Study of Relativistic Equations with Position-Dependent Mass

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Study of Relativistic Equations with Position-Dependent Mass**" submitted by **Smt. Sutapa Sur** who got her name registered on 16th May, 2016 (INDEX NO : 111/16/Maths/24) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of **Prof. Dr. Swapna Debnath** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated to
My Father
Late Bipul Chandra Sur

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Preface

This doctoral study is focused on the effects of potential with position-dependent mass at the quantum level for getting further insight into the relativistic quantum mechanical system. To continue this study two relativistic equations are considered here: the Klein-Gordon equation that describes spin-0 particle, for example π meson and the spin$-\frac{1}{2}$ particle due to Dirac that describes an electron. The presentation is interspersed with some of my own research material, where I have felt that it elucidates the presentation. In order to understand and describe processes on the atomic scale (here atomic spectra), I choose some physical potential for relativistic wave equation within the framework of Frobenious Method, Asymptotic Iteration Method, Nikiforov-Uvarov method, Laplace Transform Approach etc..

Effective masses occur in the context of transport phenomena in crystals e.g., semiconductors, where the electrons are not completely free, but interact with the potential of the lattice. To serve the purpose of the study I have considered the potentials like: q-deformed modified Eckart plus Hylleraas potential, generalised asymmetric Manning-Rosen potential, double ring shaped Coulomb potential and Manning-Rosen potential.

To gain a basic understanding of any subject demands a careful study of the underlying mathematical structures without getting trapped in the physically irrelevant mathematical details and technicalities. The need for a comprehensive and readable treatment of basic mathematical notions and

their physical consequences forces us to discuss the mathematical tools and results that are necessary for addressing the conceptual issues of direct relevance to the physical aspects of our study. This has been a decisive factor in the layout of this thesis. The thesis consists of **six chapters**. I give an extensive, but not comprehensive, introduction to the field of relativistic quantum mechanics along with a brief review of the relevant literature are presented in **Chapter 1**.

In **Chapter 2**, relativistic Klein-Gordan equation with position dependent mass has been solved analytically for the q-deformed modified Eckart plus Hylleraas potential. A generalised series is used to obtain the bound state solutions of the K-G equation using the Frobenious Method . The one dimensional K-G equation for the mass dependent modified Eckart plus Hylleraas potential in absence of scalar potential are studied here. The exactly normalized bound state wave function and energy expressions are obtained by using N-U method. Also, the bound state solutions are found for the Hulthén and Rosen-Morse potential.

Chapter 3 deals with the one dimensional Dirac Equation with position dependent mass and the Dirac equation has been solved in terms of the hypergeometric functions for generalised asymmetric Manning-Rosen potential containing different types of physical potential.Considering one dimensional electric current density for the Dirac particle the transmission and the reflection coefficients are obtained.The expression of the energy eigen values is obtained by using continuity conditions of the wave functions.

Chapter 4 considers the double ring shaped Coulomb potential within framework of relativistic Klein-Gordon equation. The bound state solution is obtained for inverse square potential from Radial part in terms of confluent hypergeometric function .Energy eigen value for isotropic harmonic oscillator and ring shaped oscillator with its solution in terms of Gauss hypergeometric function are also obtained from the angular part.

In **Chapter 5**, we have studied the quantum mechanical system within the framework of position-dependent mass for Manning-Rosen potential with the help of Laplace transform method combining with Point Canonical transformation. The general solutions are obtained via Pekeris approximation appropriate for potential analogs to Manning-Rosen potential. The

bound state solutions are obtained in an analytical form.

In **Chapter 6**, we have given some necessary appendices for Special functions connecting to relativistic wave equation and deformed hyperbolic functions which play a crucial role for mathematical understanding of the subatomic world.

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1 Introduction

We are on the verge of a century since the relativistic quantum mechanics started it's journey . The journey began with the attempts to use the correspondence principle in order to derive a relativistic wave equation intended to replace the Schrödinger equation. In 1926, the great Austrian-Irish physicist Erwin Schrödinger himself proposes the relativistic version of his non-relativistic equation (namely, Schrödinger equation)[1]. In the same year (i.e. 1926), two other physicists Oskar Klein and Walter Gordon proposed a revolutionary equation (which is now known as Klein-Gordon equation)[2,3] to describe spin-zero particle where the interaction potential is not so strong to create particle-antiparticle pairs. Although, the Klein-Gordon equation was initially dismissed due to its inability to led to positive probability density.

In 1928, British physicist Paul Dirac proposed a relativistic wave equation named upon him which pertains to particles with $1/2-$spin and is able to describe most of the single particle properties of fermions [4]. This equation also possesses solutions with negative energy similar to Klein-Gordon equation. In 1930, he postulated that the states of negative energy should be occupied to prevent transitions of an electron into lower lying states of energy[5].

Quantum mechanics is the theory that describes the dynamics of matter at the microscopic scale. This is the only valid framework to describe the microphysical world. It is vital for understanding the physics of solids, lasers, semiconductor and superconductor devices, plasmas, etc. In short, quantum mechanics is the founding basis of all modern physics: solid state,

molecular, atomic, nuclear, and particle physics, optics, thermodynamics, statistical mechanics, e.t.c.

Quantum theory works extremely well, and this represents its experimental justification. It has a very penetrating qualitative as well as quantitative prediction power; this prediction power has been verified by a rich collection of experiments. So the accurate prediction power of quantum theory gives irrefutable evidence to the validity of the postulates upon which the theory is built.

The theory of quantum mechanics has become an accepted component of modern science due to a large number of experimental verifications of its theoretical predictions. For a consistent quantum mechanical theory, there need to be restrictions or assumptions. All but one of these assumptions are physical requirements. For example, the energy spectrum is required to be real because all measurements of the energy of a system yield real results. Another axiom requires that the energy spectrum be bounded below so that the system has a stable lowest-energy state. Yet another axiom requires that the time evolution of a quantum system be unitary (probability-conserving) because the expected result of a probability measurement of a state cannot grow or decay in time. A quantum theory of elementary particles must also satisfy the physical axioms of Lorentz covariance and causality. However, there is one axiom that stands out because it is mathematical rather than physical in character, and this is the requirement that the Hamiltonian $\mathcal{H}$, which is the operator that expresses the dynamics of the quantum system, be Hermitian.

In quantum mechanics the standard formalism is based on the requirement that all observable properties of a dynamic nature are associated with the real eigenvalues of a Hermitian Hamiltonian. The motivation of the non-Hermitian quantum mechanics formalism are:

- to be able to address questions that can be answered within this formalism.
- the desire to tackle problems that can also be solved within the conventional Hermitian framework, but only with extreme difficulty, whereas the non-Hermitian quantum mechanics formalism enables a much simpler and more elegant solution.

In the context of quantum mechanics, Hermiticity is required for a Hamiltonian to have a real energy spectrum. The Hermiticity of H is expressed by the equation

$$H = H\dagger \tag{1.1}$$

where the Dirac Hermitian conjugation symbol † represents the combined operations of matrix transposition and complex conjugation. The mathematical symmetry condition (1.1) is physically obscure but very convenient because it implies that the eigenvalues of H are real and that the time-evolution operator e^{-iHt} is unitary. Hamiltonians that are non-Hermitian have traditionally been used to describe dissipative processes, such as the phenomenon of radioactive decay. However, these non-Hermitian Hamiltonians are only approximate, phenomenological descriptions of physical processes. They cannot be regarded as fundamental because they violate the requirement of unitarity. A non-Hermitian Hamiltonian whose purpose is to describe a particle that undergoes radioactive decay predicts that the probability of finding the particle gradually decreases in time. Of course, a particle cannot just disappear because this would violate the conservation of probability; rather, the particle transforms into other particles. Thus, a non-Hermitian Hamiltonian that describes radioactive decay can at best be a simplified, phenomenological, and non-fundamental description of the decay process because it ignores the precise nature of the decay products. In his book on quantum field theory Barton gives the standard reasons for why a non-Hermitian Hamiltonian cannot provide a fundamental description of nature [6]: "A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation." The purpose of this paper is to describe at an elementary level the breakthroughs that have been made in the past decade which show that while the symmetry condition (1.1) is sufficient to guarantee that the energy spectrum is real and that time evolution is unitary, the condition of Dirac Hermiticity is not necessary. It is possible to describe natural processes by means of non-Hermitian Hamiltonians.

To describe the particle dynamics in relativistic quantum mechanics

with some typical potential the Klein-Gordon wave equation is frequently used in literature. The exact solutions of the wave equations (non-relativistic or relativistic) contain all the necessary information regarding the quantum system under consideration and so they are very important. However, analytical solutions are possible only in a few simple cases such as the hydrogen atom and the harmonic oscillator [7]. If we consider the case where the interaction potential is not strong enough to create particle-antiparticle pairs, we can apply the Klein-Gordon equation to the treatment of a zero-spin particle. The near realization of these symmetries may explain degeneracies in some heavy meson spectra (spin symmetry) or in single particle energy levels in nuclei (pseudospin symmetry), when physical systems are described by relativistic theories with scalar and vector potentials [8-12].

Quantum mechanical system with position dependent mass is a very useful model in many applied branches of modern physics, e.g. semiconductor heterostructure [13], Quantum liquids [14], quantum wells and quantum dots [15], ^{3}He clusters [16], compositionally graded crystals [17] etc. In these cases, the wave function actually provides a macroscopic description of the motion of carrier electrons with position dependent mass. Consequently the study of such equation with position dependent mass becomes relevant for deeper understanding on the non-trivial quantum effects observed in the nanostructures. A lot of studies have been performed to obtain the solutions of the Schrodinger, Klein-Gordon and Dirac equations in the presence of variable mass having suitable mass distribution function for different potentials [18-19]. For example, Aygun et al. [20], Jia et al. [21], Antia et al. [22] and Souza Dutra considered position-dependent effective mass [23].

In this study our main concern is to focus on the effects of position-dependent mass at the quantum level for getting further insight into the relativistic quantum mechanical system. The energy spectra and corresponding wave function for Klein-Gordon and Dirac equation for different potentials have been studied by using different methods. We propose to investigate the solutions for Klein-Gordon and Dirac equation by introducing PDM for various potentials.

1.1 Effective-mass Quantum system

The mass distribution of a quantum system, which can be measured as the expectation values of certain observables, has two possible existent forms: it is either real or effective. The distribution is real means that it exists throughout space at the same time. The distribution is effective means that there is only a localized particle with the total mass and charge of the system at every instant, and the time average of its motion during an infinitesimal time interval forms the effective distribution. Moreover, since the integral of the formed mass density in any region is required to be equal to the expectation value of the total mass in the region, the motion of the particle must be ergodic.

For the effective mass distribution, no gravitational self-interactions exists, as there is only a localized particle at every instant. This is consistent with the superposition principle of quantum mechanics. By contrast, if the mass distribution is real, then there will exist gravitational self-interactions of the real distribution, as the distribution exists throughout the space at the same time. The existence of the gravitational self-interactions is inconsistent with the superposition principle of quantum mechanics for real distribution of mass.

In a word, the superposition principle of quantum mechanics requires that the mass distribution of a quantum system is not real but effective; at every instant there is only a localized particle with the total mass of the system, while during an infinitesimal time interval the time average of the ergodic motion of the particle forms the effective mass distribution, and the mass density in each position is proportional to the modulus square of the wave function of the system.

Effective masses occur in the context of transport phenomena in crystals, where the electrons are not completely free, but interact with the potential of the lattice. The quantum dynamics of such electrons can be modeled by an effective mass, the behaviour of which is determined by the band curvature. The effective-mass wave equations are more complicated to be solvable for potentials of physical interest than constant mass cases. The cause behind this is, the effective mass function must be chosen to be

physically meaningful, gives the equation a more complicated form than in the constant mass case.

A quantum mechanical particle endowed with a position-dependent effective mass constitutes an interesting and useful model for the study of many physical problems.The effective mass approximation is a very important method in semiconductor physics to study dynamic and static properties of charge carriers without complexity due to the lattice potential of the material. The effective-mass approach has also been used as a computational method to deal with nonuniform crystals.

To determine the electronic properties of semiconductors effective-mass theory is an important and extensively used tool. The theory is well established for homogeneous materials with small perturbations [24]. The effective-mass theory has also been applied to non-uniform materials in which the carrier effective mass depends on position.

The concept of effective mass also plays an important role within the strictures of the energy density functional approach to the quantum many body problem. The energy density functional formalism has yielded reasonable theoretical predictions of many experimental properties for several quantum many body systems. Within the energy density functional approach, the non-local terms of the associated potential can be often expressed as a position dependence on an appropriate effective mass. Besides its practical applications, the study of quantum mechanical systems with a position dependent mass also raises interesting conceptual problems of a fundamental nature.

1.2 Hamiltonian in quantum mechanics

In quantum mechanics, the Hamiltonian of a system is an operator corresponding to the total energy of that system, including both kinetic energy and potential energy. The mathematical formulation of quantum mechanics is built upon the concept of an operator. Physical pure states in quantum mechanics are represented as unit-norm vectors (probabilities are normalized to one) in a special complex Hilbert space.

Physical observable in quantum mechanics are described by Hermitian

(also, called self-adjoint) operators . An operator $\mathcal{A}$ that is self adjoint has the very reasonable property that its effect on the vectors of the Hilbert space in which it is defined is independent of what vector it acted on first. Using the standard Dirac bra and ket notation we can write this as $\langle\phi|\mathcal{A}\psi\rangle = \langle\mathcal{A}\phi|\psi\rangle$. But the specific matrix properties enforced by self-adjoincy depend on the definition of an inner product used, and there are infinite ways to define an inner product on a vector space. Vectors in the Hilbert space are written in terms of the basis vectors, and if the basis vectors are orthonormal then the inner product is just the standard one: $\langle\phi|\psi\rangle = \sum_i \phi_i^* \psi_i = \phi^\dagger \psi$. So an operator that is Hermitian is self-adjoint with respect to a given inner product rule, and in the case of the standard Hermitian inner product this means the matrix representation of the operator is equal to its complex conjugate transpose. In elementary courses on quantum mechanics one learns that a quantum theory is specified by the Hamiltonian operator that acts on a Hilbert space. The Hamiltonian $\mathcal{H}$ does three things:

- The Hamiltonian determines the energy eigenstates $|E_n\rangle$. These states are the eigenstates of the Hamiltonian operator and they solve the time-independent Schrödinger equation $\mathcal{H}|E_n\rangle = E_n|E_n\rangle$. The energy eigenstates span the Hilbert space of physical state vectors. The eigenvalues En are the energy levels of the quantum theory. In principle, one can observe or measure these energy levels. The outcome of such a physical measurement is a real number, so it is essential that these energy eigenvalues be real.

- The Hamiltonian $\mathcal{H}$ determines the time evolution in the theory. States $|t\rangle$ in the Schrödinger picture evolve in time according to the time-dependent Schrödinger equation $\mathcal{H}|t\rangle = -i\frac{d}{dt}|t\rangle$, whose formal solution is $|t\rangle = e^{i\mathcal{H}t}|0\rangle$. Operators $\mathcal{A}(t)$ in the Heisenberg picture evolve according to the time-dependent Schrödinger equation $\frac{d}{dt}\mathcal{A}(t) = -i[\mathcal{A}(t), \mathcal{H}]$, whose formal solution is $\mathcal{A}(t) = e^{i\mathcal{H}t}\mathcal{A}(0)e^{-i\mathcal{H}t}$.

- The Hamiltonian incorporates the symmetries of the theory. A quantum theory may have two kinds of symmetries: continuous symmetries, such as Lorentz invariance, and discrete symmetries, such as parity invariance and time reversal invariance. A quantum theory is

symmetric under a transformation represented by an operator $\mathcal{A}$ if $\mathcal{A}$ commutes with the Hamiltonian that describes the quantum theory: $[\mathcal{A}, \mathcal{H}] = 0$. It should be Noted that the commutation relation between the symmetry transformation represented by a linear operator $\mathcal{A}$ and the Hamiltonian implies that the eigenstates of $\mathcal{H}$ are also eigenstates of $\mathcal{A}$. Two important discrete symmetry operators are parity (space reflection), which is represented by the symbol $\mathcal{P}$, and time reversal, which is represented by the symbol $\mathcal{T}$. The operators $\mathcal{P}$ and $\mathcal{T}$ are defined by their effects on the dynamical variables $\hat{x}$ (the position operator) and $\hat{p}$ (the momentum operator). The operator $\mathcal{P}$ is linear and has the effect of changing the sign of the momentum operator $\hat{p}$ and the position operator $\hat{x}$: $\hat{p} \longrightarrow -\hat{p}$ and $\hat{x} \longrightarrow -\hat{x}$. The operator $\mathcal{T}$ is anti-linear and has the effect $\hat{p} \longrightarrow -\hat{p}$, $\hat{x} \longrightarrow \hat{x}$, and $i \longrightarrow -i$. Note that $\mathcal{P}$ changes the sign of i because (like $\mathcal{P}$) $\mathcal{T}$ is required to preserve the fundamental commutation relation $[\hat{x}, \hat{p}] = i$ of the dynamical variables in quantum mechanics.

Quantum mechanics is an association between states in a mathematical Hilbert space and experimentally measurable probabilities. The norm of a vector in the Hilbert space must be positive because this norm is a probability and a probability must be real and positive. Furthermore, the inner product between any two different vectors in the Hilbert space must be constant in time because probability is conserved. The requirement that the probability not change with time is called unitarity. Unitarity is a fundamental property of any quantum theory and must not be violated. To summarize the discussion so far, the two crucial properties of any quantum theory are that the energy levels must be real and that the time evolution must be unitary. There is a simple mathematical condition on the Hamiltonian that guarantees the reality of the energy eigenvalues and the unitarity of the time evolution; namely, that the Hamiltonian be real and symmetric.

The Hamiltonian related to variable mass must clearly incorporate the spatial variation of the conduction band edge $V_c(r)$. Morrow et al. [25] considers the general form of Hamiltonians with position dependent mass is

$$H = \frac{1}{2} m^{\alpha} p m^{\beta} p m^{\alpha} + V_c(r)$$

With $2\alpha + \beta = -1$, which constitutes the Hermitian Hamiltonians for inhomogeneous material.

The most general Hamiltonian for effective-mass system, as originally proposed by von Roos [13] is

$$H = -\frac{\hbar^2}{4}[m^{\delta}(r)\nabla m^{k}(r)\nabla m^{\lambda}(r) + m^{\lambda}(r)\nabla m^{k}(r)\nabla m^{\delta}(r)] + V(r)$$

whose classical limit is identical to the first one, and the parameters are constrained by the condition $\delta + k + \lambda = -1$.

In our study, we have used the Hamiltonian for the position dependent mass $m = m(r)$, of the form:

$$\mathcal{H} = \frac{1}{4(a+1)}\left\{ a\Big[\frac{1}{m}\mathbf{P}^2 + \mathbf{P}^2\frac{1}{m}\Big] + m^{\alpha}\mathbf{P}\mathbf{m}^{\beta}\mathbf{P}\mathbf{m}^{\gamma} + m^{\gamma}\mathbf{P}\mathbf{m}^{\beta}\mathbf{P}\mathbf{m}^{\alpha} \right\} + \mathbf{V}(\mathbf{r})$$

where $\mathcal{P}$ denotes the momentum operator and $\mathcal{V}(\nabla)$ is an arbitrary potential. also α, β, γ and a are the ambiguity parameters satisfying the constrain $\alpha + \beta + \gamma = -1$ and r is the radial coordinate.

1.2.1 Hermitian Hamiltonian

In quantum mechanics the standard formalism is based on the requirement that all observable properties of a dynamical system are associated with the real eigenvalues of a Hermitian Hamiltonian. The operator, $\mathcal{H}$ that expresses the dynamics of the quantum system requires to be Hermitian.

In 1945 the great British physicist P.A.M. Dirac done revolutionary work [26] on relativistic quantum mechanics to discover a wave equation that was first order in space and time derivatives and consistent with special relativity. A key assumption made by Dirac was that the corresponding Hamiltonian would be Hermitian. In this way Dirac was led to his celebrated equation which predicted antimatter and describes both electrons and quarks. It should turn out to describe neutrinos as well, the Dirac theory would govern all known fermionic matter in nature. The physical quantum theory must have bounded energy spectrum and acquire a Hilbert space of state vectors which are confer with an inner product having a positive norm and unitary time evolution. The simplest condition on the Hamiltonian $\mathcal{H}$ which guarantees that the quantum theory satisfies these requirements is that the $\mathcal{H}$ be real and symmetric.

For a given Hamiltonian $\mathcal{H}$, one can write down the time-independent wave equation associated with $\mathcal{H}$ and calculate the eigenfunctions $\psi_n(x)$ and eigenvalues E_n. Usually the calculations of eigenvalues and eigenfunctions are performed numerically or analytically. The eigenfunctions of Hermitian Hamiltonian $\mathcal{H}$, will be orthogonal with respect to the standard Hermitian inner product:

$$\langle \psi, \phi \rangle = \int [\psi(x)]^* \phi(x) dx$$

Two eigenfunctions $\psi_m(x)$ and $\phi_n(x)$ associated with different eigenvalues $E_m \neq E_n$ of $\mathcal{H}$ are said to be orthogonal if

$$\langle \psi_m, \phi_n \rangle = 0$$

For Hermitian Hamiltonian $\mathcal{H}$, the norm of any vector is positive and so we can normalize the eigenfunctions for which the norm is unity then

$$\langle \psi_m, \phi_n \rangle = 1$$

For a hermitian Hamiltonian the time evolution operator $e^{-i\mathcal{H}t}$ is unitary and it automatically preserves the inner product

$$\langle \chi(t), \chi(t) \rangle = \langle \chi(0) e^{-i\mathcal{H}t}, e^{-i\mathcal{H}t} \chi(0) \rangle = \langle \chi(0), \chi(0) \rangle$$

It is a theorem in the Hilbert space for linear operators which states that any (finite norm) vector . can be expressed as a linear combination of eigenfunctions of $\mathcal{H}$

$$\chi = \sum_{n=0}^{\infty} a_n \psi_n$$

In other words, we can say that the eigenfunctions of a hermitian Hamiltonian are complete. The formal statement of completeness in co-ordinate space is the reconstruction of the unit operator as a sum over the eigenfunctions

$$\sum_{n=0}^{\infty} [\psi_n(x)]^* \phi_n(y) = \delta(x - y)$$

An observable is represented by a linear hermitian operator. The outcome of a measurement is one of the real eigenvalues of this operator. The other topics such as classical and semiclassical limit of quantum theory, probability and current density for perturbative and non-perturbative calculations can also be considered for hermitian Hamiltonians.

1.2.2 Non-Hermitian Hamiltonian

Quantum mechanical systems are traditionally described by self-adjoint Hamiltonians which fulfil the natural axioms of Quantum mechanics, namely that the possible outcomes of the energy measurements correspond to real eigenvalues of the Hamiltonian, that generates a unitary (probability-conserving) evolution of the system. Still, non-self adjoint (in other words non-Hermitian) operators may be encountered in the course of quantum-mechanical analysis as a technical tool, facilitating calculations, as it happens e.g. in the method of complex scaling often used in molecular physics, or in the calculation of adiabatic transition probabilities . when an open system is described they can also arise in quantum theory as a consequence of using approximative methods, as it is in the case of a radioactive decay, dissipation in semiconductor physics or description of systems with repeated interactions.

The motivation of the non-Hermitian quantum mechanics formalism are:

- to be able to address questions that can be answered within this formalism.
- the desire to tackle problems that can also be solved within the conventional Hermitian framework, but only with extreme difficulty, whereas the non-Hermitian quantum mechanics formalism enables a much simpler and more elegant solution.

Moreover it provides the insight that is required to predict novel physical phenomena and to design the corresponding experiments.

By non-Hermitian operators we mean those operators which are not self-adjoint and consequently do not necessarily have real spectra. Non-Hermitian Hamiltonians appear frequently in the study of quantum systems and are usually interpreted as effective Hamiltonians associated with dissipative models when they posses complex spectra. However, also non-Hermitian Hamiltonians whose spectra were believed to be real have emerged sporadically in the literature. This illustrates that whereas Hermitian operators must have real eigenvalues, so that complex characteristic values can only appear in non-Hermitian systems, the latter can also present real spectra. In other words, by restricting their investigations to self-adjoint operators physicists miss out potentially significant setups. From this ob-

servation rises the interest in determining under what conditions will non-Hermitian Hamiltonians generate real eigenvalues.

Non-Hermitian Hamiltonians appear most frequently in the physics literature to describe dissipative systems, regarded as a result of an effective description. This picture arises when one has a complicated physical configuration and judges that it is sensible to isolate a handful of degrees of freedom from the totality. The outlined approach represents an initial step to have a tractable configuration formed only by the elements of interest as opposed to the larger number, possibly infinite, composing the complete setup. However, both subsystems, the smaller depicted structure and its complement, are usually inherently coupled so that there will occur exchanges between them. Therefore from the perspective of one of the subsystems one will find either loss or gain of particle number, energy or any other physical quantity. Dissipation thus emanates in a natural way and is an indication that the model used cannot be considered fundamental as a complementary description becomes necessary for taking the remaining environment into account.

In classical mechanics it is long understood the universality of dissipation, as a consequence of microscopic phenomena of increasing disorder. When the system is not in thermodynamical equilibrium and energy and matter can be exchanged between the system, considered open, and the environment, dissipation leads to heating. When extending to quantum theory, ideas concerning dissipative systems appear promptly in the study of open and closed channels in atomic physics.

Dissipation is a common feature in realistic problems not only at the classical level but also at the quantum level. Metastable states, those in equilibrium but susceptible to fall into lower-energy states with only slight interaction. Thus, dissipative phenomena constitute a very important aspect of non-Hermitian quantum systems and these concepts can hardly be dissociated. But the physical description of systems by Hamiltonians which are not self-adjoint go beyond the scope of open systems. Non-Hermitian Hamiltonians can be used as well to formulate a consistent conservative theory, with real energy and unitary evolution.

It has been understood recently that the reality of the spectra can be explained in terms of a concept known as unbroken $\mathcal{PT}$-symmetry , whose

name stems from the physical operation of simultaneous space and time reflection. In the occasion when these ideas were proposed a whole class of non-Hermitian Hamiltonians with real spectra, generalizing in a way the harmonic oscillator, were identified and analyzed. The simultaneous invariance of the Hamiltonian and its eigenfunctions under parity and time reversal transformations is a specific example of an anti-linear symmetry for which spectral properties had already been established in a generic manner . $\mathcal{PT}$-symmetry however was conjectured to be a reasonable candidate to substitute the Hermiticity postulate of quantum mechanics and despite being a more appealing formulation of the quantum theory, it has become more evident that it is not fundamental. However, in practical terms one is usually not in a position to know all eigenfunctions and eigenvalues for a Hamiltonian and therefore $\mathcal{PT}$ -symmetry furnishes a convenient mechanism to single out possibly relevant models just by examining the form of the Hamiltonian.

A broader approach to the microscopic world must instead deal with the use of various metrics characterizing different Hilbert spaces. The inherent freedom of choosing the metric in the quantum formalism may be used to redefine isospectral observable partners which are Hermitian with respect to a nontrivial metric [27]. This opens up the possibility that a non-Hermitian set of observables might be regarded as Hermitian with respect to a new metric. Given a Hamiltonian which is not Hermitian, the construction of the associated metric will guarantee not only the reality of its spectrum but also the existence of a consistent quantum framework in which one has unitary time evolution. This last property, essential in order to maintain the probabilistic interpretation of quantum mechanics since it assures conservation of probability, cannot be established simply from the reality of the spectrum. Such notions are the cornerstone of the framework known to some as Quasi-Hermiticity and to others as Pseudo-Hermiticity.

Many interesting non-Hermitian systems have been proposed, most of them theoretically but also experimentally. It still remains somewhat unclear how setups of this kind can be best employed in a laboratory although initial attempts have been made. Perhaps the most controversial consists of the quantum brachistochrone problem for non-Hermitian systems, investigated initially in [28]. Not surprisingly it is also one of the most exciting

problems in the area. It consists of determining under which conditions the evolution of a system between two pre-defined states occurs in the least amount of time. By introducing $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians as the generators of the time evolution it was observed that in principle transitions faster than in Hermitian quantum mechanics could take place. Shortly after it was shown that such phenomenon could also happen for non-$\mathcal{PT}$-symmetric systems, even for dissipative ones. The key feature of this peculiar behaviour is that one uses the eigenstates of an equivalent Hermitian Hamiltonian as the initial and/or final states regarding the non-Hermitian evolution. As all interesting problems, it generated not only answers but also more fundamental questions, such as the intriguing possibility of mixing Hermitian and non-Hermitian frameworks.

