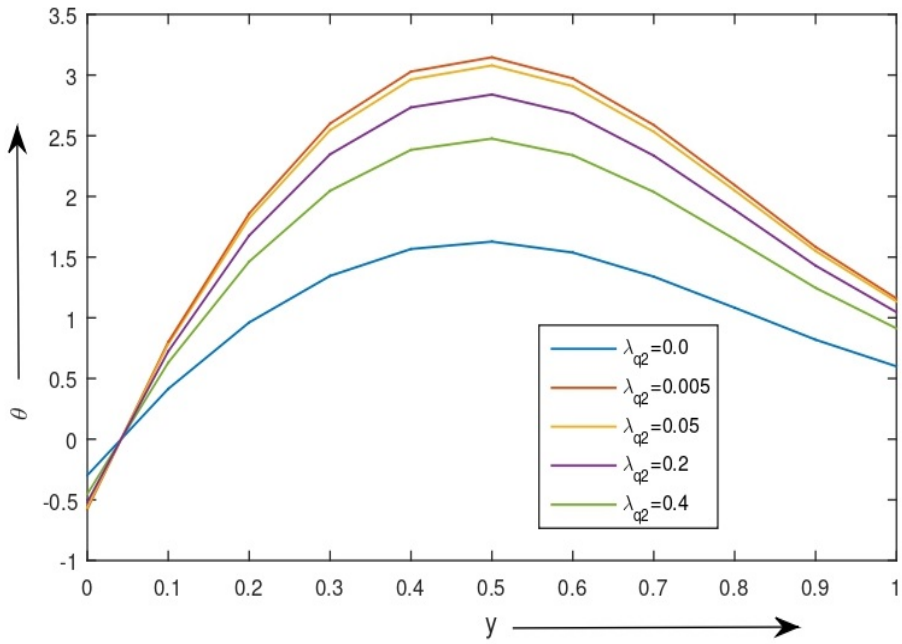


Fig. 2: Distribution of temperature(θ)

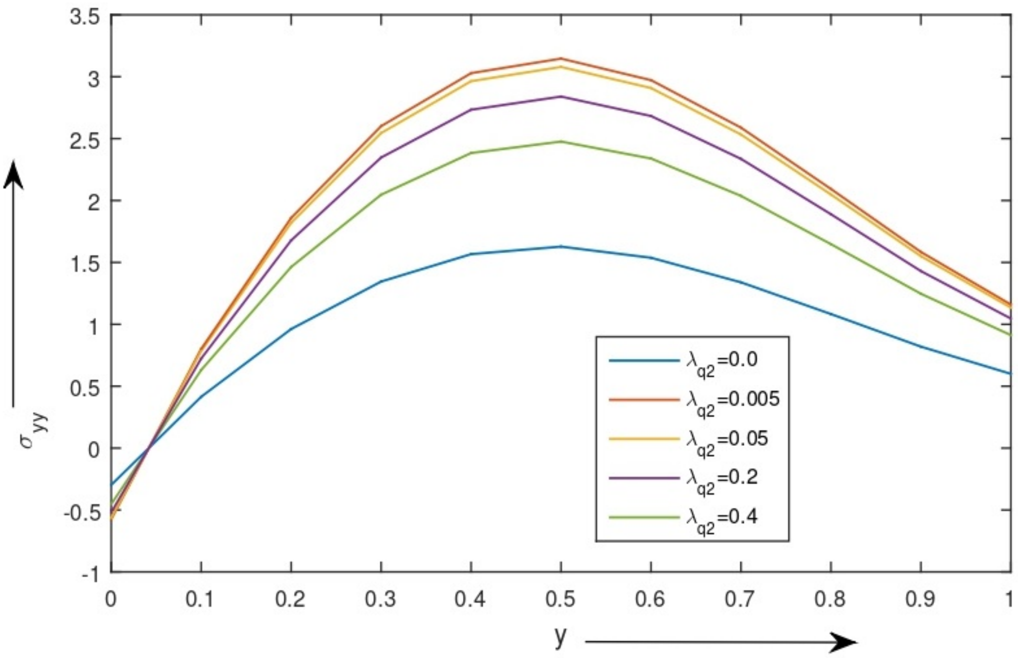


Fig.3: Distribution of stress component(σ_{yy})

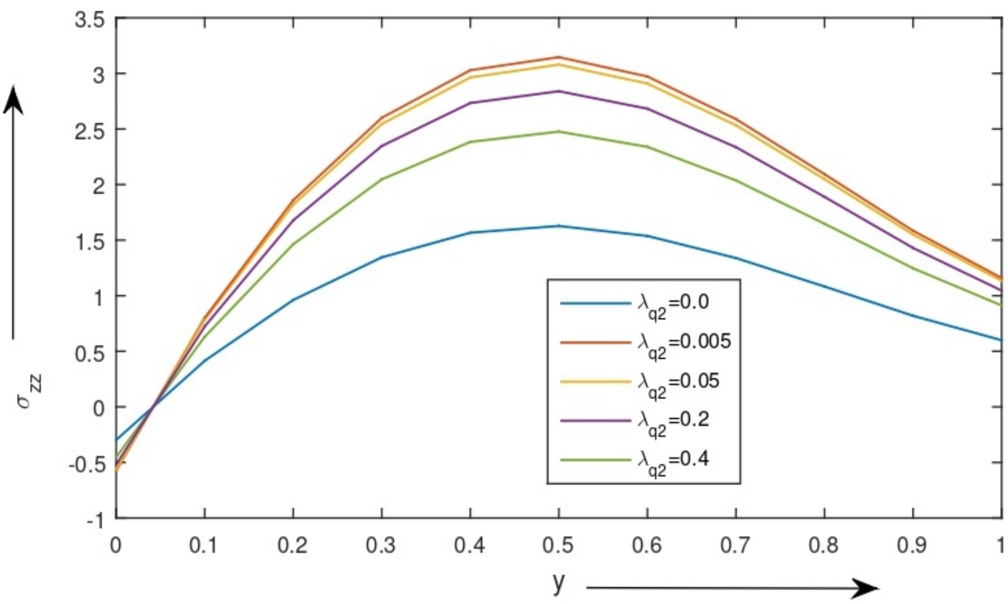


Fig.4: Distribution of stress component(σ_{zz})

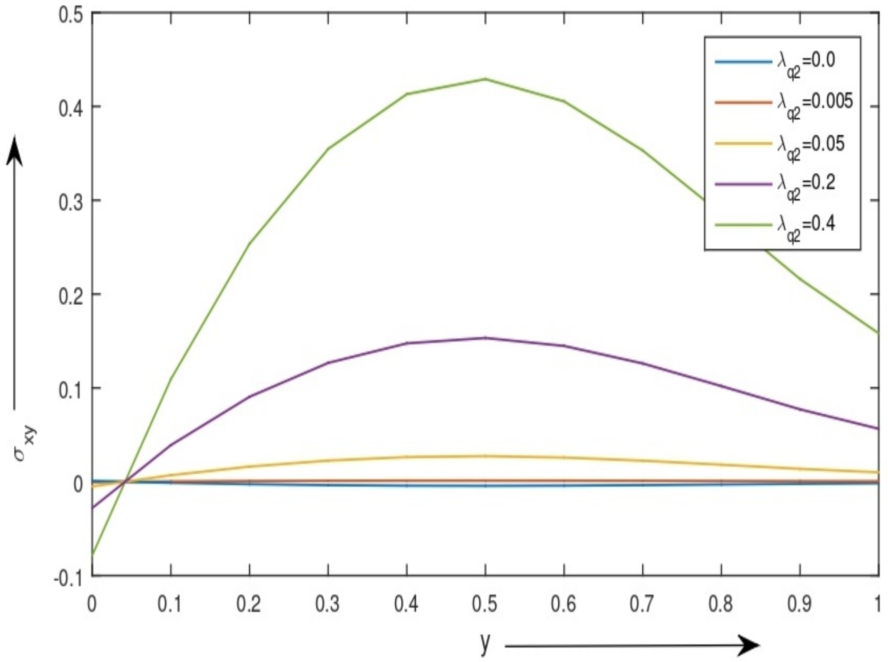


Fig.5: Distribution of stress component(σ_{xy})

Figs. 1-5: The variations of physical fields and stress components with respect to distance for fixed values of λ_{q2}

Effect of thermoelastic non-local parameter(λ_{q2})

Figs. 1-5 illustrate the variations of stress components(σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}), temperature distribution (θ) with distance(y) for fixed values of time ($t = 2.0$), magnetic intensity ($H_0 = 0.3$), $\tau_q = 0.015$, $\tau_\theta = 0.015$ and the five fixed values of λ_{q1} and λ_{q2}. These five fixed values of λ_{q1} and λ_{q2} are 0.0, 0.005, 0.05, 0.2 and 0.4. We are now trying to characterize the variations of physical field variables under the non-local model. Fig.1 predicts the stress component (σ_{xx}) for the fixed value of $\lambda_{q1} = 0$ and $\lambda_{q2} = 0$, in DPL, is propagated with maximum amplitude rather than the other four values of λ_{q1} and λ_{q2}. The stress component (σ_{xx}) shows compressive in nature within the domain $0 \leq y \leq 0.35$ after that it becomes extensive in nature. The characteristics of stress(σ_{xx}) does not significantly alter for the values of $\lambda_{q2} = 0.005$ and 0.05. The variation of the stress components and the physical field variables are completely different in DPL model (i.e.$\lambda_{q2} = 0$) compared to another four values of λ_{q2}. Figs. 2-5 predicts that the graphs intersect each other at $y = 0.05$ for different values of non-local parameter(λ_{q2}) and attain maximum value in the middle plane of the medium. Shearing stress component (σ_{xy}) (Fig.5) behaves in a almost similar manner for the values $\lambda_{q2} = 0.0$ and 0.005. Therefore, we now conclude that the characteristics of all the physical field variables and stress components show almost the same behavior in the generalized thermoelastic model except DPL model.

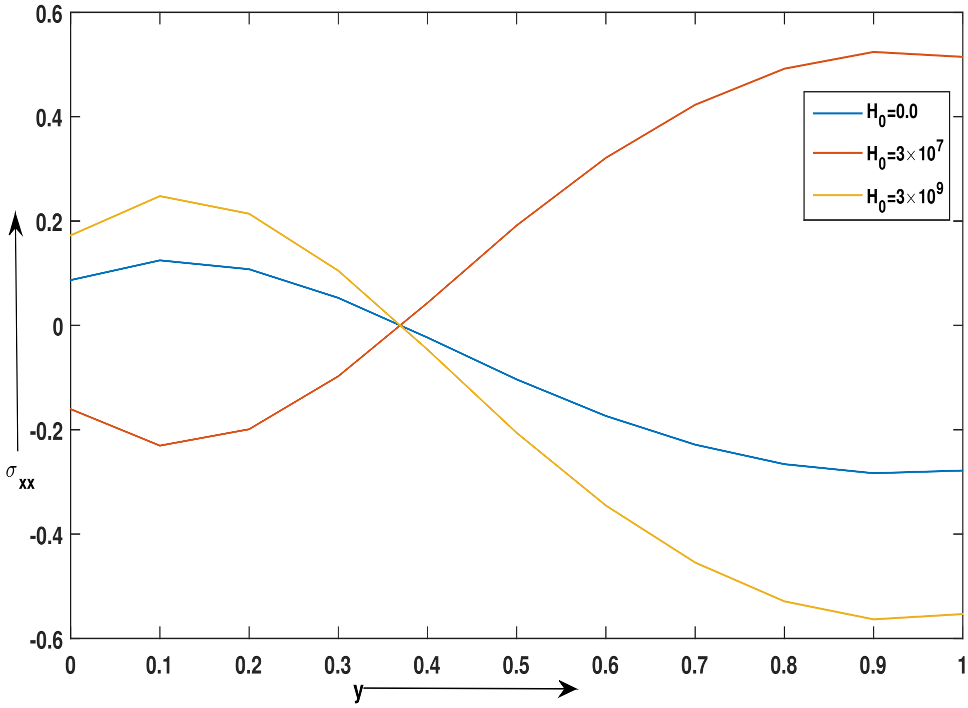


Fig. 6 : Distribution of stress components(σ_{xx})

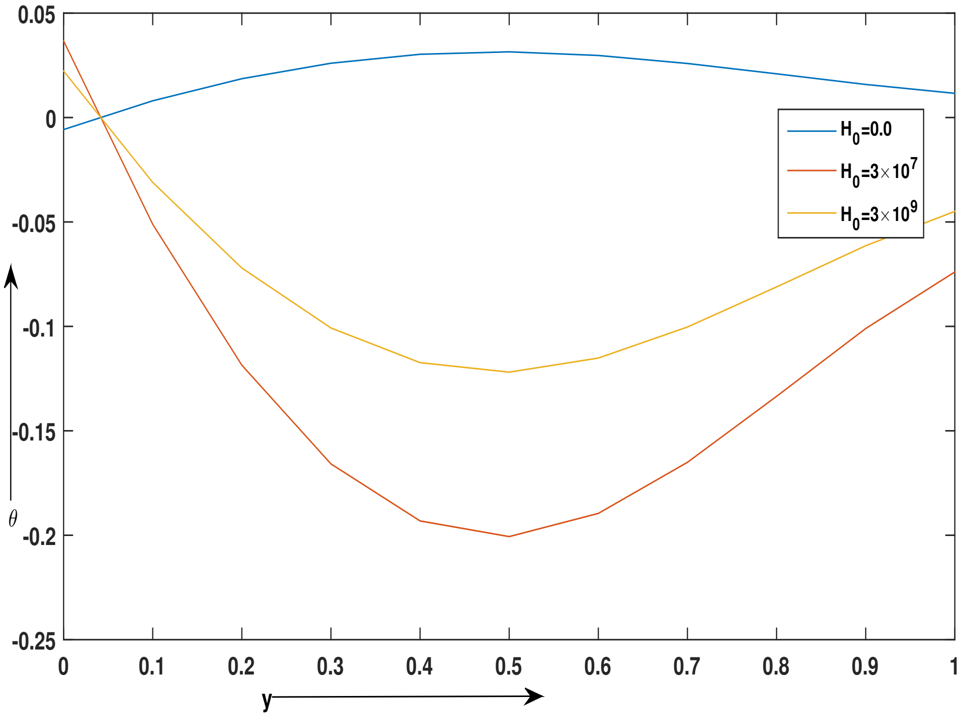


Fig. 7 : Distribution of temperature(θ)

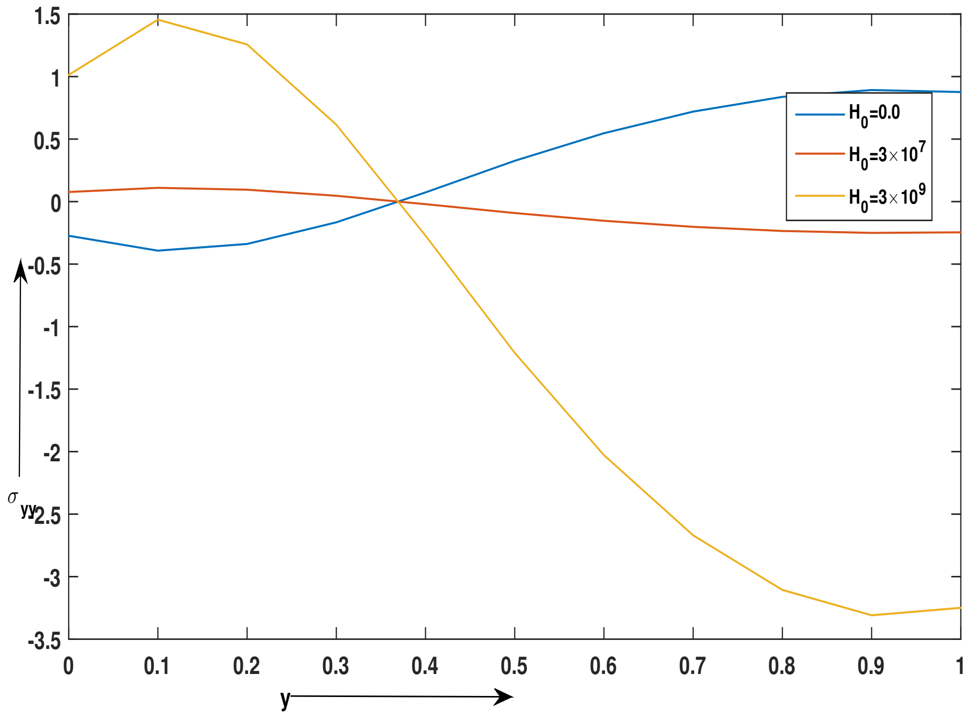


Fig. 8 : Distribution of stress components(σ_{yy})

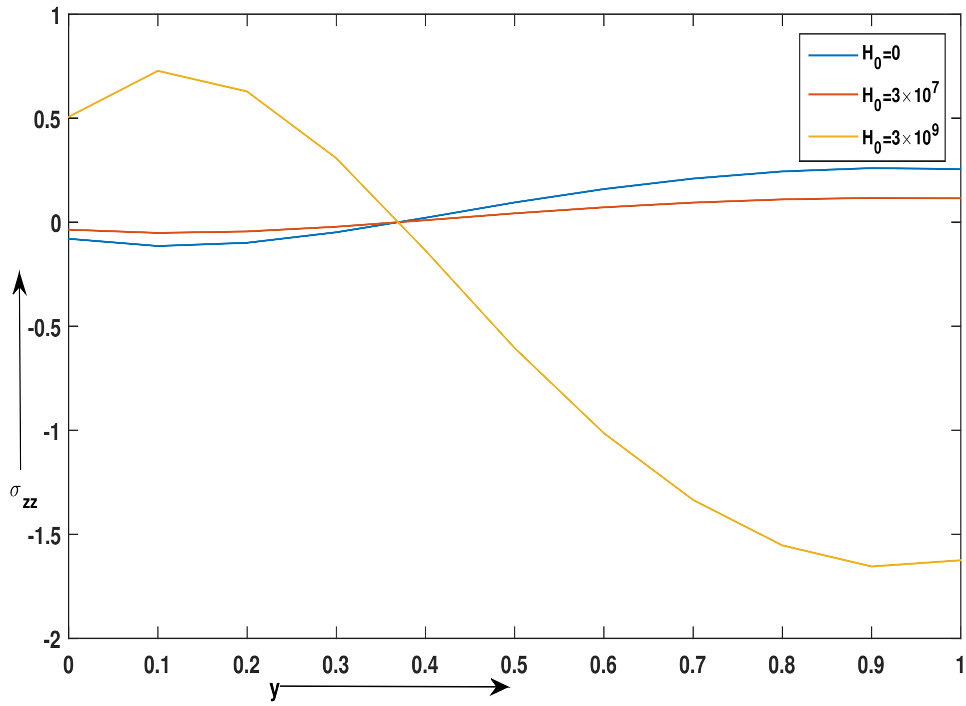


Fig. 9 : Distribution of stress components(σ_{zz})

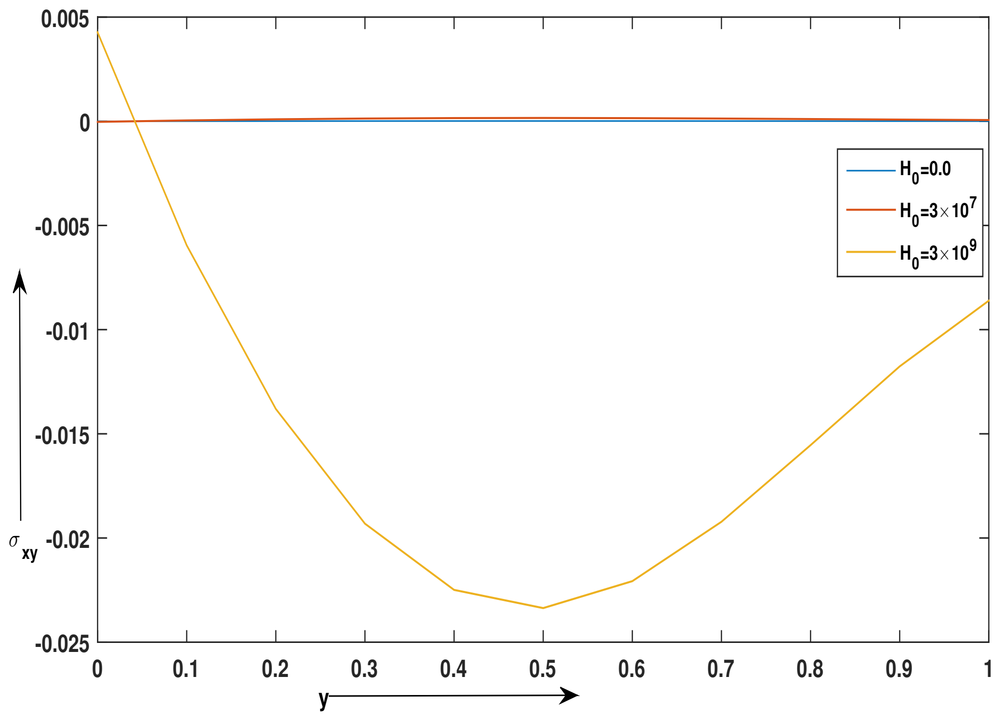


Fig. 10 : Distribution of stress components(σ_{xy})

Figs. 6-10: The variations of physical fields and stress components with respect to distance for fixed values of H_0

Effect of magnetic field(H_0)

Figs. 6-10 depict the variations of physical field variables such as temperature distribution (θ) and stress components(σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}) with distance(y) of the medium at a fixed values of time ($t = 2.0$), $\tau_q = 0.015$, $\tau_\theta = 0.015$, $\lambda_{q1} = \lambda_{q2} = 0.005$ and the three fixed values of H_0. The characteristics of field variables along with the stress components are studied for three fixed values of magnetic field($H_0 = 0, 3 \times 10^7$ and 3×10^9) under the non-local parametric value. Figs. 7, 9 and 10, temperature distribution and the stress components (σ_{zz}, σ_{xy}) are not significantly characterized at the value of $H_0 = 0$. In this case, the distribution of these stress components and temperature distribution almost coincide with the horizontal axis. Whereas, the normal stress components (σ_{xx}, σ_{yy})(Figs. 6 ,8) more prominently behave at $H_0 = 0$. The characteristics of stress component (σ_{xx}) and the temperature(Figs. 6, 7) are more significant for the value of $H_0 = 3 \times 10^7$. The deflection of this stress component as well as temperature are very much prominent at $H_0 = 3 \times 10^9$. The normal stress components (Figs. 6, 8, 9) are vanished at the distance $y = 0.35$ of the medium and the temperature (Fig.7) and the shearing stress component (σ_{xy})(Fig.10) vanish at $y = 0.05$. Specifically, it is concluded that the stress components and temperature distribution are significantly characterized with the effect of magnetic field intensities.