The non-Hermitian operators are traditionally employed in the effective description of physical systems displaying decay or dissipative behavior . The main quality of non-Hermitian operators that motivated these applications is that a generic non-Hermitian operator has complex eigenvalues whose imaginary part may be associated with decay rates. Although, there is a class of non-Hermitian operators that, similar to Hermitian operators, have a real spectrum.

Research into non-Hermitian Hamiltonians follows two paths. The first involves trying to find a physically reasonable alternative to Hermitian quantum mechanics. The second uses non-Hermitian Hamiltonians to formulate new computational schemes in the hope of making problems more tractable. The value of a non-Hermitian Hamiltonian comes from the fact that if it provides a physically reasonable and consistent theory, then it will enrich the number of solvable systems available to physicists. Bender and associates developed an approach to study non-Hermitian Hamiltonians which considers space-time reflection symmetry (called, $\mathcal{PT}$-symmetry)[29-32].

According to C.M.Bender and associates:

- the $\mathcal{PT}$-symmetric and the $\mathcal{CPT}$-symmetric together form a physically reasonable alternative to Hermitian quantum mechanics.
- all Hermitian Hamiltonians are $\mathcal{PT}$-symmetric.
- $\mathcal{PT}$-symmetric Hamiltonians can be used to define real energy eigen-

values for a quantum system. In addition, unitary time evolution can be defined if the $\mathcal{CPT}$-inner product is used to define a Hilbert space.

A more recent attempt at generalizing Quantum Mechanics is due to Bender and his collaborators who adopted all its axioms except the one that restricted the Hamiltonian to be hermitian. They replaced the latter condition with the requirement that the Hamiltonian must have an exact $\mathcal{PT}$-symmetric which in particular assured the reality of its spectrum. Bender and associate in [33] and R.M. Singh [34-36] also suggested that,$\mathcal{PT}$-symmetric version of a non-hermitian Hamiltonian possesses real eigenvalue even if concerned potentials possess complex parameters.

In recent years, special attention started to be paid to non-Hermitian Hamiltonian in the area of the so-called $\mathcal{PT}$-symmetric quantum mechanics. Motivated by the numerical observation of purely real spectrum of an imaginary cubic oscillator Hamiltonian it blossomed into a large and rapidly developing field. It is said that Hamiltonian $\mathcal{H}$ is $\mathcal{PT}$-symmetric, when it commutes with the operator $\mathcal{PT}$, i.e.

$$[\mathcal{H}; \mathcal{PT}] = 0;$$

in operator sense, where the operator $\mathcal{PT}$ stands for simultaneous spatial reflection $\mathcal{P}$ and time reversal $\mathcal{T}$. Physical relevance of $\mathcal{PT}$-symmetrical models have been suggested in electromagnetism , nuclear physics , optics , scattering , solid state physics or in superconductivity. The above condition that the Hamiltonian is $\mathcal{PT}-$ symmetric is a physical condition because $\mathcal{P}$ and $\mathcal{T}$ are elements of the homogeneous Lorentz group of spatial rotations and Lorentz boosts. The real Lorentz group consists of four parts [37]:

- The first part, called the proper orthochronous Lorentz group, is a subgroup of the Lorentz group whose elements are continuously connected to the identity.
- The second part consists of all of the elements of the proper orthochronous Lorentz group multiplied by the parity operator $\mathcal{P}$.
- The third part consists of all of the elements of the proper orthochronous Lorentz group multiplied by the time-reversal operator $\mathcal{T}$.
- The fourth part consists of all of the elements of the proper orthochronous Lorentz group multiplied by $\mathcal{PT}$.

Note that parts 2nd to 4th are not subgroups of the Lorentz group because they do not contain the identity element. These four parts of the Lorentz group are disconnected because there is no continuous path in group space from one part to another. The Lorentz invariance is a physical requirement of a theory that the theory must be invariant under Lorentz transformations belonging to the proper, orthochronous Lorentz group. We know that the physical world is not invariant under the full homogeneous Lorentz group because it has been demonstrated experimentally that there exist weak processes that do not respect parity symmetry and other weak processes that do not respect time-reversal symmetry. The real Lorentz group can be extended to the complex Lorentz group with a crucial assumption that the eigenvalues of the Hamiltonian are real and bounded below. The complex Lorentz group consists of two and not four disconnected parts. In the complex Lorentz group there exists a continuous path in group space from the elements of the real proper, orthochronous Lorentz group to the elements of 4th part of the real Lorentz group. There also exists a continuous path in group space from the elements of 2nd part to the elements of 3rd part of the real Lorentz group. The most important consequence of the discovery that non-Hermitian $\mathcal{PT}-$symmetric Hamiltonians can define acceptable theories of quantum mechanics is that we now can construct many new kinds of Hamiltonians that only a decade ago would have been rejected as being unphysical because they violate the axiom of Hermiticity. It is commonly accepted that the physical relevance of $\mathcal{PT}$-symmetric Hamiltonian holds only in case when they are similar to self-adjoint operators. This is closely related to the so-called quasi-Hermiticity

$$\Theta\mathcal{H} = \mathcal{H}^{*}\Theta;$$

where Θ is positive operator often called metric operator. The central idea is that a non-Hermitian but quasi-Hermitian operator can be taken as self-adjoint with respect to the modified scalar product $(.;\Theta.)$.

From a mathematical point of view, it is challenging to study non-Hermitian operators, since many powerful techniques usable in the self-adjoint case are not available, among others the spectral theorem and the Min-max principle. One often has to seek new techniques of investigation of these Hamiltonians, generalize the classic techniques and often resort to

perturbation methods.

1.2.3 Pseudo-Hermitian Hamiltonian

A quantum theory of elementary particles must also satisfy the physical axioms of Lorentz covariance and causality. However, there is one axiom that stands out because it is mathematical rather than physical in character, and this is the requirement that the Hamiltonian $\mathcal{H}$, which is the operator that expresses the dynamics of the quantum system, be Hermitian.

In search of the alternative to Hermitian Hamiltonian, the theory of non-Hermitian Hamiltonian is developed to provide a physically reasonable and consistent theory in the hope of making problems more tractable. An alternative approach was developed by Mostafazadeh [38] who works with pseudo-Hermitian Hamiltonians which have real and positive eigenvalues.

Mostafazadeh claimed that it is more practical to use the pseudo-Hermitian framework, rather than working in the $\mathcal{PT}$-symmetric framework directly. The fact that a $\mathcal{PT}$-symmetric theory can be constructed in the pseudo-Hermitian framework adds weight to Mostafazdeh's claim that pseudo-Hermitian quantum mechanics is the more general of the two. Mostafazadeh introduced an additional theorem which states: "if H is a non-Hermitian Hamiltonian with a discrete spectrum and has a complete biorthonormal eigen basis, then H is pseudo-Hermitian if and only if one of the following conditions hold":

- The spectrum of $\mathcal{H}$ is real.
- The complex eigenvalues come in complex conjugate pairs and the multiplicity of the eigenvalue pairs is the same.

It has also been shown that all pseudo-Hermitian Hamiltonians satisfy the above points.

It is claimed that the necessary and sufficient condition for the reality of energy spectrum of any Hamiltonian is that the Hamiltonian admits a complete set of bi-orthonormal eigenvectors. A Hamiltonian $\mathcal{H}$ is pseudo-Hermitian if it obeys the similarity transformation

$$\eta\mathcal{H}\eta^{-1} = \mathcal{H}\dagger$$

where η is a Hermitian invertible linear operator. Mostafazadeh has pointed out that all the $\mathcal{PT}$-symmetric Hamiltonians regarded so far are actually $\mathcal{P}$-pseudo Hermitian, namely

$$\mathcal{PHP}^{-1} = \mathcal{H}\dagger$$

Furthermore, by highlighting the concept of pseudo-Hermiticity he addressed that pseudo-Hermitian is a generalization of Hermiticity. This theory explains many unusual features of $\mathcal{PT}$-symmetric quantum mechanics, and also implies the redefinition of the orthogonality condition as

$$(E_i^* - E_j) \int_{-\infty}^{\infty} \psi_i^*(x)\eta\psi_j(x)dx = 0 \tag{1.2}$$

This construction contains both conventional Hermiticity ($\eta = 1$) and $\mathcal{PT}$-symmetry ($\eta = \mathcal{P}$). For the conventional Hermiticity the above equation becomes

$$(E_i^* - E_j) \int_{-\infty}^{\infty} \psi_i^*(x)\psi_j(x)dx = 0 \tag{1.3}$$

and for $\eta = \mathcal{P}$ the equation becomes

$$(E_i^* - E_j) \int_{-\infty}^{\infty} \psi_i^{\mathcal{PT}}(x)\psi_j(x)dx = 0 \tag{1.4}$$

1.3 Symmetry in quantum mechanics

Symmetries in quantum mechanics describe features of space-time and particles which are unchanged under some transformation, in the context of quantum mechanics and with applications in the mathematical formulation of the standard model and condensed matter physics. Symmetries in physics are fundamentally important constraints to formulate physical theories and models. For mathematical formulation of a system or deeper insight into the physical nature of a problem, symmetries in the system plays a crucial role. They lead to the characteristic patterns in the energy spectrum of any physical system. Symmetries are useful in physical sciences to describe crystal structures or to classify fundamental particles since with the determination of the symmetry properties of a system many of its features may be extracted. A very important theorem, due to Coleman and Mandula [39], states that the only conserved quantities in a physical theory must

be invariant under Lorentz transformations, i.e. those space-time transformations relating the coordinates of objects in different reference frames according to Special Relativity. As a consequence, as long as some requirements are fulfilled, realistic theories can only involve symmetries which do not mix internal symmetry groups, like spin, with *Poincaŕe* group symmetries, which are composed of translations, rotations and Lorentz transformations. The Coleman-Mandula theorem therefore correspond to important restrictions in nature.A surprising combination of space-time with internal symmetries can be achieved by super-symmetry, e.g. [40], where symmetries are not only formulated in terms of commutation relations but also with anti-commutation relations. One of their interesting properties is that if you apply successive symmetries to an object the composed action will still be a symmetry, allowing them to considered to form a group. Actually, the branch of science responsible for providing a general fabric to study symmetries is known as group theory.

The Standard Model of particle physics has three related natural near-symmetries. These state that the universe in which we live should be indistinguishable from one where a certain type of change is introduced.

- $\mathcal{C}$*-symmetry*: In physics, charge conjugation is a transformation that switches all particles with their corresponding antiparticles, thus changing the sign of all charges: not only electric charge but also the charges relevant to other forces. The term $\mathcal{C}$-symmetry is an abbreviation of the phrase "charge conjugation symmetry", and is used in discussions of the symmetry of physical laws under charge-conjugation.
- $\mathcal{P}$*-symmetry*: In quantum mechanics, a parity transformation is the flip in the sign of one spatial coordinate.It can also be thought of as a test for chirality of a physical phenomenon, in that a parity inversion transforms a phenomenon into its mirror image. All fundamental interactions of elementary particles, with the exception of the weak interaction, are symmetric under parity. The weak interaction is chiral and thus provides a means for probing chirality in physics. In interactions that are symmetric under parity, such as electromagnetism in atomic and molecular physics, parity serves as a powerful controlling principle underlying quantum transitions.

- $\mathcal{T}$*-symmetry*: this symmetry mandates that the laws of physics affecting the interactions of particles behave the exact same ways whether you run the clock forwards or backwards in time.

These symmetries are near-symmetries because each is broken in the present-day universe. But all of the fundamental interactions, every single one, always obeys the combination of all three of these symmetries: $\mathcal{CPT}$-symmetry. It states that any physical system made of particles that moves forwards in time will obey the same laws as the identical physical system made of antiparticles, reflected in a mirror, that moves backwards in time. It should hold for all physical phenomena, even ones we have yet to discover. In the following subsections, we kept the knowledge about some symmetries related to quantum mechanical system.

$\mathcal{PT}$-symmetry

$\mathcal{PT}$-symmetric quantum mechanical systems are invariant under the simultaneous action of the $\mathcal{P}$ space and $\mathcal{T}$ time inversion operations. These systems possess non-Hermitian Hamiltonians, still they have some characteristics similar to Hermitian problems. The most notable of these is their discrete energy spectrum, which can be partly or completely real. Typically the transition from the fully real energy spectrum to the complex one occurs when the non-Hermitian component of the Hamiltonian exceeds a certain critical limit, and it can be interpreted as the spontaneous breakdown of $\mathcal{PT}$-symmetry in that the energy eigenstates cease to be eigenstates of the $\mathcal{PT}$ operator then.

Another typical feature $\mathcal{PT}$-symmetric systems have in common with Hermitian problems is that their basis states form an orthogonal set provided that the inner product is redefined as $< \Psi|\Phi > \mathcal{PT} =< \Psi|\mathcal{P}\Phi >$. Similarly to the Hermitian setting, the so-called pseudo-norm defined by this inner product is conserved, however, a major difference is that it turned out to possess indefinite sign, and this raised the question of the probabilistic interpretation of $\mathcal{PT}$-symmetric systems. $\mathcal{PT}$-symmetry also manifests itself in scattering aspects in that the reflection coefficient exhibits handedness.

It is possible to describe natural processes by means of non-Hermitian

Hamiltonians i.e. the Hermiticity requirement (1.1) may be replaced by the analogous but physically transparent condition of space-time reflection symmetry ($\mathcal{PT}$-symmetry)

$$\mathcal{H} = \mathcal{H}^{\mathcal{PT}} \tag{1.5}$$

without violating any of the physical axioms of quantum mechanics. If $\mathcal{H}$ satisfies (1.5), it is said to be $\mathcal{PT}$-symmetric. The notation used in this context is as follows: The space-reflection operator, or parity operator, is represented by the symbol $\mathcal{P}$. The effect of $\mathcal{P}$ on the quantum-mechanical coordinate operator $\hat{x}$and the momentum operator $\hat{p}$ is to change their signs:

$$\mathcal{P}\hat{x}\mathcal{P} = -\hat{x} \qquad and \qquad \mathcal{P}\hat{p}\mathcal{P} = -\hat{p}. \tag{1.6}$$

Note that $\mathcal{P}$ is a linear operator and that it leaves invariant the fundamental commutation relation (the Heisenberg algebra) of quantum mechanics,

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\mathcal{I}, \tag{1.7}$$

where $\mathcal{I}$ is the identity matrix. The time-reversal operator is represented by the symbol $\mathcal{T}$. This operator leaves $\hat{x}$ invariant but changes the sign of $\hat{p}$:

$$\mathcal{T}\hat{x}\mathcal{T} = \hat{x} \qquad and \qquad \mathcal{T}\hat{p}\mathcal{T} = -\hat{p}. \tag{1.8}$$

Like the parity operator $\mathcal{P}$, the time-reversal operator $\mathcal{T}$ leaves the commutation relation (1.7) invariant, but this requires that T reverse the sign of the complex number i:

$$\mathcal{T}i\mathcal{T} = -i. \tag{1.9}$$

Equation (1.9) demonstrates that $\mathcal{T}$ is not a linear operator; $\mathcal{T}$ is said to be antilinear. Also, since $\mathcal{P}$ and $\mathcal{T}$ are reflection operators, their squares are the unit operator:

$$\mathcal{P}^2 = \mathcal{T}^2 = 1. \tag{1.10}$$

Finally, the $\mathcal{P}$ and $\mathcal{T}$ operators commute:

$$\mathcal{PT} - \mathcal{TP} = 0. \tag{1.11}$$

In terms of the $\mathcal{P}$ and $\mathcal{T}$ operators, we define the $\mathcal{PT}$ -reflected Hamiltonian $\mathcal{H}^{\mathcal{PT}}$ in (1.5) as $\mathcal{H}^{\mathcal{PT}} = (\mathcal{PT})\mathcal{H}(\mathcal{PT})$. Thus, if a Hamiltonian is $\mathcal{PT}$ symmetric [that is, if it satisfies (1.5)], then the $\mathcal{PT}$ operator commutes with $\mathcal{H}$:

$$\mathcal{H}(\mathcal{PT}) - (\mathcal{PT})\mathcal{H} = 0. \tag{1.12}$$

A $\mathcal{PT}$-symmetric Hamiltonian need not be Hermitian; that is, it need not satisfy the Hermiticity symmetry condition (1.1). Thus, it is possible to have a fully consistent quantum theory whose dynamics is described by a non-Hermitian Hamiltonian. Some examples of such non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians are

$$\mathcal{H} = \hat{p}^2 + i\hat{x}^3 \tag{1.13}$$

and

$$\mathcal{H} = \hat{p}^2 - \hat{x}^4 \tag{1.14}$$

It is amazing indeed that the eigenvalues of these strange-looking Hamiltonians are all real and positive and that these two Hamiltonians specify a unitary time evolution even though they are non-Hermitian. The Hamiltonians in (1.13) and (1.14) are special cases of the general parametric family of $\mathcal{PT}$-symmetric Hamiltonians

$$\mathcal{H} = \hat{p}^2 + \hat{x}^2(i\hat{x})^\epsilon \tag{1.15}$$

where the parameter ϵ is real. These Hamiltonians are all $\mathcal{PT}$-symmetric because they satisfy the condition in (1.5). It was shown in 1998 that when $\epsilon \geq 0$ all of the eigenvalues of these Hamiltonians are entirely real and positive, but when $\epsilon < 0$ there are complex eigenvalues [32]. We say that

$\epsilon \geq 0$ is the parametric region of unbroken $\mathcal{PT}$-symmetry and that $\epsilon < 0$ is the parametric region of broken $\mathcal{PT}$ symmetry.

Perhaps the simplest $\mathcal{PT}$-symmetric Hamiltonian contains a one-dimensional Klein-Gordon operator with a complex potential satisfying the $\mathcal{V} * (-x) = \mathcal{V}(x)$ relation. A number of such problems have been described by numerical and perturbational techniques, but the exact analytical solution of several potentials have also been given.

$\mathcal{PT}$-symmetry was put into a more general context when it was found that it is a special case of pseudo-hermiticity, and this explains most of the peculiar features of $\mathcal{PT}$-symmetric systems. It was shown that $\mathcal{PT}$-symmetric, and in general, pseudo-hermitian systems can be mapped into equivalent Hermitian ones, although this mapping is technically not straightforward in general. A recent significant result was the experimental verification of the existence of $\mathcal{PT}$-symmetric systems which also exhibit the spontaneous breakdown of $\mathcal{PT}$-symmetry .

Our first results concerning $\mathcal{PT}$-symmetric potentials was the systematic exploration of conditions under which shape-invariant potentials possess real and complex energy spectrum. These studies revealed that the $\mathcal{PT}$-symmetric Coulomb potential cannot be defined on the real x axis, rather one has to define an integration path in the complex x plane . This result also raised several further questions concerning the definition of the $\mathcal{PT}$-symmetric Coulomb potential, which have been settled only recently both for bound and scattering states.

Based on the results for one-dimensional exactly solvable $\mathcal{PT}$-symmetric potentials, we also discussed such potentials in two and three spatial dimensions, which can be factorized into one-dimensional problems by means of separating the radial and angular variables.

An alternative formalism of quantum mechanics in which the mathematical axiom of the hermiticity (1.1) is replaced by the physical transparent condition of space-time reflection symmetry ($\mathcal{PT}$-symmetry) i.e.

$$\mathcal{H} = \mathcal{H}^{\mathcal{PT}} \quad and \quad \mathcal{H}^{\mathcal{PT}} = (\mathcal{PT})\mathcal{H}(\mathcal{PT})$$

The Hamiltonians described by the equation (1.15) with a real parameter ϵ are all $\mathcal{PT}$-symmetric because they satisfy the condition $\mathcal{H} = \mathcal{H}^{\mathcal{PT}}$. It was shown by Bender et.al in 1998 [32] that when $\epsilon \geq 0$ all the eigenvalues

of these Hamiltonians are entirely real and positive, but when $\epsilon < 0$ there are complex eigenvalues. From this Bender concluded that $\epsilon \geq 0$ is the parametric region of the unbroken $\mathcal{PT}$-symmetry and that $\epsilon < 0$ is the parametric region of the broken $\mathcal{PT}$-symmetry. No general condition has been found for the breakdown of $\mathcal{PT}$-symmetry, but it has been observed that it usually characterizes strongly the non-Hermitian problems.

The $\mathcal{PT}$ must be 'unbroken' in the sense that it should be possible to find eigenvectors of the Hamiltonian that are invariant under $\mathcal{PT}$. This is crucial as it ensures the eigenvalues of $\mathcal{H}$ are real; since this is a subtlety arising from the prominent role of parity and time-reversal in $\mathcal{PT}$ quantum mechanics. If $\mathcal{PT}$ is unbroken then the eigenvalues of $\mathcal{H}$ must all be real, and the converse, if the eigenvalues $\mathcal{H}$ are real then $\mathcal{PT}$ is unbroken. If $\mathcal{H}$ and $\mathcal{A}$ commute and are linear operators then there exists a set of simultaneous eigenvectors. However since $\mathcal{PT}$ is not a linear operator we have no reason to believe that $\mathcal{H}$ and $\mathcal{PT}$ should have simultaneous eigenvectors.

$\mathcal{CPT}$-symmetry

In an attempt at devising a probabilistic interpretation for quantum systems with Hamiltonians having exact $\mathcal{PT}$-symmetry, Bender and his collaborators have introduced a generic symmetry of these Hamiltonians that they term as the 'charge-conjugation' symmetry [29]. Using the generator $\mathcal{C}$ of this symmetry, they were able to introduce a positive-definite inner product that they called the $\mathcal{CPT}$-inner product.

The description of the 'charge conjugation' operator $\mathcal{C}$ of the $\mathcal{PT}$-symmetric QM [29] provided by the theory of pseudo-Hermitian Hamiltonians and the fact that general pseudo-Hermitian Hamiltonians have generic antilinear symmetries [41-43] raise the natural question whether one could associate to a general pseudo-Hermitian Hamiltonian a linear symmetry generator C and an antilinear symmetry generator $\mathcal{PT}$ that would respectively generalize $\mathcal{C}$ and $\mathcal{PT}$. This question is answered in [44]. It turns out that for a given diagonalizable pseudo-Hermitian Hamiltonian $\mathcal{H}$ one may introduce generalized parity ($\mathcal{P}$), time-reversal ($\mathcal{T}$) and charge-conjugation ($\mathcal{C}$) operators and establish the $\mathcal{PT}$-, $\mathcal{C}$-, and $\mathcal{CPT}$-symmetries of $\mathcal{H}$, [44]. It must be noted that the use of the term 'charge-conjugation' in the above

discussions solely rests on the fact that similarly to the ordinary charge-conjugation operator of relativistic QM, $\mathcal{C}$ is a Hermitian involution, i.e., $\mathcal{C}^\dagger = \mathcal{C} = \mathcal{C}^{-1}$. It is important to note that because $\mathcal{C}$ is a linear operator, it is actually a $\mathbb{Z}_2$-grading operator for the Hilbert space [45].

$\mathcal{CPT}$ is the only combination of $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ that is observed to be an exact symmetry of nature at the fundamental level. The $\mathcal{CPT}$ theorem says that $\mathcal{CPT}$-symmetry holds for all physical phenomena. The $\mathcal{CPT}$-symmetry implies that a "mirror-image" of our universe; with all objects having their positions reflected through an arbitrary point (corresponding to a parity inversion), all momenta reversed (corresponding to a time inversion) and with all matter replaced by antimatter (corresponding to a charge inversion) would evolve under exactly our physical laws. The $\mathcal{CPT}$ transformation turns our universe into its "mirror image" and vice versa. It is recognized to be a fundamental property of physical laws.

In order to preserve this symmetry, every violation of the combined symmetry of two of its components must have a corresponding violation in the third component; in fact, mathematically, these are the same thing.

In physics, we have to be willing to challenge our assumptions, and to probe all possibilities, no matter how unlikely they seem. But our default should be that the laws of physics that have stood up to every experimental test, that compose a self-consistent theoretical framework, and that accurately describe our reality, are indeed correct until proven otherwise. In this case, it means that the laws of physics are the same everywhere and for all observers until proven otherwise.

Sometimes, particles may behave differently than antiparticles, physical systems may behave differently than their mirror-image reflections and physical systems behave differently depending on whether the clock runs forwards or backwards. But, as a consequence of the $\mathcal{CPT}$ theorem, particles moving forwards in time must behave the same as antiparticles reflected in a mirror moving backwards in time. That's the one symmetry, as long as the physical laws that we know of are correct, that must always hold.

1.4 Relativistic Wave Equation

Efforts to formulate a relativistic quantum mechanics began with attempts to use the correspondence principle in order to derive a relativistic wave equation intended to replace the *Schrödinger* equation. The first such equation was due to *Schrödinger*, Gordon (1926), and Klein (1927) . This scalar wave equation of second order, which is now known as the Klein-Gordon equation, was initially dismissed, since it led to negative probability densities. The year 1928 saw the publication of the Dirac equation. This equation pertains to particles with spin-1/2 and is able to describe many of the single-particle properties of fermions. The Dirac equation, like the Klein-Gordon equation, possesses solutions with negative energy, which, in the framework of wave mechanics, leads to difficulties.

To prevent transitions of an electron into lower lying states of negative energy, in 1930 Dirac postulated that the states of negative energy should all be occupied. Missing particles in these otherwise occupied states represent particles with opposite charge (antiparticles). This necessarily leads to a many-particle theory, or to a quantum field theory. By reinterpreting the Klein-Gordon equation as the basis of a field theory, Pauli and Weisskopf showed that this could describe mesons with spin zero, e.g., p mesons. The field theories based upon the Dirac and Klein-Gordon equations correspond to the Maxwell equations for the electromagnetic field, and the d'Alembert equation for the four-potential. The *Schrödinger* equation, as well as the other axioms of quantum theory, remain unchanged. Only the Hamiltonian is changed and now represents a quantized field. The elementary particles are excitations of the fields (mesons, electrons, photons, etc.).

It will be instructive to now follow the historical development rather than begin immediately with quantum field theory. For one thing, it is conceptually easier to investigate the properties of the Dirac equation in its interpretation as a single-particle wave equation. Furthermore, it is exactly these single-particle solutions that are needed as basis states for expanding the field operators. At low energies one can neglect decay processes and thus, here, the quantum field theory gives the same physical predictions as the elementary single-particle theory.

Schrödinger and others soon recognized that the source of discrepancy between the relativistic wave equation and observations was the neglect of the spin of the electron (the Klein-Gordon equation describes spin 0 particles) and the journey of relativistic wave equation commenced. Now the one-dimensional K-G equation takes the beautiful form for a spinless particle of rest mass $\mathcal{M}$ in the natural units ($\hbar = c = 1$) as

$$\Psi''(x) + [(E_n - \mathcal{V}(x))^2 - (\mathcal{M} + \mathcal{S}(x))^2]\Psi(x) = 0 \tag{1.16}$$

where n,E_n,$\mathcal{V}(x)$ and $\mathcal{S}(x)$ are the quantum number, relativistic energy of the particle , vector and scaler potentials respectively.

And the Dirac equation of a nucleon with mass $\mathcal{M}$ moving in moving in an attractive scalar potential $\mathcal{S}(r)$ and a repulsive vector potential $\mathcal{V}(r)$ for spin-$\frac{1}{2}$ particles in the relativistic unit ($\hbar = c = 1$) is

$$[\alpha.p + \beta(\mathcal{M} + \mathcal{S}(r))]\psi(r) = [E - \mathcal{V}(r)]\psi(r) \tag{1.17}$$

where E is the relativistic energy of the system, $p = -i\nabla$ is the three dimensional momentum operator and M is the mass of the fermionic particle. α, β are the 4×4 Dirac matrices given as

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{1.18}$$

where I is a 2×2 unit matrix and σ_i are the Pauli three-vector matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.19}$$

Another form of the Dirac equation for relativistic free-particle (in natural units $\hbar$ =c=1) is

$$[i\gamma^\mu \partial_\mu - m(x)]\psi(x) = 0 \tag{1.20}$$

where $m(x)$ is the Dirac particle mass, depends on one spatially coordinate x. To obtain the one-dimensional Dirac equation for the external potential $\mathcal{V}(x)$ we consider the gamma matrices γ_x and $\gamma_0\gamma$ in terms of Pauli matrices $i\sigma(x)$ and $i\sigma(z)$ respectively,

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} - [E - \mathcal{V}(x)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + m(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \times \begin{pmatrix} U_1(x) \\ U_2(x) \end{pmatrix} = 0 \tag{1.21}$$

where $U_1(x)$ and $U_2(x)$ are decomposed into upper and lower components of the two-component wave function $\psi(x)$.

1.4.1 Solvability

In quantum mechanics, the state of a system is characterized by a wave function Ψ which is the solution of the concern wave equation i.e. the second order linear partial differential equation of the form:

$$\mathcal{H}_{op}\Psi = E\Psi \tag{1.22}$$

where, $\mathcal{H}_{op}$ (may be Hermitian or, non-Hermitian) is a second order linear differential operator and E is the eigenvalue of the partial differential equation. In general, the form of the operator includes a term that depends on the corresponding interaction potential and the solutions to the equation of the system are associated with the set of energy eigenvalues.

In some models imply that the amount of conserved quantities equals the degrees of freedom in the system roughly and some models refer to a situation in which the spectra can be determined explicitly. Some models whose spectral properties, i.e., the eigenvalues and eigenfunctions of the Hamiltonian characterizing the quantum system under consideration, can be given in an explicit and closed form are denoted by $exactly \quad solvable$ models. The concept of solvability has also been extended to $quasi-exactly \quad solvable$ and $conditionally \quad exactly \quad solvable$ models. Quantum mechanical Hamiltonians are said to be $quasi-exactly \quad solvable$ if a finite portion of the energy spectrum and associated eigenfunctions can be found exactly and in closed form. And if the Hamiltonians have exact solutions only under certain conditions within the potential parameters are of $conditionally \quad exactly \quad solvable$ models.

1.4.2 Normalization of wave function

The wave function Ψ can be interpreted as the probability amplitude of the particle's position, i.e. $|\Psi|^2$ is the probability density. For this to be possible, Ψ must be square-integrable and all such square-integrable wave functions are called normalizable. And, the probability that a particle will

be found somewhere in a region is unity. That implies, for a normalized wavefunction

$$\int |\Psi|^2 d\tau = 1 \tag{1.23}$$

taking the integral over the entire region.The above equation gives the condition of normalization.

The normalization is accomplished by adjusting the numerical coefficient of Ψ so that the integral is unity. Normalization does not alter the fact that Ψ is a solution of the wave equation. If the region of space of a wave function is arbitrarily large and finite then the normalization condition converges when integrated over the finite volume of the region. This suggests that normalization is always possible. The coefficient of Ψ that makes normalization possible must be time invariant so that Ψ may satisfy the wave equation. Therefore, if normalization condition is satisfied at one instant of time, it should always be satisfied.

1.5 A brief review of literature

Effective-mass theory is an important and extensively used tool for the determination of electronic properties of semiconductors. The theory is well established for homogeneous materials with small perturbations. With the recent interest in superlattices and quantum wells, effective-mass theory has also been applied to non-uniform materials in which the carrier effective mass depends on position. Effective masses occur in the context of transport phenomena in crystals (e.g., semiconductors), where the electrons are not completely free, but interact with the potential of the lattice. The quantum dynamics of such electrons can be modeled by employing a position-dependent (effective) mass, the behaviour of which is determined by the band curvature. The Hamiltonian dynamics systems have been played an important role not only in modern physics, but also in mathematics, mechanics, engineering science, and social sciences, especially in nonlinear science, celestial mechanics, and spacecraft attitude dynamics. But traditional Hamiltonian systems theory is defined in even dimensionality space where good characters have on the structure, so also limit its

application. The principle of symmetry is a higher level of law in physics, and conserved quantities of the dynamical systems can better reveal the profound physical laws. The conserved quantity of a physical system has a close relation with its symmetry. The motivation for obtaining exact solutions of the wave equation with position dependent mass comes from the wide range of applications of these solutions in various areas of material science and condensed matter.

1.5.1 Methods involved in our study

There have been many mathematical methods developed in the past to obtain energy eigenvalues and eigenfunctions. In literature, there exist various way to study the quantum mechanical system with position-dependent mass for relativistic wave equations. These methods are mainly used to obtain the eigenvalues and the corresponding eigenfunctions for the explanation of phenomena. The methods followed to this study are given below:

Nikiforov-Uvarov Method

The N-U method [46-55] is based on solving a second order linear differential equation by reducing it to a generalized hypergeometric type. In both relativistic and non-relativistic quantum mechanics, the wave equation with a given potential can be solved by this method by reducing the one dimensional K-G equation to an equation of the form :

$$\Psi''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\Psi'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\Psi(x) = 0 \tag{1.24}$$

Where $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials of degree atmost 2 and $\tilde{\tau}(x)$ is a polynomial of degree atmost 1 . In order to find a particular solution to equation(1.24) , we set the following wave function as a multiple of two independent parts

$$\Psi(x) = \Phi(x)y(x) \tag{1.25}$$

Thus equation (1.24) reduces to a hyper-geometric type equation of the form :

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$$

Where $\tau(x) = \tilde{\tau}(x) + 2\pi(x)$ satisfies the condition $\tau'(x) < 0$ and $\pi(x)$ is defined as

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{(\frac{\sigma'(x) - \tilde{\tau}(x)}{2})^2 - \tilde{\sigma}(x) + K\sigma(x)} \qquad (1.26)$$

in which K is a parameter . Determining K is the essential point in calculation of $\pi(x)$. Since $\pi(x)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (1.26) can be put into order to be the square of a polynomial of first degree [46], which is possible only if its discriminant is zero. So, we obtain K by setting the discriminant of the square root equal to zero . Therefore, one gets a general quadratic equation for K . By using

$$\lambda = K + \pi'(x) = -n\tau'(x) - \frac{n(n-1)}{2}\sigma''(x) \qquad (1.27)$$

The values of K can used for the calculation of energy eigenvalues . Polynomial solutions $y_n(x)$ are given by the Rodrigues relation

$$y_n(x) = \frac{B_n}{\rho(x)}(\frac{d}{dx})^n[\sigma^n(x)\rho(x)] \qquad (1.28)$$

in which B_n is a normalization constant and $\rho(x)$ is the weight function satisfying

$$\rho(x) = \frac{1}{\sigma(x)} exp \int \frac{\tau(x)}{\sigma(x)} dx \qquad (1.29)$$

on the other hand , second part of the wave function $\phi(x)$ in relation (1.25) is given by

$$\phi(x) = exp \int \frac{\pi(x)}{\sigma(x)} dx \qquad (1.30)$$

Frobenius method

In this section we have worked on a method of obtaining one solution of the linear, second-order,homogeneous ODE. The method [56,57,98], a series expansion, will always work, provided the point of expansion is no worse than a regular singular point. In physics this very gentle condition is almost always satisfied.

In mathematics, the method of Frobenius, named after Ferdinand Georg Frobenius, is a way to find an infinite series solution for a second-order ordinary differential equation of the form

$$x^2 f'' + p(x) x f' + q(x) f = 0 \tag{1.31}$$

with $f' \equiv \frac{df}{dx}$ and $f'' \equiv \frac{d^2 f}{dx^2}$ in the vicinity of the regular singular point $x = 0$. One can divide by x^2 to obtain a differential equation of the form

$$f'' + \frac{p(x)}{x} f' + \frac{q(x)}{x^2} f = 0 \tag{1.32}$$

which will not be solvable with regular power series methods if either $p(x)/x$ or $q(x)/x^2$ are not analytic at $x = 0$. The Frobenius method enables one to create a power series solution to such a differential equation, provided that $p(x)$ and $q(x)$ are themselves analytic at 0 or, being analytic elsewhere, both their limits at 0 exist (and are finite).

The method of Frobenius is to seek a power series solution of the form

$$f(x) = x^r \sum_{k=0}^{\infty} A_k x^k, \qquad (A_0 \neq 0) \tag{1.33}$$

Applying 1st and 2nd order differentiation in equation (1.33):

$$f'(x) = \sum_{k=0}^{\infty} (k+r) A_k x^{k+r-1} \quad \textit{and} \quad f''(x) = \sum_{k=0}^{\infty} (k+r-1)(k+r) A_k x^{k+r-2} \tag{1.34}$$

Substituting equation (1.33) and (1.34) into (1.31), one can obtain

$$[r(r-1) + p(x)r + q(x)] A_0 x^r +$$

$$\sum_{k=1}^{\infty} [(k+r-1)(k+r) + p(x)(k+r) + q(x)] A_k x^{k+r} = 0 \tag{1.35}$$

The expression $r(r-1) + p(0)r + q(0) = I(r)$ is known as the indicial polynomial, which is quadratic in r. The general definition of the indicial polynomial is the coefficient of the lowest power of x in the infinite series. In this case it happens to be that this is the rth coefficient but, it is possible for the lowest possible exponent to be $r-2$, $r-1$ or, something else depending on the given differential equation. This detail is important to keep in mind. In the process of synchronizing all the series of the differential equation to

start at the same index value (which in the above expression is $k = 1$), one can end up with complicated expressions. However, in solving for the indicial roots attention is focused only on the coefficient of the lowest power of x.

Using this, the general expression of the coefficient of x^{k+r} is

$$I(k+r)A_k + \sum_{j=0}^{k-1} \frac{(j+r)p^{k-j}(0) + q^{k-j}(0)}{(k-j)!} A_j,$$

These coefficients must be zero, since they should be solutions of the differential equation, so

$$I(k+r)A_k + \sum_{j=0}^{k-1} \frac{(j+r)p^{k-j}(0) + q^{k-j}(0)}{(k-j)!} A_j = 0$$

This equation implies that

$$A_k = -\frac{1}{I(k+r)} \sum_{j=0}^{k-1} \frac{(j+r)p^{k-j}(0) + q^{k-j}(0)}{(k-j)!} A_j \tag{1.36}$$

The series solution with A_k above,

$$F_r(x) = \sum_{k=0}^{\infty} A_k x^{k+r} \tag{1.37}$$

Must satisfy,

$$x^2 F_r''(x) + p(x)xF_r'(x) + q(x)F_r(x) = I(r)x^r \tag{1.38}$$

If we choose one of the roots to the indicial polynomial for r in $F_r(x)$, we gain a solution to the differential equation. If the difference between the roots is not an integer, we get another, linearly independent solution in the other root.

Asymptotic Iteration Method

The AIM method [58-65] is based on solving a second order differential equation of the form :

$$f_n''(x) = \lambda_0(x) f_n'(x) + s_0(x) f_n(x) \tag{1.39}$$

Where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to x. The variables, $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To find a

general solution to this equation, we differentiate(1.31) with respect to x and find

$$f_n'''(x) = \lambda_1(x) f_n'(x) + s_1(x) f_n(x) \tag{1.40}$$

Where $\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x)$,

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x). \tag{1.41}$$

Similarly, the second derivative of (1.31) yields

$$f_n^4(x) = \lambda_2(x) f_n'(x) + s_2(x) f_n(x), \tag{1.42}$$

Where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{1.43}$$

Equation (1.31) can be easily iterated up to $(k+1)th$ and $(k+2)th$ derivatives,$k = 1, 2, 3, ...$ Therefore, we have the reccurence relations

$$\begin{aligned} f_n^{(k+1)}(x) &= \lambda_{k-1}(x) f_n'(x) + s_{k-1}(x) f_n(x), \\ f_n^{(k+2)}(x) &= \lambda_k(x) f_n'(x) + s_k(x) f_n(x), \end{aligned} \tag{1.44}$$

Where

$$\begin{aligned} \lambda_k(x) &= \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x). \end{aligned} \tag{1.45}$$

From the ratio of the (k+2)th and (k+1)th derivatives, we have

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{f_n^{(k+2)}(x)}{f_n^{(k+1)}(x)} = \frac{\lambda_k(x)[f_n'(x) + \frac{s_k(x)}{\lambda_k(x)} f_n(x)]}{\lambda_{k-1}(x)[f_n'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} f_n(x)]}. \tag{1.46}$$

For sufficiently large k, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x) \tag{1.47}$$

which is the "asymptotic" aspect of the method, then, (1.38) reduces to

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{1.48}$$

which yields

$$f_n^{(k+1)}(x) = C_1 exp(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)} dx) = C_1 \lambda_{k-1}(x) exp(\int [\alpha(x) + \lambda_0(x)] dx), \quad (1.49)$$

where C_1 is the integration constant and the right hand side of (1.41) is obtained by using (1.39) and (1.40). By inserting (1.41) into (1.36), the first-order differential equation is obtained as

$$f_n'(x) + \alpha(x) f_n(x) = C_1 exp(\int [\alpha(x) + \lambda_0(x)] dx). \quad (1.50)$$

This first-order differential equation can easily be solved and the general solution of (1.31) can be obtained as:

$$f_n(x) = exp(-\int^x \alpha(x_1) dx_1)[C_2 + C_1 \int^x exp(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)] dx_2) dx_1] \quad (1.51)$$

For a given potential, the radial Klein-Gordon equation is converted to the form of (1.31). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by (1.37). The termination condition of the method in (1.39) can be arranged as

$$\Delta_k(x) = \lambda_k(x) s_{k-1}(x) - \lambda_{k-1}(x) s_k(x) = 0, \quad (1.52)$$

where k shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of (1.44) and the radial quantum number n is equal to the iteration number k for this case. For nontrivial potentials that have no exact solutions, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential and the approximate energy eigenvalues are obtained from the roots of (1.44) for sufficiently great values of k with iteration for which k is always greater than n in these numerical solutions.

The general solution of (1.31) is given by (1.43). The first part of (1.43) gives us the polynomial solutions that are convergent and physical, whereas the second part of (1.43) gives us non-physical solutions that are divergent. Although (1.43) is the general solution of (1.31), we take the coefficient of the second part (C_1) as zero, in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wave function generator for exactly solvable potentials :

$$f_n(x) = C_2 exp(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)} dx_1), \quad (1.53)$$

where n represents the principal quantum number.

Laplace Transform Approach

The Laplace transform is an integral transform named upon the French mathematician Laplace (who proposed the transform in 1782). It helps in solving the differential equations with boundary values without finding the general solution and values of the arbitrary constants. The method [66-69] of Laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with differential equations with discontinuous forcing functions. With the use of different properties of Laplace transform and Inverse Laplace transform one can solve many important problem of physics with very simple way. Thus we will learn here to use the approach for solving the differential equations.

The Laplace transforms of different functions can be found in most of the mathematics and engineering books and hence, a very few part is included here relevant to my study.

Suppose the differential equation contain a term of the form $t^m y^{(n)}(t)$ i.e., $t^m \frac{d^n y(t)}{dt^n}$. Then the Laplace transform of the term is represented by

$$L\left\{t^m \frac{d^n y(t)}{dt^n}\right\} = (-1)^m \frac{d^m}{ds^m} L\left\{y^{(n)}(t)\right\} \tag{1.54}$$

So,

$$L\{ty''(t)\} = (-1)\frac{d}{ds} L\{y''(t)\} \tag{1.55}$$

Again, another important theorem for Laplace transform of first order and second order derivative for continuous $y(t)$ and $y'(t)$ with $t \geq 0$ of exponential order σ as $t \to \infty$ and if $y'(t)$ and $y''(t)$ is of class A, then Laplace transform of $y'(t)$ and $y''(t)$ for $s > \sigma$ are given by

$$L\{y'(t)\} = sL\{y(t)\} - y(0) \tag{1.56}$$

and

$$L\{y''(t)\} = s^2 L\{y(t)\} - sy(0) - y'(0) \tag{1.57}$$

One of the most important formula used in our calculation is : if $y(t)$ is a function of class $\mathcal{A}$, then

$$L\{t^n y(t)\} = (-1)^n \frac{d^n f(s)}{ds^n} \tag{1.58}$$

where, $f(s) = L\{y(t)\} = \int_0^\infty y(t)e^{-ts}dt$ and $n = 1, 2, 3, ...$

After conversion of second order differential equation to a first order one, we further apply the inverse Laplace transform to obtain the wave function. The relevant formulas for inverse Laplace transform are followed from reference [70].

Point Canonical Transformation Approach

The procedure in point canonical transformation [71-75] is illustrated below:

Let us consider the stationary Schrödinger equation

$$\frac{1}{2M}\Phi''(x) + (E - U)\Phi(x) = 0 \tag{1.59}$$

where the mass M is a positive constant, $U = U(x)$ denotes the potential with constant energy E, and $\Phi = \Phi(x)$ is the wave function. Now, consider the effective mass time-dependent Schrödinger equation (TDSE)

$$i\Psi_t + \frac{1}{2m}\Psi_{xx} - \frac{m_x}{2m^2}\Psi_x - V\Psi = 0 \tag{1.60}$$

where $m = m(x, t)$ is the real-valued and positive mass, $V = V(x, t)$ stands for the potential, and $\Psi = \Psi(x, t)$ is the wave function. Suppose that the potential V in the effective mass TDSE can be expressed in the form

$$V = UI_3^2 + \frac{2AB}{M^{\frac{3}{2}}}I_1 - 2\big(\frac{A}{M}I_1\big)^2 - \frac{B}{2\sqrt{M}}I_2 - \frac{A}{M}I_1I_2 - \frac{1}{8}I_2^2 - \frac{7m_x^2}{32m^3} + \frac{m_{xx}}{8m^2}$$

$$+ \int \left(\left(\frac{A\sqrt{m}}{M} + \frac{m_t}{4\sqrt{M}} \right) I_2 + \frac{\sqrt{m}}{2}(I_2)_t + \left(\frac{2A'\sqrt{m}}{M} + \frac{Am_t}{M\sqrt{m}} \right) I_1 \right.$$

$$\left. + \frac{B'\sqrt{m}}{\sqrt{M}} + \frac{Bm_t}{2\sqrt{mM}} \right) dx + i\left(c' - \frac{A}{M} \right) - \frac{B^2}{2M} \tag{1.61}$$

where the following abbreviations have been used:

$$U = U\Big(\sqrt{\frac{1}{M}}I_3 I_1 + \frac{1}{M}\int I_3 B dt\Big), \tag{1.62}$$

$$I_1 = \int \sqrt{m}dx, \quad I_2 = \int \frac{m_t}{\sqrt{m}}dx, \quad I_3 = exp\Big(\frac{2}{M}\int A dt\Big). \tag{1.63}$$

Then the solutions of the Schrödinger (1.59) and (1.60) are related to each other via the following point canonical transformation:

$$\Psi(x,t) = exp\Big(f(x,t) - iEV(t)\Big)\Phi(u(x,t)), \tag{1.64}$$

where the functions f , u and v read as follows:

$$f = \int \Big(\frac{m_x}{4m} - i\Big(\frac{\sqrt{m}}{2}\Big(\int \frac{m_t}{\sqrt{m}}dx\Big) + \frac{2A\sqrt{m}}{M}\Big(\int \sqrt{m}dx\Big) + \frac{B\sqrt{m}}{M}\Big)\Big)dx$$

$$+C \tag{1.65}$$

$$u = \sqrt{\frac{1}{M}}exp\Big(\frac{2}{M}\int A dt\Big)\Big(\int \sqrt{m}dx\Big) + \frac{1}{M}exp\Big(\frac{2}{M}\int^t A dt'\Big)B dt, \tag{1.66}$$

$$v = \int exp\Big(\frac{4}{M}\int A dt\Big)dt \tag{1.67}$$

with arbitrary, real-valued $A = A(t)$, $B = B(t)$, and $C = C(t)$. The latter statement is proved in a straightforward manner by substitution of (1.64) with $(1.65) - (1.67)$ into the effective mass TDSE (1.60). Thus, each effective mass TDSE with a potential of the form (1.61) can be reduced to a stationary Schrödinger equation by means of the point canonical transformation (1.64). Hence, if the stationary Schrödinger (1.59) is solvable, so is its effective mass counterpart (1.60).

To apply the scheme we need to follow three steps. First, to choose a potential U, such that the associated stationary Schrödinger equation (1.59) is exactly solvable, and calculate its solution. In the second step, to consider the effective mass m and fix the free parameters A, B, c that appear in (1.61) to compute the transformation components f , u, v that are defined in $(1.65) - (1.67)$. In the final step, we obtain the time-dependent potential (1.61) and its solution (1.64).

Pekeris-type approximation

A Pekeris-type approximation is a mapping for passing from (3+1)− dimensional KG and Dirac equations with an arbitrary spherical non-minimal coupling to an associated Schrödinger-like equation and the family of potentials for which this mapping corresponds to a Schrödinger equation with non-minimal coupling can be obtained. The process [76-77] is based on the expansion of the centrifugal term in powers of $y - y_e$, where $y = f(r)$ is a suitable function of r and $y_e = f(r_e)$. The function f is chosen in such a way that the replacement of the dimensionless quantity $(\frac{r_e}{r})^2$ in the centrifugal term by the first three terms in the expansion yields an exactly solvable approximate radial equation.

To obtain an approximation transforming equation into an exactly solvable approximate equation for any value of the azimuthal quantum number l, one can use the linear independence of the terms in the potential function.

Let us consider an arbitrary spherical potential $U(r)$ and y is the function of the potential coupling $U(r)$ related to Pekeris expansion to deduce the family of potentials from which a mapping onto a Schrödinger equation with non-minimal coupling emerges. More generally, if $U(r)$ represents an spherical radial potential, we can define the dimensionless variable

$$y = \gamma U(r) + 1 = f^{-1}(\gamma(r - r_e)) \tag{1.68}$$

with γ a real parameter having units of distance^{-1}, $f^{-1}(x) = \gamma U(x/\gamma + r_e) + 1$ and $f(x) = \gamma U^{-1}((x-1)/\gamma) - \gamma r_e$. In order to provide the method, we assume that U has a differentiable inverse U^{-1} and then f and f^{-1} result also differentiable. Thus, from (1.68) it follows approximated expressions for (r_e/r) and $(r_e/r)^2$ up to terms of order 2 around $y = 1(r = r_e)$

$$\frac{r_e}{r} = \left(\frac{1 + f(y)}{\gamma r_e}\right)^{-1} \approx \frac{1}{\frac{f(1)}{\gamma r_e} + 1} + \sum_{i=1}^{2} a_i (y-1)^i$$

$$\left(\frac{r_e}{r}\right)^2 = \left(\frac{1 + f(y)}{\gamma r_e}\right)^{-2} \approx \frac{1}{(\frac{f(1)}{\gamma r_e} + 1)^2} + \sum_{j=1}^{2} a_j (y-1)^j$$

$$a_1 = -\frac{\gamma r_e f'(1)}{\left(\gamma r_e + f(1)\right)^2}$$

$$a_2 = -\frac{\gamma r_e\Big(\gamma r_e f''(1) + f(1)f''(1) - 2(f'(1))^2\Big)}{2\Big(\gamma r_e + f(1)\Big)^3}$$

$$b_1 = -\frac{2(y-1)\gamma^2 r_e^2 f'(1)}{\Big(\gamma r_e + f(1)\Big)^3}$$

$$b_2 = -\frac{\gamma^2 r_e^2\Big(\gamma r_e f''(1) + f(1)f''(1) - 3(f'(1))^2\Big)}{\Big(\gamma r_e + f(1)\Big)^4} \tag{1.69}$$

To avoid terms of the type $\propto U^3$ in the term U/r, we can still make $a_2 = 0$. Then, using (1.69) the effective potential can be recasted in terms of $y - 1 = \gamma U$ for $r \sim r_e$ as

$$U_{eff}(r) \approx \left(\frac{m\omega^2}{2\gamma^2} - [1 + f(j,l)]\frac{\hbar\omega}{\gamma r_e}a_1 + \frac{\hbar^2 l(l+1)}{2r_e^2}b_2\right)(y-1)^2$$

$$+\Big(- [1 + f(j,l)]\frac{\hbar\omega}{\gamma r_e} + \frac{\hbar^2 l(l+1)}{2r_e^2}b_1\Big)(y-1) + \frac{\hbar^2 l(l+1)}{2r_e^2} - \frac{\hbar\omega}{2\gamma}\frac{d(y-1)}{dr}$$

$$= A\gamma^2\left[U(r) + \frac{B}{2\gamma A}\right]^2 + C - \frac{B^2}{4A} - \frac{\hbar\omega}{2}\frac{dU(r)}{dr} \tag{1.70}$$

with appropriate identifications for the constants A,B,C. These constants are determined by

$$A = \frac{m\omega^2}{2\gamma^2} - [1 + f(j,l)]\frac{\hbar\omega}{\gamma r_e}a_1 + \frac{\hbar^2 l(l+1)}{2r_e^2}b_2$$

$$B = -[1 + f(j,l)]\frac{\hbar\omega}{\gamma r_e} + \frac{\hbar^2 l(l+1)}{2r_e^2}b_1$$

$$C = \frac{\hbar^2 l(l+1)}{2r_e^2} \tag{1.71}$$

which together with (1.69) and $f(x) = \gamma U^{-1}((x-1)/\gamma) - \gamma r_e$ give a complete Pekeris mapping.

1.5.2 Potentials considered in our study

In nature, it is clearly not true that all electron wavefunctions spread out more and more as time passes. If we have an electron in an atom, the electron stays in the atom unless disturbed by some outside influence. Similarly, for an electron in a metal, the wavefunction might spread out through the metal, but the electron will not escape without some outside influence. The physical difference in these situations from the case of free electrons is that electrons in an atom or in a metal have forces acting on them. Equivalently, the electrons have a potential energy that is different for different locations. To learn how wavefunctions evolve in the presence of these forces or potentials, we need to understand how to modify the wave equation when they are present.

To understand the subatomic world local potentials are being used since the introduction of quantum mechanics. Most of these (like the harmonic oscillator) represent approximations of the actual physical situation, while Some of them (like the Coulomb potential) do not differ essentially from the forces observed in nature. The most elementary examples introduced at the dawn of quantum mechanics still form essential part of any quantum mechanical course, and also play a fundamental role in the formulation of most physical models of the microscopic world. The potential shape, defined by the potential type and the parameters in it is usually chosen in a way that reflects the physical picture of our intuition associates with the problem; therefore we can define attractive or repulsive, short-range or long-range potentials, etc.

The energy eigenvalues, the bound-state wave functions and the scattering matrix can be determined in closed analytical form for some potential(exactly solvable potentials). The range of these potentials has been extended considerably in the recent years by investigations inspired by some novel symmetry-based approaches. The concept of solvability has also been extended: one can talk about conditionally exactly or quasi-exactly solvable potentials too, in addition to the "classical" exactly solvable examples. Due to these developments more and more interactions can be modeled by making advantage of the increasingly flexible potential shapes offered by solvable potentials. Their solutions can be applied directly, or can be com-

bined with numerical calculations. In the simplest case analytical calculations can aid numerical studies in areas where numerical techniques might not be safely controlled. When physical system crosses a critical point, for certain singular potentials or complex potentials, or in situations, we can obtain the bound-state wave functions with arbitrary node numbers as per requirement. As the next level of complexity, analytical solutions can supply a basis for numerical calculations. This makes exactly solvable problems indispensable even in the age of rapidly developing computational resources. Besides their role in describing realistic physical problems, solvable quantum mechanical potentials also represent an interesting field of investigation in their own right. This is largely due to the mathematical elegance and beauty associated with the symmetries of these problems.

The problem of the relativistic wave equations with spatially dependent masses has been attracting much intention in the literature. Systems with position-dependent mass have been found to be very useful in studying the physical properties of various microstructures, such as semiconductor heterostructure. Quantum liquids, quantum wells and quantum dots, ^{3}He clusters, compositionally graded crystals etc. A lot of studies have been performed to obtain the solutions of the Schrodinger, Klein-Gordon and Dirac equations with position-dependent mass for different potentials.

Our activity in this field mainly concerned the exact,analytical, scattering state and bound state solution of the relativistic wave equation (i.e. of one-dimensional Klein-Gordon equation and Dirac equation). We gave a systematic treatment of shape-invariant potentials , which contain the most well-known textbook examples for solvable potentials. The potentials considered in our study are given bellow:

q-deformed modified Eckart plus Hylleraas Potential

The q-deformed modified Eckart plus Hylleraas Potential [78-82] of the form see figure 2.1:

$$V(x) = \frac{V_0}{b}\left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}}\right) - V_1\frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2\frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2}$$

Where q is the shape parameter.

We prefer to use the mass function equals to the rest mass along with the

vector part of the potential as

$$m(x) = m_0 + \frac{V_0}{b}\left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}}\right) - V_1\frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2\frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2}$$

to obtain an exactly solvable Schrödinger-like equation in absence of scalar potential . The mass function should also be a physical distribution , so we restrict ourself in the range $0 \leq x < \infty$, which gives the finite mass values as follows :

$$m(x) = \begin{cases} m_0 + \frac{V_0}{b}(a-1) - V_1 + V_2 & (for q \to 0) \quad , x \to 0 \\ m_0 + \frac{V_0 a}{b} & , x \to \infty \end{cases}$$

Actually, this distribution corresponds to shifted scalar potential function in the problem.

Generalized Asymmetric Manning-Rosen potential

The generalized Maning-Rosen (GAMR) potential [83] taken for our study is of the following form see figure 3.1,

$$\mathcal{V}(x) = \Theta(-x)\left[\frac{Ae^{2\alpha(x+x_0)}}{(\Lambda + \Delta e^{\alpha(x+x_0)})^2} + \frac{Be^{2\alpha(x+x_0)}}{(\Lambda + \Delta e^{\alpha(x+x_0)})}\right]$$

$$+\Theta(x)\left[\frac{Ce^{-2\beta(x-\tilde{x_0})}}{(\tilde{\Lambda} + \tilde{\Delta}e^{-\beta(x-\tilde{x_0})})^2} + \frac{De^{-\beta(x-\tilde{x_0})}}{(\tilde{\Lambda} + \tilde{\Delta}e^{-\beta(x-\tilde{x_0})})}\right] \tag{1.72}$$

Where $\Theta(x)$ is Heaviside step function and all the parameters are real.