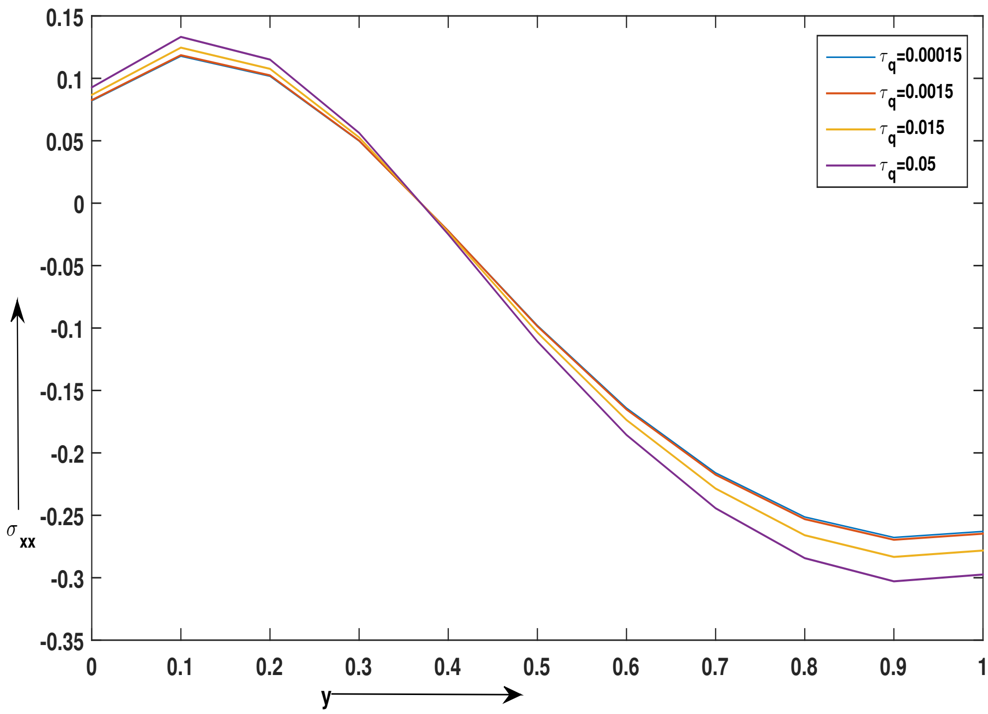


Fig. 11 : Distribution of stress components(σ_{xx})

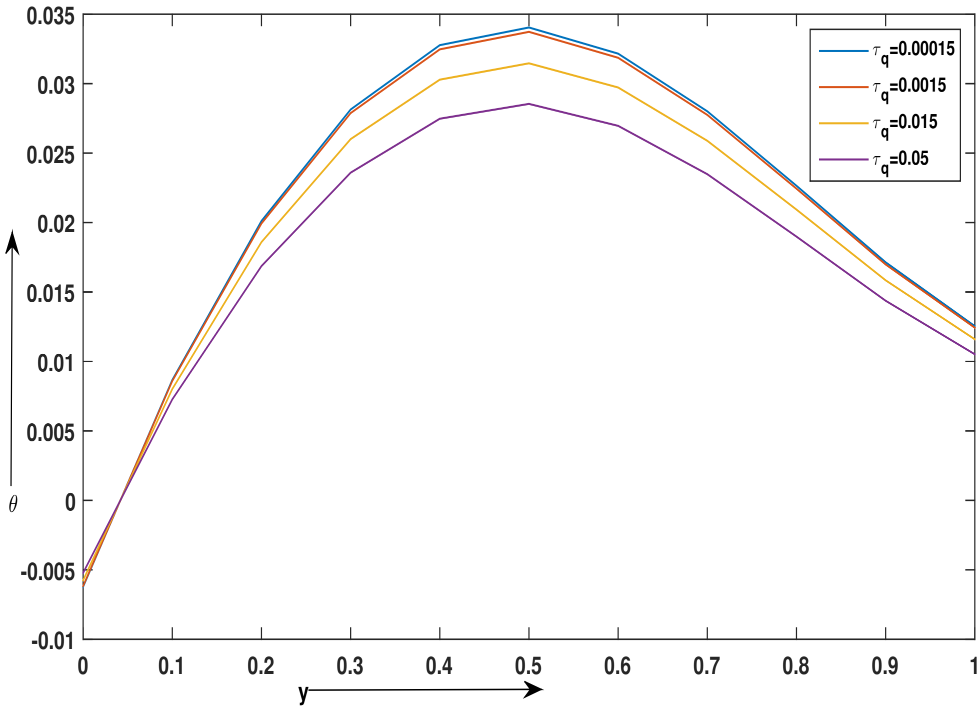


Fig. 12 : Distribution of temperature(θ)

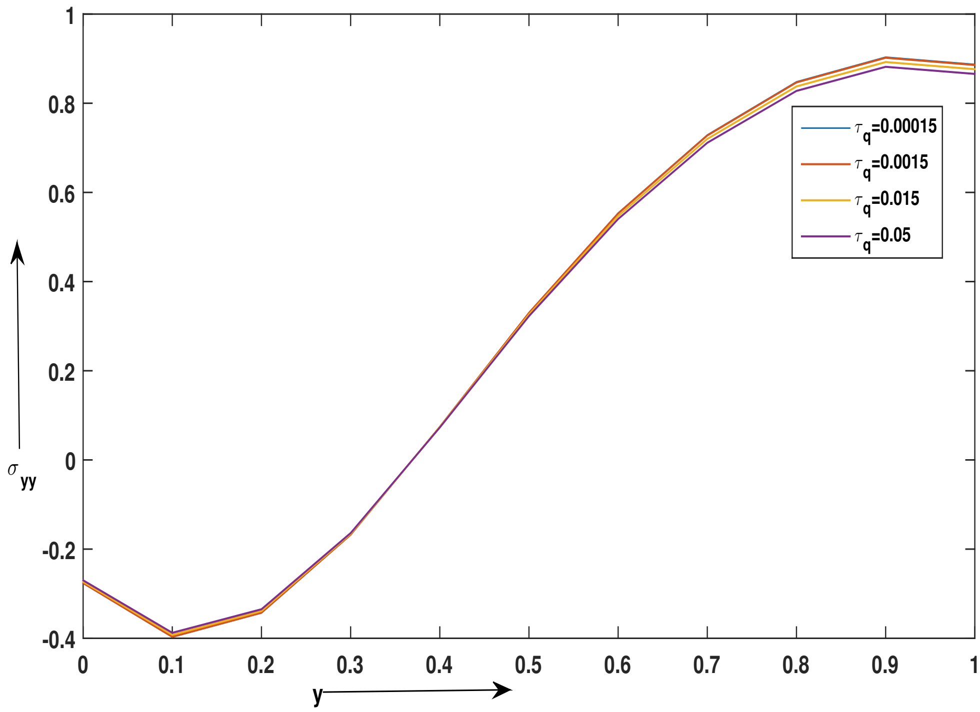


Fig. 13 : Distribution of stress components(σ_{yy})

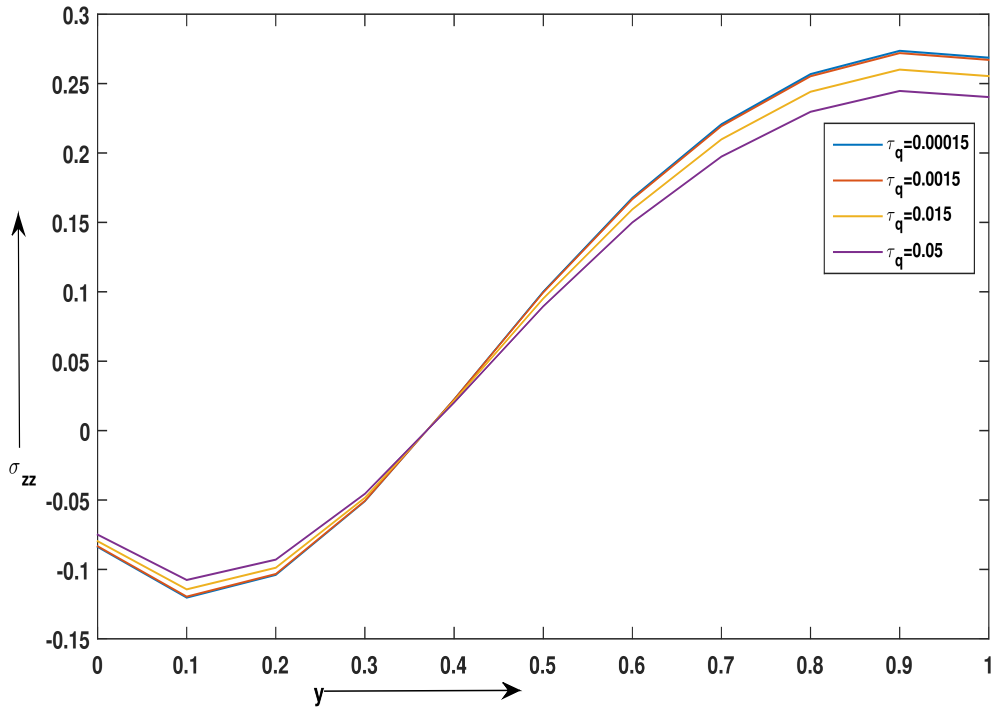


Fig. 14 : Distribution of stress components(σ_{zz})

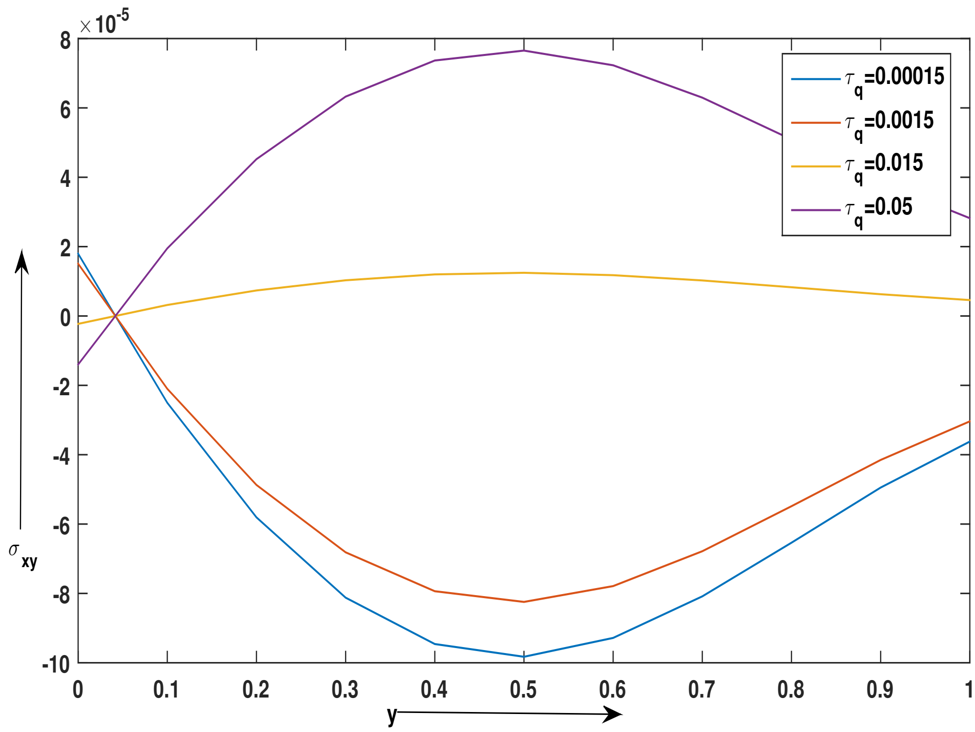


Fig. 15 : Distribution of stress components(σ_{xy})

Figs. 11-15: The variations of physical fields and stress components with respect to distance for fixed values of τ_q

Effect of (τ_q)

Figs. 11-15 depict the variations of physical field variables such as temperature distribution (θ) and stress components(σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}) with distance(y) of the medium at a fixed values of time ($t = 2.0$), $\lambda_{q2} = 0.005$, $H_0 = 0.3$ and the four fixed values of τ_q. The characteristics of field variables are studied for four fixed values of ($\tau_q = 0.00015, 0.0015, 0.015$ and 0.05). Fig. 11 shows that the characteristics of normal stress component (σ_{xx}) have almost same behavior for the four fixed values of (τ_q). It vanishes at $y = 0.35$ and then it becomes compressive in nature. Fig. 13 and 14 show that the behaviors of the stress components σ_{xx} and σ_{yy} are almost same whereas the shearing stress component (σ_{xy}) behaves in a completely different way compared to the normal stresses. It significantly propagates for fixed values of τ_q and vanishes at $y = 0.05$. Fig.12 shows that the temperature distribution is more clear at the middle of the medium and the absolute value attains maximum at this point. The variations of the temperature distribution are almost same for the values of $\tau_q = 0.00015$ and 0.0015. These figures clearly depict the more dependence of the parameter τ_q.

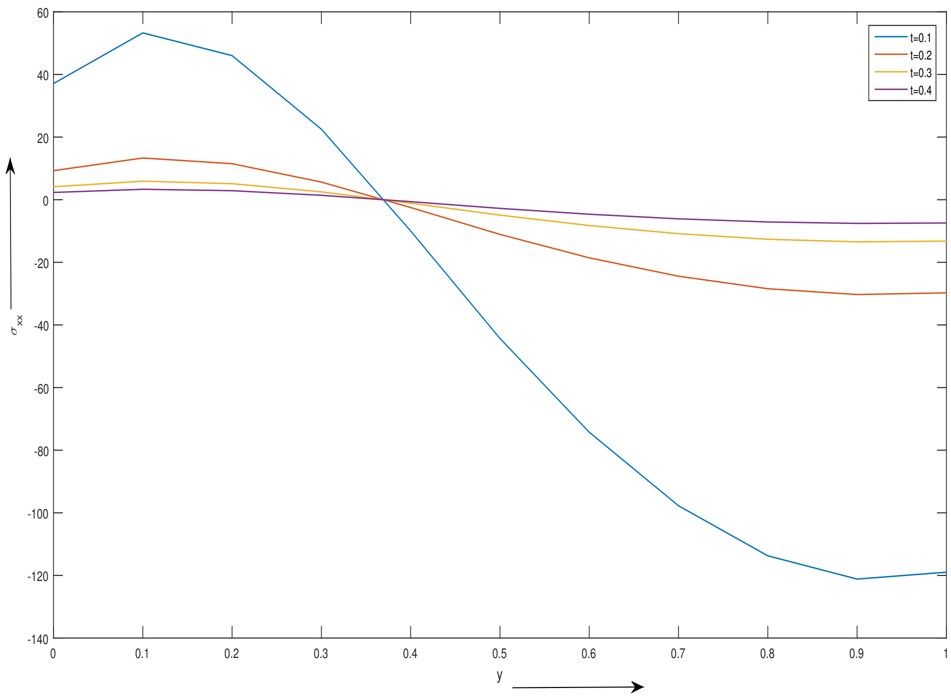


Fig. 16 : Distribution of stress components(σ_{xx})

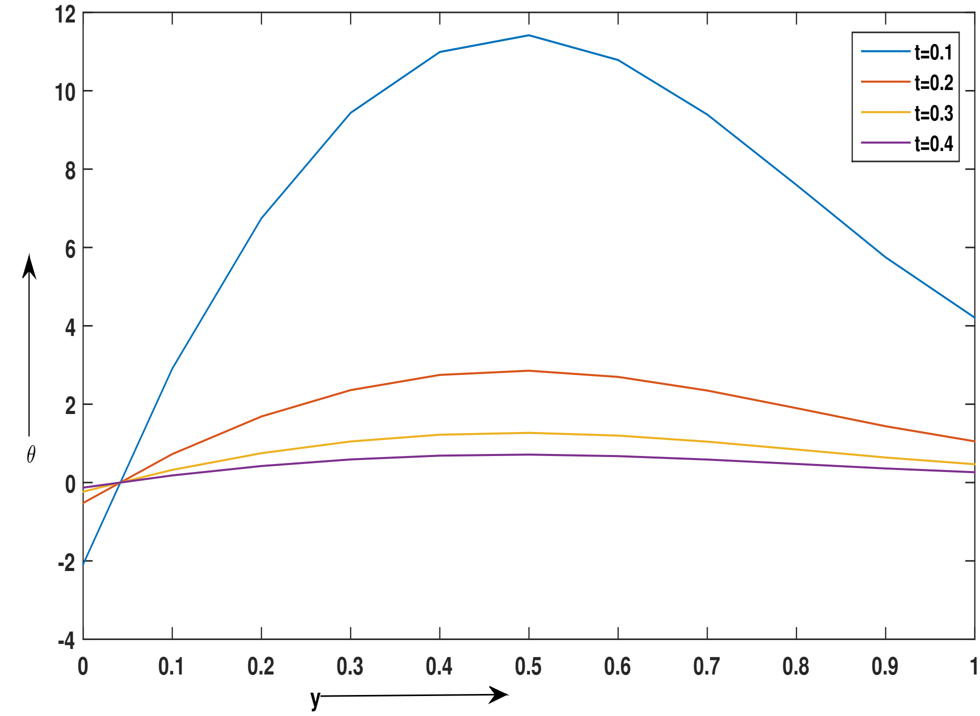


Fig. 17 : Distribution of temperature(θ)

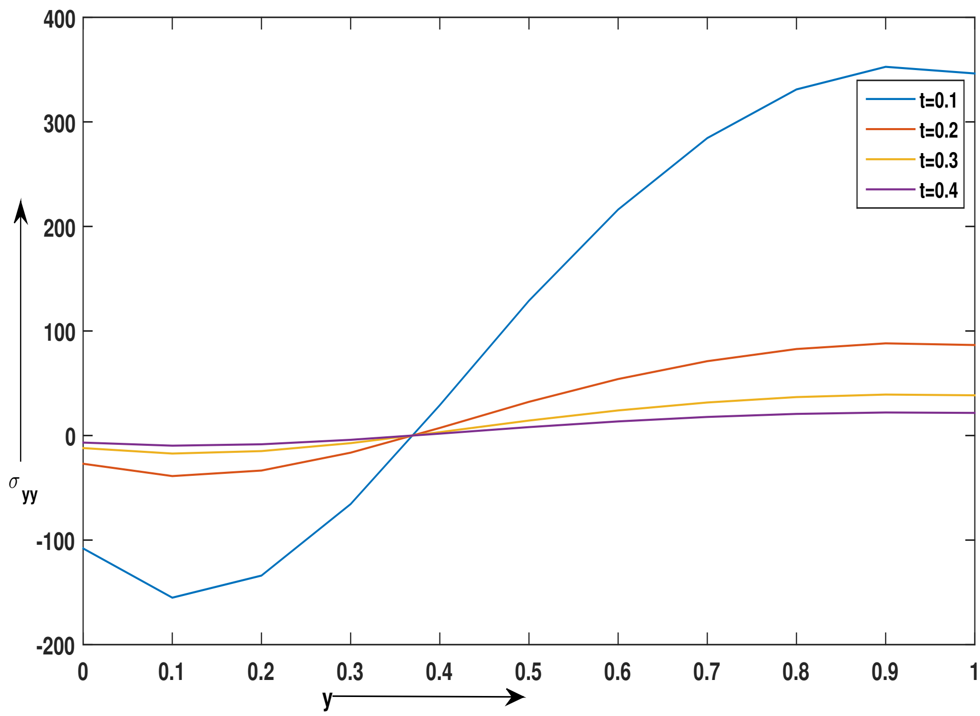


Fig. 18 : Distribution of stress components(σ_{yy})

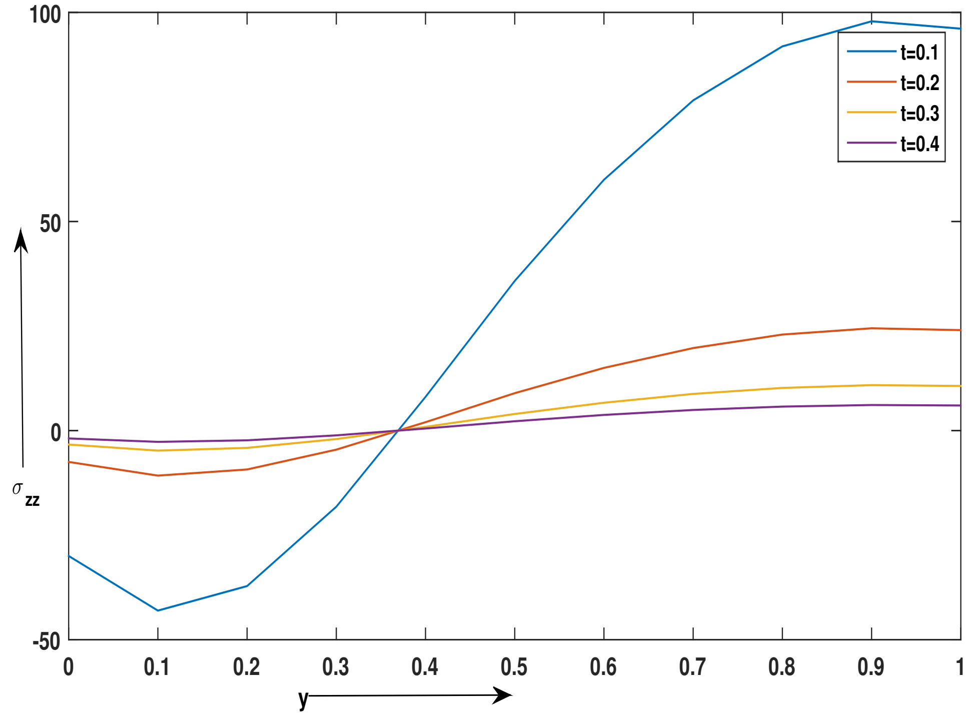


Fig. 19 : Distribution of stress components(σ_{zz})

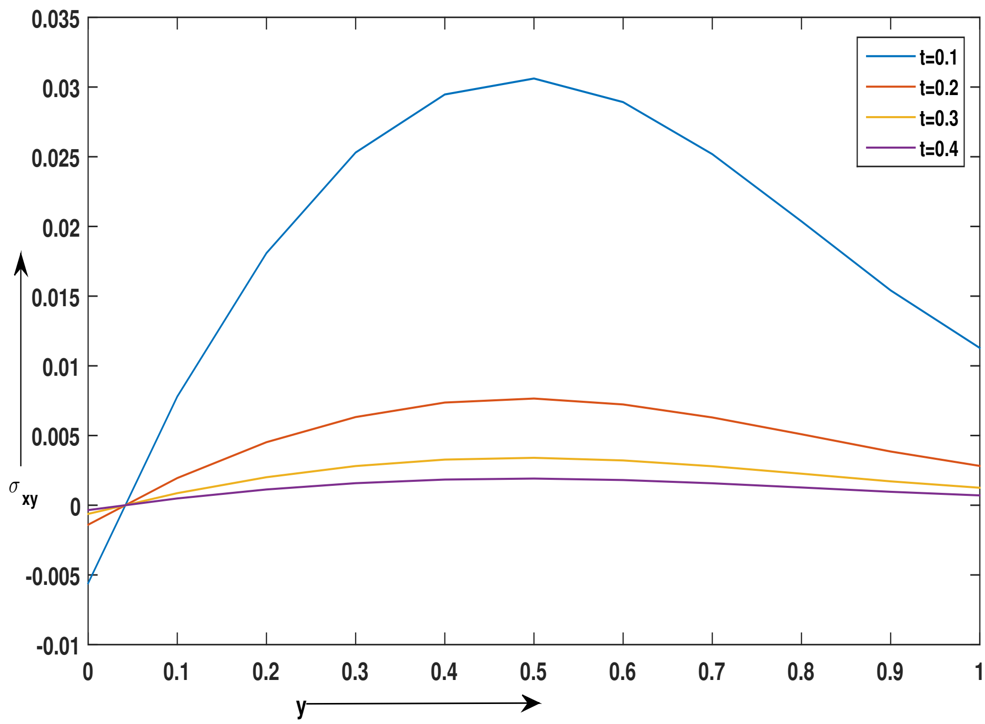


Fig. 20 : Distribution of stress components(σ_{xy})

Figs. 16-20: The variations of physical fields and stress components with respect to distance for fixed values of time t

Effect of time(t)

Figs. 16-20 depict the variations of physical field variables such as temperature distribution (θ) and stress components(σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}) with distance (y) of the medium at a fixed values $H_0 = 0.3$, $\tau_q = \tau_\theta = 0.05$, $\lambda_{q1} = \lambda_{q2} = 0.005$ and the four fixed values of time $t = 0.1, 0.2, 0.3, 0.4$. Fig. 16 shows that variations of the normal stress component (σ_{xx}) are more significant for fixed values of time $t = 0.1$ and $t = 0.2$. The absolute value attains maximum for $t = 0.1$ and the variations are almost close for $t = 0.3$ and $t = 0.4$. It vanishes at $y = 0.35$ and then it becomes compressive. Fig. 17 gives the more clear variation of the temperature distribution for $t = 0.1$ and also attains maximum at $y = 0.5$ that is the middle portion of the medium. Fig. 18 and 19 present almost the similar variations of the normal stresses (σ_{yy}, σ_{zz}). The variations of the normal stress components are more clear at $t = 0.1$ and it vanishes at $y = 0.35$. Fig. 20 also shows the more prominent variation of the shearing stress (σ_{xy}) for the value of $t = 0.1$. The characteristics of field variables along with the stress components are studied for four fixed values of time. The absolute value of θ and σ_{xy} decreases as time t increases whenever $0 \leq y \leq 0.05$ and it increases as time t decreases whenever $0.05 \leq y \leq 1$. It also attains the maximum at $y = 0.5$. Similarly, the absolute value of σ_{xx}, σ_{yy} and σ_{zz} decreases with the increase of time t whenever $0 \leq y \leq 0.4$ and it increases with the decreases of time t whenever $0.4 \leq y \leq 1$. Then, it also attains the maximum at $y = 1$.

3.1.6 Conclusions

An analytical formulation for an isotropic, homogeneous half-space semiconducting medium with the effect of phase-lag of heat flux(τ_q) and the temperature gradient(τ_θ) is presented and illustrated graphically for fixed values of physical parameters. A model has been developed incorporating non-local(λ_{qk}) behavior with dual-phase lagging which have the significant effects on the harmonic functions. We also obtained the interactions between thermal temperature and carrier intensity in heat conduction equation and the coupled plasma wave equation respectively. Numerical analysis is carried out for the semiconducting medium silicon and all the physical field variables are satisfied by the thermoelastic and plasma wave equation as well as the prescribed boundary conditions. The analysis and results in this problem will be very much significant in studying the uses of semiconductors such as diodes, triodes and modern electronics devices.

The graphical representations(Figs.1-20) are self-explanatory for representing and analyzing the different results, yet the following conclusions may be added:

(i) According to this theory, we make a generalized thermoelasticity for the Fourier's law of heat equation with non-local and dual-phase lagging parameters. These parameters indicate its effectiveness to propagate heat in a conducting medium.

(ii) Significant difference in the physical field variables are observed for DPL and non-local theory. The definition of non-local theory is more intuitionistic to understand the physical applicability.

(iii) The presence of carrier intensity accelerates the development of generalized thermoelasticity which is very much applicable to the modern techniques.