The shape of GAMR potential varies according to the values of the parameters. It becomes a potential barrier if A, B, C and D are positive and it becomes a potential well if A, B, C and D are negative.

The mass function is chosen here as $m(x) = m_0 + f(x)$, where $f(x) = \mathcal{V}(x)$ and m_0 is the rest mass.

Double Ring Shaped Coulomb potential

The double ring shaped Coulomb potential [84] is a 3-dimensional Coulomb potential surrounded by a double ring shaped inverse square potential.

Mainly the double ring shaped Coulomb potential is a non-central potential in spherical coordinate and can be written as:-

$$\mathcal{V}(r,\theta) = -\frac{A}{r} + \frac{B}{r^2 sin^2\Theta} + \frac{C}{r^2 cos^2\Theta} \tag{1.73}$$

where, $A = \eta\sigma^2 e^2$, $B = \frac{\hbar^2\eta^2\sigma^2}{2\mu}$, $C = \frac{\hbar^2}{2\mu}a$ where,$a \geq 0$

μ is the mass of the particle, η and σ are positive real parameters which range from 1 to 10 and r and Θ are the spherical coordinates. When we put $B = 0$ and $C = 0$ then $\mathcal{V}(r)$ reduces to coulomb potential and when $B = 0$ then $\mathcal{V}(r)$ reduces to ring shaped Hartmann potential. Specially the ring shaped non central potentials are used to describe the molecular structure and the interaction between the deformed nuclie and specially the molecular structure of benzene.

Manning Rosen Potential

The Manning Rosen Potential [85] is see figure 5.1:

$$\mathbf{V}(\mathbf{r}) = \frac{-V_1(1+qe^{-\alpha r})}{(1+qe^{-\alpha r})} + \frac{V_2 e^{-\alpha r}}{(1-qe^{-\alpha r})^2}. \tag{1.74}$$

Here r_0 is the equilibrium position of molecules and potential acting range is determined by the dimensionless parameter α and V_1 and V_2 are two general potential parameters.

Here we use the mass function $m = \frac{m_0}{(1-qe^{-\alpha r})^2}$, where m_0 is the rest mass and α determines the potential range.

1.6 Scope of the thesis

In recent years, the effects of the position-dependent mass on the energy spectra and corresponding eigenvalues of wave equations has been received a great attention. A very few study appeared in literature on wave equation having position-dependent mass but began in a haphazard and diffuse fashion. There are numerous early examples of isolated and disconnected discoveries of such study.

In our study, we have observed several issues:

- The results may have many applications in nuclear physics especially in chemical and molecular physics. For example, it is most useful in exploration of the resonance phenomena, where particles are temporarily trapped by the potential.
- The outcomes of our study may be able to shed light on the most outstanding problem in condensed matter physics, the theory of high temperature superconductivity.
- The knowledge of the study can be useful to decide whether some of the examples of quantum mechanical system which have appeared in the literature in the past do indeed constitute consistent quantum systems or if the same applies to newly proposed models, opening a vast universe of possibilities.
- Our study about the relativistic wave equations having position-dependent mass can be useful in studying the physical properties of various microstructures (such as semiconductor heterostructure, Quantum liquids, quantum wells and quantum dots, ^3He clusters, compositionally graded crystals etc).
- Our study about the relativistic wave equations having position-dependent mass can be helpful for the study of the multi-band transport system in nano-devices in the subject area of nanoscience and nanotechnology.
- Our study can be helpful in shedding new light on the conceptual foundations of quantum physics and also at the core of the new field of Quantum Information Theory, which foresees important technological developments through concepts such as "entanglement", "teleportation" and "quantum computation".
- Our study about the relativistic wave equations having position-dependent mass can be helpful for the study of the description of the dynamics of an anisotropic DNA molecule in biophysics with some real coupling constants.
- Our study provides a better understanding of the approximation schemes used in the context of the past and allows for the improvement to a more reliable quantitative description of scattering problem.
- The purpose of this thesis is to give an elementary introduction to this

exciting and active field of research. In writing this thesis, my hope is that the rate of new discoveries and the development of the field will continue at such a rapid pace that this review will soon become obsolete.

1.7 Conclusions

In this thesis we have searched for exact results for theories described by relativistic wave equations with position-dependent mass in Quantum Physics. Although it is widely believed that such systems can only be used to describe effective models, we have argued that they have achieved a more fundamental status in recent years. The motivation for obtaining solutions of the wave equation with position dependent mass comes from the wide range of applications of these solutions in various areas of material science and condensed matter. The main concern of our study primarily focused on obtaining the energy spectrum and wave function for a given mass distribution. Then we gave remarks on the effects of position-dependent mass into the corresponding system which we eager to describe.

Relativistic wave equations with position dependent mass plays an important role in the study of electronic properties of semi-conductors in homogeneous crystals, quantum dots, He clusters, quantum liquids etc. For some relevant mass distributions some phenomenological potentials have been solved in recent times, but there are only few papers that give the solution of the relativistic wave equation with position-dependent mass in quantum mechanics.

In order to investigate nuclear shell structure, the study of spin and pseudospin symmetric solutions of the Dirac equation has been an important area of research in nuclear physics. The concept of spin and pseudospin symmetry with nuclear shell model has been used widely in explaining a number of phenomena in nuclear physics and related areas. The spin symmetry is relevant for meson and the pseudospin symmetry has been used to study the structure of the deformed nuclei, to construct an effective shell model coupling scheme and identical bands and triaxality observed in nuclei.

To predict physical phenomena and to design the corresponding experiments, the understanding of this thesis is very helpful. The results are most useful in exploration of the resonance phenomena, where particles are temporarily trapped by the potential. One of the most striking phenomena in nature is resonance phenomena. These are associated with metastable states of a system that has sufficient energy to break up into two or more subsystems. These systems can be nuclei, atoms, molecules, solids, nano-structured materials and condensates. In this thesis many physical problems have been solved to provide a better understanding to explain the algorithms for calculating the resonance measurable quantities and to illustrate the applications of the formalism in physics, chemistry and technology.

In nuclear physics, the shape form of the potential plays an important role particularly when studying the structure of deformed nuclei or the interaction between them. Therefore, q-deformed potential is used with the shape parameter 'q' along with screening parameter 'α'.

Further developments regarding this problem have been intensively reported. Besides geometric analysis establishing, as expected, the equivalence between a completely Hermitian formulation and a completely non-Hermitian formulation, more recent results indicate that the quantum brachistochrone can be realized as subsystem of a larger Hermitian system living in a higher-dimensional Hilbert space. This is an example where theoretical advances come before those accomplished by experiments so that more direct evidences to clarify this problem still need to be obtained.

This work confirms the potential importance of studying effective mass quantum system, motivated by the recent interest in mass dependent quantum theories. The scope of such models is immense, covering possibly all fields of theoretical physics, with some illustrations presented here. I have no doubts that the investigations carried out by me are of relevance to the scientific community and hope it can inspire further developments.

2

Relativistic Equation with PDM for q-deformed modified Eckart plus Hylleraas potential

2.1 Introduction

Quantum Mechanical phenomena are described by Schrödinger equation which dictates the dynamics of quantum systems represented by Hamiltonian Operator. Solutions of Klein-Gordan Equation for some physical potential have important applications in Molecular Physics, Quantum Chemistry, Nuclear physics, condensed matter Physics, high energy physics. The study of potentials such as Hulthén [86-87], Morse [88], Rosen-Morse [64], Pseudo-harmonic [89], Pöschl-Teller [90-91], Kratzer-Fuez [68], generalized Wood Saxon [54,92-94], ring-shaped Hartmann [55] and the corresponding wave functions has been performed using various methods.

Recently, there has been renewed interest in solving Quantum Mechanical systems within the frame work of Nikiforov-Uvarov method[46-55].This technique is successfully used to solve Schrödinger, Klein-Gordan, Dirac and Duffin-Kemmer-Petieu Equqtions.

In nuclear physics, the shape form of the potential also plays an important role particularly when studying the structure of deformed nuclei or the interaction between them. Therefore , our aim , in the present work is to investigate analytical bound state

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solutions of the Klein-Gordon equation with q-deformed modified Eckart plus Hylleraas potential[78-82] in the Frobenius method [57,95] as well as in N-U method. Also, we will show that, when the deformation parameter q takes a particular value ($q = 1$), the obtained results lead to the solutions of the same problem for modified Eckart plus Hylleraas potential.

The number of solvable potentials in ordinary quantum mechanics is in fact limited, but when we assume the particle mass has a nontrivial space distribution the mathematical difficulties grow even more. In last few years, an increasing attention has been paid to the study of quantum systems with position-dependent mass (PDM). This interest is due to the wide applicability of such models in different areas of physics. A well-known example is the so-called effective-mass approximation theory in condensed matter physics, which has been extensively used for the determination of electronic properties of semiconductors, and for the description of the properties of hetero-junctions and quantum dots. In addition, PDM-models have been successfully applied in the field of molecular and atomic physics. Besides, we should also cote the recent new proposed scenarios based on the theory of general relativity, where systems with PDM arise from another origin, quite different from the condensed matter one. Furthermore, the inclusion of relativistic effects was found to be very significant when condensed systems with heavy atoms or heavy ion doping are considered. Even more, the recent discovery of graphene, a two-dimensional allotrope of carbon, which is receiving a lot of attention, has shown that electrons in this material behave as massless fermions. All these considerations have fostered an intense research activity in the field of relativistic quantum mechanics with PDM.

In recent years , the solutions of the non-relativistic wave equation with position-dependent mass have been a topic of great interest[96-100] , but there are only few papers that give the solution of the relativistic wave equation with position-dependent mass in quantum mechanics. Exact solution of the Dirac equation with position-dependent mass in the Coulomb field [101], Kepler problem in Dirac theory for a particle whose potential and mass are inversely proportional to the distance from the force center [102], the approximate solution of the one-dimensional Dirac equations with spatially dependent mass for the generalized Hulthén potential [103], the exact solution of the one-dimensional K-G equation with spatially dependent mass for the inversely linear potential [104] are some papers on relativistic wave equations with position dependent mass.

Our focus is to study the quantum systems with Position Dependent Effective

Mass (PDEM). PDEM Klein-Gordan Equation plays an important role in the study of electronic properties of semi-conductors in homogeneous crystals,quantum dots,He clusters, quantum liquids etc.

Exact solutions of effective mass Klein-Gordan Equations are difficult to obtain, as such, approximate numerical techniques are often used.

In this chapter, relativistic Klein-Gordan equation with Position Dependent Mass has been solved analytically for the q-deformed modified Eckart plus Hylleraas potential. A generalized series is used to obtain the bound state solutions of the K-G equation using the Frobenious Method . The one dimensional K-G equation for the mass dependent modified Eckart plus Hylleraas potential in absence of scalar potential are studied in this paper. The exactly normalized bound state wave function and energy expressions are obtained by using N-U method.

2.2 Brief discussion of Klein Gordan Equation with position dependent mass

In this study, the one dimensional K-G equation for a spinless particle of mass m in the natural units $\hbar = c = 1$ can be expressed

$$\Psi''(x) + [(E - \mathcal{V}(x))^2 - (m + S(x))^2]\Psi(x) = 0 \tag{2.1}$$

where E , $\mathcal{V}(x)$ and $S(x)$ are the relativistic energy of the particle , vector and scalar potentials respectively. Now considering the q-deformed modified Eckart plus Hylleraas Potential of the form:

$$\mathcal{V}(x) = \frac{V_0}{b}\left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}}\right) - V_1\frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2\frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2} \tag{2.2}$$

Where q is the shape parameter.

We prefer to use the mass function equals to the rest mass along with the vector part of the potential as

$$m(x) = m_0 + \frac{V_0}{b}\left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}}\right) - V_1\frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2\frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2} \tag{2.3}$$

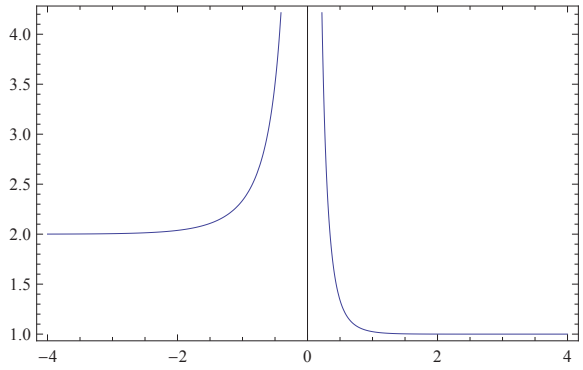

Figure 2.1: The modified Eckart plus Hylleraas Potential with unit value of α, a, b, q.

to obtain an exactly solvable Schrödinger-like equation in absence of scalar potential . The mass function should also be a physical distribution , so we restrict ourself in the range $0 \leq x < \infty$, which gives the finite mass values as follows :

$$m(x) = \begin{cases} m_0 + \frac{V_0}{b}(a-1) - V_1 + V_2 \quad (for q \to 0) \quad , x \to 0 \\ m_0 + \frac{V_0 a}{b} \qquad\qquad\qquad\qquad , x \to \infty \end{cases}$$

Actually, this distribution corresponds to shifted scalar potential function in the problem. Substituting equation (2.3) in equation (2.1) we have

$$\Psi''(x) + \left[(E^2 - m_0^2) - 2(E + m_0) \left\{ \frac{V_0}{b} \left(\frac{a - e^{-2\alpha x}}{1 - q e^{-2\alpha x}} \right) - V_1 \right.\right. \\ \left.\left. \frac{e^{-2\alpha x}}{1 - q e^{-2\alpha x}} + V_2 \frac{e^{-2\alpha x}}{(1 - q e^{-2\alpha x})^2} \right\} \right] \Psi(x) = 0 \qquad (2.4)$$

2.3 Application of Nikiforov-Uvarov Method

Introducing a new variable $s = e^{-2\alpha x}$ it is straight forward to show that (2.4) takes the form:

$$\Psi''(s) + \frac{1 - qs}{s(1 - qs)} \Psi'(s) + \frac{1}{s^2(1 - qs)^2} \left[s^2 q^2 (\epsilon^2 - \gamma^2 - \zeta^2) \right. \\ \left. + 2qs(\gamma^2 - \epsilon^2) + (\epsilon^2 - \omega^2) \right] \Psi(s) = 0 \qquad (2.5)$$

Where we use the notations $\frac{E^2-m_0^2}{4\alpha^2} = \epsilon^2$,$\gamma^2 = \frac{E+m_0}{4\alpha^2 q}\{2V_1 + 2\frac{V_0}{b}(aq+1) - 2V_2\}$, $\zeta^2 = \frac{E+m_0}{4\alpha^2 q}\{(V_1+V_2) - \frac{V_0}{b}(aq-1)\}$ and $2\frac{V_0}{b}\frac{E+m_0}{4\alpha^2} = \omega^2$ comparing equation (2.5) with equation (1.24) we have

$$\begin{aligned}
\tilde{\tau}(s) &= 1 - qs; \\
\sigma(s) &= s(1-qs); \\
\tilde{\sigma}(s) &= s^2q^2(\epsilon^2 - \gamma^2 - \zeta^2) + 2qs(\gamma^2 - \epsilon^2) + (\epsilon^2 - \omega^2);
\end{aligned} \tag{2.6}$$

Substituting equation (2.6) the relation (1.26) we get

$$\pi(s) = -\frac{qs}{2} \pm \sqrt{q^2s^2(\frac{1}{4} + \gamma^2 + \zeta^2 - \epsilon^2 - k_1) + qs(k_1 - 2\gamma^2 + 2\epsilon^2) + (\omega^2 - \epsilon^2)} \tag{2.7}$$

where k_1 satisfies the relation $k = k_1 q$ Further the discriminant of the upper expression under the square root has to be set equal to zero. Therefore, we obtain

$$\Delta = q^2(k_1 + 2\epsilon^2 - 2\gamma^2)^2 - 4q^2(\frac{1}{4} + \gamma^2 + \zeta^2 - \epsilon^2 - k_1)(\omega^2 - \epsilon^2) \tag{2.8}$$

Solving equation (2.8) for constant k_1 , we obtain the double roots as , $k_1\prime, k_1\prime\prime = 2(\gamma^2 - \omega^2) \pm 2\xi\eta$, where $\xi^2 = \omega^2 - \epsilon^2$ and $\eta^2 = (\frac{1}{4} + \zeta^2 + \omega^2 - \gamma^2)$.

Thus substituting these values for each k_1 into equation (2.7) , we obtain

$$\pi(s) = -\frac{qs}{2} \pm \begin{cases} (\xi - \eta)qs - \xi; & for \quad k_1\prime = 2(\gamma^2 - \omega^2) + 2\xi\eta \\ (\xi + \eta)qs - \xi; & for \quad k_1\prime\prime = 2(\gamma^2 - \omega^2) - 2\xi\eta \end{cases} \tag{2.9}$$

By choosing an appropriate value for k in $\pi(s)$ which satisfies the condition $\tau'(s) < 0$, one gets $\pi(s) = -qs(\xi + \eta + \frac{1}{2}) + \xi$ for $k = 2(\gamma^2 - \omega^2) - 2\xi\eta$; giving the function:

$$\tau(s) = 1 - 2qs[1 + (\xi + \eta)] + 2\xi \tag{2.10}$$

If we consider $\lambda = k + \Pi'$ defined in (1.27) we obtain

$$\lambda = q[2(\gamma^2 - \omega^2) - 2\xi\eta - \frac{1}{2} - (\xi + \eta)] \tag{2.11}$$

Again using equation (1.27) , we have:

$$\lambda_n = q[n^2 + n + 2n(\xi + \eta)] \tag{2.12}$$

Using the condition $\lambda = \lambda_n$ one obtains the eigen values of ϵ from the following equation:

$$\omega^2 - \epsilon^2 = \left[\frac{8(\gamma^2 - \omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{2.13}$$

From (1.29) it can be shown that the weight function $\rho(s)$ is $\rho(s) = s^{2\xi}(1-qs)^{2\eta}$ and by substituting $\rho(s)$ into the Rodrigues relation (1.28) one gets

$$y_n(s) = \frac{B_n}{s^{2\xi}(1-qs)^{2\eta}}(\frac{d}{ds})^n[s^n(1-qs)^n s^{2\xi}(1-s)^{2\eta}] = \frac{B_n}{s^{2\xi}(1-qs)^{2\eta}} P_n^{(2\xi,2\eta)}(s) \quad (2.14)$$

where $P_n^{(2\xi,2\eta)}(s)$ stands for Jacobi polynomial [105,106] and B_n is the normalizing constant. The other part of the wave function is simply found from (1.30) as ,

$$\phi(s) = s^{\xi}(1-qs)^{(\frac{1}{2}+\eta)} \quad (2.15)$$

Finally , the wave function is obtained as follows

$$\psi(s) = B_n s^{-\xi}(1-qs)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \quad (2.16)$$

2.4 Application of Frobenius Method

consider the same Klein-Gordan equation and the same Eckart plus modified Hylleraas Potential given in section 2.2. After development , we get the following equation:

$$\psi''(s) + \frac{1}{s}\psi'(s) + \frac{1}{s^2}\left[\epsilon^2 - 2\beta^2\left\{\frac{V_0}{b}\frac{a-s}{1-qs} - V_1\frac{s}{1-qs} + V_2\frac{s}{(1-qs)^2}\right\}\right](s) = 0 \quad (2.17)$$

where we use the notations $\frac{E^2-m_0^2}{4\alpha^2} = \epsilon^2$ and $\frac{E+m_0}{4\alpha^2} = \beta^2$
Comparing (2.17) with the equation (1.32) we have, $P(s) = 1$ and
$Q(s) = \left[\epsilon^2 - 2\beta^2\left\{ - V_1\frac{s}{1-qs} + \frac{V_0}{b}\frac{a-s}{1-qs} + V_2\frac{s}{(1-qs)^2}\right\}\right]$
Putting these values the equation (2.17) becomes ,

$$\psi''(s) + \frac{P(s)}{s}\psi'(s) + \frac{Q(s)}{s^2}\psi(s) = 0 \quad (2.18)$$

By using Fuck's theorem , we can write :

$$\psi(s) = \sum_{k=0}^{\infty} a_k s^{k+r}, \quad with \quad a_0 \neq 0 \quad (2.19)$$

Differentiation gives us:

$$\psi''(s) = \sum_{k=0}^{\infty}(k+r-1)(k+r)a_k s^{k+r-2} \quad and \quad \psi'(s) = \sum_{k=0}^{\infty}(k+r)a_k s^{k+r-1} \quad (2.20)$$

Putting equation (2.20) in equation (2.18) one obtains:

$$\sum_{k=0}^{\infty} a_k s^k\{[(k+r)^2+\epsilon^2-2\frac{V_0}{b}a\beta^2]+s^2[q^2\{(k+r)^2+\epsilon^2\}-2qV_1\beta^2-2q\frac{V_0}{b}\beta^2] \\ +s[-2q(k+r)^2-2q\epsilon^2+2V_1\beta^2+2\frac{V_0}{b}(qa+1)\beta^2-2V_2\beta^2]\}=0 \quad (2.21)$$

By effecting a change of variable we obtain:

$$a_0[(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)]+\sum_{n=1}^{\infty}s^n[a_n\{(q^2+1)\{(n+r)^2+\epsilon^2\} \\ -2V_1\beta^2q-2\frac{V_0}{b}\beta^2(a+q)\}+a_{n-1}\{2V_1\beta^2+2\frac{V_0}{b}(aq+1)\beta^2-2V_2\beta^2 \\ -2q\{(n+r-1)^2+\epsilon^2\}\}]=0 \quad (2.22)$$

By solving the indicial equation $I=a_0[(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)]$,we obtain

$$(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)=0$$

$$i.e. \quad r=\pm\sqrt{\frac{-\epsilon^2(q^2+1)+2qV_1\beta^2+2\frac{V_0}{b}\beta^2(a+q)}{q^2+1}}=\pm\nu \quad (2.23)$$

For $r=\nu$ we have:

$$a_n=\prod_{i=1}^{n}\frac{2q\{(i+\nu-1)^2+\epsilon^2\}-2V_1\beta^2-2\frac{V_0}{b}(aq+1)\beta^2+2V_2\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\}-2V_1q\beta^2-2\frac{V_0}{b}(a+q)\beta^2}a_0 \quad , \quad n=1,2,....(2.24)$$

So it gets a representation of the solution

$$a_k=\prod_{i=1}^{k}\frac{2q\{(i+\nu-1)^2+\epsilon^2\}-2V_1\beta^2-2\frac{V_0}{b}(aq+1)\beta^2+2V_2\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\}-2V_1q\beta^2-2\frac{V_0}{b}(a+q)\beta^2}a_0 \quad , \quad k=1,2,....(2.25)$$

Using the relations (2.23) and (2.13) , we obtain the energy eigenvalue associated with the wave function. We can express the solutions obtained based on the Jacobi polynomial [106]:this result is more accurate.The coefficients of the solution being assessed explicitly,we seek the bounded solutions.We will only retain the negative value .

2.5 Discussion

In this subsection we consider some special cases of the potential in consideration:
(I) Hulthén Potential:

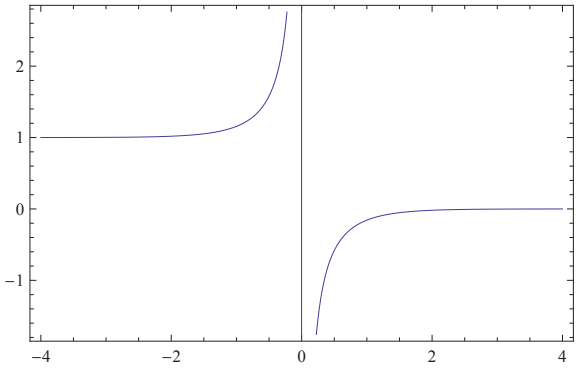

Figure 2.2: The Hulthén Potential with unit value of α, q.

If we set $V_0 = V_2 = 0$ and $a = 0$ and $b = 1$,the potential in (2.2) reduces to

$$\mathcal{V}(x) = -V_1 \frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \tag{2.26}$$

which is the Hulthén potential.

Furthermore we get the eigen values ϵ from the equation

$$\epsilon^2 = -\left[\frac{8\gamma^2 - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{2.27}$$

and the eigen function is

$$\psi(s) = B_n s^{-\xi}(1 - qs)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{2.28}$$

where $\gamma^2 = 2\zeta^2, \omega^2 = 0, \eta^2 = (\frac{1}{4} - \zeta^2), \xi^2 = -\epsilon^2$.

Again, applying Frobenius method we obtain

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2 + \epsilon^2\} - 2V_1\beta^2}{(q^2+1)\{(i+\nu)^2 + \epsilon^2\} - 2V_1 q\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{2.29}$$

(II) Rosen-Morse Potential:

If we set $V_1 = V_2 = 0$ and $a = -1$ and $b = 1$,the potential in (2.2) reduces to

$$\mathcal{V}(x) = -V_0 \frac{1 + e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \tag{2.30}$$

which is the Rosen-Morse potential [106].

Furthermore we get the eigen values ϵ from the equation

$$\epsilon^2 = \omega^2 - \left[\frac{8(\gamma^2 - \omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{2.31}$$

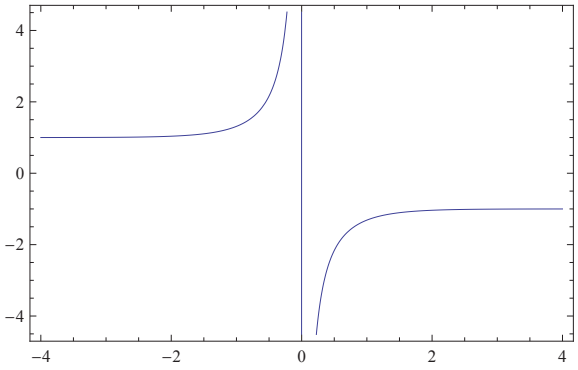

Figure 2.3: The Rosen-Morse Potential with unit value of α, q.

and the eigen function is

$$\psi(s) = B_n s^{-\xi}(1-qs)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{2.32}$$

where $\gamma^2 = 2\frac{V_0(aq+1)}{b}\frac{E+m_0}{4\alpha^2 q}$,$\zeta^2 = \frac{-V_0(aq-1)}{b}\frac{E+m_0}{4\alpha^2 q}$, $\omega^2 = 2\frac{V_0}{b}\frac{E+m_0}{4\alpha^2 q}$, $\eta^2 = (\frac{1}{4}+\omega^2+\zeta^2-\gamma^2)$, $\xi^2 = \omega^2 - \epsilon^2$.

Again, applying Frobenius method we obtain

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2+\epsilon^2\} - 2\frac{V_0}{b}(aq+1)\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\} - 2\frac{V_0}{b}(a+q)\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{2.33}$$

(III) shape parameter $q = 1$:

For N-U method we have the wave function as

$$\psi(s) = B_n s^{-\xi}(1-s)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{2.34}$$

one obtains the eigen values of ϵ from the following equation:

$$\omega^2 - \epsilon^2 = \left[\frac{8(\gamma^2-\omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1)+2\eta}\right]^2 \tag{2.35}$$

For Frobenius method, we have

$$a_k = \prod_{i=1}^{k} \frac{\{(i+\nu-1)^2+\epsilon^2\} - V_1\beta^2 - \frac{V_0}{b}(a+1)\beta^2 + V_2\beta^2}{\{(i+\nu)^2+\epsilon^2\} - V_1\beta^2 - \frac{V_0}{b}(a+1)\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{2.36}$$

2.6 Conclusions

In this chapter, the exact solution of the effective mass K-G equation for the modified Eckart plus Hylleraas potential in absence of Lorentz scalar potential. The eigen values and eigen functions are obtained using the Frobenius method as well as Nikiforov-Uvarov method. We gave a schematic graphical representation of the modified Eckart

plus Hylleraas potential with a shape parameter 'q' and also the graphical representation of Hulthén and Rosen-Morse Potential. The eigen values of the potential reduces to that of well known potentials viz., Hulthén Potential in equation (2.26) and Rosen-Morse Potential in equatioon (2.30), when we make appropriate choices of parameter a, b, V_0, V_1, V_2 . Finally we also obtain the wave function which is expressed in terms of the Jacobi Polynomials.