3.1.7 Appendix

$$L_{21}=\begin{bmatrix} c_{41} & 0 & c_{43} \\ c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63} \end{bmatrix} \text{ and } L_{22}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{55} & c_{56} \\ 0 & c_{65} & c_{66} \end{bmatrix},$$

$$y_i=\begin{bmatrix} f_{11}f_{12}-f_{13}f_{14} \\ c_{61}f_{14}-c_{51}f_{12} \\ c_{51}f_{13}-c_{61}f_{11} \\ \lambda_i y_1 \\ \lambda_i y_2 \\ \lambda_i y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$c_{41}=\frac{p+\xi^2\varepsilon_3+\varepsilon_4}{\varepsilon_3}$, $c_{43}=-\frac{\varepsilon_5}{\varepsilon_3}$, $c_{51}=-\frac{1}{m_1}(\xi^2+\frac{\varepsilon_2}{1+\tau_\theta p})+\frac{1}{m_1\varepsilon_3}(p+\xi^2\varepsilon_3+\varepsilon_4)$,

$c_{52}=\frac{1}{m_1}\{m_2p^2+m_1\xi^2+(1-i\xi\lambda_{q1}+p\tau_q)\}\frac{\varepsilon_1 p}{1+\tau_\theta p}$, $c_{53}=\frac{1}{m_1}\{1-i\xi\lambda_{q1}+p\tau_q\}\frac{p}{1+\tau_\theta p}-\frac{\varepsilon_5}{\varepsilon_3}$,

$c_{55}=\frac{\lambda_{q2}\varepsilon_1 p}{m_1(1+\tau_\theta p)}$, $c_{56}=\frac{\lambda_{q2}p}{m_1(1+\tau_\theta p)}$, $c_{61}=-\frac{\varepsilon_2}{(1+\tau_\theta p)}$, $c_{62}=\frac{\varepsilon_1 p(1-i\xi\lambda_{q1}+p\tau_q)}{(1+\tau_\theta p)}$,

$c_{63}=\frac{(1-i\xi\lambda_{q1}+p\tau_q)p+(1+\tau_\theta p)\xi^2}{(1+\tau_\theta p)}$, $c_{65}=\frac{\varepsilon_1 p\lambda_{q2}}{(1+\tau_\theta p)}$, $c_{66}=\frac{p\lambda_{q2}}{(1+\tau_\theta p)}$,

$f_{11}=c_{52}+(c_{55}-\lambda_i)\lambda_i$, $f_{12}=c_{63}+(c_{66}-\lambda_i)\lambda_i$, $f_{13}=c_{62}+c_{65}\lambda_i$, $f_{14}=c_{53}+c_{56}\lambda_i$,

$G_{11}=0$, $G_{12}=y_{12}e^{-\lambda_1 y}$, $G_{13}=y_{14}e^{-\lambda_3 y}$, $G_{14}=y_{16}e^{-\lambda_5 y}$, $G_{21}=e^{-k_5 y}$,

$G_{22}=\frac{i\xi(a_3y_{22}-y_{32}-y_{12})e^{-\lambda_1 y}}{\beta_0(\lambda_1^2-k_5^2)}$, $G_{23}=\frac{i\xi(a_3y_{24}-y_{34}-y_{14})e^{-\lambda_3 y}}{\beta_0(\lambda_3^2-k_5^2)}$, $G_{24}=\frac{i\xi(a_3x_{26}-y_{36}-y_{16})e^{-\lambda_5 y}}{\beta_0(\lambda_5^2-k_5^2)}$,

$G_{31}=\frac{-i\xi}{k_5}e^{-k_5y}$, $G_{32}=[\frac{\xi^2(a_3y_{22}-y_{32}-y_{12})}{\beta_0\lambda_1(\lambda_1^2-k_5^2)}-\frac{y_{22}}{\lambda_1}]e^{-\lambda_1 y}$, $G_{33}=[\frac{\xi^2(a_3x_{24}-y_{34}-y_{14})}{\beta_0\lambda_3(\lambda_3^2-k_5^2)}-\frac{y_{24}}{\lambda_3}]e^{-\lambda_3 y}$,

$G_{34}=[\frac{\xi^2(a_3x_{26}-y_{36}-y_{16})}{\beta_0\lambda_5(\lambda_5^2-k_5^2)}-\frac{y_{26}}{\lambda_5}]e^{-\lambda_5 y}$, $G_{41}=0$, $G_{42}=y_{32}e^{-\lambda_1 y}$,$G_{43}=y_{34}e^{-\lambda_3 y}$,

$G_{44}=y_{36}e^{-\lambda_5 y}$, $G_{51}=2i\xi(1-\frac{1}{\beta_0})e^{-k_5y}$,

$G_{52}=[\frac{2(1-\beta_0)\xi^2(a_3y_{22}-y_{32}-y_{12})}{\beta_0^2(\lambda_1^2-k_5^2)}-(\frac{1}{\beta_0}-2)y_{22}-\frac{y_{32}}{\beta_0}-\frac{y_{12}}{\beta_0}]e^{-\lambda_1 y}$,

$G_{53}=[\frac{2(1-\beta_0)\xi^2(a_3y_{24}-y_{34}-y_{14})}{\beta_0^2(\lambda_3^2-k_5^2)}-(\frac{1}{\beta_0}-2)y_{24}-\frac{y_{34}}{\beta_0}-\frac{y_{14}}{\beta_0}]e^{-\lambda_3 y}$,

$G_{54}=[\frac{2(1-\beta_0)\xi^2(a_3y_{26}-y_{36}-y_{16})}{\beta_0^2(\lambda_5^2-k_5^2)}-(\frac{1}{\beta_0}-2)y_{26}-\frac{y_{36}}{\beta_0}-\frac{y_{16}}{\beta_0}]e^{-\lambda_5 y}$,

$G_{61}=2i\xi(1-\frac{1}{\beta_0})e^{-k_5y}$, $G_{62}=[\frac{2(-1+\beta_0)\xi^2(a_3y_{22}-y_{32}-y_{12})}{\beta_0^2(\lambda_1^2-k_5^2)}+\frac{y_{22}}{\beta_0}-\frac{y_{32}}{\beta_0}-\frac{y_{12}}{\beta_0}]e^{-\lambda_1 y}$,

$G_{63}=[\frac{2(-1+\beta_0)\xi^2(a_3y_{24}-y_{34}-y_{14})}{\beta_0^2(\lambda_3^2-k_5^2)}+\frac{y_{24}}{\beta_0}-\frac{y_{34}}{\beta_0}-\frac{y_{14}}{\beta_0}]e^{-\lambda_3 y}$,

$G_{64}=[\frac{2(-1+\beta_0)\xi^2(a_3y_{26}-y_{36}-y_{16})}{\beta_0^2(\lambda_5^2-k_5^2)}+\frac{y_{26}}{\beta_0}-\frac{y_{36}}{\beta_0}-\frac{y_{16}}{\beta_0}]e^{-\lambda_5 y}$,

$G_{71}=0$, $G_{72}=\frac{\{(1-2\beta_0)y_{22}-y_{32}-y_{12}\}e^{-\lambda_1 y}}{\beta_0}$, $G_{73}=\frac{\{(1-2\beta_0)y_{24}-y_{34}-y_{14}\}e^{-\lambda_3 y}}{\beta_0}$,

$G_{74}=\frac{\{(1-2\beta_0)y_{26}-y_{36}-y_{16}\}e^{-\lambda_5 y}}{\beta_0}$, $G_{81}=-(k_5+\frac{\xi^2}{k_5})e^{-k_5y}$,

$G_{82}=[-\frac{i\xi}{\beta_0(\lambda_1^2-k_5^2)}(\lambda_1+\frac{\xi^2}{\lambda_1})(a_3y_{22}-y_{32}-y_{12})+\frac{i\xi y_{22}}{\lambda_1}]e^{-\lambda_1 y}$,

$G_{83}=[-\frac{i\xi}{\beta_0(\lambda_3^2-k_5^2)}(\lambda_3+\frac{\xi^2}{\lambda_3})(a_3y_{24}-y_{34}-y_{14})+\frac{i\xi y_{24}}{\lambda_3}]e^{-\lambda_3 y}$,

$G_{84}=[-\frac{i\xi}{\beta_0(\lambda_5^2-k_5^2)}(\lambda_5+\frac{\xi^2}{\lambda_5})(a_3y_{26}-y_{36}-y_{16})+\frac{i\xi y_{26}}{\lambda_5}]e^{-\lambda_5 y}$.

3.2 Wave propagation in a non-local magneto-thermoelastic medium permeated by heat source

3.2.1 Introduction

The increasing attention has been fully focussed on the widely spread branch of thermoelasticity and magneto-thermoelasticity. Material science researchers and engineers are more concerned with the thermal effects on elasto-static and elasto-dynamical objects. In high temperature field like nuclear reactor, thermoelasticity as well as electro-magneto thermoelasticity play a crucial role. Biot[30] first introduced coupled theory modifying the conventional Fourier's law of heat conduction equation. Coupled theory of heat conduction depicts two physical phenomena (i) infinite speed in propagation of heat waves and (ii) absence of elastic term. These two phenomena contradicts to the actual observation and raised the ambiguity about this type of theory. Lord and Shulman [80] introduced generalized thermoelasticity and energy equation was modified by introducing thermal relaxation time parameter. This theory of generalized thermoelasticity is called L-S theory. Green and Lindsay[58] modified L-S theory by introducing two relaxation time parameters in the equation of motion and energy equation which are known as Temperature Rate Dependent Thermoelasticity[TRDTE]. Dhaliwal and Sherief[44] proved the uniqueness of the solution of this type of thermoelastic theory. Green and Naghdi[59], [60], [61] developed another three models I, II, III of thermoelasticity which are - thermoelasticity with energy dissipation (TEWEDT) and without energy dissipation (TEWDEDT). A generalized electromagneto-thermoelastic interaction for a thin cylindrical semiconducting medium is discussed by Das et al.[41]. Also, Ghosh et al. [54] studied the interaction in an anisotropic three-dimensional elastic slab due to prescribed surface temperature in presence of electro-magnetic field. A thermoelastic problem with non-linear heat equation was studied by Das et al.[40]. Abbas[2] illustrated a solution for a generalised magneto-thermoelastic problem for a non-homogeneous and isotropic annular cylinder by using the finite element method.
Eringen [46] proposed the nonlocal theory for an elastic medium which discussed the long gap in between the limitation of the classical continuum hypothesis and the theory of atomic structure. The nonlocal thermoelastic model is developed to determine the thermoelastic behavior of the nanostructure of the continuum. The governing equations with temperature-dependent thermal relaxation parameters are solved by using the harmonic plane waves and integral transforms (Laplace and Fourier transforms). However,

the generalized thermoelastic theory may be considered in some specific cases, such as the external characteristic length to the internal characteristic length of the structural approaches. Although, the structural approaches of the molecular dynamic method is successfully predicting the thermal and mechanical properties of nanomaterial systems with limited number of molecules and atoms. It requires lengthy computational effort for nanomaterial systems. To overcome this tremendous computational efforts, it is better to introduce an additional material length scale parameters to the size effect in extending generalized thermoelastic theory. According to the Eringen's nonlocal thermoelastic theory, the stress at a point depends on the strain at the point as well as the strains at other points within the domain. Therefore, the nonlocal stress field is determined from the convolution of the local strain field and a smoothing kernel function for a thermoelastic model. Gupta and Mukhopadhyay [62] studied generalized thermoelasticity with the help of nonlocal and dual-phase lag theory. Recently, Tzou[113], [117] developed dual-phase-lag heat equation theory by incorporating two-phase-lag parameters associated with the temperature gradient and the heat flux vector. Two phase lag parameters are related to the fastest transient effects of the thermal inertial and the micro-structural interaction properties of the medium. This theory is known as dual-phase-lag (DPL) model. Later, three-phase-lag (TPL) model was developed by Roychoudhury [102]. Tzou and Guo [118] developed a new approach known as non-local theory which also helps to explore the impact of stress and strain for all the points of the body at a material point. Zhang and He [140] studied the nono-local thermoelastic problem with moving heat source. Molla et al. [86] illustrated the propagation of waves in generalized thermo-viscoelastic medium with nono-local heat transfer. Mondal et al. [87] analyzed the wave propagation for thermoelastic materials on Eringen's non-local thermoelasticity. The Moore-Gibson-Thompson model with nonlocal thermal response for circular cylindrical cavity was studied by Mohammed et al. [85].

In this problem, we consider a generalized thermoelastic interaction for a two dimensional isotropic medium under non-local heat conduction theory in the presence of electromagnetic fields and heat source. The governing equations are transformed by using the harmonic plane waves and then a vector matrix differential equation is formulated which is solved by eigenvalue method. Finally, the displacement, stress components and temperature distribution are presented and compared graphically with two theories such as non-local dual-phase-lag (NLDPL) and non-local Lord-Shulman (NLLS) theory under the variations of different physical parameters. In presence of electro-magnetic fields, the significant effects of non-local variables and phase lagging parameters on displacements, temperature distribution and stress components are analytically studied. Comparisons are also made graphically for normal stress component with and without heat-flux and concluding remarks are drawn.

3.2.2 Basic equations for theoretical model

Due to size effect with heat conduction, it is required to modify the classical heat conduction theory as it is essentially non-local at micro and/or nano-scale. Tzou [113] suggested the modification introducing material's characteristic length. As in Eringen [46] and Challamel et al. [32], the governing equations for an isotropic and homogeneous elastic medium in context of non-local Lord-Shulman(NLLS) model are as -

Equation of motion: In the presence of electro-magnetic field, we have the transportation equation of motion for the isotropic body as

$$\sigma_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{3.30}$$

where, F_i is the Lorentz force components of the form $\mathbf{F}_i = \mu_0(\mathbf{J} \times \mathbf{H})_{\mathbf{i}}$.
Stress-strain-temperature relation:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\theta\delta_{ij} \tag{3.31}$$

Energy Equation: In presence of heat source, we consider the energy equation as-

$$Q - q_{i,i} = \rho T_0 \frac{\partial S}{\partial t} \tag{3.32}$$

Entropy Equation:

$$T_0\rho S = \rho C_e\theta + \gamma T_0 e_{kk} \tag{3.33}$$

Heat Conduction Equation: As in Tzou and Guo [118], we consider the modified non-local heat conduction equation as

$$(1 + (\lambda_q)_k \frac{\partial}{\partial x_k} + \tau_q \frac{\partial}{\partial t})q_i = -K_\theta(1 + \tau_\theta \frac{\partial}{\partial t})\theta_{,i} \tag{3.34}$$

where, T_0 is the uniform reference temperature, S is the entropy per unit mass, c_e is specific heat, ρ denotes the constant mass density, λ and μ are Lamè's constants, K_θ is thermal conductivity, $\alpha = \frac{3\lambda+2\mu}{\alpha_\theta}$, α_θ is the co-efficient of linear thermal expansion, τ_θ represents phase lag of temperature gradient, τ_q is phase lag of heat flux and $(\lambda_q)_k$ is the component of non-local length vector.

From equations (3.30)-(3.34), we obtain the modified heat transportation equation and the equation of motion as-

$$K_\theta(1+\tau_\theta\frac{\partial}{\partial t})\theta_{,ii} = (1+(\lambda_q)_k)\frac{\partial}{\partial x_k}+\tau_q\frac{\partial}{\partial t}(\rho c_e\frac{\partial\theta}{\partial t})(\rho c_e\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial}{\partial t}(u_{k,k})-Q) \quad (3.35)$$

$$\rho\frac{\partial^2 u_i}{\partial t^2} = (\lambda+\mu)u_{j,ji}+\mu u_{i,jj}-\gamma\theta_{,i}+F_i \quad (3.36)$$

As in Ghosh and Lahiri [51], the medium in presence of electro-magnetic field must follow the Maxwell's field equations which are as following

$$\text{curl }\mathbf{h} = \mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}, \quad (3.37)$$

$$\text{div }\mathbf{h}=0, \qquad \mathbf{E}=-\mu_0(\dot{\mathbf{u}}\times\mathbf{H})$$

$$\text{curl }\mathbf{E} = -\mu_0\frac{\partial\mathbf{h}}{\partial t}$$

$$\mathbf{B}=\mu_{\mathbf{0}}(\mathbf{H}+\mathbf{h}), \qquad \mathbf{D}=\varepsilon_{\mathbf{0}}\mathbf{E}$$

where $\vec{\mathbf{H}} = \vec{\mathbf{H}}_0+\vec{\mathbf{h}}$, $\vec{\mathbf{H}}$ is the total magnetic field vector, $\vec{\mathbf{J}}$ is the electric current density, ε_0 is the electric permeability and μ_0 is the magnetic permeability.

The three theories such as the non-local classical dynamical coupled(NLCDC) theory, the first generalized thermoelastic theory i.e., non-local Lord and Shulman's(NLLS) theory and second generalized heat transportation with the non-local dual-phase-lag(NLDPL) theory proposed by Tzou [116], [114] prescribed by the modified heat transportation equation (3.35) which are as follows

(i) The non-local classical dynamical coupled (NLCDC) theory

$\tau_\theta = \tau_q = 0$

(ii) The first generalized thermoelastic theory i.e., non-local Lord and Shulman's (NLLS) theory

$\tau_\theta = 0, \quad \tau_q = \tau_0 > 0$

(iii) The second generalized heat transportation with non-local dual-phase-lag(NLDPL) theory

$\tau_q \geq \tau_\theta > 0$

3.2.3 Formulation and solution of the problem

We now consider here an isotropic, homogeneous and perfectly conducting thermoelastic two-dimensional half-space($x \geq 0$) in cartesian co-ordinate system (Figure 3.1)

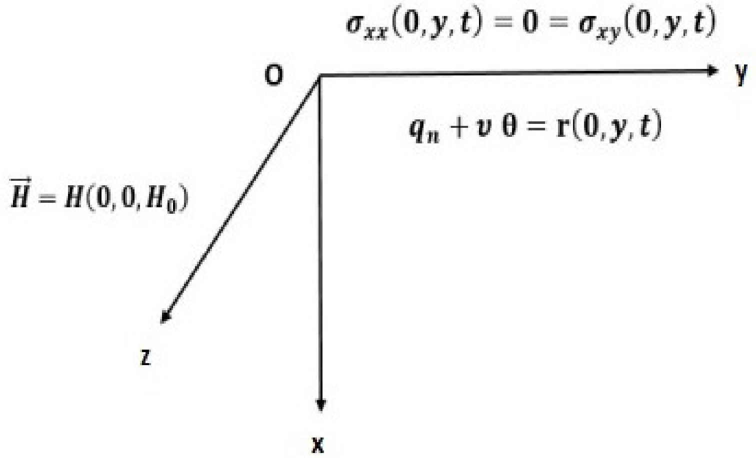


Figure 3.1: Schematic diagram of the theoretical model.

in presence of the constant magnetic field $\mathbf{H} = (\mathbf{0}, \mathbf{0}, \mathbf{H_0} + \mathbf{h})$, where $\mathbf{H_0} = (0, 0, H_0)$ which is acting in the direction of z-axis). An induced magnetic field $\mathbf{h} = (0, 0, h_0)$ and an induced electric field, $\mathbf{E} = (E_1, E_2, 0)$ has been produced according to the application of magnetic field $\mathbf{H}$ which satisfy the linearized equations of electromagnetism. By using equation (3.37), the expressions of Lorentz forces are

$$F_1 = \mu_0 H_0^2 (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}) - \epsilon_0 \mu_0^2 H_0^2 \ddot{u} \tag{3.38}$$

and

$$F_2 = \mu_0 H_0^2 (\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}) - \epsilon_0 \mu_0^2 H_0^2 \ddot{v} \tag{3.39}$$

By using the equations (3.38) and (3.39), equation (3.36) gives the equation of motion componentwise as

$$(\rho + \epsilon_0 \mu_0^2 H_0^2) \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu + \mu_0 H_0^2) \frac{\partial^2 u}{\partial x^2} + (\lambda + \mu + \mu_0 H_0^2) \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 u}{\partial y^2} - \gamma \frac{\partial \theta}{\partial x} \tag{3.40}$$

and

$$(\rho + \epsilon_0 \mu_0^2 H_0^2) \frac{\partial^2 v}{\partial t^2} = (\lambda + 2\mu + \mu_0 H_0^2) \frac{\partial^2 v}{\partial y^2} + (\lambda + \mu + \mu_0 H_0^2) \frac{\partial^2 u}{\partial x \partial y} + \mu \frac{\partial^2 v}{\partial x^2} - \gamma \frac{\partial \theta}{\partial y} \tag{3.41}$$

Equation (3.35) gives the modified heat transportation equation as

$$K_\theta (1 + \tau_\theta \frac{\partial}{\partial t})(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}) = (1 + \lambda_{q_1} \frac{\partial}{\partial x} + \lambda_{q_2} \frac{\partial}{\partial y} + \tau_q \frac{\partial}{\partial t})(\rho c_e \frac{\partial \theta}{\partial t} + \gamma T_0 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - Q) \tag{3.42}$$

The stress components are derived from the equation (3.31) which are as of the following

$$\sigma_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y} - \gamma \theta \tag{3.43}$$

$$\sigma_{yy} = (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda\frac{\partial u}{\partial x} - \gamma\theta \tag{3.44}$$

$$\sigma_{xy} = \mu(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{3.45}$$

We now introduce the following variables to get the non-dimensional form of the above mentioned governing equations

$$\begin{gathered}(x^*, y^*) = c_0\eta_0(x, y), \quad (u^*, v^*) = c_0\eta_0(u, v)\\ (t^*, \tau^*) = c_0^2\eta_0(t, \tau), \quad \sigma_{ij}^* = \tfrac{1}{\rho c_0^2}\sigma ij\\ \theta^* = \tfrac{1}{\rho c_0^2}\gamma\theta, \quad Q^* = \tfrac{\rho^2 c_e c_0^4 \eta_0}{\gamma}Q\\ h^* = \tfrac{1}{H_0 h}, \quad J_i^* = \tfrac{1}{\eta_0 c_0 H_0}J_i\end{gathered} \tag{3.46}$$

Introducing the non-dimensional variables in equations (3.40)-(3.45), we obtain (omitting the asterisks for convention)the equations of motion, heat transportation and stress components as

$$\alpha_0^2\frac{\partial^2 u}{\partial t^2} = \beta_0^2\frac{\partial^2 u}{\partial x^2} + (\beta_0^2 - 1)\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} - \beta_0^2\frac{\partial\theta}{\partial x} \tag{3.47}$$

$$\alpha_0^2\frac{\partial^2 v}{\partial t^2} = \beta_0^2\frac{\partial^2 v}{\partial y^2} + (\beta_0^2 - 1)\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2} - \beta_0^2\frac{\partial\theta}{\partial y} \tag{3.48}$$

$$(1+\tau_\theta\frac{\partial}{\partial t})(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}) = (1+\lambda_{q_1}\frac{\partial}{\partial x}+\lambda_{q_2}\frac{\partial}{\partial y}+\tau_q\frac{\partial}{\partial t})(\frac{\partial\theta}{\partial t}+\epsilon(\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 v}{\partial y\partial t})-Q) \tag{3.49}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} + (1 - \frac{2}{\beta^2})\frac{\partial v}{\partial y} - \theta \tag{3.50}$$

$$\sigma_{yy} = \frac{\partial v}{\partial y} + (1 - \frac{2}{\beta^2})\frac{\partial u}{\partial x} - \theta \tag{3.51}$$

$$\sigma_{xy} = \frac{1}{\beta^2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{3.52}$$

where
$\alpha_0^2 = (\frac{\rho+\epsilon_0\mu_0^2H_0^2}{\mu})c_0^2$ *and* $\beta_0^2 = \frac{\lambda+2\mu+\mu_0H_0^2}{\mu} = \frac{\rho c_0^2}{\mu}$,
$\eta_0 = \frac{\rho c_e}{K_\theta}$ *and* $\epsilon = \frac{\gamma^2 T_0}{\rho^2 c_e c_0^2}$,
$c_0^2 = \frac{\lambda+2\mu}{\rho}$ *and* $\beta^2 = \frac{\rho c_0^2}{\mu} = \frac{\lambda+2\mu}{\mu}$
As in Ghosh et al.[54], the physical variables can be decomposed by using normal modes as in the following form -

$$\xi(x, y, t) = \xi(x)e^{\omega t + iay} \tag{3.53}$$

where $\xi = [u \quad v \quad \sigma_{ij} \quad \theta]$, $\xi^* = [u^* \quad v^* \quad {\sigma_{ij}}^* \quad \theta^*]$, ω is the angular frequency and a is the wave numbers along y-direction.

By using equation (3.53), equations (3.47)-(3.49) can be written in the form of vector matrix differential equation

$$\frac{dv}{dx} = Av + f \tag{3.54}$$

where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $v = \begin{bmatrix} u & v & \theta & u' & v' & \theta' \end{bmatrix}^T$,

$f = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & C_{67}Q \end{bmatrix}^T$, A_{11} is the null matrix, A_{12} is the identity matrix of order 3, A_{21} and A_{22} are given in Appendix.

3.2.4 Solution of the vector-matrix differential equation:

Let $\lambda = \lambda_i\,, (i = 1(1)6)$ be the roots of the characteristic equation which is given by

$$|A - \lambda I| = 0 \tag{3.55}$$

and these roots i.e. eigenvalues of the coefficient matrix A, $\lambda = \lambda_i$ are in the form of $\lambda = \pm\lambda_i\,, (i = 1(1)3)$. Also, $X(\lambda)$ is the eigenvector corresponding to the eigenvalue λ of the matrix A, which is given as

$$X(\lambda) = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \lambda\delta_1 \quad \lambda\delta_2 \; \lambda\delta_3]^T \tag{3.56}$$

where the analytic expressions of $\delta_i, (i = 1(1)3)$ are given in the Appendix.
Let $V_i, (i = 1(1)6)$ be the eigenvectors corresponding to the eigenvalues $\lambda = \lambda_i\,(i = 1(1)6)$ which is denoted by $V_i = \begin{bmatrix} (X)_{\lambda=\lambda_{\frac{i+1}{2}},(i=1(2)5)} \\ (X)_{\lambda=\lambda_{\frac{-i}{2}},(i=2(2)6)} \end{bmatrix}$.
For regularity condition as $x \to +\infty$ (Ghosh and Lahiri [51]), the general solution of the differential equation (3.54) for isotropic half-space is

$$v = \sum_{i=1}^{3} V_i \; y_i \tag{3.57}$$

where

$$y_r = A_r e^{\lambda_r x} + e^{\lambda_r x}\int_{-\infty}^{\infty} H_r e^{-\lambda_r x} dx \tag{3.58}$$

$$\mathit{and} \quad H_r = V^{-1} f \quad \mathit{where} \quad V = (V_i), i = 1(1)6 \tag{3.59}$$

where A_r are constants which are to be evaluated by using boundary conditions. With the help of the equations (3.57)-(3.59), we can find the analytical solutions of the displacement components and the temperature distribution which are as follows

$$(u, v, \theta) = \textstyle\sum_{j=1}^{3}(x_{1j}, x_{2j}, x_{3j})(A_j e^{\lambda_j x} - \frac{H_i}{\lambda_i}) \tag{3.60}$$

Using this analytic solution, equation (3.31) gives the expressions of stress components which are as follows

$$(\sigma_{11}, \sigma_{22}, \sigma_{12}) = \sum_{j=1}^{3} A_j (R_{1j}, R_{2j}, R_{3j})(x) - d_k \tag{3.61}$$

where, $x_{ij}(x), R_{ij}(x)$ and d_k $(i, j = 1(1)3$ and $k = 1(2)5)$ are given in the Appendix II.

3.2.5 Boundary conditions

To obtain the values of A_r, we consider the following boundary conditions
Mechanical Condition: The boundary surface of the medium is traction-free i.e.,

$$\begin{aligned} \sigma_{11}(0, y, t) = 0 \\ \sigma_{12}(0, y, t) = 0 \end{aligned} \tag{3.62}$$

Thermal Condition: The boundary surface is also experienced a thermal load with constant intensity i.e.,

$$q_n + \nu\theta = r^*(0, y, t) \tag{3.63}$$

where q_n is the normal components of the heat flux vector, $r^*(0, y, t)$ is the intensity of applied heat source and ν is the Biot's number.

Using above mentioned boundary conditions prescribed in equations (3.62) and (3.63), we now get the following linear equations

$$\begin{aligned} \textstyle\sum_{j=1}^{3} A_j S_{1j} = d_1 \\ \textstyle\sum_{j=1}^{3} A_j S_{3j} = d_3 \\ \textstyle\sum_{j=1}^{3} A_j S_{5j} = d_5 \end{aligned} \tag{3.64}$$

From equation (3.64), we can obtain the values of the arbitrary parameters A_r such as $A_r = \frac{D_i}{D}$ where $D_i,\ D\ \ (i, r = 1(1)3);\ \ d_i's,\ S'_{ij}s\ \ (i = 1(2)5)$ are given in the Appendix II.

3.2.6 Numerical results

Several graphs have been presented for comparing the numerical results in the context of NLCDC, NLLS and NLDPL theories. For analyzing the effect of wave propagation, we now consider $\omega = \omega_0 + i\varsigma$, ω is a complex number. As in Ghosh and Lahiri [51], copper(Cu) is taken for numerical computation and the values of the material constants are as follows:

$\lambda = 7.76\ X\ 10^{10} kgm^{-1}s^{-2}$	$\mu = 3.86\ X\ 10^{10} kgm^{-1}s^{-2}$	$\rho = 8954 kgm^{-3}$
$c_e = 383.1 Jkg_{-1}K^{-1}$	$K = 386 Wm^{-1}K^{-1}$	$T_0 = 293K$
$\epsilon_0 = \frac{10^{-9}}{36\pi}$	$\mu_0 = 4\pi \times 10^{-7}$	$H_0 = \frac{10^7}{4\pi}$
$c_0 = 2200.0$	$\tau = 5 \times 10^{-5}$	$\gamma = 210 \times 10^4$
$r* = 20$	$Q = 0.5$	

The numerical computations have been carried out with the help of suitable computer programming language MATLAB R2016a and the analysis are studied on the basis of the several graphical representations.

3.2.7 Graphical representation and analysis

In order to study the characteristics of various components of the field variables like-stresses, displacement and temperature, several graphs have been drawn with respect to different values of the space variable (x, y), time t, and heat source(Q). Along with the graphical representation, the corresponding analysis are also made.

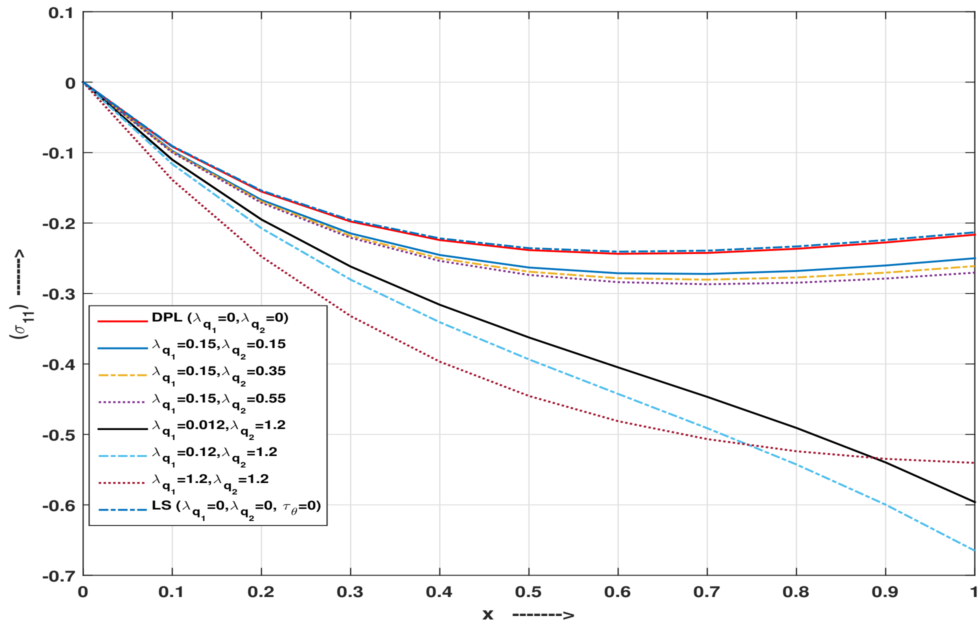


Figure 2: Effect of non-local variables(λ_{q1} and λ_{q2}) on stress component (σ_{11})

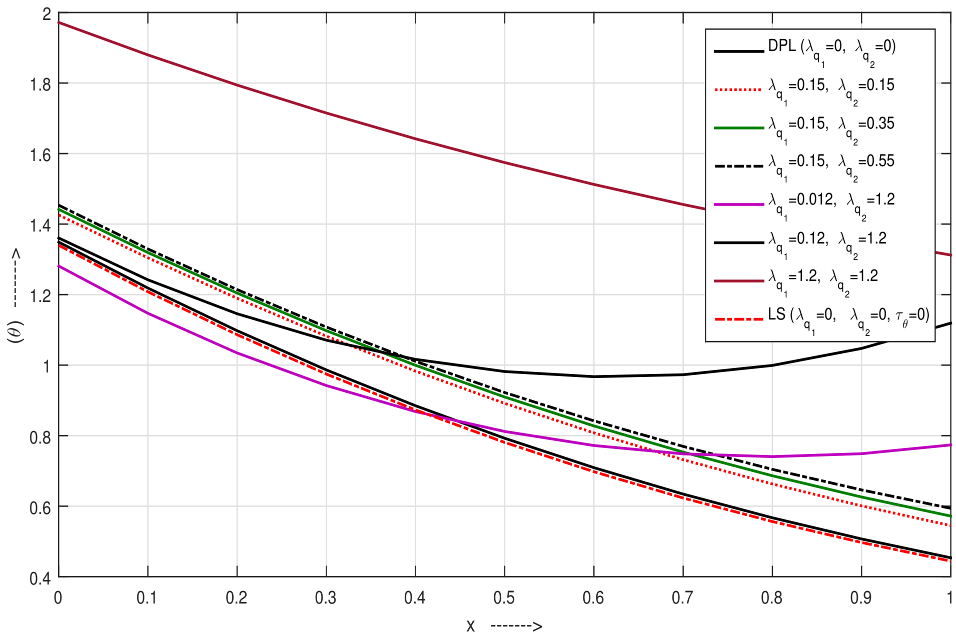


Figure 3: Effect of non-local variables(λ_{q1} and λ_{q2}) on temperature(θ)

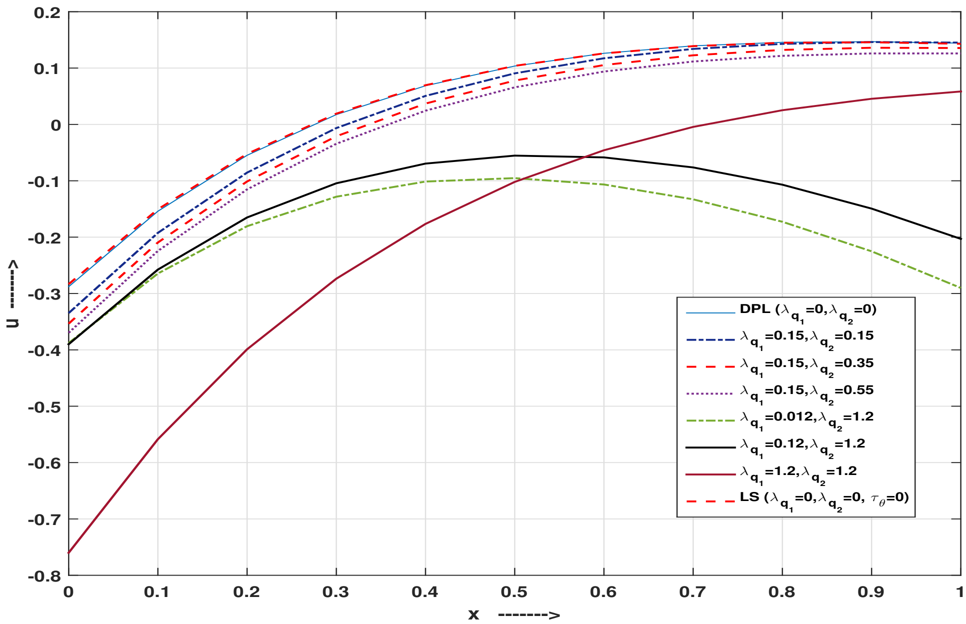


Figure 4: Effect of non-local variables(λ_{q_1} and λ_{q_2}) on displacement(u)

Effect of thermoelastic non-local parameters(λ_{q1} and λ_{q2})

Figures 2-4 illustrate the variations of stress components(σ_{11}), temperature distribution (θ) and displacement component(u) with distance(x) for fixed values of non-local parameters λ_{q1} and λ_{q2} in presence of heat-source(Q). Comparisions are also depicted with well established model of non-local dual phase-lag(NLDPL) and non-local Lord-Shulman(NLLS) theory.

Figure 2 shows that the normal stress component(σ_{11}) is compressive in nature within the whole region $0 \leq x \leq 1.0$. The absolute values of this normal stress component increases when space variable x increases. The absolute value of the normal stress component is maximum when ratio of λ_{q2} and λ_{q1} is greater than 1 and the significant variation of σ_{11} occurs within the region $0.2 \leq x \leq 1.0$ when ratio of λ_{q2} and λ_{q1} is less than and equal to 1. The absolute value of this stress component σ_{11} increases when λ_{q1} decreases at the middle portion of the space variable of the medium.

Figure 3 illustrates that the effect of non-local variables (λ_{q1} and λ_{q2}) on temperature(θ) for fixed values of physical field variables. The absolute values of the temperature distribution(θ) decrease as x increases. The absolute value of temperature attains maximum at $x = 0$ for $\lambda_{q1} = \lambda_{q2} = 1.2$. The variations of temperature are almost parallel for $\lambda_{q1} = 0.012$ and $\lambda_{q1} = 0.12$. These variation of temperature gradually

decreases when x increases for different values of λ_{q1} and λ_{q2}.
Figure 4 predicts that the significant variation of displacement component(u) within the region $0 \leq x \leq 1.0$ when ratio of λ_{q1} and λ_{q2} is less than and equal to 1. The absolute value of displacement component(u) attains at $x = 1$ for $\lambda_{q1} = 1.2$.
It has been clearly observed that the characteristic curves for NLLS model and NLDPL model are very much significant in all of the above cases.

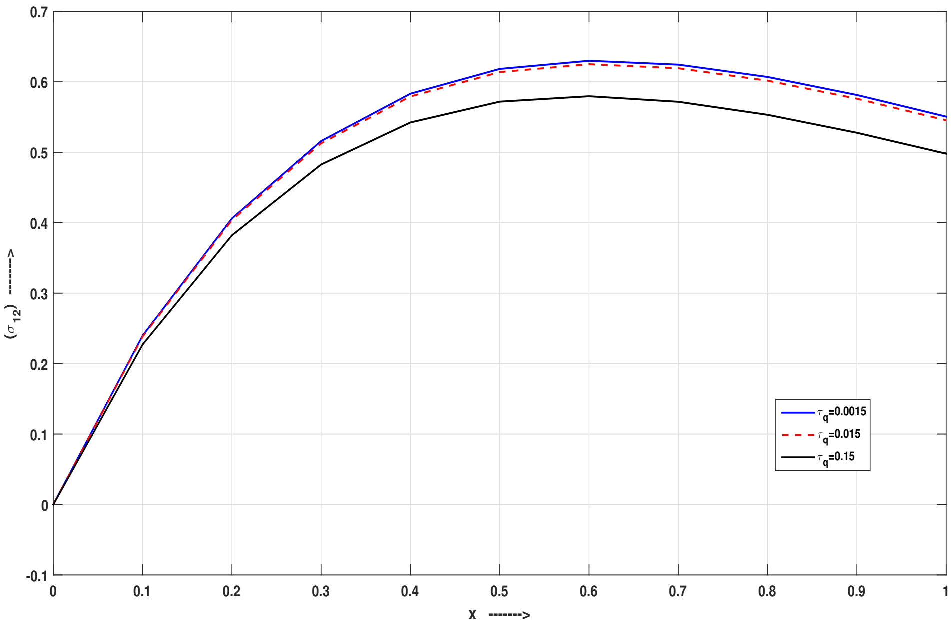


Figure 5: Effect of τ_q on stress component(σ_{12})

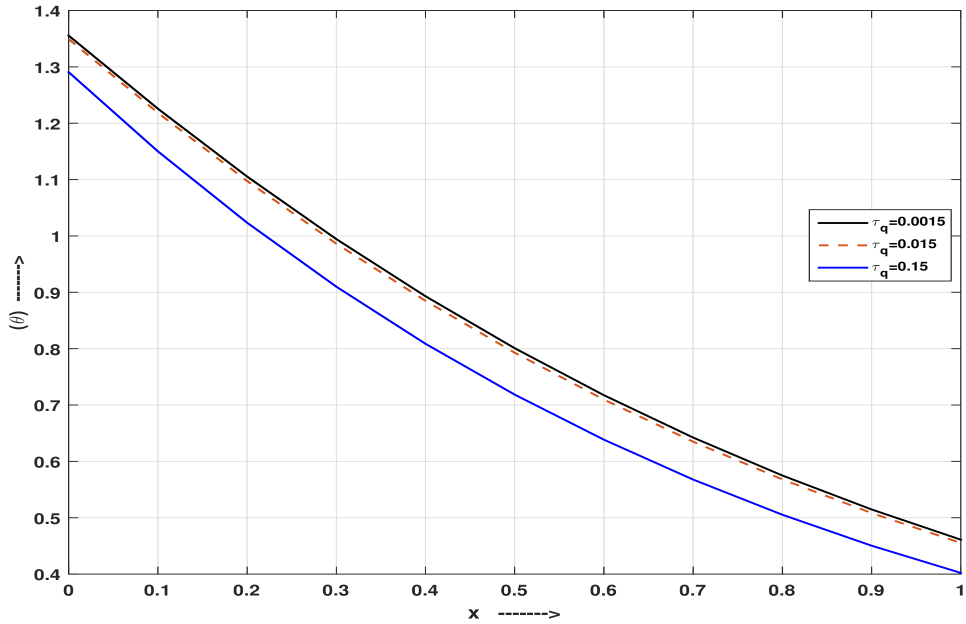


Figure 6: Effect of τ_q on temperature (θ)

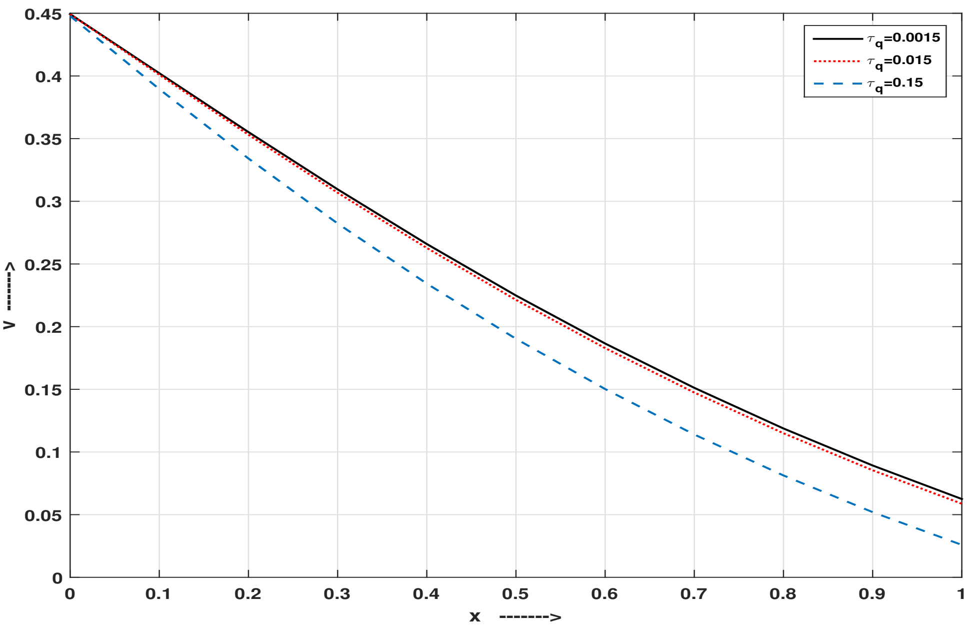


Figure 7: Effect of τ_q on displacement(v)

Effect of heat flux lagging parameter(τ_q)

Figures 5-7 depict the variations of stress component(σ_{12}), temperature distribution (θ) and displacement component (v) with distance(x) for different values of magnetic intensity ($H_0 = 10^7/4\pi$), $\tau_\theta = 0.02$, non-local parameters ($\lambda_{q1} = 0.012$, λ_{q2}=0.015) and $\tau_q = 0.0015, 0.015, 0.15$ in presence of the heat-source(Q). Figures 5, 6 and 7 depict that the variations of stress component(σ_{12}), temperature distribution (θ) and displacement component (v) are almost same for the values of $\tau_q = 0.0015$ and 0.015. These figures also give the different variations for the values of $\tau_q = 0.15$. The characteristics of these variations are extensive in nature in the whole region $0 \leq x \leq 1.0$.
Figure 5 shows that the variations of shearing stress component (σ_{12}) attains maximum at the middle portion of the medium. The effect of σ_{12} is more prominent within the region $0.3 \leq x \leq 1.0$ for different values of τ_q.
Figures 6 and 7 clearly say that the temperature distribution and displacement component of the medium gradually decrease as x increases. The absolute value temperature and displacement attains maximum at $x = 0$.
The effect of $\tau_q(= 0.15)$ is more prominent for stress component (σ_{12}), temperature distribution (θ) and displacement component (v).

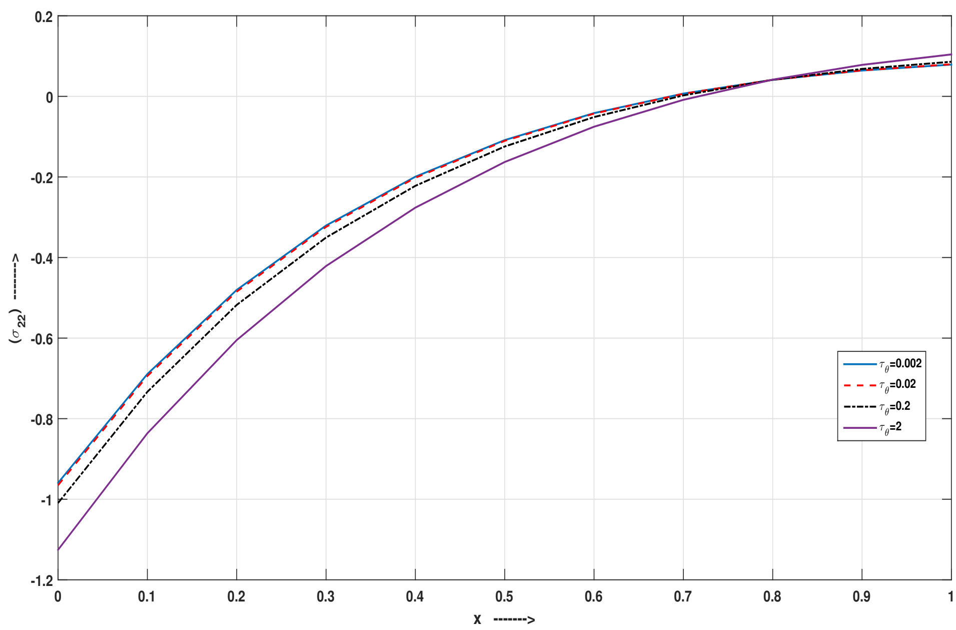


Figure 8: Effect of τ_θ on stress component(σ_{22})

Effect of phase lagging of temperature gradient parameter(τ_θ)

Figure 8 depicts the distributions of stress component(σ_{22}) with distance(x) for fixed values for fixed values of magnetic intensity ($H_0 = 10^7/4\pi$), $\tau_q = 0.0015$, non-local parameters ($\lambda_{q1} = 0.012$, λ_{q2}=0.015) and the four fixed values of $\tau_\theta = 0.002, 0.02, 0.2, 2.0$ in presence of heat-source(Q). The normal stress σ_{22} is compressive for different values of τ_θ as shown in the figure. The characteristics of σ_{22} are identical for different values of τ_θ. The effect of the phase-lag parameter τ_θ for the different values of physical variables have been illustrated in this figure and clearly declares that the this phase lag parameter plays an important role for studying the characteristics of this type of material.