3

Massive Dirac Particle in Generalized Asymmetric Manning-Rosen Potential

3.1 Introduction

In recent years the study of Quantum Mechanical Systems within the framework of position dependent mass(PDM) has received much attention in the literature[107-111].Quantum particles with PDM constitute useful models for the study of many physical problems, for example determination of electronic properties of semiconductor hetero-structure[112] , the properties of hetero-junctions, quantum dots[113,114] ,^{3}He clusters [115], metal clusters [116], the properties of heterojuncions [107], quantum wells [14], semi-conductors [117], the study of condensed matter physics of impurities in crystals [17] and the density of energy in many body problems.The investigation of relativistic effect is important in the study of heavy atoms or heavy ions[107] . For this type of particles Dirac equation where the mass becomes a function, plays an important role.

Dirac equation has been used for the study of relativistic heavy ion collisions, heavy ion spectroscopy, laser matter interaction and specially in higher energy physics and

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condensed matter physics[118]. In physics it is very important to understand the structure of nucleus, atoms, molecules and the material objects.In order to get a complete information about a Quantum Mechanical system it is needed to study the scattering and the bound states. The scattering and bound states in non-relativistic and relativistic Quantum Mechanics with a potential are studied in order to describe the behaviour and the interaction between atoms and particles. Thus it is important to create a model which contain potential concepts. Some potentials that describes the interaction between nuclie and nuclie-particle and the structures of diatomic and polyatomic molecules are Kratzer-fuez[68], Rosen-Morse[106], Wood-Saxon[94], Morse[88], Hulthén[119], Cusp[120], Eckart potentials[81,82] etc.

In this article we have considered Generalised Asymmetric Manning Rosen (GAMR) potential[83]. The GAMR potential was first proposed by Manning and Rosen [85] in 1932 to define the vibrational behaviour of diatomic molecules. After that it has been used to describe the interaction between two atoms of a diatomic molecule. Some potentials can be generalized to describe the interactions consisting of more than one process.

The GAMR potential taken for our study is of the following form,

$$\mathcal{V}(x) = \Theta(-x)\left[\frac{Ae^{2\alpha(x+x_0)}}{(\Lambda+\Delta e^{\alpha(x+x_0)})^2} + \frac{Be^{2\alpha(x+x_0)}}{(\Lambda+\Delta e^{\alpha(x+x_0)})}\right]$$

$$+\Theta(x)\left[\frac{Ce^{-2\beta(x-\tilde{x_0})}}{(\tilde{\Lambda}+\tilde{\Delta}e^{-\beta(x-\tilde{x_0})})^2} + \frac{De^{-\beta(x-\tilde{x_0})}}{(\tilde{\Lambda}+\tilde{\Delta}e^{-\beta(x-\tilde{x_0})})}\right] \tag{3.1}$$

Where $\Theta(x)$ is Heaviside step function and all the parameters are real.

The shape of GAMR potential varies according to the values of the parameters. It becomes a potential barrier if A, B, C and D are positive and it becomes a potential well if A, B, C and D are negative.

The mass function is chosen here as $m(x) = m_0 + f(x)$, where $f(x) = \mathcal{V}(x)$and m_0 is the rest mass.

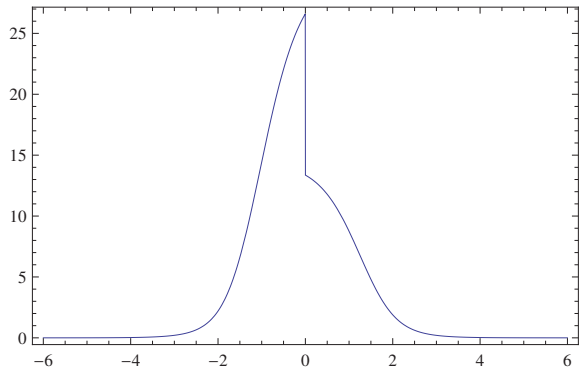

Figure 3.1: The Generalised Asymmetric Manning Rosen Potential for $\alpha = \beta = 2$,$x_0 = \tilde{x_0} = 1$,$A = B = C = D = 1$,$\Lambda = \tilde{\Lambda} = 0.1$ and $\Delta = 0.2$, $\tilde{\Delta} = 0.3$ (for smaller values of αx_0 and $\beta \tilde{x_0}$)

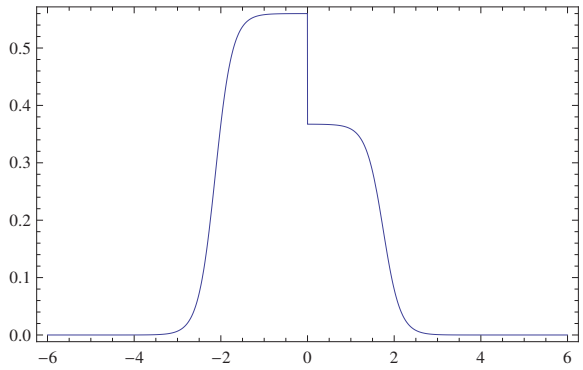

Figure 3.2: The Generalised Asymmetric Manning Rosen Potential for $\alpha = \beta = 5$,$x_0 = \tilde{x_0} = 2$,$A = B = C = D = 1$,$\Lambda = 1$,$\tilde{\Lambda} = 10$ and $\Delta = 2.5$, $\tilde{\Delta} = 3.5$ (for larger values of αx_0 and $\beta \tilde{x_0}$)

3.2 Dirac equation with position dependent mass

The Dirac equation for relativistic free-particle[121] (in natural units $\hbar$ =c=1) is as

$$[i\gamma^{\mu}\partial_{\mu} - m(x)]\psi(x) = 0 \tag{3.2}$$

where $m(x)$ is the Dirac particle mass, depends on one spatially coordinate x. To obtain the one-dimensional Dirac equation for the external potential $\mathcal{V}(x)$ we consider the gamma matrices γ_x and $\gamma_0\gamma$ in terms of Pauli matrices $i\sigma(x)$ and $i\sigma(z)$ respectively,

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} - [E - \mathcal{V}(x)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + m(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \times \begin{pmatrix} U_1(x) \\ U_2(x) \end{pmatrix} = 0 \tag{3.3}$$

where $U_1(x)$ and $U_2(x)$ are decomposed into upper and lower components of the two-component wave function $\psi(x)$ and (3.3) turns into the following two-coupled differential equations :

$$\frac{dU_1(x)}{dx} = -\Big[m(x) + E - \mathcal{V}(x)\Big]U_2(x)$$

$$\frac{dU_2(x)}{dx} = -\Big[m(x) - E + \mathcal{V}(x)\Big]U_1(x) \tag{3.4}$$

According to Flügge[122] the following expressions are obtained

$$\Theta(x) = U_1(x) + iU_2(x)$$

$$\Phi(x) = U_1(x) - iU_2(x) \tag{3.5}$$

By putting (3.5) into (3.4) we get the following equations:-

$$\frac{d\Theta(x)}{dx} = i[E - \mathcal{V}(x)]\Theta(x) - im(x)\Phi(x) \tag{3.6}$$

$$\frac{d\Phi(x)}{dx} = -i[E - \mathcal{V}(x)]\Phi(x) + im(x)\Theta(x) \tag{3.7}$$

From equations (3.6) and (3.7) for $\Theta(x)$ and $\Phi(x)$ we get the following two distinct second order differential equations as follows:

$$\frac{d^2\Theta(x)}{dx^2} - \frac{1}{m(x)}\frac{dm(x)}{dx}\frac{d\Theta(x)}{dx} + \Bigg\{[E-\mathcal{V}(x)]^2 - m(x)^2 + i\frac{d\mathcal{V}(x)}{dx} + i[E-\mathcal{V}(x)]\frac{1}{m(x)}\frac{dm(x)}{dx}\Bigg\}\Theta(x) = 0 \tag{3.8}$$

$$\frac{d^2\Phi(x)}{dx^2} - \frac{1}{m(x)}\frac{dm(x)}{dx}\frac{d\Phi(x)}{dx} + \Bigg\{[E-\mathcal{V}(x)]^2 - m(x)^2 - i\frac{d\mathcal{V}(x)}{dx} - i[E-\mathcal{V}(x)]\frac{1}{m(x)}\frac{dm(x)}{dx}\Bigg\}\Phi(x) = 0 \tag{3.9}$$

The mass function for the Dirac particle is chosen as

$$m(x) = m_0 + f(x), \tag{3.10}$$

where the function $f(x)$ is given by

$$f(x) = \Theta(-x)\left[\frac{Ae^{2\alpha(x+x_0)}}{(\Lambda+\Delta e^{\alpha(x+x_0)})^2} + \frac{Be^{2\alpha(x+x_0)}}{(\Lambda+\Delta e^{\alpha(x+x_0)})}\right]$$

$$+\Theta(x)\left[\frac{Ce^{-2\beta(x-\tilde{x_0})}}{(\tilde{\Lambda}+\tilde{\Delta}e^{-\beta(x-\tilde{x_0})})^2} + \frac{De^{-\beta(x-\tilde{x_0})}}{(\tilde{\Lambda}+\tilde{\Delta}e^{-\beta(x-\tilde{x_0})})}\right]$$

and the rest mass for the Dirac particle is m_0 ; therefore the derivative term of the mass function $m(x)$ is ignored in (3.8) and (3.9) .
So the equations (3.8) and (3.9) becomes respectively as:

$$\frac{d^2\Theta(x)}{dx^2} + \Bigg\{[E-\mathcal{V}(x)]^2 - m(x)^2 + i\frac{d\mathcal{V}(x)}{dx}\Bigg\}\Theta(x) = 0 \tag{3.11}$$

$$\frac{d^2\Phi(x)}{dx^2} + \Bigg\{[E-\mathcal{V}(x)]^2 - m(x)^2 - i\frac{d\mathcal{V}(x)}{dx}\Bigg\}\Phi(x) = 0 \tag{3.12}$$

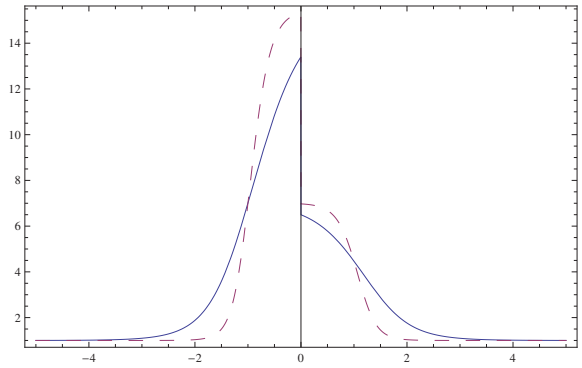

Figure 3.3: The mass variation for Generalised Asymmetric Manning Rosen Potential for $\alpha = \beta = 5$(for larger values of αx_0 and $\beta \tilde{x_0}$) ,$x_0 = \tilde{x_0} = 1$,$A = B = C = D = 1$,$\Lambda = \tilde{\Lambda} = 0.2$ and $\Delta = 0.3$, $\tilde{\Delta} = 0.5$, $\alpha = \beta = 2$,(for smaller values of αx_0 and $\beta \tilde{x_0}$) and taking $m_0 = 1$

3.3 Scattering State Solution of Dirac Equation

As we have to find out the solutions for region $x < 0$ and $x > 0$ thus we consider a new variable $y = -\frac{\Delta}{\Lambda} e^{\alpha(x+x_0)}$ in (3.11)for region $x < 0$. Then the equation (3.11) becomes,

$$\alpha^2 y^2 \frac{d^2\Theta_L(y)}{dy^2} + \alpha^2 y \frac{d\Theta_L(y)}{dy} + \left\{ (E^2 - {m_0}^2) - 2(E + m_0)\left(\frac{Ay^2}{\Delta^2(1-y)^2} - \frac{By}{\Delta(1-y)} \right) \right\} \Theta_L(y) = 0 \quad (3.13)$$

We consider a trial wave function ,

$$\Theta_L(y) = y^{\xi}(1-y)^{\lambda}\omega(y) \quad (3.14)$$

Then equation (3.13) reduces to the Gaussian Differential Equation[123] given in following equation,

$$y(1-y)\frac{d^2\omega}{dy^2} + \left\{ 1 + 2\xi - (2\xi + 2\lambda + 1)y \right\} \frac{d\omega}{dy} - (\xi + \lambda + \eta)(\xi + \lambda - \eta)\omega = 0 \quad (3.15)$$

The parameters ξ , λ and η are given by,

$$\xi = \frac{i\rho}{\alpha}, \quad where \qquad \rho = \sqrt{E^2 - {m_0}^2}$$

$$\lambda = \frac{1}{2} + \sqrt{\frac{2A(E+m_0)}{\alpha^2\Delta^2} + \frac{1}{4}}$$

$$\eta = \sqrt{-\frac{(E^2 - {m_0}^2)}{\alpha^2} + \frac{2(E+m_0)}{\Delta\alpha^2}[\frac{A}{\Delta} + B]} \tag{3.16}$$

The solution of (3.15) is found in the form of hypergeometric function as

$$\omega(y) = N_1 {}_2F_1(\xi + \lambda + \eta, \xi + \lambda - \eta, 1 + 2\xi; y)$$

$$+ N_2 y^{-2\xi} {}_2F_1(-\xi + \lambda + \eta, -\xi + \lambda - \eta, 1 - 2\xi; y) \tag{3.17}$$

Then the whole left-hand solution of equation (3.11) i.e. the solution of equation (3.13) becomes,

$$\begin{aligned}\Theta_L(y) &= N_1 y^{\xi}(1-y)^{\lambda} {}_2F_1(\xi + \lambda + \eta, \xi + \lambda - \eta, 1 + 2\xi; y) \\ &+ N_2 y^{-\xi}(1-y)^{\lambda} {}_2F_1(-\xi + \lambda + \eta, -\xi + \lambda - \eta, 1 - 2\xi; y)\end{aligned} \tag{3.18}$$

Now we choose a new variable $z = -\frac{\tilde{\Delta}}{\tilde{\Lambda}} e^{-\beta(x - \tilde{x_0})}$ for equation (3.11) for region $x > 0$ and then equation (3.11) becomes,

$$\beta^2 z^2 \frac{d^2\Theta_R(z)}{dz^2} + \beta^2 z \frac{d\Theta_R(z)}{dz}$$

$$+ \left\{ (E^2 - {m_0}^2) - 2(E+m_0)\left(\frac{Cz^2}{\tilde{\Delta}^2(1-z)^2} - \frac{Dz}{\tilde{\Delta}(1-z)} \right) \right\} \Theta_R(z) = 0 \tag{3.19}$$

We consider a trial wave function ,

$$\Theta_R(z) = z^{\tilde{\xi}}(1-z)^{\tilde{-\lambda}} \nu(z) \tag{3.20}$$

Then equation (3.19) reduces to the Gaussian Differential Equation given in following equation,

$$z(1-z)\frac{d^2\nu}{dz^2} + \left\{ 1 + 2\tilde{\xi} - (2\tilde{\xi} + 2\tilde{\lambda} + 1)z \right\} \frac{d\nu}{dz} - (\tilde{\xi} - \tilde{\lambda} + \tilde{\eta})(\tilde{\xi} - \tilde{\lambda} - \tilde{\eta})\nu = 0 \tag{3.21}$$

The parameters $\tilde{\xi}$, $\tilde{\lambda}$ and $\tilde{\eta}$ are given by,

$$\tilde{\xi} = \frac{i\rho}{\beta}, \quad where \qquad \rho = \sqrt{E^2 - {m_0}^2}$$

$$\tilde{\lambda} = -\frac{1}{2} + \sqrt{\frac{2C(E+m_0)}{\beta^2\tilde{\Delta}^2} + \frac{1}{4}}$$

$$\tilde{\eta} = \sqrt{-\frac{(E^2 - {m_0}^2)}{\beta^2} + \frac{2(E+m_0)}{\tilde{\Delta}\beta^2}[\frac{C}{\tilde{\Delta}} - D]} \tag{3.22}$$

The solution of (3.21) is found in the form of hypergeometric function as

$$\nu(z) = N_3 2F_1(\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, \tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 + 2\tilde{\xi}; z)$$

$$+ N_4 z^{-2\tilde{\xi}} 2F_1(-\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, -\tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 - 2\tilde{\xi}; z) \tag{3.23}$$

Then the whole right-hand solution of equation (3.11) i.e. the solution of equation (3.19) becomes,

$$\begin{aligned}\Theta_R(z) = {} & N_3 z^{\tilde{\xi}}(1-z)^{-\tilde{\lambda}} 2F_1(\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, \tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 + 2\tilde{\xi}; z) \\ & + N_4 z^{-\tilde{\xi}}(1-z)^{-\tilde{\lambda}} 2F_1(-\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, -\tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 - 2\tilde{\xi}; z)\end{aligned} \tag{3.24}$$

3.4 Transmission and Reflection Coefficient for Electric Current Density

By using the asymptotic behaviours as $x \to -\infty$ and as $x \to +\infty$ for the solutions given in equation(3.18) and equation(3.24) we get the transmission(T) and reflection(R) coefficients.

As $x \to -\infty$ the left-hand solution i.e. equation (3.18) becomes,

$$\Theta_L(x) = N_1(\frac{\Delta}{\Lambda})^{\frac{i\rho}{\alpha}} e^{\frac{-\pi\rho}{\alpha}} e^{i\rho(x+x_0)} + N_2(\frac{\Delta}{\Lambda})^{\frac{-i\rho}{\alpha}} e^{\frac{\pi\rho}{\alpha}} e^{-i\rho(x+x_0)} \tag{3.25}$$

and as $x \to +\infty$ the right hand solution i.e. equation (3.24) becomes,

$$\Theta_R(x) = N_4(\frac{\tilde{\Delta}}{\tilde{\Lambda}})^{-\frac{i\rho}{\beta}} e^{\frac{\pi\rho}{\beta}} e^{i\rho(x-\tilde{x_0})} \tag{3.26}$$

So to find the electric current density $J(x)$ for Dirac particle

$$J(x) = \frac{1}{2}[|\Theta(x)|^2 - |\Phi(x)|^2] \tag{3.27}$$

By putting equation (3.25) and (3.26) in equation (3.6) we have ,
the asymptotic behaviours for left hand solution $\Phi_L(x)$ becomes,

$$\Phi_L(x) = N_1(\frac{\Delta}{\Lambda})^{\frac{i\rho}{\alpha}} e^{\frac{-\pi\rho}{\alpha}} e^{i\rho(x+x_0)} \frac{(E-\rho)}{m(x)} + N_2(\frac{\Delta}{\Lambda})^{\frac{-i\rho}{\alpha}} e^{\frac{\pi\rho}{\alpha}} e^{-i\rho(x+x_0)} \frac{(E+\rho)}{m(x)} \tag{3.28}$$

and the asymptotic behaviours for right hand solution $\Phi_R(x)$ becomes,

$$\Phi_R(x) = N_4(\frac{\tilde{\Delta}}{\tilde{\Lambda}})^{-\frac{i\rho}{\beta}} e^{\frac{\pi\rho}{\beta}} e^{i\rho(x-\tilde{x_0})} \frac{(E-\rho)}{m(x)} \tag{3.29}$$

Using equations (3.25),(3.26),(3.28) and (3.29) in equation (3.27) we get left hand value of $J(x)$ i.e. $J_L(x)$ and the right hand value of $J(x)$ i.e. $J_R(x)$.
From $J_L(x)$ and $J_R(x)$ we can calculate the value of J_{trans}(transmitted current) , J_{inc}(incident current) and J_{ref}(reflected current) .Finally the reflection coefficient(R) and the transmission coefficient(T) are found as the following equations respectively:

$$R = \frac{J_{ref}}{J_{inc}} = \left|\frac{N_2}{N_1}\right|^2 \frac{E+\rho}{E-\rho} e^{\frac{4\pi\rho}{\alpha}} \tag{3.30}$$

$$T = \frac{J_{trans}}{J_{inc}} = \left|\frac{N_4}{N_1}\right|^2 e^{2\pi\rho(\frac{1}{\alpha}+\frac{1}{\beta})} \tag{3.31}$$

where R and T satisfy the condition $R + T = 1$. For the clear view of the coefficients we have to use continuity condition

$$\Theta_L(x=0) = \Theta_R(x=0) \qquad and \tag{3.32}$$

$$\frac{d\Theta_L}{dx}\Big|_{x=0} = \frac{d\Theta_R}{dx}\Big|_{x=0} \tag{3.33}$$

using the continuity condition we obtain the following result:-

$$\frac{N_4}{N_1} = \frac{D_2K_2[(R_1+R_2)D_1 + R_3D_4] - D_1K_1[(R_4+R_5)D_2 + R_6D_5]}{D_2K_2[(R_7+R_8)D_3 + R_9D_6] - D_3K_3[(R_4+R_5)D_2 + R_6D_5]}$$

$$\frac{N_2}{N_1} = \frac{D_1K_1[(R_7+R_8)D_3 + R_9D_6] - D_3K_3[(R_1+R_2)D_1 + R_3D_4]}{D_3K_3[(R_4+R_5)D_2 + R_6D_5] - D_2K_2[(R_7+R_8)D_3 + R_9D_6]} \tag{3.34}$$

where the abbreviations are given below:-
$\sigma = -\frac{\Delta}{\Lambda}$
$\tilde{\sigma} = -\frac{\tilde{\Delta}}{\tilde{\Lambda}}$
$K_1 = \sigma^{\xi} e^{\alpha\xi x_0}(1 - \sigma e^{\alpha x_0})^{\lambda}$

$K_2 = \sigma^{-\xi} e^{-\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}$
$K_3 = \tilde{\sigma}^{-\tilde{\xi}} e^{-\beta\tilde{\xi}\tilde{x_0}}(1-\tilde{\sigma} e^{\beta\tilde{x_0}})^{-\tilde{\lambda}}$
$D_1 = 2F_1(\xi+\lambda+\eta, \xi+\lambda-\eta, 1+2\xi; \sigma e^{\alpha x_0})$
$D_2 = 2F_1(-\xi+\lambda+\eta, -\xi+\lambda-\eta, 1-2\xi; \sigma e^{\alpha x_0})$
$D_3 = 2F_1(-\tilde{\xi}-\tilde{\lambda}+\tilde{\eta}, -\tilde{\xi}-\tilde{\lambda}-\tilde{\eta}, 1-2\tilde{\xi}; \tilde{\sigma} e^{\beta\tilde{x_0}})$
$D_4 = 2F_1(\xi+\lambda+\eta+1, \xi+\lambda-\eta+1, 2+2\xi; \sigma e^{\alpha x_0})$
$D_5 = 2F_1(-\xi+\lambda+\eta+1, -\xi+\lambda-\eta+1, 2-2\xi; \sigma e^{\alpha x_0})$
$D_6 = 2F_1(-\tilde{\xi}-\tilde{\lambda}+\tilde{\eta}+1, -\tilde{\xi}-\tilde{\lambda}-\tilde{\eta}+1, 2-2\tilde{\xi}; \tilde{\sigma} e^{\beta\tilde{x_0}})$
$R_1 = \sigma^{\xi}(\alpha\xi) e^{\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}$
$R_2 = \sigma^{(\xi+1)}(-\lambda\alpha) e^{\alpha(\xi+1)x_0}(1-\sigma e^{\alpha x_0})^{\lambda-1}$
$R_3 = \sigma^{\xi} e^{\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}\frac{(\xi+\lambda+\eta)(\xi+\lambda-\eta)}{(1+2\xi)}$
$R_4 = \sigma^{-\xi}(-\alpha\xi) e^{-\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}$
$R_5 = \sigma^{(-\xi+1)}(-\lambda\alpha) e^{-\alpha(1-\xi)x_0}(1-\sigma e^{\alpha x_0})^{(\lambda-1)}$
$R_6 = \sigma^{-\xi} e^{-\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}\frac{(-\xi+\lambda+\eta)(-\xi+\lambda-\eta)}{(1-2\xi)}$
$R_7 = \tilde{\sigma}^{-\tilde{\xi}}(\beta\tilde{\xi}) e^{-\beta\tilde{\xi}\tilde{x_0}}(1-\tilde{\sigma} e^{\beta\tilde{x_0}})^{-\tilde{\lambda}}$
$R_8 = \tilde{\sigma}^{(-\tilde{\xi}+1)}(\tilde{\lambda}\beta) e^{-\beta(\tilde{\xi}-1)\tilde{x_0}}(1-\tilde{\sigma} e^{\beta\tilde{x_0}})^{-\tilde{\lambda}}$
$R_9 = \tilde{\sigma}^{-\tilde{\xi}} e^{-\beta\tilde{\xi}\tilde{x_0}}(1-\tilde{\sigma} e^{\beta\tilde{x_0}})^{-\tilde{\lambda}}\frac{(-\tilde{\xi}-\tilde{\lambda}+\tilde{\eta})(-\tilde{\xi}-\tilde{\lambda}-\tilde{\eta})}{(1-2\tilde{\xi})}$

3.5 Bound State and Ground State Solution for Energy Eigen Value

CaseI:-Solution for Negative Region($x<0$)

We put $V(x) \to -V(x)$ in equation (3.1) to obtain the shape of the potential as a potential well and for the potential well we get the bound states solution.Thus we consider a new variable $y = -\frac{\Delta}{\Lambda} e^{\alpha(x+x_0)}$ in (3.11)for region $x<0$. Then the equation (3.11) becomes,

$$\alpha^2 y^2 \frac{d^2\Theta_L(y)}{dy^2} + \alpha^2 y \frac{d\Theta_L(y)}{dy}$$

$$+\left\{(E^2 - {m_0}^2) - 2(E+m_0)\left(-\frac{Ay^2}{\Delta^2(1-y)^2} + \frac{By}{\Delta(1-y)}\right)\right\}\Theta_L(y) = 0 \quad (3.35)$$

We consider a trial wave function ,

$$\Theta_L(y) = y^{\xi_1}(1-y)^{\lambda_1}P(y) \tag{3.36}$$

Then equation (3.35) reduces to the Gaussian Differential Equation given in following equation,

$$y(1-y)\frac{d^2P}{dy^2} + \left\{1 + 2\xi_1 - (2\xi_1 + 2\lambda_1 + 1)y\right\}\frac{dP}{dy}$$

$$- (\xi_1 + \lambda_1 + \eta_1)(\xi_1 + \lambda_1 - \eta_1)P = 0 \tag{3.37}$$

The parameters ξ_1 , λ_1 and η_1 are given by,

$$\xi_1 = \frac{i\rho}{\alpha}, \quad where \qquad \rho = \sqrt{E^2 - {m_0}^2}$$

$$\lambda_1 = \frac{1}{2} + \sqrt{\frac{-2A(E+m_0)}{\alpha^2\Delta^2} + \frac{1}{4}}$$

$$\eta_1 = \sqrt{-\frac{(E^2 - {m_0}^2)}{\alpha^2} + \frac{2(E+m_0)}{\Delta\alpha^2}[\frac{-A}{\Delta} - B]} \tag{3.38}$$

The solution of (3.37) is found in the form of hypergeometric function as

$$P(y) = A_1 2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; y) + \tag{3.39}$$

$$A_2 y^{-2\xi_1} 2F_1(-\xi_1 + \lambda_1 + \eta_1, -\xi_1 + \lambda_1 - \eta_1, 1 - 2\xi_1; y) \tag{3.40}$$

Then the left hand solution of equation (3.11) i.e. the solution of equation (3.36) becomes,

$$\Theta_L(y) = A_1 y^{\xi_1}(1-y)^{\lambda_1} 2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; y)$$

$$+ A_2 y^{-\xi_1}(1-y)^{\lambda_1} 2F_1(-\xi_1 + \lambda_1 + \eta_1, -\xi_1 + \lambda_1 - \eta_1, 1 - 2\xi_1; y) \tag{3.41}$$

CaseII:-Solution for Positive Region($x > 0$)

Now we choose a new variable $z = -\frac{\tilde{\Delta}}{\tilde{\Lambda}}e^{-\beta(x-\tilde{x_0})}$ for equation (3.11) for region $x > 0$ and then equation (3.11) becomes,

$$\beta^2 z^2 \frac{d^2\Theta_R(z)}{dz^2} + \beta^2 z \frac{d\Theta_R(z)}{dz}$$

$$+\left\{(E^2-{m_0}^2)-2(E+m_0)\left(\frac{-Cz^2}{\tilde{\Delta}^2(1-z)^2}-\frac{-Dz}{\tilde{\Delta}(1-z)}\right)\right\}\Theta_R(z)=0 \quad (3.42)$$

We consider a trial wave function ,

$$\Theta_R(z)=z^{\tilde{\xi_1}}(1-z)^{-\tilde{\lambda_1}}Q(z) \quad (3.43)$$

Then equation (3.41) reduces to the Gaussian Differential Equation given in following equation,

$$z(1-z)\frac{d^2Q}{dz^2}+\left\{1+2\tilde{\xi_1}-(2\tilde{\xi_1}+2\tilde{\lambda_1}+1)z\right\}\frac{dQ}{dz}$$

$$-(\tilde{\xi_1}-\tilde{\lambda_1}+\tilde{\eta_1})(\tilde{\xi_1}-\tilde{\lambda_1}-\tilde{\eta_1})Q=0 \quad (3.44)$$

The paramters $\tilde{\xi_1}$, $\tilde{\lambda_1}$ and $\tilde{\eta_1}$ are given by,

$$\tilde{\xi_1}=\frac{i\rho}{\beta}, \quad where \qquad \rho=\sqrt{E^2-{m_0}^2}$$

$$\tilde{\lambda_1}=-\frac{1}{2}+\sqrt{\frac{-2C(E+m_0)}{\beta^2\tilde{\Delta}^2}+\frac{1}{4}}$$

$$\tilde{\eta_1}=\sqrt{-\frac{(E^2-{m_0}^2)}{\beta^2}+\frac{2(E+m_0)}{\tilde{\Delta}\beta^2}[-\frac{C}{\tilde{\Delta}}-D]} \quad (3.45)$$

The solution of (3.43) is found in the form of hypergeometric function as

$$Q(z)=A_3 2F_1(\tilde{\xi_1}-\tilde{\lambda_1}+\tilde{\eta_1},\tilde{\xi_1}-\tilde{\lambda_1}-\tilde{\eta_1},1+2\tilde{\xi_1};z)+$$
$$A_4 z^{-2\tilde{\xi_1}}2F_1(-\tilde{\xi_1}-\tilde{\lambda_1}+\tilde{\eta_1},-\tilde{\xi_1}-\tilde{\lambda_1}-\tilde{\eta_1},1-2\tilde{\xi_1};z)$$

$$(3.46)$$

Then the whole right-hand solution of equation (3.11) i.e. the solution of equation of equation (3.42) becomes,

$$\Theta_R(z)=A_3 z^{\tilde{\xi_1}}(1-z)^{-\tilde{\lambda_1}}2F_1(\tilde{\xi_1}-\tilde{\lambda_1}+\tilde{\eta_1},\tilde{\xi_1}-\tilde{\lambda_1}-\tilde{\eta_1},1+2\tilde{\xi_1};z)$$
$$+A_4 z^{-\tilde{\xi_1}}(1-z)^{-\tilde{\lambda_1}}2F_1(-\tilde{\xi_1}-\tilde{\lambda_1}+\tilde{\eta_1},-\tilde{\xi_1}-\tilde{\lambda_1}-\tilde{\eta_1},1-2\tilde{\xi_1};z) \quad (3.47)$$

By the boundary condition i.e. as $x \to \mp\infty$, $y \to 0$ and $z \to 0$ and the wave functions go to zero at infinity we get from equation (3.40) and (3.46) $A_2 = A_4 = 0$.
Then equation (3.40) becomes,

$$\Theta_L(y) = A_1 y^{\xi_1}(1-y)^{\lambda_1} 2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; y) \tag{3.48}$$

and the equation (3.46) becomes,

$$\Theta_R(z) = A_3 z^{\tilde{\xi_1}}(1-z)^{-\tilde{\lambda_1}} 2F_1(\tilde{\xi_1} - \tilde{\lambda_1} + \tilde{\eta_1}, \tilde{\xi_1} - \tilde{\lambda_1} - \tilde{\eta_1}, 1 + 2\tilde{\xi_1}; z) \tag{3.49}$$

CaseIII:-Solution for Ground state($x = 0$)
Then by using the continuity condition given by

$$\Theta_L(x=0) = \Theta_R(x=0) \qquad and \tag{3.50}$$

$$\frac{d\Theta_L}{dx}\Big|_{x=0} = \frac{d\Theta_R}{dx}\Big|_{x=0} \tag{3.51}$$

using the continuity condition we obtain from equation(3.47) and (3.48), the expression for the energy eigen values given by:-

$$[(C_5 - C_6)C_1 + C_7 C_3]C_{11}C_1 - [(C_8 - C_9)C_2 + C_{10}C_4]C_{12}C_2 = 0 \tag{3.52}$$

where the abbreviations are:-
$S_1 = -\frac{\Delta}{\Lambda}$
$S_2 = -\frac{\tilde{\Delta}}{\tilde{\Lambda}}$
$C_1 = 2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; Se^{\alpha x_0})$
$C_2 = 2F_1(\tilde{\xi_1} - \tilde{\lambda_1} + \tilde{\eta_1}, \tilde{\xi_1} - \tilde{\lambda_1} - \tilde{\eta_1}, 1 + 2\tilde{\xi_1}); \tilde{S_1}e^{\beta\tilde{x_0}})$
$C_3 = 2F_1(\xi_1 + \lambda_1 + \eta_1 + 1, \xi_1 + \lambda_1 - \eta_1 + 1, 2 + 2\xi_1; Se^{\alpha x_0})$
$C_4 = 2F_1(\tilde{\xi_1} - \tilde{\lambda_1} + \tilde{\eta_1} + 1, \tilde{\xi_1} - \tilde{\lambda_1} - \tilde{\eta_1} + 1, 2 + 2\tilde{\xi_1}); \tilde{S_1}e^{\beta\tilde{x_0}})$
$C_5 = \alpha\xi_1 S_1^{\xi_1} e^{\alpha\xi_1 x_0}(1 - S_1 e^{\alpha x_0})^{\lambda_1}$
$C_6 = \alpha\lambda_1 S_1^{\xi_1+1} e^{\alpha(\xi_1+1)x_0}(1 - S_1 e^{\alpha x_0})^{\lambda_1 - 1}$
$C_7 = S_1^{\xi_1} e^{\alpha\xi_1 x_0}(1 - S_1 e^{\alpha x_0})^{\lambda_1} \frac{(\xi_1+\lambda_1+\eta_1)(\xi_1+\lambda_1-\eta_1)}{(1+2\xi_1)}$
$C_8 = -\beta\tilde{\xi_1}\tilde{S_1}^{\tilde{\xi_1}} e^{\beta\tilde{\xi_1}\tilde{x_0}}(1 - \tilde{S_1}e^{\beta\tilde{x_0}})^{-\tilde{\lambda_1}}$
$C_9 = \beta\tilde{\lambda_1}\tilde{S_1}^{\tilde{\xi_1}} e^{(\tilde{\xi_1}+1)\beta\tilde{x_0}}(1 - \tilde{S_1}e^{\beta\tilde{x_0}})^{-\tilde{\lambda_1}-1}$
$C_{10} = \tilde{S_1}^{\tilde{\xi_1}} e^{\beta\tilde{\xi_1}\tilde{x_0}}(1 - \tilde{S_1}e^{\beta\tilde{x_0}})^{-\tilde{\lambda_1}} \frac{(\tilde{\xi_1}-\tilde{\lambda_1}+\tilde{\eta_1})(\tilde{\xi_1}-\tilde{\lambda_1}-\tilde{\eta_1})}{(1+2\tilde{\xi_1})}$
$C_{11} = S_1^{\xi_1} e^{\alpha\xi_1 x_0}(1 - S_1 e^{\alpha x_0})^{\lambda_1}$
$C_{12} = \tilde{S_1}^{\tilde{\xi_1}} e^{\beta\tilde{\xi_1}\tilde{x_0}}(1 - \tilde{S_1}e^{\beta\tilde{x_0}})^{-\tilde{\lambda_1}}$

3.6 Conclusions

The scattering state and the bound state solutions for the one dimensional Dirac equation with position dependent mass for GAMR potential is obtained in this study. Solving the equation for the positive and negative region we get the wave function in terms of the hypergeometric function.Bound state and ground state solutions are also obtained. For the solution at $x = 0$ we use continuity condition and finally an expression for energy eigen value is obtained. Transmission(T) and reflection(R) coefficients for electric current density for this equation are found and the unitary condition $R+T = 1$ is preserved for PDM case.For bound state we get the discrete spectrum for GAMR potential well with effective PDM , finding an exact condition for the energy eigen values. Here we gave a schematic graphical representation of the GAMR potential and the mass distribution.

4

Klein-Gordon Equation with Double Ring Shaped Coulomb Potential

4.1 Introduction

In relativistic quantum mechanics, scattering state solutions of the relativistic wave equation play an important role for some physical potentials. Recently, there has been an increased interest in finding the scattering states solutions for non-relativistic and relativistic quantum mechanics with central and non-central potentials. B.Talukdar, A. Yunus and M.R. Amin have obtained the s-wave bound state and scattering state solutions of Klein-Gordon equation for the Hulthen potential[124]. F. Dominguez-Adame and A. Rodriguez obtained the solutions for the relativistic Screened Coulomb Potential[125]. Alhaidari obtained the scattering and bound state solutions of the three-dimensional Scrödinger equation for non-central potentials (special cases considered as the Aharonov-Bohm, Hartmann and magnetic monopole potentials)[126], Chen et al. studied relativistic scattering with the Coulomb plus a new ring-shaped potential [127]. Wei et al. investigated approximately analytical scattering state solutions of the l-wave Scrödinger equation for the Eckart potential by a proper approximation to the centrifugal term[82]. Wei et al. also studied approximately analytical scattering state

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solutions of the l-wave Scrödinger equation for the Manning-Rosen potential[128]. M. Movahedi et al. investigated the relativistic scattering state solutions of the Makarov potential [129]. Arda et al. obtained the scattering solutions of the one-dimensional Schrödinger equation for the Woods-Saxon potential within the position-dependent mass formalism [130].

Particularly the Coulombic ring shaped Potential is introduced by Hartmann et al.[131] in Quantum Chemistry to describe ring shaped molecules, such as benzene. The double ring shaped Coulomb potential is a 3-dimensional Coulomb potential surrounded by a double ring shaped inverse square potential. Mainly the double ring shaped Coulomb potential is a non-central potential in spherical coordinate and can be written as:-

$$\mathcal{V}(r,\theta) = -\frac{A}{r} + \frac{B}{r^2 sin^2\Theta} + \frac{C}{r^2 cos^2\Theta}$$

where, $A = \eta\sigma^2 e^2$, $B = \frac{\hbar^2\eta^2\sigma^2}{2\mu}$, $C = \frac{\hbar^2}{2\mu}a$ where,$a \geq 0$ μ is the mass of the particle, η and σ are positive real parameters which range from 1 to 10 and r and Θ are the spherical coordinates. In equation (1) when we put $B = 0$ and $C = 0$ then $\mathcal{V}(r)$ reduces to coulomb potential and when $B = 0$ then V(r) reduces to ring shaped Hartmann potential. Specially the ring shaped non central potentials are used to describe the molecular structure[132] and the interaction between the deformed nuclie [133,202] and specially the molecular structure of benzene[134].

The method includes the super symmetric approach [86,135-136], the Variational method [137],the exact quantization rule (EQR) [128,138-141], the hypervirial perturbation [142], the shifted $\frac{1}{N}$ expansion (SE) [143-152], the modified shifted $\frac{1}{N}$ expansion (MSE) [153], smooth transformation [154], series method [155], the path integral approach [156], the standard methods [157-159], the perturbative treatment [160-166], the algebraic method [167], the shape invariant method [168], the Frobenius method [57,95], the Laplace transform approach [66-69], the asymptotic iteration method (AIM) [58-65] and the Nikiforov-Uvarov method (NU) [46-55]. However, since the Klein-Gordon equation is a second order differential equation, the NU method is more suitable for obtaining analytical solutions to such a differential equation.

4.2 Stationary Radial and Angle-dependent Klein-Gordon Equation

The stationary 3D K-G equation with the coupling of a vector potential $\mathcal{V}(r)$ and a scalar potential $S(r)$ for a particle of rest mass m_0 in the natural units $\hbar = \mathrm{c} = 1$ can be expressed as

$$\nabla^2\Psi(r,\theta,\phi) + [(E - \mathcal{V}(r,\theta,\phi))^2 - (m_0 + S(r,\theta,\phi)^2)]\Psi(r,\theta,\phi) = 0 \tag{4.1}$$

where, E,$\mathcal{V}(r)$ and $S(r)$ are the relativistic energy of the particle, vector and scaler potentials, respectively. Assuming $\mathcal{V}(r) = S(r)$ we get from Eqn. (4.1),

$$\nabla^2\Psi(r,\theta,\phi) + [(E^2 - {m_0}^2) - 2(E + m_0)V(r,\theta)]\Psi(r,\theta,\phi) = 0 \tag{4.2}$$

Now considering the double ring shaped Coulomb potential the KG-equation reduces to,

$$\nabla^2\Psi(r,\theta,\phi) + [(E^2 - {m_0}^2) - 2(E + m_0)\{-\frac{A}{r} + \frac{B}{r^2 sin^2\Theta} + \frac{C}{r^2 cos^2\Theta}\}]\Psi(r,\theta,\phi) = 0 \tag{4.3}$$

To separate the variables for the stationary wave function we assume,

$$\Psi(r,\theta,\phi) = \frac{R(r)}{r}\frac{\Theta(\theta)}{sin^{\frac{1}{2}}\theta}\Phi(\phi) \tag{4.4}$$

By following the standard procedure of separation of variables we get the component equations as follows:-

$$\frac{d^2R}{dr^2} + (\delta^2 + \frac{\lambda A}{r} - \frac{\alpha^2}{r^2})R(r) = 0 \tag{4.5}$$

$$\frac{d^2\Theta}{d\theta^2} - [\frac{\lambda B}{sin^2\theta} + \frac{\lambda C}{cos^2\theta} - \frac{1}{4} - \frac{1}{4sin^2\theta} - \alpha^2 + \frac{\beta^2}{sin^2\theta}]\Theta(\theta) = 0 \tag{4.6}$$

$$\frac{d^2\Phi}{d\phi^2} = -\beta^2\Phi(\phi) \tag{4.7}$$

where, $\lambda = 2(E + m_0)$, $\delta^2 = (E^2 - {m_0}^2)$, represents the relativistic energy of a particle and α^2 and β^2 are separation constants. Putting $\alpha^2 = l(l+1)$, which

we often encounter in various Schrödinger quantum systems, with the orbital angular momentum $l = 0,1,2,...$ and the magnetic quantum number $\beta = 0,\pm1,\pm2,...$.

The solution of equation (4.7) is the azimuthal angle solution and it is,

$$\Phi(\phi) = De^{i\beta\phi} \tag{4.8}$$

The equation (4.5) is radial equation and equation (4.6) is angle-dependent equation for the KG equation. For these two equations we use AIM in our next parts.

4.3 Solution Of the Radial and Angle-dependent Klein-Gordon Equation

4.3.1 Solution Of the Radial Klein-Gordon Equation :

To solve the equation (4.5) with AIM for $l \neq 0$, we should transform equation (4.5)to the form of equation (4.1).For bound state solution of the equation (4.5) we consider $R(0) = 0$ and $R(\infty) = 0$. Therefore for the physically acceptable radial solution we consider the radial wave function as follows :

$$R(r) = r^{(l+1)}e^{-i\delta}f(r) \tag{4.9}$$

Thus by substituting $y = -2i\delta r$ and taking $R(r)$ as in equation (4.5) the wave function reduces to ,

$$\frac{d^2f}{dy^2} - 2(\frac{1}{2} - \frac{l+1}{y})\frac{df}{dy} - (\frac{l+1}{y} - \frac{i\xi A}{y\delta})f(y) = 0 \tag{4.10}$$

where, $\xi = \frac{\lambda}{2}$ and $\lambda_0(y) = 2(\frac{1}{2} - \frac{l+1}{y})$ and $S_0(y) = \frac{l+1}{y} - \frac{i\xi A}{y\delta}$.

Calculating $\lambda_n(y)$ and $S_n(y)$ we get,

$$\begin{aligned}
\lambda_0(y) &= 2(\frac{1}{2} - \frac{l+1}{y}) \\
S_0(y) &= \frac{l+1}{y} - \frac{i\xi A}{y\delta} \\
\lambda_1(y) &= \frac{2(l+1)}{y^2} + \frac{(l+1)-a}{y} - \frac{4(l+1)}{y} + \frac{4(l+1)^2}{y^2} \\
S_1(y) &= \frac{(l+1-a)(y-2l-3)}{y^2} \\
\lambda_2(y) &= 1 - \frac{2(a-2l+2)}{y} + \frac{(a+3l+3)-2(l+1)(l+a)}{y^2} - \frac{4(l+1)(2l^2+9l+8)}{y^3} \\
S_2(y) &= \frac{1}{y^3}[(l+1-a)\{y^2 - y(a+3l+4) + (4l^2+16l+15)\}]...etc.
\end{aligned} \tag{4.11}$$

where, $a = \frac{i\xi A}{\delta}$ combining these results with the condition given in equation (1.47) yields:

$$\frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} \Rightarrow \left[\frac{i\xi A}{\delta}\right]_{n=0} = (l+1) \tag{4.12}$$

$$\frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} \Rightarrow \left[\frac{i\xi A}{\delta}\right]_{n=1} = 2(l+1) \tag{4.13}$$

. . .

$$\frac{S_n}{\lambda_n} = \frac{S_{(n+1)}}{\lambda_{(n+1)}} \Rightarrow \left[\frac{i\xi A}{\delta}\right]_{n} = (n+1)(l+1) \tag{4.14}$$

Thus the generalised term in equation (4.14) gives the energy spectrum of the KG-equation with double ring shaped Coulomb potential ,where n is radial quantum number ($n = 0, 1, 2, ..$).

For $A = 0$ in equation (4.5) we get a singular solution which corresponds to the inverse square potential which gives bound state only if the separation constant is negative specially less than $-\frac{1}{4}$. The bound states are determined by potential well type and hence the quantum number n is limited by the study of potential well.

Equation (4.10) satisfies the confluent hypergeometric function and the solution of the differential equation given in (4.10) can be written as,

$$f(r) = 1F_1(l+1-\frac{i\xi A}{\delta}, 2l+2; -2i\delta r) \tag{4.15}$$

Thus the radial part of the wave function can be written as,

$$R(r) = N_1 r^{(l+1)} e^{-i\delta} 1F_1(l+1-\frac{i\xi A}{\delta}, 2l+2; -2i\delta r) \tag{4.16}$$

where N_1 is the normalizing constant.

4.3.2 Solution Of the Angle-dependent Klein-Gordon Equation :

The angular part of the KG equation for double ring shaped coulomb potential is given by equation (4.6). We consider $\tilde{l} = l + \frac{1}{2}$ and $l(l+1) = \tilde{l}^2 - \frac{1}{4}$ and the equation (4.6) becomes ,

$$\frac{d^2\Theta}{d\theta^2} - \left[\frac{\lambda C}{cos^2\theta} + \frac{\lambda B - \frac{1}{4} + \beta^2}{sin^2\theta}\right]\Theta = -\tilde{l}^2\Theta(\theta) \tag{4.17}$$

Defining P and Q as follows,

$$P = -\frac{1}{2} \pm \sqrt{\lambda B + \beta^2}\ and \tag{4.18}$$

$$Q = -\frac{1}{2} \pm \sqrt{\lambda C + \frac{1}{4}} \tag{4.19}$$

Equation (4.17) reduces to,

$$\frac{d^2\Theta}{d\theta^2} - \left[\frac{Q(Q+1)}{cos^2\theta} + \frac{P(P+1)}{sin^2\theta}\right]\Theta = -\tilde{l}^2\Theta(\theta) \tag{4.20}$$

To solve this equation by AIM with boundary conditions i.e. $\theta(0)$ and $\theta(\pi)$ are finite we consider the following wave function,

$$\Theta(\theta) = sin^{P+1}\theta cos^{Q+1}\theta f(\theta) \tag{4.21}$$

By using this wave function in equation (4.20) we get the second order homogeneous differential equation as,

$$\frac{d^2f}{d\theta^2} = 2\{(Q+1)tan\theta - (P+1)cot\theta\}\}\frac{df}{d\theta} + [(P+Q+2)^2 - (l+\frac{1}{2})^2]f \tag{4.22}$$

Now by using AIM method we have,

$$s_0 = [(P+Q+2)^2 - (l+\frac{1}{2})^2] \tag{4.23}$$

$$\lambda_0 = 2\{(Q+1)tan\theta - (P+1)cot\theta\} \tag{4.24}$$

$$s_1 = 2[(P+Q+2)^2 - (l+\frac{1}{2})^2]\{(Q+1)tan\theta - (P+1)cot\theta\} \tag{4.25}$$

$$\lambda_1 = 2\{(Q+1)sec^2\theta + (P+1)cosec^2\theta\} + [(P+Q+2)^2 - (l+\frac{1}{2})^2] + 4[\{(Q+1)tan\theta - (P+1)cot\theta\}]^2 \tag{4.26}$$

and so on . Thus combining these results with the condition given by equation (1.47) we have ,

$$\frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} \Rightarrow \quad \tilde{l}^2 = (P+Q+2)^2 \tag{4.27}$$

$$\frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} \Rightarrow \quad \tilde{l}^2 = (P+Q+4)^2 \tag{4.28}$$

$$....etc. \tag{4.29}$$

and finally we get the generalized form as,

$$\tilde{l}^2 = (P+Q+2n+2)^2 \quad for \quad n = 0, 1, 2, 3, ... \tag{4.30}$$

By putting the value of P , Q and $\tilde{l}^2$ into the equation (4.30) we get value of l,

$$l = \sqrt{\lambda B + \beta^2} + \sqrt{\lambda C + \frac{1}{4}} + 2n + \frac{1}{2} \tag{4.31}$$

now by inserting the value of l into the generalized form of energy eigen value for the radial part of the KG-equation with double ring shaped coulomb potential given by equation (4.14), we get the relativistic energy spectrum for a bound electron from the following equation,

$$\left[\frac{i\xi A}{\delta}\right]_n = (n+1)(\sqrt{\lambda B + \beta^2} + \sqrt{\lambda C + \frac{1}{4}} + 2n + \frac{1}{2} + 1) \tag{4.32}$$

By putting $B = C = 0$ into the above equation we get the energy eigen value for isotropic harmonic oscillator and by putting $B = 0$ and $C \neq 0$ into the above equation we get the eigen values for the ring shaped oscillator.

For the eigen function of the angular part we substitute $cos^2(\theta) = x$ in the equation (4.22) and it reduces to the form :

$$x(1-x)\frac{d^2f}{dx^2} + [(Q+\frac{3}{2}) - (P+Q+3)x]\frac{df}{dx} + \frac{1}{4}[(P+Q+2)^2 - (l+\frac{1}{2})^2]f = 0$$

(4.33)

The solution of the above differential equation is of the form of Gauss hypergeometric function given by,

$$f(\theta) = 2F_1(-n, P+Q+2+n, Q+\frac{3}{2}; cos^2(\theta) \tag{4.34}$$

Therefore, using the value of $f(\theta)$ in the equation (4.21) and we get the eigen function for the angular part as follows:

$$\Theta(\theta) = N_2 sin^{P+1}\theta cos^{Q+1}\theta 2F_1(-n, P+Q+2+n, Q+\frac{3}{2}; cos^2(\theta)) \tag{4.35}$$

where N_2 is the normalizing constant.

4.4 The Wave Function

Combining equations (4.8), (4.16) and (4.35) we obtain the total wave function for the double ring shaped coulomb potential in spherical co-ordinates as

$$\Psi(r,\theta,\phi) = A_n r^l e^{-i(\delta-\beta\phi)} sin^{P+\frac{1}{2}}\theta cos^{Q+1}\theta 1F_1(l+1-\frac{i\xi A}{\delta}, 2l+2; -2i\delta r)$$

$$2F_1(-n, P+Q+2+n, Q+\frac{3}{2}; cos^2(\theta)) \tag{4.36}$$

where A_n is a normalization constant.

4.5 Conclusions

We have solved the KG-equation for the Double Ring Shaped coulomb Potential via Asymptotic Iteration Method .Using this method we get the general expression of the energy spectrums and the corresponding wave function in terms of confluent hypergeometric function multi-dimensional space .From the solution of the radial part we get the bound state solution of the inverse square potential by equating $A = 0$ in equation (4.5) and we also get a solution of the bound state of potential well type .From the solution of angular part we get the energy eigen values for isotropic harmonic oscillator and for the ring shaped oscillator by equating the coefficients in equation (4.32) with appropriate values . The double ring shaped potentials have many applications in the field of nuclear physics and quantum chemistry which are mainly used to describe the interaction between the deformed pair of nuclei in Physics and to describe the molecular structure of benzene in chemistry. Thus the non-central potential named double ring shaped Coulomb potential is used to find the quantum information in chemical and Molecular Physics.

5

Quantum mechanical system with position dependent mass

5.1 Introduction

A quantum mechanical particle endowed with a position-dependent effective mass constitutes an interesting and useful model for the study of many physical problems.The effective mass approximation is a very important method in semiconductor physics to study dynamic and static properties of charge carriers without complexity due to the lattice potential of the material. The effective-mass approach has also been used as a computational method to deal with nonuniform crystals.

To determine the electronic properties of semiconductors effective-mass theory is an important and extensively used tool. The theory is well established for homogeneous materials with small perturbations [24]. The effective-mass theory has also been applied to non-uniform materials in which the carrier effective mass depends on position.

The concept of effective mass also plays an important role within the strictures of the energy density functional approach to the quantum many body problem. The energy density functional formalism has yielded reasonable theoretical predictions of many experimental properties for several quantum many body systems. Within the energy density functional approach, the non-local terms of the associated potential can be often expressed as a position dependence on an appropriate effective mass. Besides

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its practical applications, the study of quantum mechanical systems with a position dependent mass also raises interesting conceptual problems of a fundamental nature.

It is important to create a model which contain potential concepts i.e. to describe the behaviour and interaction between atoms and particles. Potentials play important role to describe the interaction between nuclei , nuclear particle and the structures of the diatomic molecules. Various potentials are used to analyze the nature of vibration of Quantum System such as pseudo-harmonic [69,169-171], modified Eckart plus Hylleraas [78-82], morse type [88], Wood-Saxon [54,92-94], Rosen-Morse [64,106], harmonic oscillator [172] specially on lower dimensions. The solutions are also crucial in quantum soluble systems. Methods involve in literature are Nikiforov-Uvarov method [46-55], asymptotic iteration method [58-65], Point-Cannonical transformation [71-75], Lie algebraic method [173], super symmetry approach [136], factorization method [174] etc.

Here we use Manning Rosen potential(MRP) [85] to solve the Schrödinger equation with PDM. These type of potentials are used to describe the quark interactions [175] in particle and high energy physics, spectroscopy [176] in nuclear physics, binding energy and inclusive momentum distributions [177] in atomic physics, the inter and intra molecular interactions and atomic pair correlations in molecular physics/chemistry [178-179].

S.M.Ikdhair [180] has considered a mass function $m = \frac{m_0}{(1-\delta e^{\frac{-\alpha(r-r_0)}{r_0}})^2}$, where m_0 is the rest mass and δ is a free parameter and $0 \leq \delta < 1$ to deal with the q-deformed morse potential. Here we use a similar mass function $m = \frac{m_0}{(1-qe^{-\alpha r})^2}$, where m_0 is the rest mass and α determines the inverse range of potential. To investigate the behaviour of MRP within the frame work of Schrödinger equation we use Pekeris approximation [77] and applying some simple constraints we can construct mass function such that the equation can be solved by LTA.

One of the most effective and different method to solve Schrödinger equation with PDM for a hyperbolic potential is Laplace transform method (LTM). LTM is an integral transform method which has been used by many authors [66-67]. It is a powerful method and it helps us to solve second order differential equation by converting them into a simpler form whose solutions may be obtained easily. Thus LTM is a very effective method to solve the radial equations.

Throughout this chapter we have applied Laplace transform method by converting the wave equation in suitable form for the method with the help of point canonical

transformation as well as Pekeris approximation.