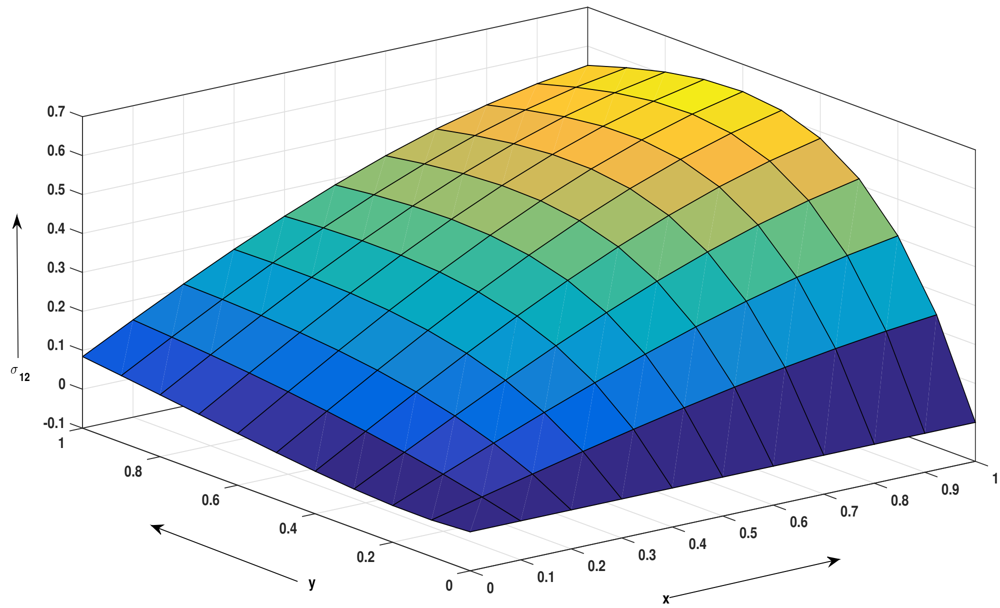


Figure 9: Graph of Stress(σ_{12})

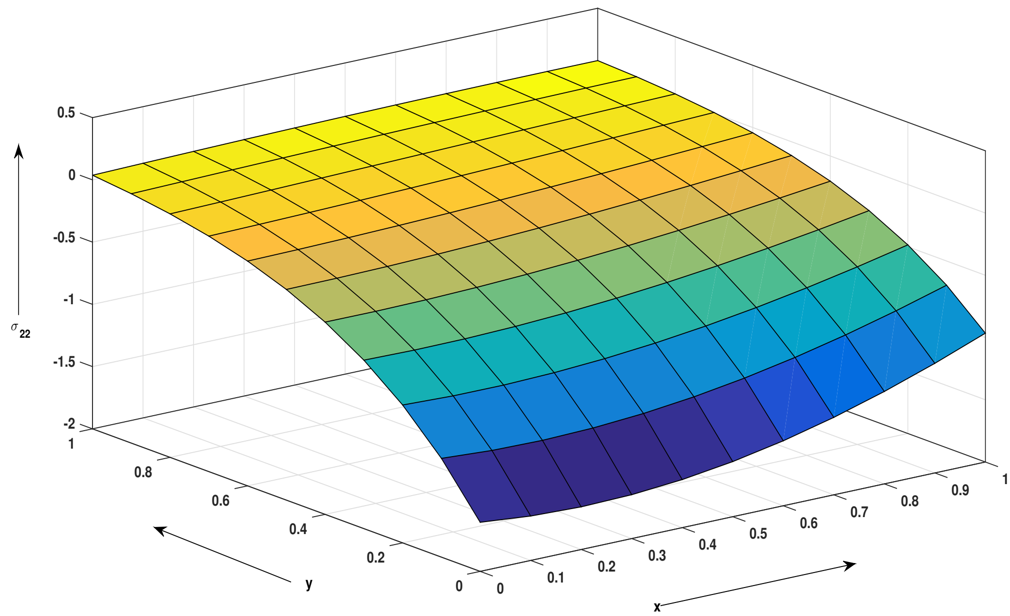


Figure 10: Graph of Stress (σ_{22})

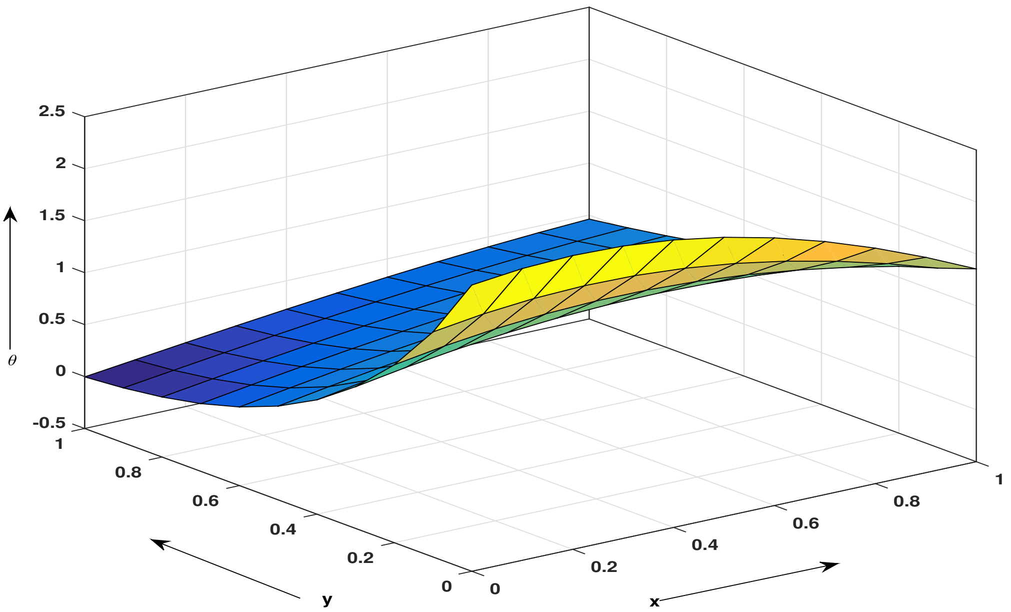


Figure 11: Graph of Temperature (θ)

Three-dimensional Variation of stress components and temperature distribution

Figures 9-11 represent the effects of the two space variable (x and y) of the stress components (σ_{12}, σ_{22}) and temperature distribution (θ) in presence of the heat-source(Q). The nature of the field variables are also studied for fixed values of magnetic intensity ($H_0 = 10^7/4\pi$), $\tau_q = 0.0015$, $\tau_\theta = 0.02$ and non-local parameters ($\lambda_{q1} = 0.012$, λ_{q2}=0.015).

Figure 9 represents the shearing stress component σ_{12} is compressive for different space variables x and y. The absolute value of shearing stress component(σ_{12} increases and attains maximum at $x = 1$ and $y = 0.6$.

Figure 10 also predicts the characteristic of the graph of the normal stress component(σ_{22}) increases as y increases and it attains the maximum at $y = 1$.

Figure 11 illustrates that the absolute values of temperature distribution (θ) decreases as y increases.

Three-variational variations of the field variables and the stress components are represented graphically and the nature of these graphs are wave-like propagation. This propagations are highly dependent on the space variables as well as time variable.

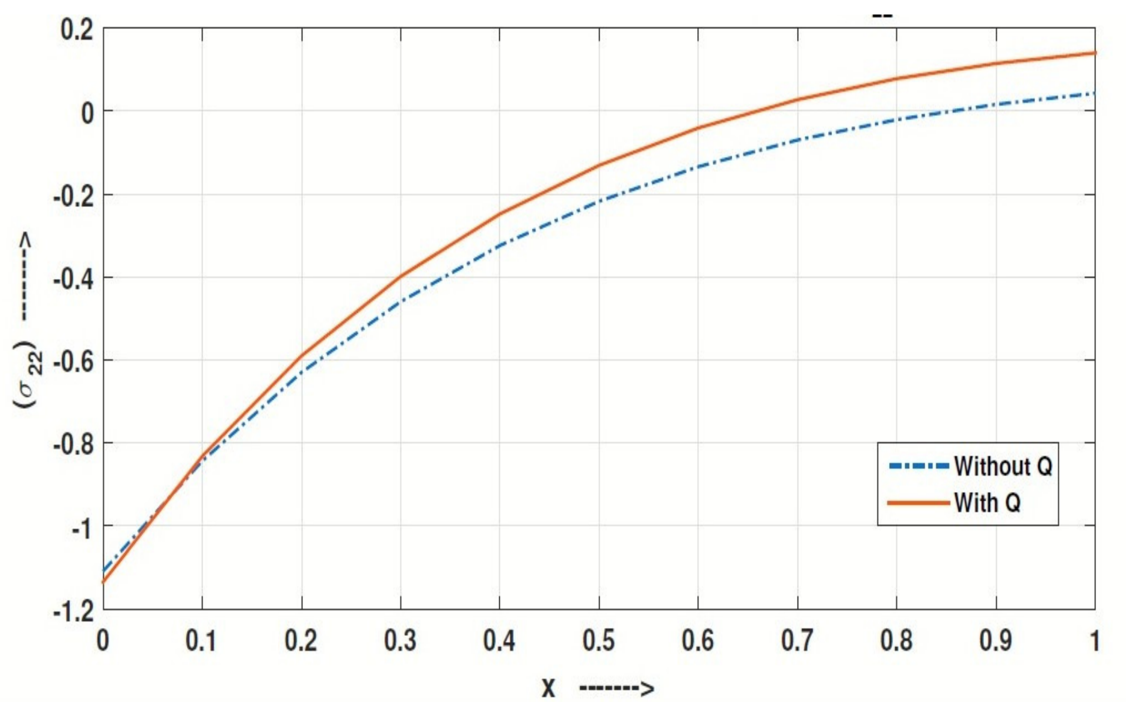


Figure 12: Effect of heat source (Q) on stress component (σ_{22})

Effect of the external heat source(Q)

Figure 12 represents that the effects of the external heat source (Q) on the normal stress component (σ_{22}) with the space variable x in presence of the external heat source Q and without heat source. The nature of this normal stress component is also studied for fixed values of magnetic intensity ($H_0 = 10^7/4\pi$), $\tau_q = 0.0015$, $\tau_\theta = 2.0$ and the non-local parameters ($\lambda_{q1} = 0.012$, λ_{q2}=0.015). The absolute strength of the external heat source is considered here for 0.5 unit. The figure shows that the significant distributions of the normal stress component(σ_{22}) within the region $0.2 \leq x \leq 1.0$. The normal stress σ_{22} is extensive within the region $0.7 \leq x \leq 1.0$ for fixed value of y in the presence of heat source whereas it is compressive without heat source. The figure depicts that the two graphs, with and without heat source intersects to each other at $x = 0.05$, after that the absolute values of the normal stress component decreases as x increases and it attains minimum at $x = 0.15$. The presence and absence of external heat source on the field variables are clearly characterized by these graphical representations.

3.2.8 Conclusion

The result that is presented here is more significant for future investigation regarding the non-local thermoelastic model. The importance of this model is inclusion of the non-local variables along with the phase-lag parameters in the heat transportation process which also enhances the thermoscopic effects at a macroscopic level. Non-local response in dual phase-lag model has been extended to predict the significant effect of external heat source. Here, it is observed that the significant effects of the non-local variables (λ_{q_1} and λ_{q_2}) on the displacement components, stress components and temperature distribution of the medium. Experimental results are underway to support the NLDPL and NLLS model with phase-lag variables(τ_q and τ_θ) proposed herewith. The primary emphasis has been given on the additional thermal disturbances due to external heat source. Inclusion of variables(τ_q and τ_θ) in the heat transportation equation caused the term of "non-Fourier" character which may be needed for the effect of micro-scale heat transfer. This results and analysis provide the natural method to get well-posed nonlocal elastic problems for application to nano-structure. Therefore, the nonlocal stress-strain model has been extensively adopted in the various problems such as the bending and buckling of nano-beams, and the problem of the nonlocal magneto-thermoelastic behavior for nano-structure size is widely applied in nowadays. The results obtained here is significant for future investigation regarding non-local thermoelasticity. The non-local variables in the heat conduction process are included in this model to observe the enhancement of the thermoscopic effects at a macroscopic level. We have formulated a two dimensional problem here to investigate the effect of non local variables in heat conduction equation for the generalized thermoelasticity theory. Here, we observe the significant effects of the non-local variables (λ_{q_1} and λ_{q_2}) on displacement, stress components and temperature of the medium. The variations in characteristic curves due to NLDPL and NLLS models has been depicted graphically. The effects of phase lag variables (τ_q and τ_θ) have also been observed with or without heat source.

3.2.9 Appendix

$$C_{41} = \frac{a^2+\alpha_0^2\omega_1^2}{\beta_0^2}, \quad C_{42} = 0, \quad C_{43} = 0,$$
$$C_{44} = 0, \quad C_{45} = \frac{ia(1-\beta_0^2)}{\beta_0^2}, \quad C_{46} = 1,$$
$$C_{51} = 0, \quad C_{52} = \alpha_0^2\beta_1^2 + a^2\beta_0^2, \quad C_{53} = ia\beta_0^2,$$
$$C_{54} = 0, \quad C_{55} = ia(1-\beta_0^2), \quad C_{56} = 0$$
$$C_{61} = C_{41}\epsilon\lambda_{q_1}\omega_1,$$
$$C_{62} = C_{42}\epsilon\lambda_{q_1}\omega_1 + \epsilon(ia\omega_1 - \lambda_{q_2}a^2\omega_1 + \tau_q\omega_1^2 ia),$$
$$C_{63} = C_{43}\epsilon\lambda_{q_1}\omega_1 + \omega_1 + \tau_q\omega_1^2 + \lambda_{q_2}\omega_1 ia + a^2 + a^2\tau_\theta\omega_1,$$
$$C_{64} = C_{44}\epsilon\lambda_{q_1}\omega_1 + \epsilon(\omega_1 + ia\lambda_{q_2}\omega_1 + \tau_q\omega_1^2),$$
$$C_{65} = C_{45}\epsilon\lambda_{q_1}\omega_1 + \epsilon\lambda_{q_1}ia\omega_1, \; C_{67} = -(1 + ia\lambda_{q_2} + \omega_1\tau_q)$$

$$A_{21} = \left[\ C_{ij}\ \right]_{i=4(1)6, j=1(1)3}, A_{22} = \left[\ C_{ij}\ \right]_{(i,j)=4(1)6}$$

For j=1(1)6, k=1(1)5

$$R_{1j}(x) = [\lambda_j x_{1j} + ia(1-\tfrac{2}{\beta^2})x_{2j} - x_{3j}]e^{-\lambda_j x}, \quad N_{1j} = \tfrac{1}{\lambda_j}[ia(1-\tfrac{2}{\beta^2})x_{2j}(x) - x_{3j}],$$
$$R_{2j}(x) = [\lambda_j x_{2j} + ia(1-\tfrac{2}{\beta^2})x_{1j} - x_{3j}]e^{-\lambda_j x} \quad N_{2j} = \tfrac{1}{\lambda_j}[iax_{2j} - x_{3j}],$$
$$R_{3j}(x) = [\tfrac{1}{\beta^2}\lambda_j x_{1j} + iax_{2j}]e^{-\lambda_j x}, \quad N_{3j} = \tfrac{1}{\lambda_j}[iax_{2j}],$$
$$R_{4j}(x) = x_{2j}e^{-\lambda_j x}, \quad N_{4j} = \tfrac{1}{\lambda_j}[x_{3j}],$$
$$R_{5j}(x) = (\nu - \lambda_j)x_{3j}e^{-\lambda_j x}, \quad N_{5j} = \tfrac{1}{\lambda_j}[\nu],$$
$$R_{kj}(0) = S_{kj}$$

$d_1 = \sum_{j=1}^{6} A_j R_{1j}(0) = \sum_{j=1}^{6} H_j N_{1j}$,
$d_3 = \sum_{j=1}^{6} A_j R_{3j}(0) = \sum_{j=1}^{6} H_j N_{3j}$
and $d_5 = r^* + \sum_{j=1}^{6} H_j N_{5j}$

$$D = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{31} & S_{32} & S_{33} \\ S_{51} & S_{52} & S_{53} \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & S_{12} & S_{13} \\ d_3 & S_{32} & S_{33} \\ d_5 & S_{52} & S_{53} \end{vmatrix}, \quad D_2 = \begin{vmatrix} S_{11} & d_1 & S_{13} \\ S_{31} & d_3 & S_{33} \\ S_{51} & d_5 & S_{53} \end{vmatrix},$$
$$D_3 = \begin{vmatrix} S_{11} & S_{12} & d_1 \\ S_{31} & S_{32} & d_3 \\ S_{51} & S_{52} & d_5 \end{vmatrix}$$

$$
\begin{aligned}
where, \quad & \delta_1 = f_{12}f_{23} - f_{13}f_{22}, \\
& \delta_2 = f_{13}f_{21} - f_{11}f_{23} \\
and \quad & \delta_3 = f_{11}f_{22} - f_{12}f_{21}
\end{aligned}
$$

$f_{11} = C_{41} + C_{44}\lambda_i - \lambda_i^2; f_{12} = C_{42} + C_{45}\lambda_i; f_{13} = C_{43} + C_{46}\lambda_i; f_{21} = C_{51} + C_{54}\lambda_i; f_{22} = C_{52} + C_{55}\lambda_i - \lambda_i^2; f_{23} = C_{53} + C_{56}\lambda_i; f_{31} = C_{61} + C_{64}\lambda_i; f_{32} = C_{62} + C_{65}\lambda_i; f_{33} = C_{63} + C_{66}\lambda_i - \lambda_i^2; x_{ij} = \delta_j, (i, j = 1(1)3).$

4

STUDY OF THERMOELASTIC BEHAVIOUR IN CURVILINEAR CO-ORDINATE SYSTEM

PROBLEMS :

- PROBLEM -5 **Analysis of Multi-Phase Lag Gradients for a Spherical Cavity due to Prescribed Internal Pressure.**

  This paper has been communicated.

- PROBLEM -6 **Vibrations of a Circular Cylinder in Generalized Thermoelasticity.**

4.1 Analysis of multi-phase lag gradients for a spherical cavity due to prescribed internal pressure

4.1.1 Introduction

The phenomenon of finite speed of heat wave propagation was modified applying phase-lag gradient (τ_t and τ_q) and relaxation time parameter to conventional heat equation due to Fourier's law in generalized thermoelasticity as in Lord-Shulman [80] and Green-Lindsay [58]. As an extension Green and Nagdhi [59], [60], [61] proposed three different models viz. G-N – Type I, Type II and Type III. Basically classical theory of thermoelasticity (CTE) and G-N – Type I model exhibits the same characteristics of infinite spit of heat propagation in classical coupled thermoelasticity theory. Type II and Type III are related to non dissipation and dissipation of energy respectively.

In the history of thermoelasticity Tzou [113], [117] introduced the dual phase-lag model to investigate the effect of lagging behavior within the thermoelastic medium. Later Roy Choudhuri [102] proposed the concept of three-phase-lag model in the conventional thermal equation. Several researchers like Quintanilla and Racke [99], Ghosh et al. [49] were able to find the solutions of the heat conduction equation associated with the three-phase-lag model in their recent studies. Moreover Zenkour [128] recently came up with a refined two-temperature multi-phase-lag model involving the heat flux vector and the temperature gradient which is applicable to generalized thermoelastic medium. Researchers like Bagri and Eslami [24]-[26], Kar and Kanoria [69]-[70], Das et al. [43] solved several problems applying the above mentioned theories of generalized thermoelasticity.

Chandrasekharaiah and Keshavan [34] solved a thermoelastic problem regarding an unbounded solid with cylindrical cavity, whereas a theoretical study was done by Misra et al. [82], [83] to generate stress in elastic and viscoelastic solids containing a spherical and a circular cylindrical hole respectively. Sinha and Elsibai [109] presented thermoelastic interaction in an infinite solid having a spherical hole. Sherief and Saleh [108] were concerned with a one-dimensional problem to figure out thermal stresses and temperature in an unbounded solid with a spherical cavity considering a sudden variation in the temperature. Abd-Alla et al. [6] analyzed the thermoelastic interaction in an orthotropic solid having a spherical cavity. Mukhopadhyay [88] dealt with a thermo-viscoelastic problem in an infinite solid consisting of a spherical cavity under a periodic loading and temperature remaining unchanged. Moreover, Mukhopadhyay [89] studied the thermoealstic behaviour without energy dissipation in the same medium considering a stressed free cavity surface under a thermal shock. Abd-Alla et al. [7] analyzed a viscoelastic medium containing a spherical cavity to figure out the field quantities in the said medium. Mukhopadhyay [90] discussed the thermally induced vibration in an unbounded continuum with a spherical cavity. Rakshit Kundu and Mukhopadhyay [100] evaluated the field quantities while analyzing a viscoelastic medium containing a spherical hole. Youssef [124] studied the thermoelastic behavior of an infinite solid having a cylindrical cavity. Abbas [1] solved a thermoelastic problem in an unbounded solid containing either a spherical or cylindrical hole. Aouadi [22] dealt with a one dimensional issue in a infinite solid with a spherical cavity in context of generalized thermal diffusion. Itu et al. [67] studied composite circular plates through radial ribs to examine improved rigidity.

Ghosh and Kanoria [55], [56] solved a thermoelastic problem to determine the field quantities in an isotropic medium containing a spherical cavity. They also did the same for a functionally graded spherically infinite continuum having a spherical hole. Abbas and Abd-Alla [4] considered a infinite solid having a cylindrical cavity to study its thermoelastic behaviour. G-N Type III model was applied by Mukhopadhyay and Kumar [92] to examine the thermoelastic behaviour of a unbounded body with cylindrical cavity. Xia et al. [121] examined the dynamic nature of an infinite body having a cylindrical cavity under thermal shock. Youssef [125] came up with two solutions discussing the thermoelastic interactions in an infinite solid respectively having a spherical and a cylindrical hole. Allam et al. [21] presented the electromagneto-thermoelastic interactions of an infinite body containing a spherical cavity applying the generalised thermoelasticity theory proposed by Green-Nagdhi.

The refined two-temperature multi-phase-lag model, proposed by Zenkour [128] has drawn attention of many researchers and has been used in various studies of thermoelastic analysis of different structures. Sardar et al. [103] dealt with a three-dimensional problem on coupled thermoelasticity in an anisotropic half-space using two-temperature multi-phase-lag model. In a recent work of ours (Lahiri et al. [75]), we have analyzed the dimensional variations of stress, strain and temperature of a two-dimensional homogeneous isotropic solid based on multi-phase-lag model. In this problem, an unbounded

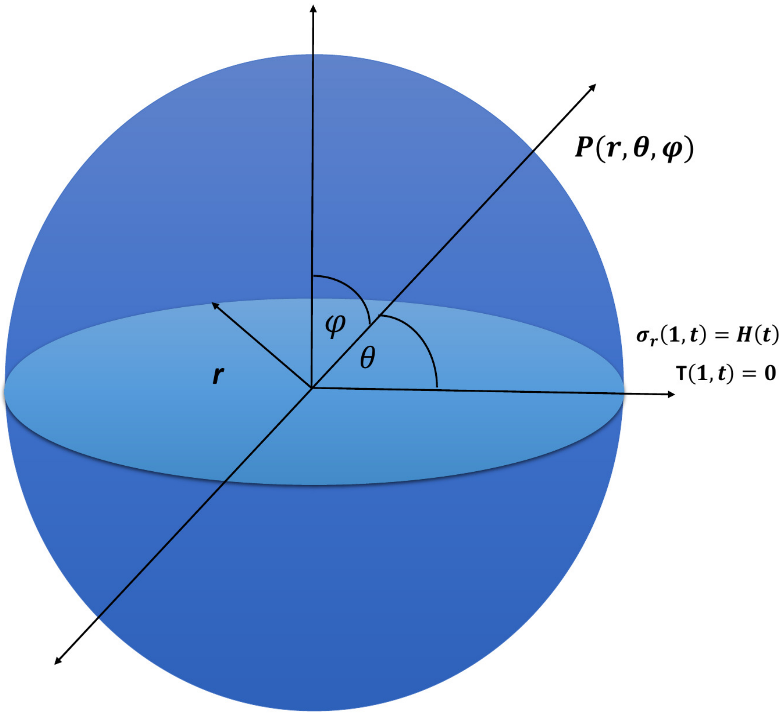


Figure 1: Schematic diagram

thermoelastic body with a spherical cavity equipped with two-temperature multi-phase-lag (RPL) thermoelastic model, which is solved using vector matrix method. Some special cases drawn from RPL have also been investigated through graphical analysis.

4.1.2 Formulation of the problem

In the curvilinear coordinate system, we have worked on a multi-phase lag model with a spherical cavity. We have considered the unbounded isotropic thermoelastic medium here. The displacement has radial component $u(r,t)$ only here. There are three principal stresses which acts along radial direction (σ_r), cross-radial direction (σ_θ) and transverse direction (σ_ϕ). Here, we consider the case of the stresses along radial direction (σ_r) and along transverse direction $(\sigma_\phi = \sigma_\theta)$.

In absence of the body forces, the displacement equation of motion for spherical symmetry is obtained as

$$\rho C_1^2 \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \beta\left(\frac{\partial T}{\partial r} + \alpha\frac{\partial^2 T}{\partial t \partial r}\right) = \rho\frac{\partial^2 u}{\partial t^2} \tag{4.1}$$

and the heat conduction equation in context of multi-phase lag thermoelasticity is given by

$$\left(1+\sum_{n=1}^{N}\frac{\tau_T^n}{n!}\frac{\partial^n}{\partial t^n}\right)K\nabla^2 T = \left[\overline{R}+\tau_0\frac{\partial}{\partial t}+\sum_{n=1}^{N}\frac{\tau_q^{n+1}}{(n+1)!}\frac{\partial^{n+1}}{\partial t^{n+1}}\right]$$
$$\left[\rho C\left(\frac{\partial T}{\partial t}+\alpha_0\frac{\partial^2 T}{\partial t^2}\right)+\beta T_0\left(\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)\right] \quad (4.2)$$

The constitutive relation between the stresses σ_r and σ_ϕ can be written as

$$\sigma_r = \rho C_1^2\frac{\partial u}{\partial r}+\frac{2\rho C_1^3 u}{r(1-C_1)}-\beta\left(T+\alpha\frac{\partial T}{\partial t}\right) \quad (4.3)$$

and

$$\sigma_\phi = \frac{\rho C_1^3}{1-C_1}\frac{\partial u}{\partial r}+\frac{\rho C_1^3 u}{r(1-C_1)}-\beta\left(T+\alpha\frac{\partial T}{\partial t}\right) \quad (4.4)$$

To make the equations dimension free, we introduce the following variables

$$r' = \frac{r}{a} \qquad t' = \frac{Kt}{\rho C a^2} \qquad u' = \frac{\rho C_1^2 u}{\beta a\theta_0}$$
$$T' = \frac{T-T_0}{T_0} \qquad (\tau',\tau_0',\tau_T',\tau_q') = \frac{K}{\rho C a^2}(\tau,\tau_0,\tau_T,\tau_q) \qquad (\alpha',\alpha_0') = \frac{K}{\rho C a^2}(\alpha,\alpha_0)$$
$$(\sigma_r',\sigma_\phi') = \frac{\rho C_1^2}{2\mu T_0}(\sigma_r,\sigma_\phi)$$

Introducing non-dimensional variables in the equations (4.1)-(4.4), we obtain (dropping the primes)

$$\frac{\partial}{\partial r}\left[\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)-\left(1+\alpha\frac{\partial}{\partial t}T\right)\right] = \delta^2\frac{\partial^2 u}{\partial t^2} \quad (4.5)$$

$$\left(1+\sum_{n=1}^{N}\frac{\tau_T^n}{n!}\frac{\partial^n}{\partial t^n}\right)\left(\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}\right) = \left[\overline{R}+\tau_0\frac{\partial}{\partial t}+\sum_{n=1}^{N}\frac{\tau_q^{n+1}}{(n+1)!}\frac{\partial^{n+1}}{\partial t^{n+1}}\right]$$
$$\left[\left(1+\alpha_0\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial t}+\epsilon\left(1+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)\right] \quad (4.6)$$

$$\sigma_r = \frac{1}{1-2\nu}\left[(1-\nu)\frac{\partial u}{\partial r}+\frac{2\nu u}{r}-(1-\nu)\left(1+\alpha\frac{\partial u}{\partial t}\right)T\right] \quad (4.7)$$

$$\sigma_\phi = \frac{1}{1-2\nu}\left[\nu\frac{\partial u}{\partial r}+\frac{u}{r}-(1-\nu)\left(1+\alpha\frac{\partial}{\partial t}\right)T\right] \quad (4.8)$$

where $\delta = \frac{K}{\rho C C_1 a}$ is a dimension-free inertial parameter and ν is the poisson ratio of the material.