5.2 Hamiltonian with position dependent mass for Manning Rosen Potential:

The most general form of Hamiltonian for the position dependent mass $m = m(r)$, is given by,

$$\mathcal{H} = \frac{1}{4(a+1)}\left\{ a\Big[\frac{1}{m}\mathbf{P}^2 + \mathbf{P}^2\frac{1}{m}\Big] + m^{\alpha}\mathbf{P}\mathbf{m}^{\beta}\mathbf{P}\mathbf{m}^{\gamma} + m^{\gamma}\mathbf{P}\mathbf{m}^{\beta}\mathbf{P}\mathbf{m}^{\alpha}\right\} + \mathbf{V}(\mathbf{r}) \tag{5.1}$$

where $\mathcal{P}$ denotes the momentum operator and $\mathcal{V}(\nabla)$ is an arbitrary potential. also α, β, γ and a are the ambiguity parameters satisfying the constrain $\alpha + \beta + \gamma = -1$ and r is the radial coordinate. The commutation relation by the differentiating properties of the momentum operator $\mathbf{P}$ is,

$$[\mathcal{P}, f(r)] = \mathcal{P}f - f\mathcal{P} = -i\hbar\frac{df}{dr}\hat{r} \tag{5.2}$$

where *f(r)* is the arbitrary function of the radial coordinate r. Using equation (5.2), (5.1) turns into:

$$\mathcal{H} = \frac{1}{2m}\mathcal{P}^{\in} + \frac{i\hbar}{2}\frac{1}{m^2}\frac{dm}{dr}\mathcal{P}_{\nabla} + U_{\alpha,\beta,\gamma,a}(r) \tag{5.3}$$

where

$$U_{\alpha,\beta,\gamma,a}(r) = -\frac{\hbar^2}{4m^3(a+1)}\left[(\alpha+\gamma-a)m\frac{d^2m}{dr^2} + 2(a-\alpha-\gamma-\alpha\gamma)(\frac{dm}{dr})^2\right] + \mathbf{V}(\mathbf{r}) \tag{5.4}$$

Imposing some conventional constrain on ambiguity parameters like $(\alpha + \gamma - a) = 0$ and $(a - \alpha - \gamma - \alpha\gamma) = 0$ with two possible solutions: (i) $\alpha = 0$ and $a = \gamma$ (ii) $a = \alpha$ and $\gamma = 0$, the effective potential can be reduced to $U_{\alpha,\beta,\gamma,a}(r) = V(r)$. Here, we are interested in this case where the Schrödinger equation yeilds:

$$-\frac{\hbar^2}{2m}\left[\nabla^2 - \frac{1}{m}\frac{dm}{dr}\nabla\right]\varphi(r) = [E - \mathcal{V}(\nabla)]\varphi(r) \tag{5.5}$$

The wave function can be separated to the following form :

$$\varphi(r) = \frac{1}{r}\psi(r)Y(\theta,\phi). \tag{5.6}$$

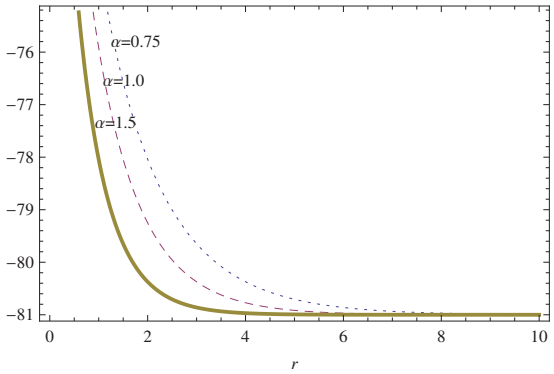

Figure 5.1: Manning-Rosen potential versus r with $V_2 = \frac{1}{3}V_1$ and $V_1 = 81\ eV$ for different values of α

Using equation (5.6) into (5.5), one can easily obtain the radial wave equation as:

$$\left[\frac{d^2}{dr^2} - \frac{1}{m}\frac{dm}{dr}\left(\frac{d}{dr} - \frac{1}{r}\right) - \frac{l(l+1)}{r^2}\right]\psi(r) = -\frac{2m}{\hbar^2}[E - \mathcal{V}(\nabla)]\psi(r) \tag{5.7}$$

To eliminate first derivative term we use the transformation based on point canonical transformation method[181]:

$$\psi(r) = \sqrt{m(r)}\phi(r). \tag{5.8}$$

Substituting equation (5.8) into equation (5.7) one obtains:

$$\frac{d^2\phi(r)}{dr^2} + \left[\frac{1}{2m}\frac{d^2m}{dr^2} - \frac{3}{4}\left(\frac{1}{m}\frac{dm}{dr}\right)^2 + \frac{1}{rm}\frac{dm}{dr} - \frac{l(l+1)}{r^2}\right]\phi(r) = -\frac{2m}{\hbar^2}[E - \mathcal{V}(\nabla)]\phi(r) \tag{5.9}$$

The Manning-Rosen potential is:

$$\mathcal{V}(\nabla) = \frac{-V_1(1 + qe^{-\alpha r})}{(1 - qe^{-\alpha r})} + \frac{V_2 e^{-2\alpha r}}{(1 - qe^{-\alpha r})^2} \tag{5.10}$$

The range of potential is determined by the dimensionless parameter α, q is the deformation parameter and V_1 and V_2 are two general potential parameters.

The solution is mainly depending on replacing the orbital centrifugal term of singularity with the help of a suitable transformation for Pekeris approximation as

$$r \to \frac{r - r_0}{r} \tag{5.11}$$

Here r_0 is the equilibrium position of molecules.

The centrifugal potential barrier term of equation (5.9) can be written using (5.11) as:

$$\frac{l(l+1)}{r^2} = \frac{l(l+1)}{{r_0}^2}\frac{1}{(1+r)^2}$$
$$\cong \frac{l(l+1)}{{r_0}^2}\left[C_0 + C_1\frac{e^{-\alpha r}}{(1-qe^{\alpha r})} + C_2\frac{e^{-2\alpha r}}{(1-qe^{\alpha r})^2}\right] \tag{5.12}$$

where,

$$C_0 = 1 - \frac{1}{\alpha}(1-q)(3+q) + \frac{3}{\alpha^2}(1-q)^2; C_1 = \frac{2}{\alpha}(1-q)^2(2+q) - \frac{6}{\alpha^2}(1-q)^3;$$
$$C_2 = -\frac{1}{\alpha}(1-q)^3(1+q) + \frac{3}{\alpha^2}(1-q)^4 \tag{5.13}$$

taking $q \to 0$,

$$C_0 = 1 - \frac{3}{\alpha} + \frac{3}{\alpha^2}; C_1 = \frac{4}{\alpha} - \frac{6}{\alpha^2}; C_2 = \frac{3}{\alpha^2} - \frac{1}{\alpha} \tag{5.14}$$

In the same way, we expand the term $\frac{1}{rm}\frac{dm}{dr}$ in equation (5.9) as follows:

$$\frac{1}{rm}\frac{dm}{dr} = \frac{1}{r_0 m}\frac{dm}{dr}\frac{1}{(1+r)} \cong \frac{1}{r_0 m}\left[B_0 + B_1\frac{e^{-\alpha r}}{(1-qe^{\alpha r})} + B_2\frac{e^{-2\alpha r}}{(1-qe^{\alpha r})^2}\right] \tag{5.15}$$

$$B_0 = 1 - \frac{2(1+q)}{\alpha(1+2q)} + \frac{(1+q)}{\alpha^2(1+4q)} - \frac{(1+q)}{2\alpha(1+2q)};$$
$$B_1 = \frac{2}{\alpha(1+2q)} - \frac{2}{\alpha^2(1+4q)}; B_2 = -\frac{1}{2\alpha(1+2q)} + \frac{1}{\alpha^2(1+4q)} \tag{5.16}$$

taking $q \to 0$,

$$B_0 = 1 + \frac{1}{\alpha^2} - \frac{3}{2\alpha}; B_1 = \frac{2}{\alpha} - \frac{2}{\alpha^2}; B_2 = -\frac{1}{2\alpha} + \frac{1}{\alpha^2} \tag{5.17}$$

By substituting equation (5.10) -(5.12) and equation (5.15) to equation (5.9) and taking,

$$y = e^{-\alpha r} \tag{5.18}$$

we have,

$$\left[y^2\frac{d^2}{dy^2} + y\frac{d}{dy} + \Lambda\right]\phi = \frac{l(l+1)}{\alpha^2{r_0}^2}\left(C_0 + C_1 y + C_2 y^2\right)\phi \tag{5.19}$$

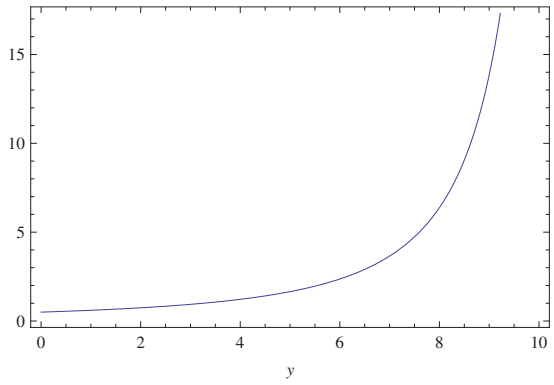

Figure 5.2: Graphical representation of mass distribution for $q = 0.9$

where,Λ is as follows:

$$\Lambda = \frac{y^2}{2m}\frac{d^2m}{dy^2} - \frac{3}{4}\frac{y^2}{m^2}(\frac{dm}{dy})^2 + \frac{y}{m}\frac{dm}{dy}\left[\frac{1}{2} - \frac{1}{\alpha r_0}(B_0 + B_1 y + B_2 y^2)\right]\frac{m}{m_0}(P^2 + Qy + Ry^2 \tag{5.20}$$

where

$$P^2 = -\frac{2(E+V_1)m_0}{\alpha^2\hbar^2}; Q = -\frac{2(2qV_1 - V_2)m_0}{\alpha^2\hbar^2}; R = -\frac{4(-qV_2 + V_1q^2)m_0}{\alpha^2\hbar^2} \tag{5.21}$$

Using the following effective mass distribution:

$$m = \frac{m_0}{(1-qy)^2} \tag{5.22}$$

where $m \to m_0$ when $q \to 0$ and m_0 is the rest mass.

Considering mass distribution given in equation (5.22) we can convert Λ as follows:

$$\Lambda = D_0 + D_1 y + D_2 y^2 \tag{5.23}$$

where

$$D_0 = -P^2 \;\; ; \;\; D_1 = -Q \;\; ; \;\; D_2 = -R \tag{5.24}$$

Substituting equation (5.23) in equation (5.19) we have,

$$y^2\frac{d^2\phi}{dy^2} + y\frac{d\phi}{dy} - [\mu^2 - \nu^2 + \eta^2 y^2]\phi = 0 \tag{5.25}$$

where

$$-\mu^2 = D_0 - \frac{l(l+1)}{\alpha^2 {r_0}^2}C_0; \nu^2 = D_1 - \frac{l(l+1)}{\alpha^2 {r_0}^2}C_1; -\eta^2 = D_2 - \frac{l(l+1)}{\alpha^2 {r_0}^2}C_2 \tag{5.26}$$

5.3 Bound State solution:

The Södinger equation with position dependent mass for Manning-Rosen Potential turns into equation (5.25) by using the mass distribution given in (5.22).We consider the function ϕ as follows to get finite solutions for large values of y:

$$\phi(y) = y^{-\mu} f(y) \tag{5.27}$$

then equation (5.25) turns into,

$$y^2 \frac{d^2 f}{dy^2} - (2\mu - 1) y \frac{df}{dy} + y(\nu^2 - \eta^2 y^2) f(y) = 0 \tag{5.28}$$

By using Laplace Transform[70] equation (5.28) can be transformed into,

$$(s^2 - \eta^2) \frac{dF}{ds} + [(2\mu + 1)s - \nu^2] F(s) = 0 \tag{5.29}$$

Therefore the equation given in equation (5.28) transform into a first order differential equation given in (5.29) and the solutions are in the form

$$F(s) = N(y + \eta)^{-(2\mu+1)} \left(\frac{y - \eta}{y + \eta} \right)^{\frac{\nu^2}{2\eta} - \frac{(2\mu+1)}{2}} \tag{5.30}$$

where N is a constant.To have a well-behaved wave function we must impose the condition,

$$\frac{\nu^2}{\eta} - (2\mu + 1) = 2n, \quad \text{where, } n = 0, \pm 1, \pm 2, ... \tag{5.31}$$

Here n is positive or negative according as the magnitude of $\frac{\nu^2}{\eta}$ is greater or smaller than the magnitude of $(2\mu + 1)$ .To get the finite solution for large y the parameter μ needs to be large enough as per equation (5.27). We can get positive value of μ for sufficiently large values of $\frac{\nu^2}{\eta}$ and for positive or negative values of n. Again we can get positive value of μ for smaller values of $\frac{\nu^2}{\eta}$, which is only possible for negative values of n.

To apply inverse Laplace Transform into (5.30) we expand it in power series as:

$$F(s) = \sum_{m=0}^{\infty} \frac{(2\eta)^m}{m!} \begin{cases} N_+ \frac{(-1)^m n!}{(n-m)!} (s + \eta)^{-(2\mu^+ + 1 + m)}, n > 0 \\ N_- \frac{(-1)^{2m} (n+m-1)!}{(n-1)!} (s + \eta)^{-(2\mu^- + 1 + m)}, n < 0 \end{cases} \tag{5.32}$$

where $N_{\pm}$ are two integrating constants and $\mu^{\pm}$ corresponds to positive or negative values of n as given in equation (5.31). Now applying inverse Laplace Transform to equation (5.32) we have,

$$f(y) = \sum_{m=0}^{\infty} \frac{(2\eta)^m y^m e^{-\eta y}}{m!} \begin{cases} N_+ \frac{(-1)^m n! y^{2\mu^+}}{(n-m)!\Gamma(2\mu^+ + 1 + m)}, n > 0 \\ N_- \frac{(n+m-1)! y^{2\mu^-}}{(n-1)!\Gamma(2\mu^- + 1 + m)}, n < 0 \end{cases} \tag{5.33}$$

By the series expansion of the confluent hypergeometric function equation (5.33) becomes,

$$f(y) = \begin{cases} N_+ y^{2\mu^+} e^{-\eta y} {}_1F_1(-\eta; 2\mu^+ + 1, 2\eta y), n > 0 \\ N_- y^{2\mu^-} e^{-\eta y} {}_1F_1(-\eta; 2\mu^- + 1, 2\eta y), n < 0 \end{cases} \tag{5.34}$$

Thus using equation (5.27) and (5.34) in the equation (5.8) the solution becomes,

$$\psi(y) = N\sqrt{m(y)} y^{2\mu} e^{-\eta y} {}_1F_1(-\eta; 2\mu + 1, 2\eta y) \tag{5.35}$$

where N is normalizing constant and the mass function $m(y)$ is given in equation (5.22). The parameter μ is obtained from equation (5.31) as,

$$\mu = \frac{\nu^2}{2\eta} - n - \frac{1}{2}. \tag{5.36}$$

where ν and η are given in equation (5.26).

5.4 Energy spectrum:

We get the bound state solution for position dependent Schrödinger equation with the Manning Rosen Potential via Laplace Transform Approach.We get the energy eigen function and now we will find out the energy eigen value for that function. Comparing equation (5.24) and (5.21), we get value of D_0. Using this value of D_0 and the value of μ (given by equation (5.36)) into equation (5.26), we get,

$$E = \frac{\alpha^2\hbar^2}{2m_0}\left[\frac{l(l+1)}{\alpha^2}C_0 - \left\{n + \frac{1}{2} - \frac{1}{2\eta}(D_1 - \frac{l(l+1)}{\alpha^2 {r_0}^2}C_1)\right\}^2\right] - V_1 \tag{5.37}$$

where

$$\eta = [\frac{l(l+1)}{\alpha^2 {r_0}^2}C_2 - D_2]^{\frac{1}{2}} \tag{5.38}$$

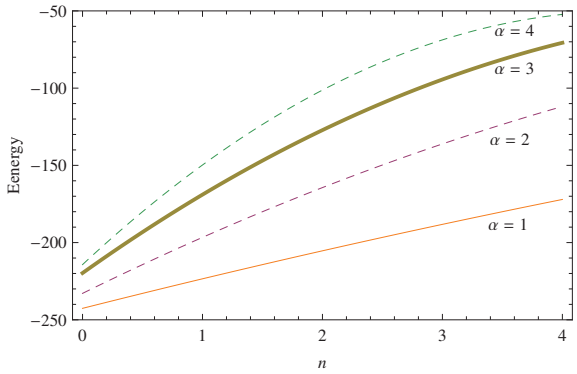

Figure 5.3: Energy spectrum for different values of α

5.5 Conclusions:

In this chapter, we have obtained the bound state solutions of the Schrödinger equation for the Manning-Rosen potential with position dependent mass.The energy equation have been obtained with the help of LTM, a powerful method to solve second order differential equation via conversion of it into a more simpler one. The eigen function have been obtained in terms of confluent hypergeometric function . The out come of this article is applicable for any kind of mass function for which one can set the condition $\Lambda = D_0 + D_1 y + D_2 y^2$. One schematic graphical representations for potential function, mass distribution and energy spectrum are presented in Fig5.1,Fig5.2 and Fig5.3 respectively.

Appendices

Appendix A:
Special functions connecting to relativistic wave equation

Hermite polynomial

The Hermite polynomial of order n is defined by

$$H_n(x) = \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r} \tag{A.1}$$

Where

$$[\tfrac{n}{2}] = \begin{cases} \frac{n}{2}, & n \quad is \quad even \\ \frac{n-1}{2}, & n \quad is \quad odd \end{cases}$$

The orthogonality property of the Hermite polynomial is

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} 0, & if \quad m \neq n \\ \sqrt{\pi} 2^n n!, & if \quad m = n \end{cases} \tag{A.2}$$

Laguerre polynomial

The Laguerre polynomial of order n is defined by

$$L_n^k(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+k)!}{(n-r)!(k+r)!r!} x^r \tag{A.3}$$

The orthogonality property of the Laguerre polynomial is

$$\int_0^\infty e^{-x}x^k L_n^k(x)L_m^k(x)dx = \begin{cases} 0, & if \quad m \neq n \\ \frac{(n+k)}{n!}, & if \quad m = n \end{cases} \tag{A.4}$$

Jacobi polynomial

The Jacobi polynomial of order n is defined by

$$J_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{r=0}^{n}\begin{pmatrix} n+\alpha \\ r \end{pmatrix}\begin{pmatrix} n+\beta \\ n-r \end{pmatrix}(x-1)^{n-r}(x+1)^r \tag{A.5}$$

The Jacobi polynomial of order n can also be expressed as

$$J_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)}\sum_{r=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix}\frac{\Gamma(n+\alpha+\beta+r+1)}{\Gamma(r+\alpha+1)}\left(\frac{x-1}{2}\right)^r \tag{A.6}$$

Where
$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$
The orthogonality property of the Jacobi polynomial is

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta J_n^{(\alpha,\beta)}(x)J_m^{(\alpha,\beta)}(x)dx = \begin{cases} 0, & if \quad m \neq n \\ \frac{2^{\alpha+\beta+1}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!(1+\alpha+\beta+n)\Gamma(1+\alpha+\beta+n)}, & if \quad m = n \end{cases} \tag{A.7}$$

Gauss Hypergeometric Function

The Gauss hypergeometric function is defined by

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r}\frac{x^r}{r} \tag{A.8}$$

An important integral formula is given by

$${}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{(\beta-1)}(1-t)^{(\gamma-\beta-1)}(1-xt)^{-\alpha}dt \tag{A.9}$$

where $Re(\alpha) > Re(\beta) > 0, \mid \gamma \mid < 1$. Now $x \to 1$. Eq. (A.9) becomes

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{A.10}$$

and

$${}_2F_1(\alpha,\beta;\beta;x) = (1-x)^{-\alpha} \tag{A.11}$$

Appendix B:
Deformed hyperbolic functions

The deformed hyperbolic functions are defined as

$$\sinh_q x = \frac{e^x - qe^{-x}}{2}, \cosh_q x = \frac{e^x + qe^{-x}}{2}, \tanh_q x = \frac{\sinh_q x}{\cosh_q x} \tag{B.1}$$

and we use the relations

$$q\mathrm{sech}_q^2 x + \tanh_q^2 x = 1, \coth_q^2 x - q\mathrm{cosech}_q^2 x = 1 \tag{B.2}$$

$$(\tanh_q x)' = q\mathrm{sech}_q^2 x, (\mathrm{cosech}_q x)' = -\mathrm{cosech}_q x \coth_q x,$$

$$(\coth_q x)' = -q\mathrm{cosech}_q^2 x \tag{B.3}$$

where prime denotes the differentiation with respect to x.

$$\cosh_q(x+iy) = \cosh_q x \cos y + i \sinh_q x \sin y \tag{B.4}$$

$$\sinh_q(x+iy) = \sinh_q x \cos y + i \cosh_q x \sin y \tag{B.5}$$

$$\sinh_q x = 2 \sinh_{\sqrt{q}} \left(\frac{x}{2}\right) \cosh_{\sqrt{q}} \left(\frac{x}{2}\right) \tag{B.6}$$

$$\cosh_q x = 2 \cosh_{\sqrt{q}}^2 \left(\frac{x}{2}\right) - \sqrt{q} = 2 \sinh_{\sqrt{q}}^2 \left(\frac{x}{2}\right) + \sqrt{q} \tag{B.7}$$

$$2 \cosh_q(x+iy) \cosh_q(x-iy) = \cosh_{q^2} 2x + q \cos 2y \tag{B.8}$$

$$2 \sinh_q(x+iy) \sinh_q(x-iy) = \cosh_{q^2} 2x - q \cos 2y \tag{B.9}$$

$$\tanh_q(x+iy) = \frac{\sinh_q x \cosh_q x + iq \sin y \cos y}{\cosh_q^2 x - q \sin^2 y} \tag{B.10}$$

$$\mathrm{sech}_q(x+iy) = \frac{\cosh_q x \cos y - i \sinh_q x \sin y}{\cosh_q^2 x - q \sin^2 y} \tag{B.11}$$

$$\mathrm{cosech}_q(x+iy) = \frac{\sinh_q x \cos y - i \cosh_q x \sin y}{\sinh_q^2 x + q \sin^2 y} \tag{B.12}$$

$$\coth_q(x+iy) = \frac{\sinh_q x \cosh_q x - iq \sin y \cos y}{\sinh_q^2 x + q \sin^2 y} \tag{B.13}$$

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List of publications

1. Relativistic Klein-Gordan Equation with Position Dependent Mass for q-deformed modified Eckart plus Hylleraas potential. *Electronic Journal of Theoretical Physics*, (2018), 14, No. 37, 79-90.

2. Massive Dirac Particle in Generalized Asymmetric Manning-Rosen Potential. *The African Review of Physics* , (2017), 12: 0007 54-60.

3. Klein-Gordon Equation with Double Ring Shaped Coulomb Potential via AIM. *Springer Nature Switzerland*,(2020), SCI 863 725-733.

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Relativistic Klein-Gordan Equation with Position Dependent Mass for q-deformed modified Eckart plus Hylleraas potential

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Abstract: Relativistic Klein-Gordan equation with Position Dependent Mass has been solved analytically for the q-deformed modified Eckart plus Hylleraas potential. A generalised series is used to obtain the bound state solutions of the K-G equation using the Frobenious Method . The one dimensional K-G equation for the mass dependent modified Eckart plus Hylleraas potential in absence of scalar potential are studied in this paper. The exactly normalized bound state wave function and energy expressions are obtained by using N-U method. Also, the bound state solutions are found for the Hulthén and Rosen-Morse potential.

Keywords: Klein-Gordan equation; q-deformed modified Eckart plus Hylleraas potential; position dependent mass; N-U method; Frobenious Method
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1 Introduction

Quantum Mechanical phenomena are described by Schrödinger equation which dictates the dynamics of quantum systems represented by Hamiltonian Operator. Solutions of Klein-Gordan Equation for some physical potential have important applications in Molecular Physics, Quantum Chemistry, Nuclear physics, condensed matter Physics, high energy physics. The study of potentials such as Hulthén [1], Morse [2], Rosen-Morse [3], Pseudo-harmonic [4], Poschl-Teller [5, 6], Kratzer-Fuez [7], generalized Wood Saxon [8], ring-shaped Hartmann [9] and the corresponding wave functions has been performed using various methods.

Recently, there has been renewed interest in solving Quantum Mechanical systems within the frame work of Nikiforov-Uvarov method[10-14].This technique is successfully

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used to solve Schrödinger, Klein-Gordan, Dirac and Duffin-Kemmer-Petieu Equqtions.

In nuclear physics, the shape form of the potential also plays an important role particularly when studying the structure of deformed nuclei or the interaction between them. Therefore , our aim , in the present work is to investigate analytical bound state solutions of the Klein-Gordon equation with q-deformed modified Eckart plus Hylleraas potential[15-19] in the Frobenius method [20] as well as in N-U method. Also, we will show that, when the deformation parameter q takes a particular value ($q = 1$), the obtained results lead to the solutions of the same problem for modified Eckart plus Hylleraas potential.

In recent years , the solutions of the non-relativistic wave equation with position-dependent mass have been a topic of great interest[21-25] , but there are only few papers that give the solution of the relativistic wave equation with position-dependent mass in quantum mechanics. Exact solution of the Dirac equation with position-dependent mass in the Coulomb field [26], Kepler problem in Dirac theory for a particle whose potential and mass are inversely proportional to the distance from the force center [27], the approximate solution of the one-dimensional Dirac equations with spatially dependent mass for the generalized Hulthen potential [28], the exact solution of the one-dimensional K-G equation with spatially dependent mass for the inversely linear potential [29] are some papers on relativistic wave equations with position dependent mass.

Our focus is to study the quantum systems with Position Dependent Effective Mass (PDEM). PDEM Klein-Gordan Equation plays an important role in the study of electronic properties of semi-conductors in homogeneous crystals,quantum dots,He clusters, quantum liquids etc. Exact solutions of effective mass Klein-Gordan Equations are difficult to obtain, as such, approximate numerical techniques are often used. Our work is generalised as follows:- In section 2 we have discussed the NU method and Frobenious method . We give a brief discussion of Klein-Gordan Equation with position-dependent mass in section 3. In section 4 we discuss the solutions of Klein-Gordan Equation by using both the methods and section 5 is left for conclusion.

2. Overview of Nikiforov-Uvarov and Frobenius Method Method

A. Overview of Nikiforov-Uvarov Method

The N-U method is based on solving a second order linear differential equation by reducing it to a generalized hypergeometric type. In both relativistic and non-relativistic quantum mechanics, the wave equation with a given potential can be solved by this method by reducing the one dimensional K-G equation to an equation of the form :

$$\Psi''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\Psi'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\Psi(x) = 0 \tag{1}$$

Where $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials of degree atmost 2 and $\tilde{\tau}(x)$ is a polynomial of degree atmost 1 . In order to find a particular solution to equation(1) , we set the following wave function as a multiple of two independent parts

$$\Psi(x) = \Phi(x)y(x) \tag{2}$$

Thus equation (1) reduces to a hyper-geometric type equation of the form :

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$$

Where $\tau(x) = \tilde{\tau}(x) + 2\pi(x)$ satisfies the condition $\tau'(x) < 0$ and $\pi(x)$ is defined as

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{(\frac{\sigma'(x) - \tilde{\tau}(x)}{2})^2 - \tilde{\sigma}(x) + K\sigma(x)} \tag{3}$$

in which K is a parameter . Determining K is the essential point in calculation of $\pi(x)$. Since $\pi(x)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (3) can be put into order to be the square of a polynomial of first degree [10], which is possible only if its discriminant is zero. So, we obtain K by setting the discriminant of the square root equal to zero . Therefore, one gets a general quadratic equation for K . By using

$$\lambda = K + \pi'(x) = -n\tau'(x) - \frac{n(n-1)}{2}\sigma''(x) \tag{4}$$

The values of K can be used for the calculation of energy eigenvalues . Polynomial solutions $y_n(x)$ are given by the Rodrigues relation

$$y_n(x) = \frac{B_n}{\rho(x)}(\frac{d}{dx})^n[\sigma^n(x)\rho(x)] \tag{5}$$

in which B_n is a normalization constant and $\rho(x)$ is the weight function satisfying

$$\rho(x) = \frac{1}{\sigma(x)} exp \int \frac{\tau(x)}{\sigma(x)} dx \tag{6}$$

on the other hand , second part of the wave function $\phi(x)$ in relation (2) is given by

$$\phi(x) = exp \int \frac{\pi(x)}{\sigma(x)} dx \tag{7}$$

B.Overview of Frobenius Method

This method finds the solutions of a differential equation in the form of series,either a whole series,a Laurent series, or even a series involving contribute exhibitors. The difference between these situations is the properties of regularity of the equation coefficients. To do this you must put the equation in the form:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \tag{8}$$

Suppose a regular singular point x_0,singular functions P(x) and Q(x) and using the Fuck's theorem, we can write the solutions of the differential equation in the form:

$$y(x) = \sum_0^\infty a_k(x - x_0)^{k+r} \tag{9}$$

The indicial equation is obtained for

$$r(r-1) + P(0)r + Q(0) = 0 \tag{10}$$

For each found values r, we determine the values a_k and then the solutions of the differential equation.

3. Brief discussion of Klein Gordan Equation with position dependent mass

The one dimensional K-G equation for a spinless particle of mass m in the natural units $\hbar = c = 1$ can be expressed

$$\Psi''(x) + [(E - V(x))^2 - (m + S(x))^2]\Psi(x) = 0 \tag{11}$$

where E , V(x) and S(x) are the relativistic energy of the particle , vector and scalar potentials respectively. Now considering the q-deformed modified Eckart plus Hylleraas Potential of the form:

$$V(x) = \frac{V_0}{b}\left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}}\right) - V_1\frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2\frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2} \tag{12}$$

Where q is the shape parameter.