4.1.3 Method of Solution: Formulation of a Vector-Matrix Differential Equation

Now applying the Laplace transform of the form

$$\overline{u}(r,p) = \int_0^\infty u(r,t)e^{-pt}dt \tag{4.9}$$

$$T(r,p) = \int_0^\infty T(r,t)e^{-pt}dt \tag{4.10}$$

$$L = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} \tag{4.11}$$

to the eqs (4.5) and (4.6), we obtain

$$L(\overline{u}) = \delta^2 p^2 \overline{u} + (1+\alpha p)\frac{d\overline{T}}{dr} \tag{4.12}$$

$$L\left(\frac{d\overline{T}}{dr}\right) = \epsilon_1 \epsilon p(1+\tau p)\delta^2 p^2 \overline{u} + \epsilon_1\{p(1+\alpha_0 p) + \epsilon p(1+\tau p)(1+\alpha p)\}\frac{d\overline{T}}{dr} \tag{4.13}$$

where L$=\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}$

Equations (4.12) and (4.13) can be expressed as a vector matrix differential equation given by

$$L\tilde{V} = \tilde{A}\tilde{V} \tag{4.14}$$

where

$$\tilde{V} = \begin{bmatrix} \overline{u} & \frac{d\overline{T}}{dr} \end{bmatrix}^T \tag{4.15}$$

$$A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{4.16}$$

and

$$C_{11} = \delta^2 p^2,$$

$$C_{12} = 1 + \alpha p,$$

$$C_{21} = \epsilon_1 \epsilon p(1+\tau p)\delta^2 p^2,$$

$$C_{22} = \epsilon_1\{p(1+\alpha_0 p) + \epsilon p(1+\tau p)(1+\alpha p)\}$$

4.1.4 Solution of the problem using Vector-Matrix Differential Equation

To solve the equation (4.14), we substitute

$$\widetilde{V} = \widetilde{X}(\lambda)\omega(r,\gamma) \tag{4.17}$$

where λ is a scalar, $\widetilde{X}$ is a vector independent of r and $\omega(r,\gamma)$ is a non-trivial solution of the scalar differential equation

$$L(\omega) = 0 \tag{4.18}$$

The solution of the above equation will be

$$w = \frac{\gamma}{r}e^{-\gamma r} + \frac{1}{r^2}e^{-\gamma r} \tag{4.19}$$

Using (4.17) and (4.19), we obtained from the equation (4.14) the following algebraic eigenvalue problem

$$\widetilde{A}\widetilde{X}(\gamma) = \gamma^2\widetilde{X}(\gamma) \tag{4.20}$$

where $\widetilde{X}(\gamma)$ is the eigenvector corresponding to the eigenvalue γ^2. The characteristic equation corresponding to the matrix $\widetilde{A}$ is given by

$$\gamma^4 - \gamma^2(C_{11} + C_{22}) + (C_{11}C_{22} - C_{12}C_{21}) = 0 \tag{4.21}$$

The roots of the characteristic equation (4.21) are of the form $\gamma = \gamma_1^2$ and $\gamma = \gamma_2^2$, where

$$\begin{aligned} \gamma_1^2 + \gamma_2^2 &= C_{11} + C_{22} \\ \gamma_1^2\gamma_2^2 &= C_{11}C22 - C_{12}C_{21} \end{aligned} \tag{4.22}$$

The eigenvectors $\widetilde{X}(\gamma_j^2),\ j = 1,2$ corresponding to the eigenvalues $\gamma_j^2,\ j = 1,2$ can be calculated as

$$\widetilde{X}_j(\gamma_j) = \begin{bmatrix} X_1(\gamma_j^2) \\ X_2(\gamma_j^2) \end{bmatrix}_{j=1,2} = \begin{bmatrix} -C_{12} \\ C_{11} - \gamma_j^2 \end{bmatrix}_{j=1,2} \tag{4.23}$$

Thus the solution of equation (4.14) can be written as

$$\widetilde{V}(r,p) = A\widetilde{X}(\gamma_1^2)\left(\frac{e^{-\gamma_1 r}}{r^2} + \frac{\gamma_1}{r}e^{-\gamma_1 r}\right) + B\widetilde{X}(\gamma_2^2)\left(\frac{e^{-\gamma_2 r}}{r^2} + \frac{\gamma_2}{r}e^{-\gamma_2 r}\right) \tag{4.24}$$

where the constants A and B can be obtained using boundary conditions. Components of $\widetilde{V}$ are given by

$$\overline{u}(r,p) = -AC_{12}\left(\frac{e^{-\gamma_1 r}}{r^2} + \frac{\gamma_1}{r}e^{-\gamma_1 r}\right) - BC_{12}\left(\frac{e^{-\gamma_2 r}}{r^2} + \frac{\gamma_2}{r}e^{-\gamma_2 r}\right) \tag{4.25}$$

and

$$\frac{d\overline{T}}{dr} = A(C_{11} - \gamma_1^2)\left(\frac{e^{-\gamma_1 r}}{r^2} + \frac{\gamma_1}{r}e^{-\gamma_1 r}\right) + B(C_{12} - \gamma_2^2)\left(\frac{e^{-\gamma_2 r}}{r^2} + \frac{\gamma_2}{r}e^{-\gamma_2 r}\right) \tag{4.26}$$

From (4.26) we obtain

$$\overline{T}(r,p) = -A(C_{11}-\gamma_1^2)\frac{e^{-\gamma_1 r}}{r} - B(C_{12}-\gamma_2^2)\frac{e^{-\gamma_2 r}}{r} \tag{4.27}$$

Applying Laplace transform given by (4.11) to the equations (4.7) and (4.8) and using (4.22) and (4.24) we obtain

$$\begin{aligned}\overline{\sigma_r} = \frac{1}{1-2\nu}\Bigg[Ae^{-\gamma_1 r}\Bigg\{(1-\nu)\left(\frac{\gamma_1^2}{r}+\frac{2\gamma_1}{r^2}+\frac{2}{r^3}\right)C_{12} - 2\nu C_{12}\left(\frac{1}{r^3}+\frac{\gamma_1}{r^2}\right)\\ +(1-\nu)(1+\alpha p)\frac{C_{11}-\gamma_1^2}{r}\Bigg\} + Be^{-\gamma_2 r}\Bigg\{(1-\nu)\left(\frac{\gamma_2^2}{r}+\frac{2\gamma_2}{r^2}+\frac{2}{r^3}\right)C_{12} - 2\nu C_{12}\left(\frac{1}{r^3}+\frac{\gamma_2}{r^2}\right)\\ +(1-\nu)(1+\alpha p)\frac{C_{11}-\gamma_2^2}{r}\Bigg\}\Bigg]\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\overline{\sigma_\phi} = \frac{C_{12}}{1-2\nu}\Bigg[Ae^{-\gamma_1 r}\Bigg\{\left(\nu\frac{\gamma_1^2}{r}+\frac{2\gamma_1}{r^2}+\frac{2}{r^3}\right) - \left(\frac{1}{r^3}+\frac{\gamma_1}{r^2}\right) + (1-\nu)\frac{C_{11}-\gamma_1^2}{r}\Bigg\}\\ + Be^{-\gamma_2 r}\Bigg\{\nu\left(\frac{\gamma_2^2}{r}+\frac{2\gamma_2}{r^2}+\frac{2}{r^3}\right) - \left(\frac{1}{r^3}+\frac{\gamma_2}{r^2}\right) + (1-\nu)\frac{C_{11}-\gamma_2^2}{r}\Bigg\}\Bigg]\end{aligned} \tag{4.29}$$

4.1.5 Boundary Condition

We now study the thermoelastic interactions when the boundary of the cavity is maintained at zero temperature. The boundary condition at the surface of the cavity $r = 1$ is taken as

$$\sigma_r(1,t) = H(t) \tag{4.30}$$

$$T(1,t) = 0, \tag{4.31}$$

where

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Taking Laplace transform of (4.28) and (4.29), we have

$$Am_{11} + Bm_{12} = m_{13} \tag{4.32}$$

$$Am_{21} + Bm_{22} = 0, \tag{4.33}$$

where

$$m_{11} = e^{-\gamma_1}\{(1-\nu)(\gamma_1^2+2\gamma_1+2)C_{12} - 2\nu C_{12}(1+\gamma_1) + (1-\nu)(1+\alpha p)(C_{11}-\gamma_1^2)\},$$
$$m_{12} = e^{-\gamma_2}\{(1-\nu)(\gamma_2^2+2\gamma_2+2)C_{12} - 2\nu C_{12}(1+\gamma_2) + (1-\nu)(1+\alpha p)(C_{11}-\gamma_2^2)\},$$

$$m_{13} = \frac{1-2\nu}{p}, \qquad m_{21} = (C_{11}-\gamma_1^2)e^{-\gamma_1}, \qquad m_{22} = (C_{11}-\gamma_1^2)e^{-\gamma_1}$$

Solving (4.32) and (4.39), we obtain

$$A = \frac{m_{13}m_{22}}{m_{11}m_{22}-m_{12}m_{21}}, \qquad B = -\frac{m_{13}m_{21}}{m_{11}m_{22}-m_{12}m_{21}}$$

4.1.6 Numerical analysis

Numerical analysis and computations have been done using the mechanical and thermal conditions as mentioned in euations (4.30) and (4.31) to study the characteristic behaviours of the physical constants with respect to space variables and time.
The numerical values (in SI unit) of constants are taken as in:

$$\tau_q = 0.01, \tau_\theta = 0.0001, \tau_0 = 0.01,$$

$$R = 1, t = 0.7, \delta = 7.3 \times 10^{-9},$$

$$p = 7, \alpha = 0.1, \alpha_0 = 0.05,$$

$$\epsilon = 2.97 \times 10^{-4}, \tau = 0.05, r = 2$$

4.1.7 Geometrical Representation and analysis

Depending upon the boundary conditions and using above mentioned numerical values, the geometrical representation of different physical variables are provided in two separate cases as follows -
Figure 2 to 5 represent the variation of non-dimensional numeric values of displacement, two stress components(σ_r, σ_ϕ) and temperature along radius for t=01, t=0.2 and t=0.3 .
Figure 6 to 9 represent the variation of non-dimensional numeric values of displacement, two stress components(σ_r, σ_ϕ) and temperature for r=1, r=1.5 and r=2 .
Figure 10 to 13 depict the three-dimensional characteristics of displacement, two stress components(σ_r, σ_ϕ) and temperature with respect to radius and time.

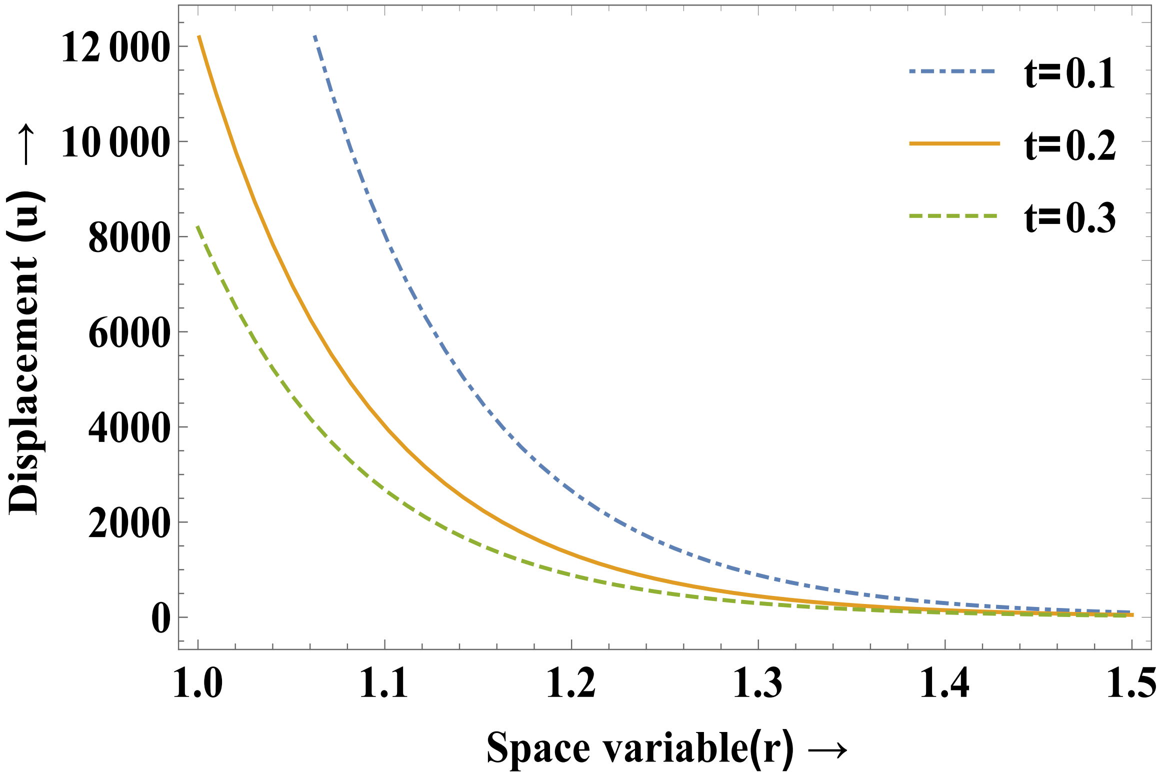


Figure 2: Distribution of $displacement(u)$ along radius(r) for different t..

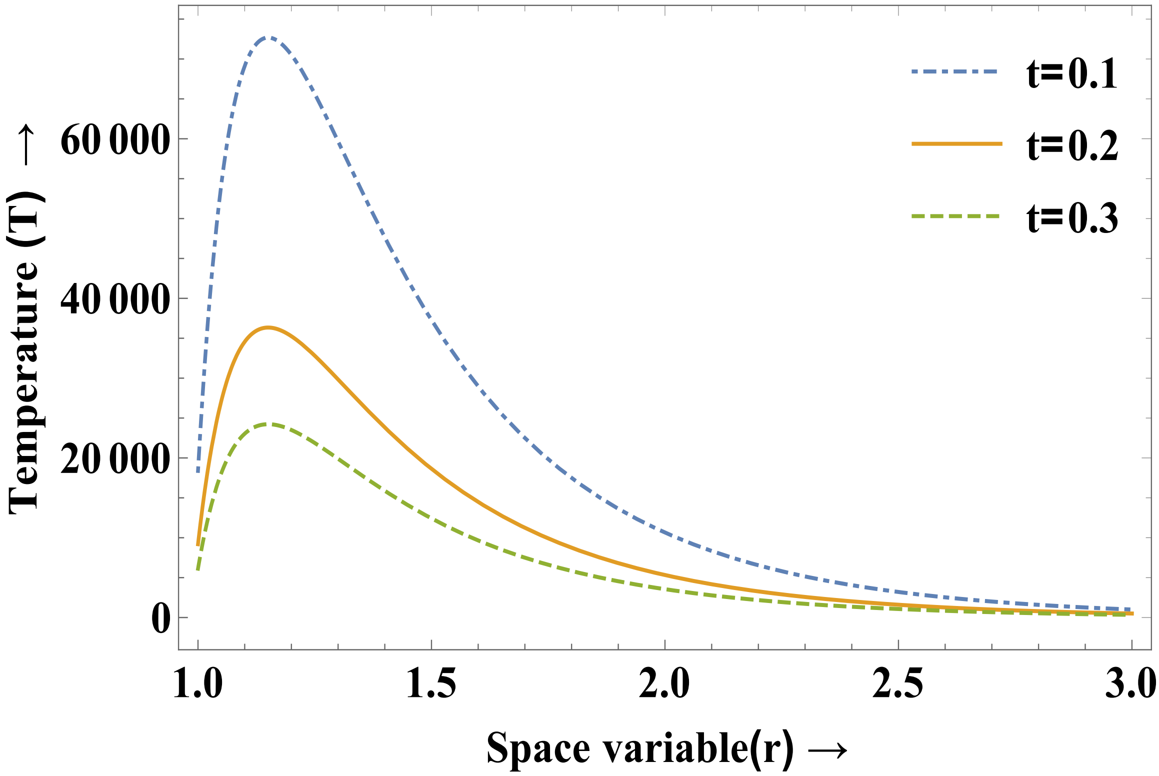


Figure 3: Distribution of $temperature(T)$ along radius(r) for different t.

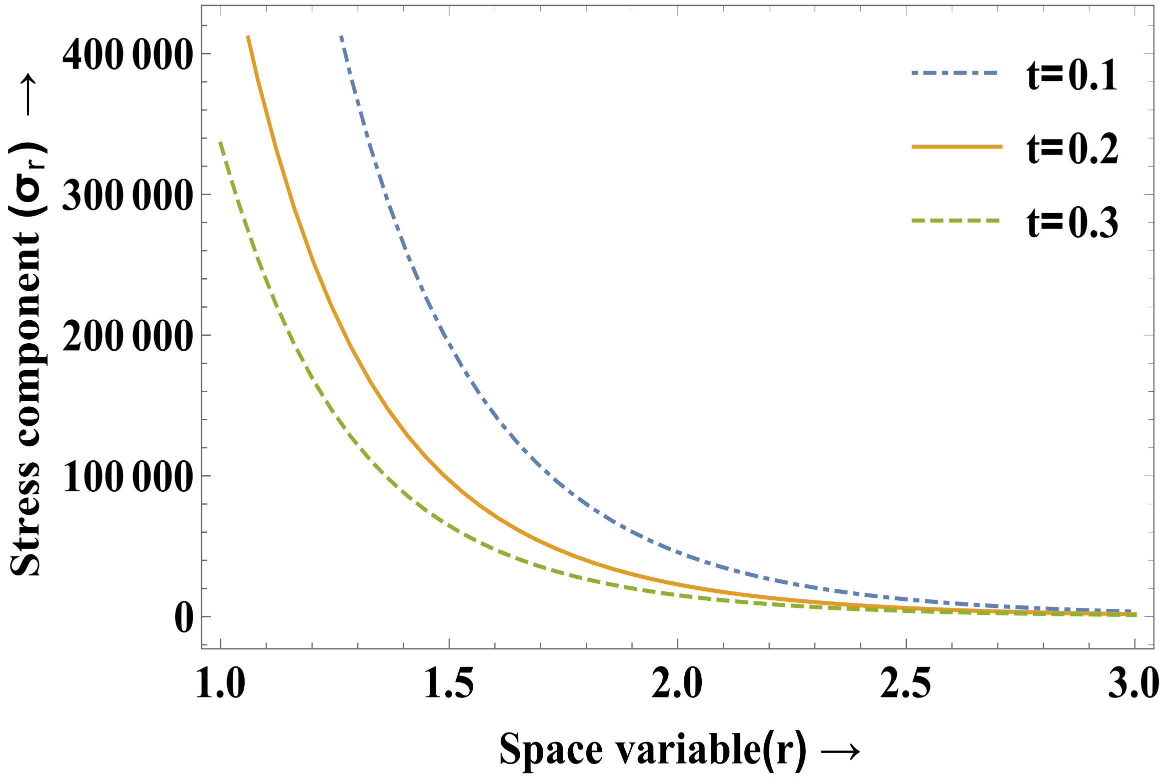


Figure 4 : Distribution of σ_r along radius(r) for different t.

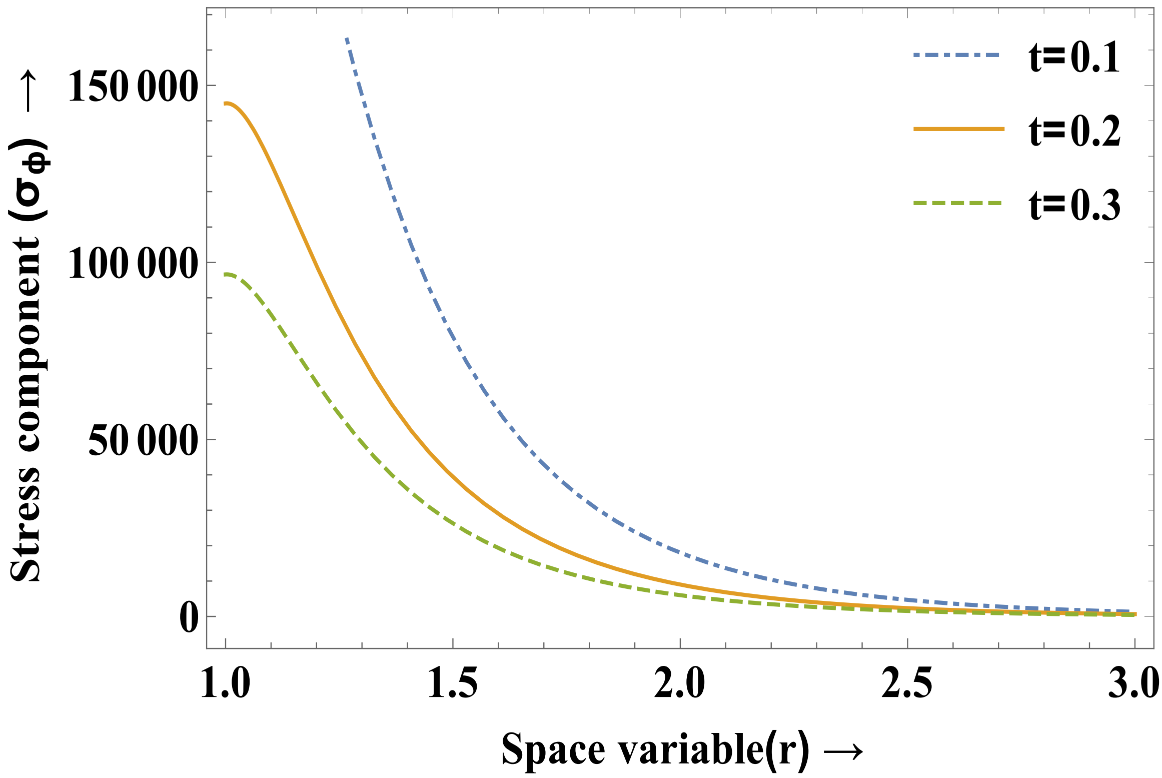


Figure 5 : Distribution of σ_ϕ along radius(r) for different t.

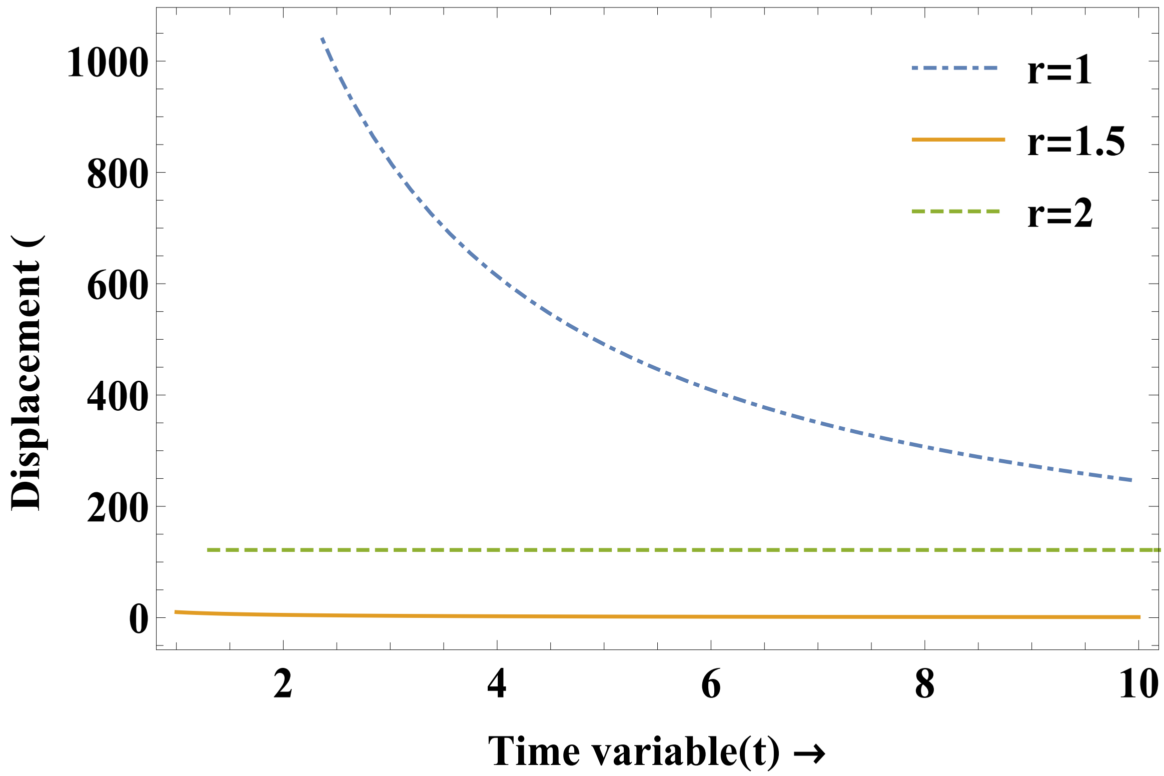


Figure 6: Representation of $displacement(u)$ w.r.to t for different values of r.

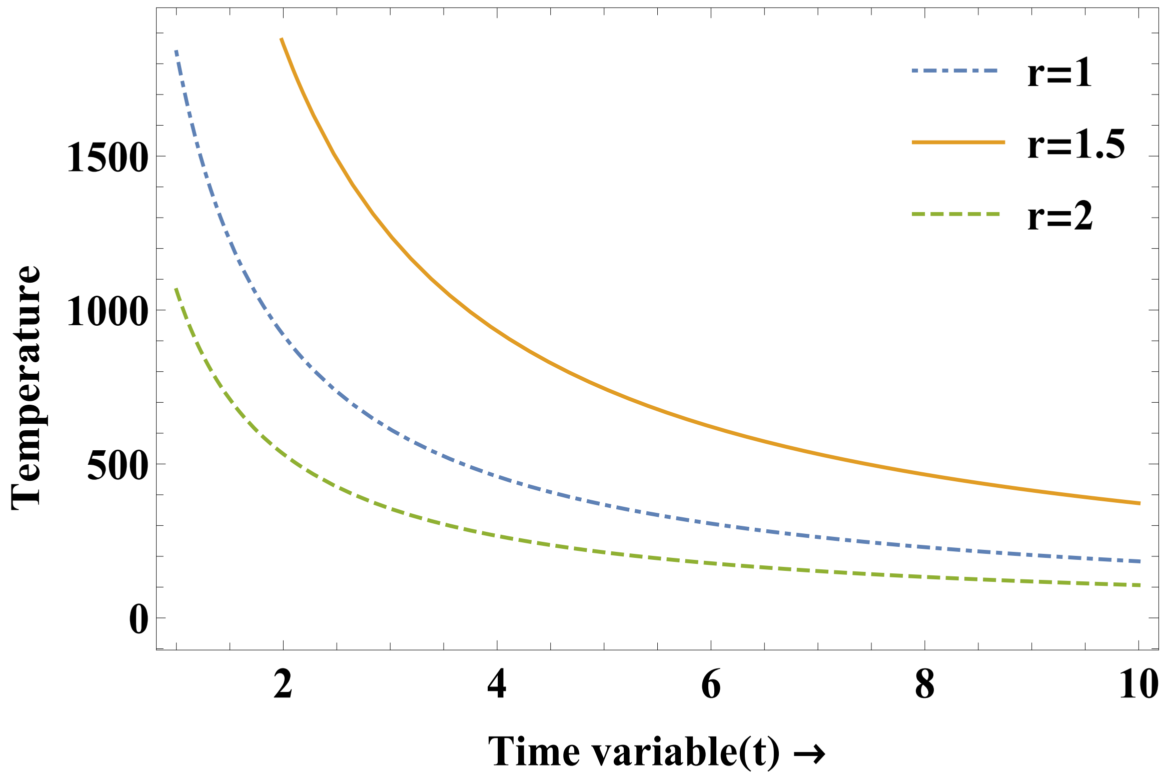


Figure 7: Representation of $temperature(T)$ w.r.to t for different values of r.

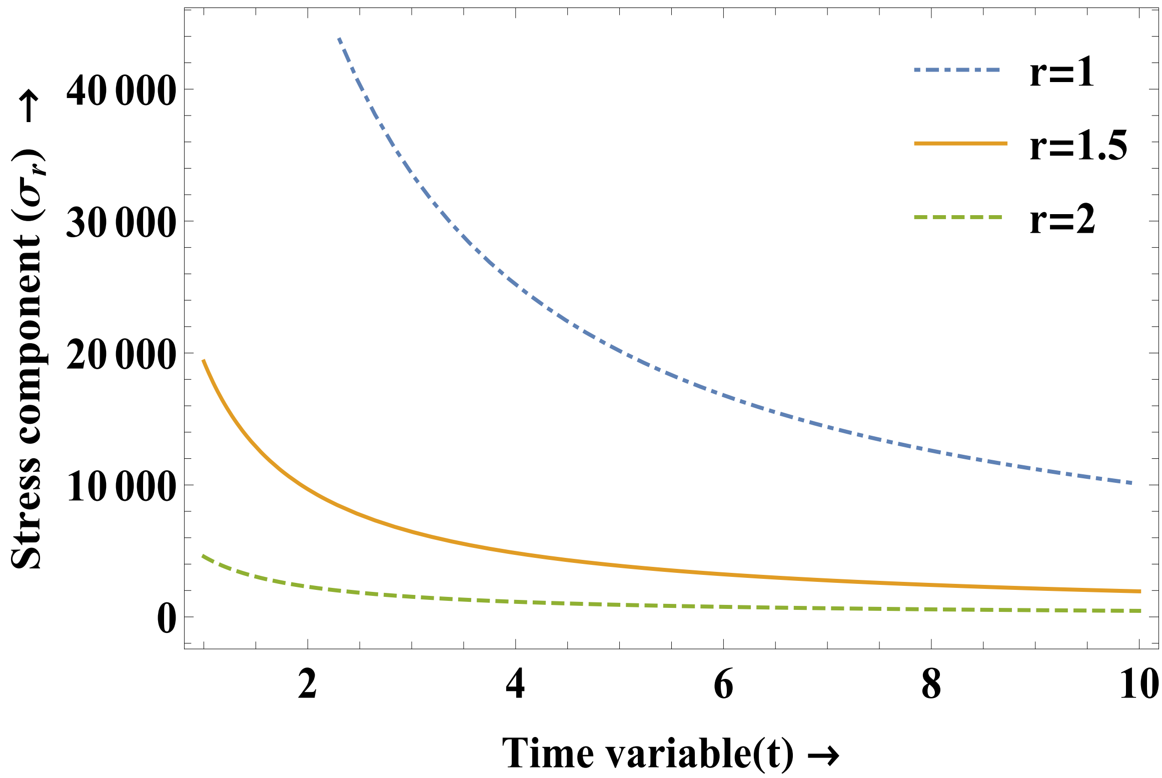


Figure 8: Representation of σ_r w.r.to t for different values of r.

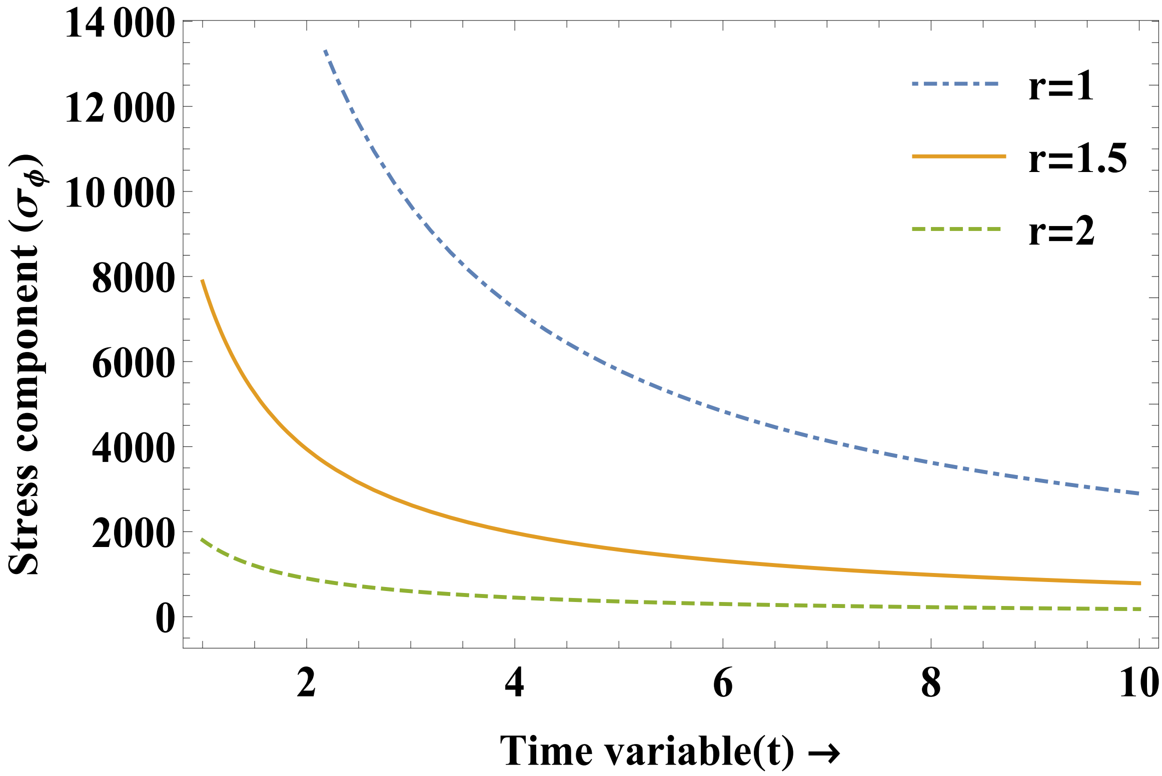


Figure 9: Representation of σ_ϕ w.r.to t for different values of r.

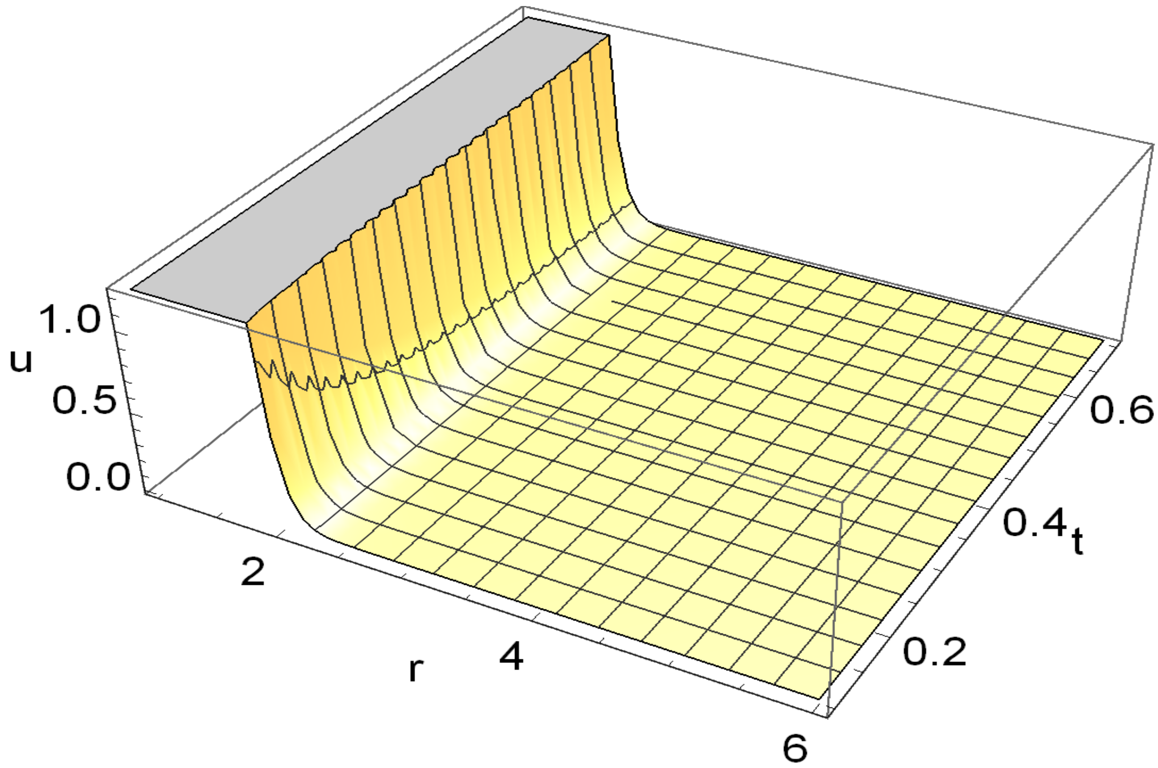


Figure 10: Distribution of $displacement(u)$ w.r.to t and r.

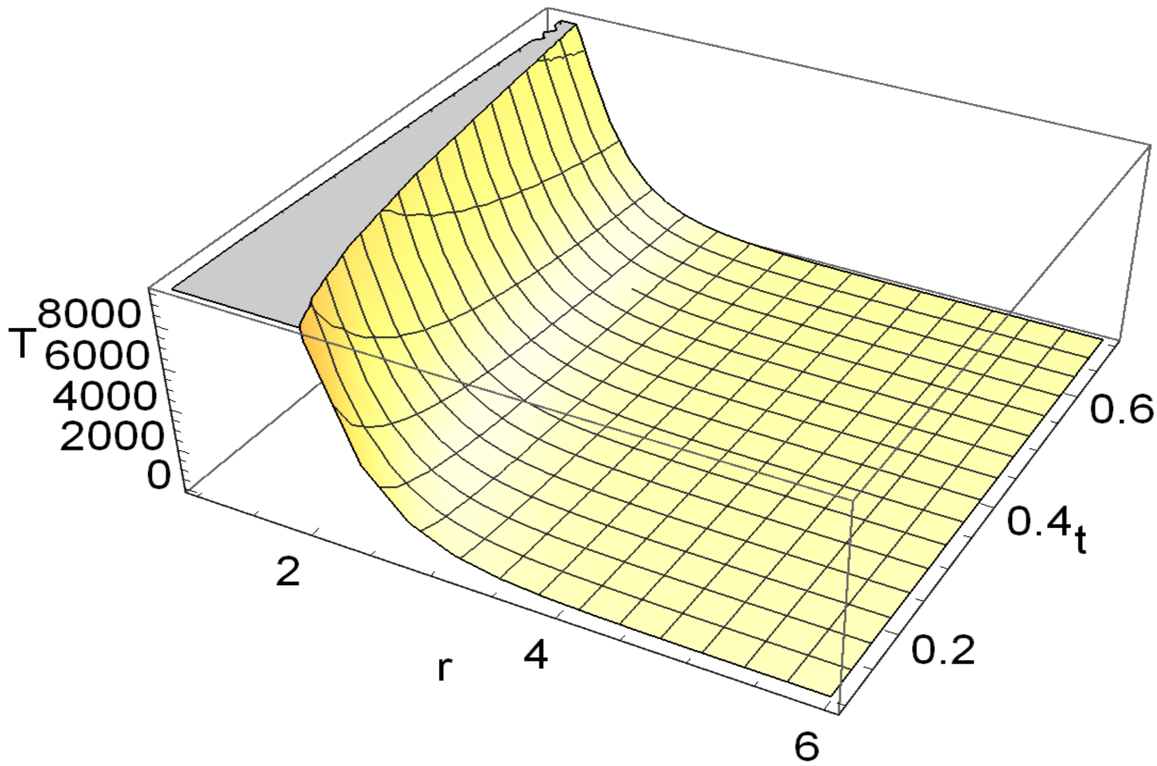


Figure 11: Distribution of $temperature(T)$ w.r.to t and r.

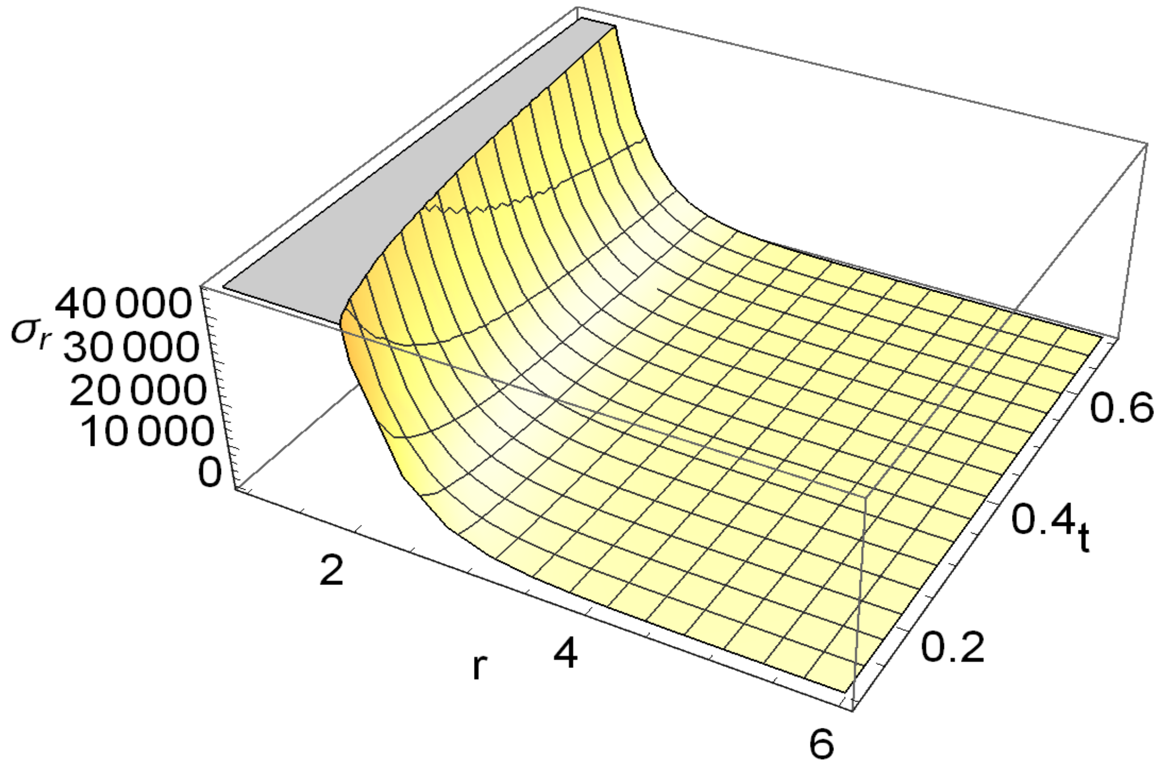


Figure 12: Distribution of σ_r w.r.to t and r.

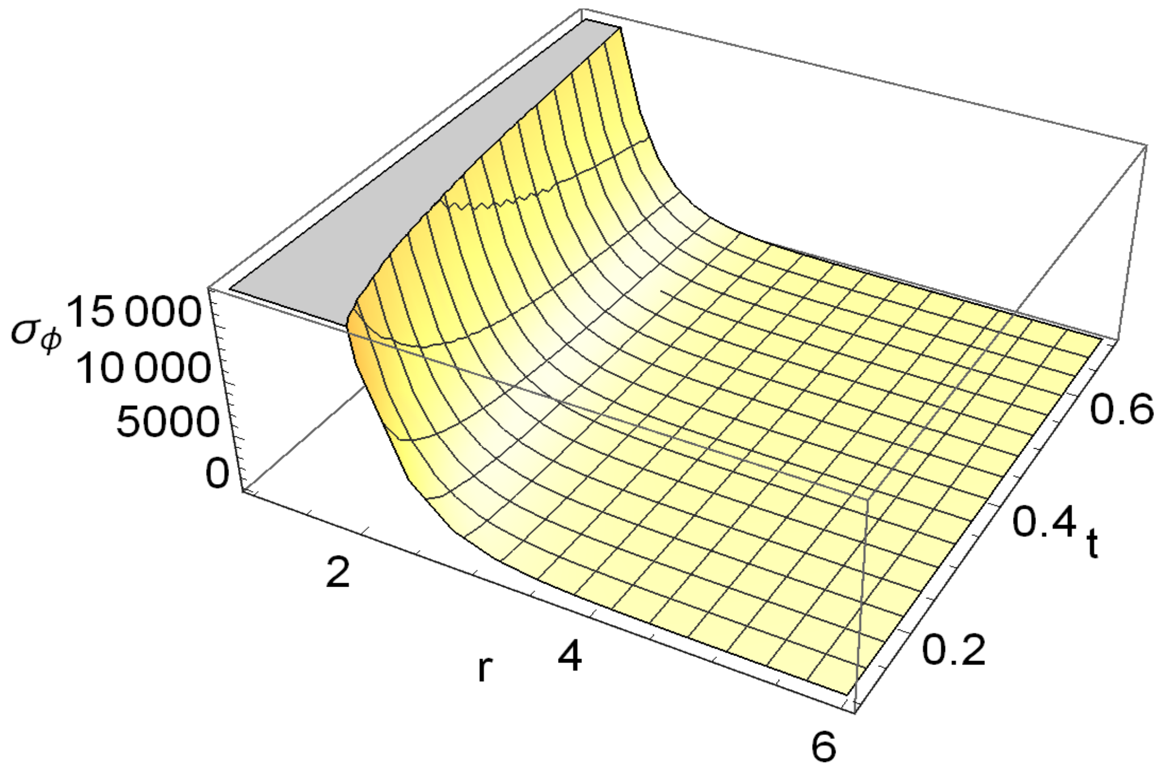


Figure 13: Distribution of σ_ϕ w.r.to t and r.

PHYSICAL VARIABLES	CTE	L-S	G-N-II	G-N-III	SPL	DPL	RPL			
							N=1	N=2	N=3	N=4
u	0.140441	0.0677516	0	0.140441	0.257153	0.00184237	0.0665107	0.0664713	0.066471	0.066471
T	1528.55	1521.42	1449.52	1528.55	1535.35	1497.62	1521.25	1521.243851	1521.243756	1521.243755
σ_r	6568.66	6536.32	6223.8	6568.66	6600.27	6431.89	6535.57	6535.542545	6535.542123	6535.542117
σ_ϕ	2585.77	2573.63	2452.08	2585.77	2597.39	2533.35	2573.35	2573.340338	2573.340179	2573.340176

Figure 14: Data table for non-dimensional numeric values of displacement, temperature and stresses due to different multi-phase lag models.

The data table displays the numeric values of displacement, two stress components(σ_r, σ_ϕ) and temperature due to different phase lag models and different multi phase lag models.

4.1.8 Conclusion

In this work, the multiphase lag concept is studied and verified successfully using the prominent mechanical and thermal boundary conditions associated to governing equations. The two and three dimensional variations of the different stress components, strain components and temperature curves has been represented graphically.

The tabular data in Fig. 14 represents the compact variations of the numerical value of different stress components, temperature and displacement components in context of different thermoelastic models compared to multiphase lag model. From the data table differentiation for the effect of different phase lag models and multi phase lags on different physical variables can be easily achieved.

4.2 Vibrations of a circular cylinder in generalized thermoelasticity

4.2.1 Introduction

The classical uncoupled theory of thermoelasticity has two shortcomings. One of them is the absence of elastic terms in the heat conduction equation. Secondly, the heat conduction equation is parabolic in nature, which indicates infinite speeds of propagation for heat waves. These two phenomena are not compatible with the physical observations.

Biot [30] introduced the theory of coupled thermoelasticity to eliminate the first paradox, though this theory fails to overcome the second shortcoming. Lord and Shulman [80] introduced a generalized theory of coupled thermoelasticity consisting of a wave type-heat conduction equation which ensures finite speeds of propagation for heat waves.

There is another generalization of theory of coupled thermoelasticity, consisting of two relaxation time parameters, which is called the theory of temperature rate dependent thermoelasticity (TRDTE). While reviewing the thermodynamics of thermoelastic solids, Müller [93] put some restrictions on a class of constitutive equations by proposing an entropy production inequality, which is generalized by Green and Laws [57]. Moreover Green and Lindsay [58] established an explicit form of these constitutive equations. Şuhubi [111] was able to obtain these equations independently. Eraby and Şuhubi [45] examined wave propagation in a cylinder. Ignaczak [65], [66] established a decomposition theorem regarding the theory of generalized thermoelasticity and studied a strong discontinuity wave considering relaxation times.

Chandrasekharaiah and Keshavan [34] solved a thermoelastic problem regarding an unbounded solid with cylindrical cavity, whereas a theoretical study was done by Misra et al. [82], [83] to generate stress in elastic and viscoelastic solids containing a spherical and a circular cylindrical hole respectively. Sinha and Elsibai [109] presented thermoelastic interaction in an infinite solid having a spherical hole. Sherief and Saleh [108] were concerned with a one-dimensional problem to figure out thermal stresses and temperature in an unbounded solid with a spherical cavity considering a sudden variation in the temperature. Abd-Alla et al. [6] analyzed the thermoelastic interaction in an orthotropic solid having a spherical cavity. Mukhopadhyay [88] dealt with a thermo-viscoelastic problem in an infinite solid consisting of a spherical cavity under a periodic loading and temperature remaining unchanged. Moreover, Mukhopadhyay [89] studied the thermoealstic behaviour without energy dissipation in the same medium considering a stressed free cavity surface under a thermal shock. Abd-Alla et al. [7] analyzed a viscoelastic medium containing a spherical cavity to figure out the field quantities in the said medium. Mukhopadhyay [90] discussed the thermally induced vibration in an unbounded continuum with a spherical cavity. Rakshit Kundu and Mukhopadhyay [100] evaluated the field quantities while analyzing a viscoelastic medium containing a spherical hole. Youssef [124] studied the thermoelastic behavior of an infinite solid having a cylindrical cavity. Abbas [1] solved a thermoelastic problem in an unbounded solid containing either a spherical or cylindrical hole. Aouadi [22] dealt with a one dimensional issue in a infinite solid with a spherical cavity in context of generalized thermal diffusion. Itu et al. [67] studied composite circular plates through radial ribs to examine improved rigidity.

In this problem, the eigenvalue approach as in Lahiri et al. [75] has been applied to analyze the longtudinal vibrations of an infinite thermoelastic medium containing a circular cylindrical cavity considering the governing equations of generalized thermoelasticity obtained by Lord and Shulman [80].

4.2.2 Formulation of the problem

The fundamental equations for a thermoelastic medium can be written as

$$\begin{aligned} (\lambda+2\mu)\nabla div\ \vec{u} - \mu\ rot\ rot\ \vec{u} - \gamma\nabla\theta &= \rho\frac{\partial^2 u}{\partial t^2} \\ K_1\ div\ \nabla\ \theta - \frac{\gamma T}{\rho c}\frac{\partial}{\partial t}div\ \vec{u} &= \frac{\partial\theta}{\partial t} \end{aligned} \tag{4.34}$$

where λ and μ are the Lame's constant, $\vec{u}$ is the displacement vector, $\gamma = \alpha_T(3\lambda + 2\mu)$, α_T is thermal expansion coefficient, θ is the differential temperature distribution, ρ is the mass density of the medium, $K_1 = \frac{K}{\rho c}$ is the diffusivity, K is the thermal conductivity, c is the specific heat per unit mass at constant strain, T is the reference temperature.

Reducing the equations in (4.34) in cylindrical polar co-ordinates (r, ϕ, z), z being the axis of the cylinder and assuming $u_\phi = 0$ and u_r, u_z and θ are functions of r and

z only, we obtain

$$(\lambda+2\mu)\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+\mu\frac{\partial^2 u_r}{\partial z^2}+(\lambda+\mu)\frac{\partial^2 u_z}{\partial r\partial z}-\gamma\frac{\partial\theta}{\partial r}=\rho\frac{\partial^2 u_r}{\partial t^2} \tag{4.35}$$

$$(\lambda+2\mu)\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}+\frac{\partial^2 u_z}{\partial z^2}\right)-\frac{\mu}{r}\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-\gamma\frac{\partial\theta}{\partial z}-\mu\left(\frac{\partial^2 u_r}{\partial z\partial r}-\frac{\partial^2 u_z}{\partial r^2}\right)=\rho\frac{\partial^2 u_z}{\partial t^2} \tag{4.36}$$

$$K_1\left(\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}\right)-\frac{\gamma T}{\rho c}\frac{\partial}{\partial t}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right)=\frac{\partial\theta}{\partial t} \tag{4.37}$$

The stress components are given by

$$\begin{aligned}\sigma_r&=\lambda\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right)+2\mu\frac{\partial u_r}{\partial r}\\ \tau_{rz}&=\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\\ \tau_{r\theta}&=0\end{aligned} \tag{4.38}$$

4.2.3 Method of Solution: Formulation of a Vector-Matrix Differential Equation

Let us assume that all the quantities are simple harmonic functions of z and t.

$$\begin{aligned}u_r&=U(r)e^{i(qz+pt)}\\ u_z&=iW(r)e^{i(qz+pt)}\\ \theta&=\Theta(r)e^{i(qz+pt)}\end{aligned} \tag{4.39}$$

If we introduce the irrotational velocity $c_1=\sqrt{\frac{\lambda+2\mu}{\rho}}$ and the equivoluminal velocity $c_2=\sqrt{\frac{\mu}{\rho}}$ and the dimensionless quantities $\tau=qr$, $\omega=\frac{P}{qc_1}$, $\beta\omega=\frac{P}{qc_2}$, $\beta=\sqrt{\frac{\lambda+2\mu}{\mu}}$ and a differential operator $L=\frac{d^2}{d\zeta^2}+\frac{1}{\zeta}\frac{d}{d\zeta}-\frac{1}{\zeta^2}$, the longitudinal vibration of a cylinder coupled with a thermal field (4.37) can be written as

$$\begin{aligned}L(U)&=-IU+G\frac{dW}{d\zeta}+H\frac{d\theta}{d\zeta}\\ L\left(\frac{dW}{d\zeta}\right)&=IFU-(E+FG)\frac{dW}{d\zeta}+H\frac{d\theta}{d\zeta}\\ L\left(\frac{d\theta}{d\zeta}\right)&=-NIU+N(G-l)\frac{dW}{d\zeta}+(M+NH)\frac{d\theta}{d\zeta}\end{aligned} \tag{4.40}$$

where

$$I = \frac{l}{\beta^2}(\beta^2\omega^2 - l),\ G = \frac{1}{\beta^2}(\beta^2 - l),\ H = \frac{\gamma}{\rho c_1^2 q}$$
$$E = \beta^2(\omega^2 - l),\ F = \beta^2 - l,\ Q = \frac{\gamma}{\rho c_2^2 q}$$
$$M = l + \frac{p(i - p\tau)}{K_1 q^2}, N = \frac{p\gamma T(i - p\tau)}{K_1 q \rho c}$$

Now the equations in (4.40) can be written in the matrix form as

$$L\bar{v}(\zeta) = \bar{A}\bar{v}(\zeta) \tag{4.41}$$

where $\bar{\lambda} = -\alpha^2$ and $\bar{X}$ is a scalar function of α.

4.2.4 Solution of the problem using Vector-Matrix Differential Equation

Let the eigenvalues of the matrix $\bar{A}$ be λ_1, λ_2 and λ_3. Then the eigenvectors $\bar{X}_j$ corresponding to the eigenvalues λ_1, λ_2 and λ_3 can be calculated as

$$\bar{X}_j = \begin{bmatrix} \frac{1}{\beta^2}(1-\beta^2)\left(\lambda_j - 1 - i\frac{c_1\omega}{K_1 q}\right) + i\omega\Gamma \\ \\ i\omega\Gamma\lambda_j - \left(\lambda_j - 1 - i\frac{c_1\omega}{K_1 q}\right)\left(\lambda_j + \frac{\beta^2\omega^2-1}{\beta^2}\right) \\ \\ i\frac{p\gamma T}{\rho c K_1 q \beta^2}(\lambda_j + \beta^2\omega^2 - 1) \\ \end{bmatrix}_{j=1,2}$$

$$\bar{X}_3 = \begin{bmatrix} \frac{1-\beta^2}{\beta^2}\left(\lambda_3 - 1 - i\frac{c_1\omega}{K_1 q} + i\omega\Gamma\right) \\ \\ i\omega\Gamma\lambda_3 - \left(\lambda_3 - 1 - i\frac{c_1\omega}{K_1 q}\right)\left(\lambda_3 + \frac{\beta^2\omega^2-1}{\beta^2}\right) \\ \\ 0 \\ \end{bmatrix}$$

The solution of Eq. (4.41) can be written as

$$\bar{v}(\zeta) = A\bar{X}_1 J_1(\alpha_1\zeta) + B\bar{X}_2 J_1(\alpha_2\zeta) + C\bar{X}_3 J_1(\alpha_3\zeta) \tag{4.42}$$

where $\alpha_j^2 = -\lambda_j$, $j = 1, 2, 3$ and A, B and C are arbitrary constants. The components of space vector $\bar{v}(\zeta)$ in (4.42) can be written as

$$\begin{aligned} U &= A\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_1^2+1+i\frac{c_1\omega}{K_1q}\right)+i\omega\Gamma\right\}J_1(\alpha_1\zeta) \\ &= B\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_2^2+1+i\frac{c_1\omega}{K_1q}\right)+i\omega\Gamma\right\}J_1(\alpha_2\zeta) \\ &= C\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_3^2+1+i\frac{c_1\omega}{K_1q}\right)+i\omega\Gamma\right\}J_1(\alpha_3\zeta) \end{aligned} \tag{4.43}$$

$$\begin{aligned} \frac{dW}{d\zeta} &= A\left\{-i\omega\Gamma\alpha_1^2-\left(\alpha_1^2+1+i\frac{c_1\omega}{K_1q}\right)\left(\alpha_1^2-\frac{\beta^2\omega^2-1}{\beta^2}\right)\right\}J_1(\alpha_1\zeta) \\ &= B\left\{-i\omega\Gamma\alpha_2^2-\left(\alpha_2^2+1+i\frac{c_1\omega}{K_1q}\right)\left(\alpha_2^2-\frac{\beta^2\omega^2-1}{\beta^2}\right)\right\}J_1(\alpha_2\zeta) \\ &= C\left\{-i\omega\Gamma\alpha_3^2-\left(\alpha_3^2+1+i\frac{c_1\omega}{K_1q}\right)\left(\alpha_3^2-\frac{\beta^2\omega^2-1}{\beta^2}\right)\right\}J_1(\alpha_3\zeta) \end{aligned} \tag{4.44}$$

$$\frac{d\Theta}{d\zeta} = -i\frac{\gamma PT}{\rho cK_1q\beta^2}(\alpha_1^2-\omega^2\beta^2+1)AJ_1(\alpha_1\zeta)-i\frac{\gamma PT}{\rho cK_1q\beta^2}(\alpha_2^2-\beta^2\omega^2+1)BJ_1(\alpha_2J) \tag{4.45}$$

The stress components are given by

$$\begin{aligned} \sigma_r &= AR_{11}+BR_{12}+CR_{13} \\ \sigma_{rz} &= AR_{21}+BR_{22}+CR_{23} \end{aligned} \tag{4.46}$$

where

$$\begin{aligned} &R_{11} \\ &= q\Bigg[\frac{1+\alpha_1^2}{\alpha_1^2}\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_1^2+1+i\frac{c_1\omega}{K_1q}\right)+i\omega\Gamma\right\}\left\{\lambda J_0(\alpha_1\zeta)+\frac{2\mu\alpha_1^2}{\alpha_1^2+1}\left(J_0(\alpha_1\zeta)-\frac{J_1(\alpha_1\zeta)}{\alpha\zeta}\right)\right\} \\ &-\frac{\gamma^2T}{i\omega\Gamma\rho c}\left\{\frac{1+\alpha_1^2}{\alpha_1^2}(\omega^2-i\omega\Gamma-1-\alpha_1^2)\right\}\left\{\frac{\beta^2-1}{\beta^1}\left(\alpha_1^2+1+\frac{ic_1\omega}{K_1q}\right)+i\omega\Gamma\right\}J_0(\alpha_1\zeta)\Bigg]e^{i(qz+pt)} \end{aligned}$$

$$\begin{aligned} &R_{12} \\ &= q\Bigg[\frac{1+\alpha_2^2}{\alpha_2^2}\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_2^2+1+i\frac{c_1\omega}{K_1q}\right)+i\omega\Gamma\right\}\left\{\lambda J_0(\alpha_2\zeta)+\frac{2\mu\alpha_2^2}{\alpha_2^2+1}\left(J_0(\alpha_2\zeta)-\frac{J_1(\alpha_2\zeta)}{\alpha_2\zeta}\right)\right\} \\ &+\frac{\gamma^2T}{i\omega\Gamma\rho c}\left\{\frac{1+\alpha_2^2}{\alpha_2^2}(\omega^2-i\omega\Gamma-\alpha_2^2-1)\right\}\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_2^2+1+i\frac{c_1\omega}{K_1q}\right)+i\omega\Gamma\right\}J_0(\alpha_2\zeta)\Bigg]e^{i(qz+pt)} \end{aligned}$$

$$R_{13} = q\left\{\frac{2\mu\alpha_3}{\alpha_3^2+1}f(-\alpha_3^2)\left(J_0(\alpha_3\zeta) - \frac{J_1(\alpha_3\zeta)}{\alpha_3\zeta}\right)\right\}e^{i(qz+pt)}$$

$$R_{21} = \left[\left\{2i\mu q\frac{\beta^2-1}{\beta^2}\left(\alpha_1^2+1+i\frac{c_1\omega}{K_1 q}\right)+i\omega\Gamma\right\}J_1(\alpha_1\zeta)\right]e^{i(qz+pt)}$$

$$R_{22} = \left[\left\{2i\mu q\frac{\beta^2-1}{\beta^2}\left(\alpha_2^2+1+i\frac{c_1\omega}{K_1 q}\right)+i\omega\Gamma\right\}J_1(\alpha_2\zeta)\right]e^{i(qz+pt)}$$

$$R_{23} = \left\{i\mu q\frac{\alpha_3^2-1}{\alpha_3^2+1}f(-\alpha_3^2)J_1(\alpha_3\zeta)\right\}e^{i(qz+pt)}$$

The temperature is given by

$$\theta = AR_{31} + BR_{32} + CR_{33} \tag{4.47}$$

where

$$R_{31} =$$
$$\left[\frac{q\gamma T}{i\omega\Gamma\rho c}\frac{1+\alpha_1^2}{\alpha_1}(\omega^2 - i\omega\Gamma - 1 - \alpha_1^2)\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_1^2+1+i\frac{c_1\omega}{K_1 q}\right)+i\omega\Gamma\right\}J_0(\alpha_1\zeta)\right]e^{i(qz+pt)}$$

$$R_{32} =$$
$$\left[\frac{q\gamma T}{i\omega\Gamma\rho c}\frac{1+\alpha_2^2}{\alpha_2}(\omega^2 - i\omega\Gamma - 1 - \alpha_2^2)\left\{\frac{\beta^2-1}{\beta^2}\left(\alpha_2^2+1+i\frac{c_1\omega}{K_1 q}\right)+i\omega\Gamma\right\}J_0(\alpha_2\zeta)\right]e^{i(qz+pt)}$$

$R_{33} = 0$

4.2.5 Boundary Conditions

Now we study the thermoelastic interactions of the cavity by considering two cases, viz.

Case 1: Lateral surface of the cylinder kept at ambient temperature.

In this case the boundary conditions are given by

$$\sigma_r = \sigma_{rz} = 0,\ \theta = \theta_0\ at\ r = a\ and\ t = 0 \tag{4.48}$$

Case 2: Surface of the cylinder impervious to heat.

In this case the boundary conditions are given by

$$\sigma_r = \sigma_{rz} = \frac{\partial\theta}{\partial r} = 0, for\phi = qa = u. \tag{4.49}$$

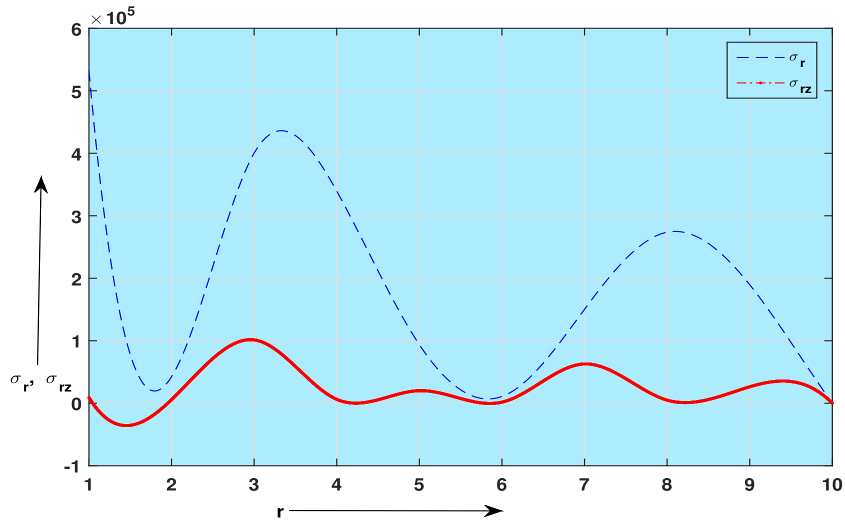
4.2.6 Numerical Analysis

We analyzed the characteristic behaviours of the physical constants with respect to the space variables numerically. The numerical values of the constants for the material aluminium in SI units are as follows

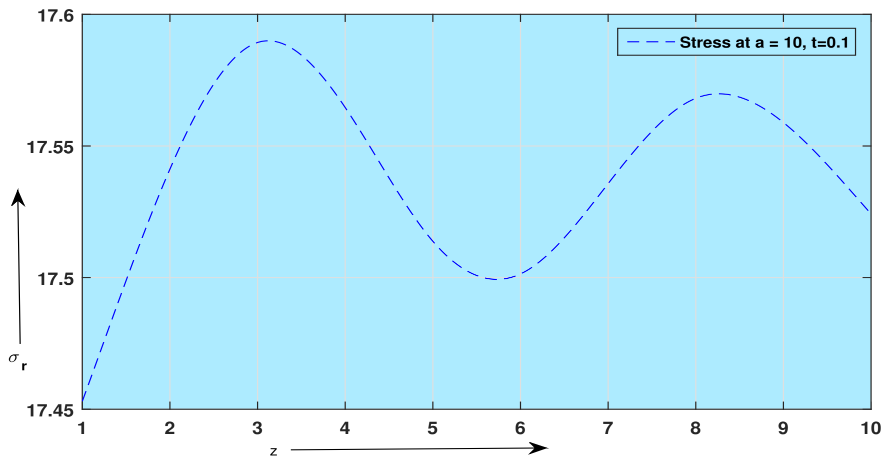
$$\begin{array}{ccc} \lambda = 4.137 \times 10^{10}\, N/m^2, & \mu = 2.75 \times 10^{10} N/m^2, & \rho = 2.7 \times 10^3 kg/m^3, \\ K_1 = 8.418 \times 10^{-5}\, m^2/s, & c = 8.96 \times 10^2\, J/kg°C, & q = 0.1 \times 10^{10}, \\ \alpha_T = 2.4 \times 10^{-5}\, °C, & m = 1.76543 \times 10^{-3}, & \Gamma = 1.6481482 \times 10^{-3}, \\ \alpha_T = 2.4 \times 10^{-5}\, °C, & m = 1.76543 \times 10^{-3}, & \Gamma = 1.6481482 \times 10^{-3}, \\ & \tau = 2.95503 \times 10^{-16} & \end{array}$$

4.2.7 Geometrical Representation

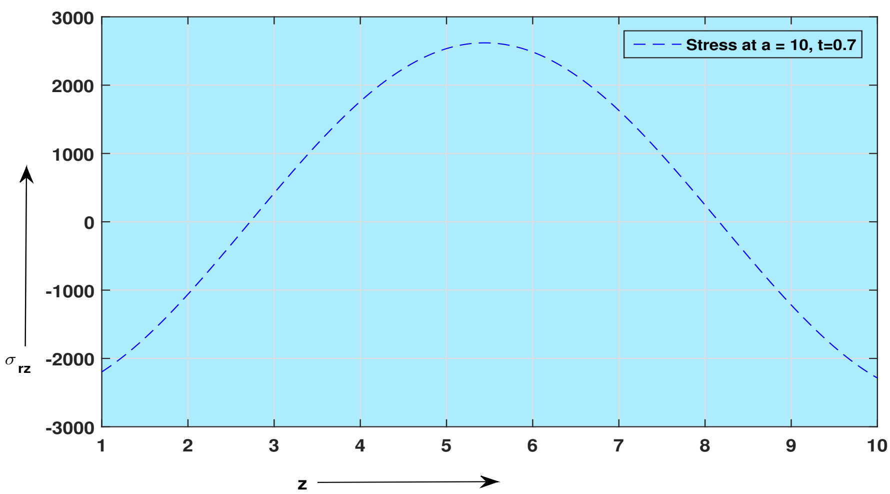
Depending upon the boundary conditions and using above mentioned numerical values, the geometrical representation of different physical variables are provided in two separate cases as follows.



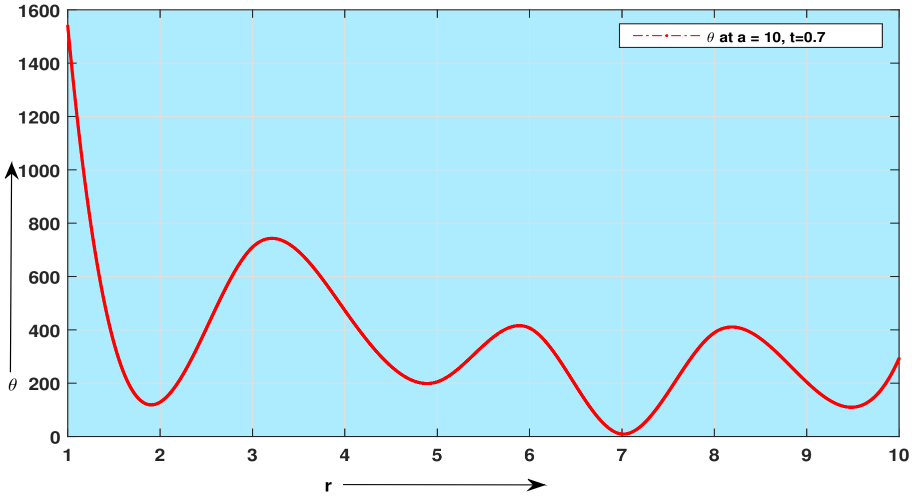
Case1: Fig.1 : Distribution of radial stress(σ_r) and cross-radial stress (σ_{rz}) along radius (r)



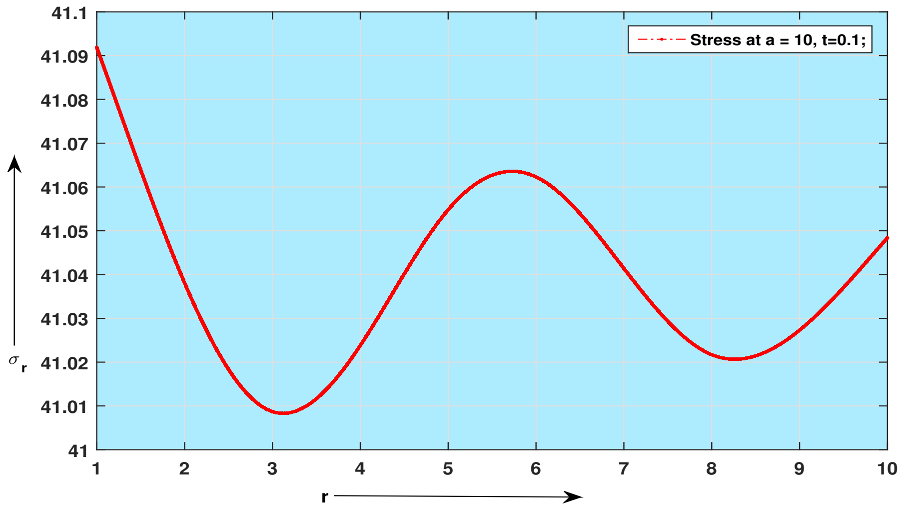
Case1: Fig.2 : Distribution of radial stress (σ_r) along axial coordinate (z)



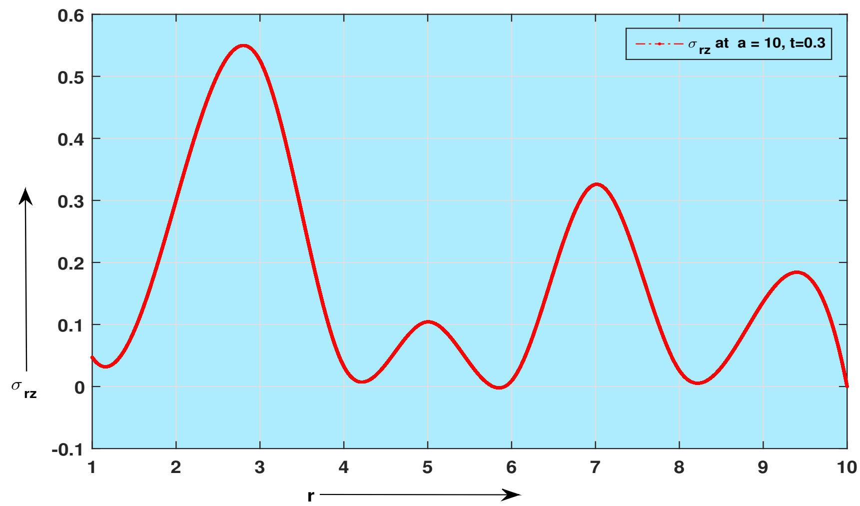
Case1: Fig.3 : Distribution of radial stress (σ_r) along axial coordinate (z)



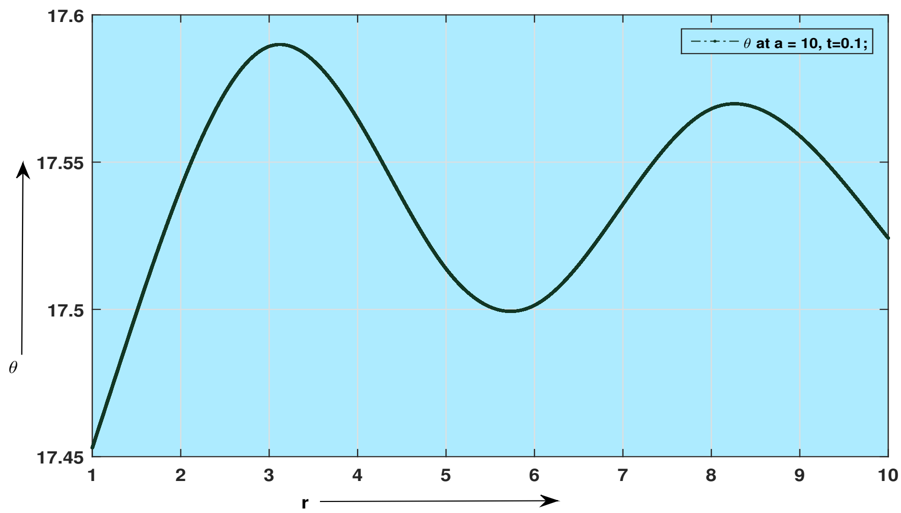
Case1: Fig.4 : Distribution of temperature (θ) along axial coordinate (r)



Case2: Fig.5 : Distribution of radial stress (σ_r) along axial co-ordinate (z)



Case2: Fig.6 : Distribution of cross-radial stress (σ_{rz}) along axial co-ordinate (z)



Case2: Fig.7 : Distribution of temperature (θ) along radius (r)

4.2.8 Conclusion

An analysis of the longitudinal vibration of a circular cylinder has been accomplished considering the generalised thermoelasticity theory proposed by Lord and Shulman. A solution for several field variables and stress components has been obtained solving a vector matrix differential equation through an eigenvalue approach. The said theory has been verified successfully by graphical illustrations. The validity and accuracy of this research work has been assured through the boundary condition provided for two different cases.

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