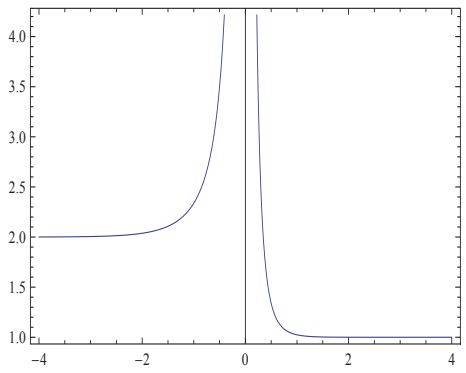

Fig.1.The modified Eckart plus Hylleraas Potential with unit value of α, a, b, q.

We prefer to use the mass function equals to the rest mass along with the vector part of the potential as

$$m(x) = m_0 + \frac{V_0}{b}\left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}}\right) - V_1\frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2\frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2} \tag{13}$$

to obtain an exactly solvable Schrödinger-like equation in absence of scalar potential . The mass function should also be a physical distribution , so we restrict ourself in the

range $0 \leq x < \infty$, which gives the finite mass values as follows :

$$m(x) = \begin{cases} m_0 + \frac{V_0}{b}(a-1) - V_1 + V_2 \ (forq \to 0) \ , x \to 0 \\ m_0 + \frac{V_0 a}{b} \qquad\qquad , x \to \infty \end{cases}$$

Actually, this distribution corresponds to shifted scalar potential function in the problem. Substituting equation (13) in equation (11) we have

$$\Psi''(x) + \left[(E^2 - m_0^2) - 2(E+m_0) \left\{ \frac{V_0}{b} \left(\frac{a - e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \right) - V_1 \right.\right. \\ \left.\left. \frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} + V_2 \frac{e^{-2\alpha x}}{(1 - qe^{-2\alpha x})^2} \right\} \right] \Psi(x) = 0 \qquad (14)$$

4A.Application of Nikiforov-Uvarov Method

Introducing a new variable $s = e^{-2\alpha x}$ it is straight forward to show that (14) takes the form:

$$\Psi''(s) + \frac{1-qs}{s(1-qs)}\Psi'(s) + \frac{1}{s^2(1-qs)^2} \Big[s^2 q^2 (\epsilon^2 - \gamma^2 - \zeta^2) \\ + 2qs(\gamma^2 - \epsilon^2) + (\epsilon^2 - \omega^2) \Big] \Psi(s) = 0 \qquad (15)$$

Where we use the notations $\frac{E^2 - m_0^2}{4\alpha^2} = \epsilon^2$,$\gamma^2 = \frac{E+m_0}{4\alpha^2 q}\{2V_1 + 2\frac{V_0}{b}(aq+1) - 2V_2\}$, $\zeta^2 = \frac{E+m_0}{4\alpha^2 q}\{(V_1+V_2) - \frac{V_0}{b}(aq-1)\}$ and $2\frac{V_0}{b}\frac{E+m_0}{4\alpha^2} = \omega^2$ comparing equation (15) with equation (1) we have

$$\begin{aligned} &\tilde{\tau}(s) = 1 - qs; \\ &\sigma(s) = s(1-qs); \\ &\tilde{\sigma}(s) = s^2 q^2 (\epsilon^2 - \gamma^2 - \zeta^2) + 2qs(\gamma^2 - \epsilon^2) + (\epsilon^2 - \omega^2); \end{aligned} \qquad (16)$$

Substituting equation (16) the relation (3) we get

$$\pi(s) = -\frac{qs}{2} \pm \sqrt{q^2 s^2 (\frac{1}{4} + \gamma^2 + \zeta^2 - \epsilon^2 - k_1) + qs(k_1 - 2\gamma^2 + 2\epsilon^2) + (\omega^2 - \epsilon^2)} \qquad (17)$$

where k_1 satisfies the relation $k = k_1 q$ Further the discriminant of the upper expression under the square root has to be set equal to zero. Therefore, we obtain

$$\Delta = q^2 (k_1 + 2\epsilon^2 - 2\gamma^2)^2 - 4q^2 (\frac{1}{4} + \gamma^2 + \zeta^2 - \epsilon^2 - k_1)(\omega^2 - \epsilon^2) \qquad (18)$$

Solving equation (18) for constant k_1 , we obtain the double roots as , $k_1\prime, k_1\prime\prime = 2(\gamma^2 - \omega^2) \pm 2\xi\eta$, where $\xi^2 = \omega^2 - \epsilon^2$ and $\eta^2 = (\frac{1}{4} + \zeta^2 + \omega^2 - \gamma^2)$.

Thus substituting these values for each k_1 into equation (17) , we obtain

$$\pi(s) = -\frac{qs}{2} \pm \begin{cases} (\xi-\eta)qs - \xi;\ for\ \ k_1\prime = 2(\gamma^2-\omega^2) + 2\xi\eta \\ (\xi+\eta)qs - \xi;\ for\ k_1\prime\prime = 2(\gamma^2-\omega^2) - 2\xi\eta \end{cases} \tag{19}$$

By choosing an appropriate value for k in $\pi(s)$ which satisfies the condition $\tau'(s) < 0$, one gets $\pi(s) = -qs(\xi+\eta+\frac{1}{2}) + \xi$ for $k = 2(\gamma^2-\omega^2) - 2\xi\eta$; giving the function:

$$\tau(s) = 1 - 2qs[1 + (\xi+\eta)] + 2\xi \tag{20}$$

If we consider $\lambda = k + \Pi'$ defined in (4) we obtain

$$\lambda = q[2(\gamma^2-\omega^2) - 2\xi\eta - \frac{1}{2} - (\xi+\eta)] \tag{21}$$

Again using equation (4) , we have:

$$\lambda_n = q[n^2 + n + 2n(\xi+\eta)] \tag{22}$$

Using the condition $\lambda = \lambda_n$ one obtains the eigen values of ϵ from the following equation:

$$\omega^2 - \epsilon^2 = \left[\frac{8(\gamma^2-\omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{23}$$

From (6) it can be shown that the weight function $\rho(s)$ is $\rho(s) = s^{2\xi}(1-qs)^{2\eta}$ and by substituting $\rho(s)$ into the Rodrigues relation (5) one gets

$$y_n(s) = \frac{B_n}{s^{2\xi}(1-qs)^{2\eta}}(\frac{d}{ds})^n[s^n(1-qs)^n s^{2\xi}(1-s)^{2\eta}] = \frac{B_n}{s^{2\xi}(1-qs)^{2\eta}} P_n^{(2\xi,2\eta)}(s) \tag{24}$$

where $P_n^{(2\xi,2\eta)}(s)$ stands for Jacobi polynomial [30] and B_n is the normalizing constant. The other part of the wave function is simply found from (7) as ,

$$\phi(s) = s^{\xi}(1-qs)^{(\frac{1}{2}+\eta)} \tag{25}$$

Finally , the wave function is obtained as follows

$$\psi(s) = B_n s^{-\xi}(1-qs)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{26}$$

4B.Application of Frobenius Method

consider the same Klein-Gordan equation and the same Eckart plus modified Hylleraas Potential given in section 3. After development , we get the following equation:

$$\psi''(s) + \frac{1}{s}\psi'(s) + \frac{1}{s^2}\left[\epsilon^2 - 2\beta^2\left\{\frac{V_0}{b}\frac{a-s}{1-qs} - V_1\frac{s}{1-qs} + V_2\frac{s}{(1-qs)^2}\right\}\right]\Psi(s) = 0 \tag{27}$$

where we use the notations $\frac{E^2-m_0^2}{4\alpha^2}=\epsilon^2$ and $\frac{E+m_0}{4\alpha^2}=\beta^2$
Comparing (27) with the equation (8) we have, $P(s)=1$ and
$Q(s)=\left[\epsilon^2-2\beta^2\left\{-V_1\frac{s}{1-qs}+\frac{V_0}{b}\frac{a-s}{1-qs}+V_2\frac{s}{(1-qs)^2}\right\}\right]$
Putting these values the equation (27) becomes ,

$$\psi''(s)+\frac{P(s)}{s}\psi'(s)+\frac{Q(s)}{s^2}\psi(s)=0 \tag{28}$$

By using Fuck's theorem , we can write :

$$\psi(s)=\sum_{k=0}^{\infty}a_k s^{k+r},\quad with\quad a_0\neq 0 \tag{29}$$

Differentiation gives us:

$$\psi''(s)=\sum_{k=0}^{\infty}(k+r-1)(k+r)a_k s^{k+r-2}\quad and\quad \psi'(s)=\sum_{k=0}^{\infty}(k+r)a_k s^{k+r-1} \tag{30}$$

Putting equation (30) in equation (28) one obtains:

$$\sum_{k=0}^{\infty}a_k s^k\{[(k+r)^2+\epsilon^2-2\frac{V_0}{b}a\beta^2]+s^2[q^2\{(k+r)^2+\epsilon^2\}-2qV_1\beta^2-2q\frac{V_0}{b}\beta^2]$$
$$+s[-2q(k+r)^2-2q\epsilon^2+2V_1\beta^2+2\frac{V_0}{b}(qa+1)\beta^2-2V_2\beta^2]\}=0 \tag{31}$$

By effecting a change of variable we obtain:

$$a_0[(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)]+\sum_{n=1}^{\infty}s^n[a_n\{(q^2+1)\{(n+r)^2+\epsilon^2\}$$
$$-2V_1\beta^2q-2\frac{V_0}{b}\beta^2(a+q)\}+a_{n-1}\{2V_1\beta^2+2\frac{V_0}{b}(aq+1)\beta^2-2V_2\beta^2$$
$$-2q\{(n+r-1)^2+\epsilon^2\}\}]=0 \tag{32}$$

By solving the indicial equation $I=a_0[(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)]$,we obtain

$$(q^2+1)(r^2+\epsilon^2)-2qV_1\beta^2-2\frac{V_0}{b}\beta^2(a+q)=0$$
$$i.e.\quad r=\pm\sqrt{\frac{-\epsilon^2(q^2+1)+2qV_1\beta^2+2\frac{V_0}{b}\beta^2(a+q)}{q^2+1}}=\pm\nu \tag{33}$$

For $r=\nu$ we have:

$$a_n=\prod_{i=1}^{n}\frac{2q\{(i+\nu-1)^2+\epsilon^2\}-2V_1\beta^2-2\frac{V_0}{b}(aq+1)\beta^2+2V_2\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\}-2V_1q\beta^2-2\frac{V_0}{b}(a+q)\beta^2}a_0\quad ,\quad n=1,2,....(34)$$

So it gets a representation of the solution

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2+\epsilon^2\} - 2V_1\beta^2 - 2\frac{V_0}{b}(aq+1)\beta^2 + 2V_2\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\} - 2V_1q\beta^2 - 2\frac{V_0}{b}(a+q)\beta^2} a_0 \quad , \quad k = 1, 2, ...(35)$$

Using the relations (33) and (23) , we obtain the energy eigenvalue associated with the wave function. We can express the solutions obtained based on the Jacobi polynomial [31]:this result is more accurate.The coefficients of the solution being assessed explicitly,we seek the bounded solutions.We will only retain the negative value .

5. Discussion

In this subsection we consider some special cases of the potential in consideration: (I) Hulthen Potential:

If we set $V_0 = V_2 = 0$ and $a = 0$ and $b = 1$,the potential in (12) reduces to

$$V(x) = -V_1 \frac{e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \tag{36}$$

which is the Hulthen potential.

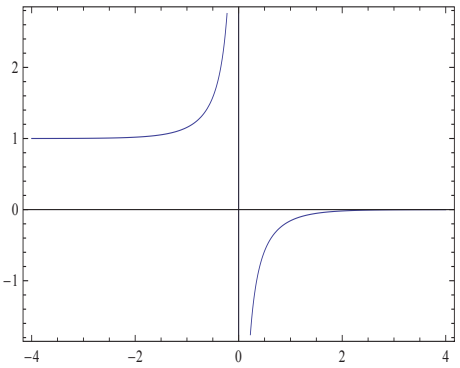

Fig.2.The Hulthén Potential with unit value of α, q.

Furthermore we get the eigen values ϵ from the equation

$$\epsilon^2 = -\left[\frac{8\gamma^2 - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{37}$$

and the eigen function is

$$\psi(s) = B_n s^{-\xi}(1-qs)^{(-\eta+\frac{1}{2})} P_n^{(2\xi,2\eta)}(s) \tag{38}$$

where $\gamma^2 = 2\zeta^2, \omega^2 = 0, \eta^2 = (\frac{1}{4} - \zeta^2), \xi^2 = -\epsilon^2$.

Again, applying Frobenius method we obtain

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2+\epsilon^2\} - 2V_1\beta^2}{(q^2+1)\{(i+\nu)^2+\epsilon^2\} - 2V_1q\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{39}$$

(II) Rosen-Morse Potential:

If we set $V_1 = V_2 = 0$ and $a = -1$ and $b = 1$,the potential in (12) reduces to

$$V(x) = -V_0 \frac{1 + e^{-2\alpha x}}{1 - qe^{-2\alpha x}} \tag{40}$$

which is the Rosen-Morse potential.

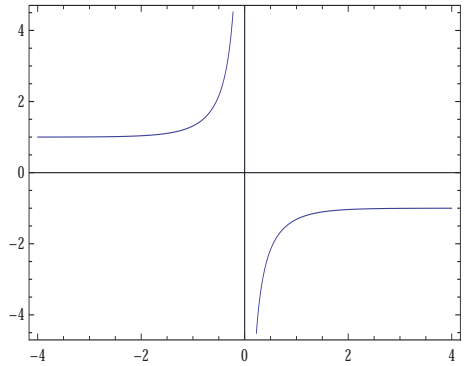

Fig.3.The Rosen-Morse Potential with unit value of α, q.

Furthermore we get the eigen values ϵ from the equation

$$\epsilon^2 = \omega^2 - \left[\frac{8(\gamma^2 - \omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{41}$$

and the eigen function is

$$\psi(s) = B_n s^{-\xi}(1 - qs)^{(-\eta + \frac{1}{2})} P_n^{(2\xi, 2\eta)}(s) \tag{42}$$

where $\gamma^2 = 2\frac{V_0(aq+1)}{b}\frac{E+m_0}{4\alpha^2 q}$,$\zeta^2 = \frac{-V_0(aq-1)}{b}\frac{E+m_0}{4\alpha^2 q}$, $\omega^2 = 2\frac{V_0}{b}\frac{E+m_0}{4\alpha^2 q}$, $\eta^2 = (\frac{1}{4} + \omega^2 + \zeta^2 - \gamma^2)$, $\xi^2 = \omega^2 - \epsilon^2$.

Again, applying Frobenius method we obtain

$$a_k = \prod_{i=1}^{k} \frac{2q\{(i+\nu-1)^2 + \epsilon^2\} - 2\frac{V_0}{b}(aq+1)\beta^2}{(q^2+1)\{(i+\nu)^2 + \epsilon^2\} - 2\frac{V_0}{b}(a+q)\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{43}$$

(III) shape parameter $q = 1$:

For N-U method we have the wave function as

$$\psi(s) = B_n s^{-\xi}(1 - s)^{(-\eta + \frac{1}{2})} P_n^{(2\xi, 2\eta)}(s) \tag{44}$$

one obtains the eigen values of ϵ from the following equation:

$$\omega^2 - \epsilon^2 = \left[\frac{8(\gamma^2 - \omega^2) - (2n+1)^2 - 1 - 2\eta(2n+1)}{4(2n+1) + 2\eta}\right]^2 \tag{45}$$

For Frobenius method, we have

$$a_k = \prod_{i=1}^{k} \frac{\{(i+\nu-1)^2 + \epsilon^2\} - V_1\beta^2 - \frac{V_0}{b}(a+1)\beta^2 + V_2\beta^2}{\{(i+\nu)^2 + \epsilon^2\} - V_1\beta^2 - \frac{V_0}{b}(a+1)\beta^2} a_0 \quad , \quad k = 1, 2, ... \tag{46}$$

5. Conclusion

In this article , the exact solution of the effective mass K-G equation for the modified Eckart plus Hylleraas potential in absence of Lorentz scalar potential. The eigen values and eigen functions are obtained using the Frobenius method as well as Nikiforov-Uvarov method. We gave a schematic graphical representation of the modified Eckart plus Hylleraas potential with a shape parameter 'q' and also the graphical representation of Hulthén and Rosen-Morse Potential. The eigen values of the potential reduces to that of well known potentials viz., Hulthén Potential in equation (36) and Rosen-Morse Potential in equatioon (40), when we make appropriate choices of parameter a, b, V_0, V_1, V_2 . Finally we also obtain the wave function which is expressed in terms of the Jacobi Polynomials.

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Massive Dirac Particle in Generalized Asymmetric Manning-Rosen Potential

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The one-dimensional Dirac Equation with position dependent mass has been solved in terms of the hyper-geometric functions for generalized asymmetric Manning-Rosen Potential containing different type of physical potential. Considering one-dimensional electric current density for the Dirac particle the transmission and the reflection coefficients are obtained. The expression for the energy-eigen values is obtained by using continuity conditions of the wave functions.

1. Introduction

In recent years the study of Quantum Mechanical Systems within the framework of position dependent mass (PDM) has received much attention in the literature [1-5]. Quantum particles with PDM constitute useful models for the study of many physical problems, for example determination of electronic properties of semiconductor hetero-structure [6], various properties of hetero-junctions , quantum dots [7] ,^{3}He clusters , metal clusters, and the density of energy in many body problems. The investigation of relativistic effect is important in the study of heavy atoms or heavy ions [1]. For this type of particles Dirac equation, where the mass becomes a function, plays an important role.

Dirac equation has been used for the study of relativistic heavy ion collisions, heavy ion spectroscopy, laser-matter interaction, and especially in higher energy physics and condensed matter physics [8]. In physics, it is very important to understand the structure of nucleus, atoms, molecules and the material objects. In order to get complete information about a quantum mechanical system, it is needed to study the scattering and the bound states. The scattering and bound states in non-relativistic and relativistic quantum mechanics with a potential are studied in order to describe the behavior and the interaction between atoms and particles. Thus it is important to create a model which contains potential concepts. Some potentials that describes the interaction between nuclei and nuclei-particle and the structures of diatomic and poly-atomic molecules are Kratzer-Fuez [9], Rosen-Morse [10], Wood-Saxon [11], Morse [12], Hulthe'n [13], Cusp [14], Deng-Fan, and Eckart potentials [15-16].

In this article, we have considered generalized asymmetric Manning-Rosen (GAMR) potential [17], the GAMR potential was first proposed by Manning and Rosen in 1933 to define the vibrational behavior of diatomic molecules. After that it has been used to describe the interaction between two atoms of a diatomic molecule. Some types of potentials can be generalized to describe interactions consisting of more than one process.

The GAMR potential taken for our study is of the following form

$$V(x) = \Theta(-\mathrm{x})\left[\frac{\mathrm{Ae}^{2\alpha(\mathrm{x}+\mathrm{x}_0)}}{(\Lambda+\Delta\mathrm{e}^{\alpha(\mathrm{x}+\mathrm{x}_0)})^2}+\frac{\mathrm{Ae}^{\alpha(\mathrm{x}+\mathrm{x}_0)}}{(\Lambda+\Delta\mathrm{e}^{\alpha(\mathrm{x}+\mathrm{x}_0)})}\right]+ \Theta(\mathrm{x})\left[\frac{\mathrm{Ce}^{-2\beta(\mathrm{x}-\bar{\mathrm{x}}_0)}}{(\tilde{\Lambda}+\tilde{\Delta}\mathrm{e}^{-\beta(\mathrm{x}-\bar{\mathrm{x}}_0)})^2}+\frac{\mathrm{Ae}^{-\beta(\mathrm{x}-\bar{\mathrm{x}}_0)}}{(\tilde{\Lambda}+\tilde{\Delta}\mathrm{e}^{-\beta(\mathrm{x}-\bar{\mathrm{x}}_0)})}\right] \quad (1)$$

Where, $\Theta(x)$ is the Heaviside step-function and all the parameters are real.

The shape of GAMR potential varies according to the values of the parameters. It becomes a potential barrier if A, B, C and D are positive and it becomes a potential well if A, B, C and D are negative.

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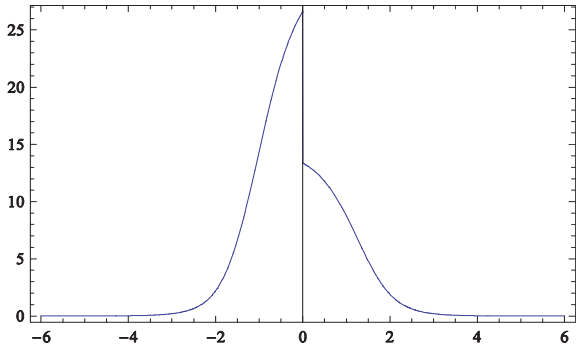

Fig.1 The generalized asymmetric Manning-Rosen potential for $\alpha = \beta = 2, x_0 = \tilde{x}_0 = 1, A = B = C = D = 1$, $\Lambda = \tilde{\Lambda} = 0.1$ and $\Delta = 0.2, \tilde{\Delta} = 0.3$ (for smaller values of αx_0 and $\beta \tilde{x}_0$)

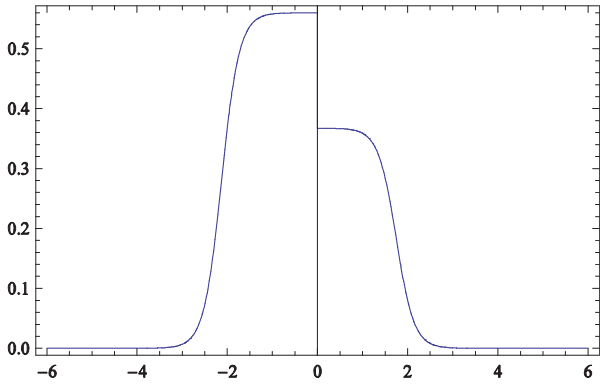

Fig.2 The generalized asymmetric Manning-Rosen potential for $\alpha = \beta = 5, x_0 = \tilde{x}_0 = 2, A = B = C = D = 1$, $\Lambda = 1$, $\tilde{\Lambda} = 10$ and $\Delta = 2.5, \tilde{\Delta} = 3.5$ (for larger values of αx_0 and $\beta \tilde{x}_0$)

The mass function is chosen here as $m(x) = m_0 + f(x)$, where f(x) =V(x) and m_0 is the rest mass.

Our work is generalized as follows. In section 2, we have discussed the relativistic Dirac equation with position dependent mass. In section 3, we discuss the scattering state solutions of Dirac equation for GAMR potential. We give a brief discussion about the scattering state solutions in section 3. Transmission and reflection coefficients are obtained in section 4 and the bound state condition for energy-eigen values is obtained in section 5. Finally, section 6 is kept for concluding remark.

2. Dirac equation with position dependent mass

The Dirac equation for relativistic free-particle [18] (in natural units $\hbar$=c=1) is as

$$[i\gamma^{\mu}\partial_{\mu} - \mathrm{m(x)}]\psi(\mathrm{x}) = 0 \qquad (2)$$

Where, m(x) is the Dirac particle mass, which depends on one spatial coordinate, x. To obtain the one-dimensional Dirac equation for the external potential V(x) we consider the gamma matrices γ_x and γ_0 in terms of Pauli matrices $i\sigma(x)$ and $i\sigma(z)$, respectively.

$$\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{d}{dx} - [E - V(x)]\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + m(x)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\} \times \begin{pmatrix}U_1(x)\\ U_2(x)\end{pmatrix} = 0 \qquad (3)$$

Where, $U_1(x)$ and $U_2(x)$ are decomposed into upper and lower components of the two-component wave function $\psi(x)$ and Eqn. (3) turns into the following two-coupled differential equations.

From equations (6) and (7) for $\Theta(x)$ and $\Phi(x)$, we get the following two distinct second order differential equations as follows

$$\frac{dU_1(x)}{dx} = -[m(x) + E - V(x)]U_2(x)$$
$$\frac{dU_2(x)}{dx} = -[m(x) - E + V(x)]U_1(x) \qquad (4)$$

According to Flügge [19], the following expressions are obtained

$$\Theta(\mathrm{x}) = U_1(\mathrm{x}) + \mathrm{i}U_2(\mathrm{x})$$
$$\Phi(\mathrm{x}) = U_1(\mathrm{x}) - \mathrm{i}U_2(\mathrm{x}) \qquad (5)$$

By putting (5) into (4) we get the following equations

$$\frac{d\Theta(\mathrm{x})}{dx} = i[E - V(x)]\Theta(\mathrm{x}) - im(x)\Phi(\mathrm{x}) \qquad (6)$$
$$\frac{d\Phi(\mathrm{x})}{dx} = -i[E - V(x)]\Phi(\mathrm{x}) + im(x)\Theta(\mathrm{x}) \qquad (7)$$

From equations (6) and (7) for $\Theta(x)$ and $\Phi(x)$ we get the following two distinct second order differential equations as follows

$$\frac{d^2\Theta(\mathrm{x})}{dx^2} - \frac{1}{m(x)}\frac{dm(x)}{dx}\frac{d\Theta(\mathrm{x})}{dx} + \left\{\left[E - V(x)\right]^2 m(x)^2 + i\frac{dV(\mathrm{x})}{dx} + i\left[E - V(x)\right]\frac{1}{m(x)}\frac{dm(x)}{dx}\right\}\Theta(\mathrm{x}) = 0 \qquad (8)$$

$$\frac{d^2\Phi(\mathrm{x})}{dx^2} - \frac{1}{m(x)}\frac{dm(x)}{dx}\frac{d\Phi(\mathrm{x})}{dx} + \left\{\left[E - V(x)\right]^2 m(x)^2 - i\frac{dV(\mathrm{x})}{dx} - i\left[E - V(x)\right]\frac{1}{m(x)}\frac{dm(x)}{dx}\right\}\Phi(\mathrm{x}) = 0 \qquad (9)$$

The mass function for the Dirac particle is chosen as

$$m(x) = m_0 + f(x); \qquad (10)$$

Where, the function f(x) is given by

$$\mathrm{f(x)} = \Theta(-\mathrm{x})\left[\frac{\mathrm{Ae}^{2\alpha(\mathrm{x}+\mathrm{x}_0)}}{\left(\Lambda + \Delta\mathrm{e}^{\alpha(\mathrm{x}+\mathrm{x}_0)}\right)^2} + \frac{\mathrm{Ae}^{\alpha(\mathrm{x}+\mathrm{x}_0)}}{\left(\Lambda + \Delta\mathrm{e}^{\alpha(\mathrm{x}+\mathrm{x}_0)}\right)}\right] + \Theta(\mathrm{x})\left[\frac{\mathrm{Ce}^{-2\beta(\mathrm{x}-\tilde{\mathrm{x}}_0)}}{\left(\tilde{\Lambda} + \tilde{\Delta}\mathrm{e}^{-\beta(\mathrm{x}-\tilde{\mathrm{x}}_0)}\right)^2} + \frac{\mathrm{Ae}^{-\beta(\mathrm{x}-\tilde{\mathrm{x}}_0)}}{\left(\tilde{\Lambda} + \tilde{\Delta}\mathrm{e}^{-\beta(\mathrm{x}-\tilde{\mathrm{x}}_0)}\right)}\right]$$

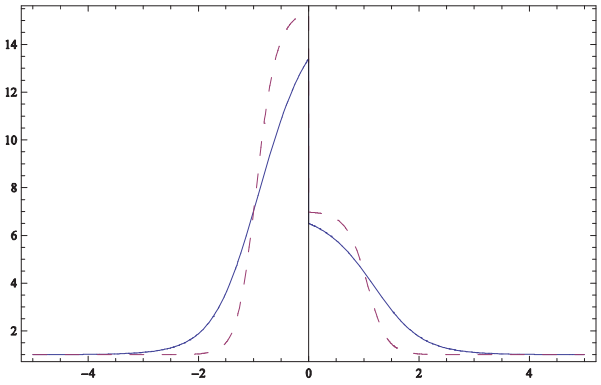

Fig.3.The mass variation for Generalized Asymmetric Manning Rosen Potential for $\alpha = \beta = 5$(for larger values of αx_0 and $\beta\tilde{x}_0$) , $x_0 = \tilde{x}_0 = 1$, $A = B = C = D = 1$, $\Lambda = \tilde{\Lambda} = 0.2$ and $\Delta = 0.3$, $\tilde{\Delta} = 0.5$, $\alpha = \beta = 2$, (for smaller values of αx_0 and $\beta\tilde{x}_0$) and taking $m_0 = 1$, the rest mass for the Dirac particle. Therefore, the derivative term of the mass function $m(x)$ is ignored in (8) and (9).

So the equations (8) and (9) becomes respectively as

$$\frac{d^2\Theta(x)}{dx^2} + \left\{[E - V(x)]^2 m(x)^2 + i\frac{dV(x)}{dx}\right\}\Theta(x) = 0 \quad (11)$$

$$\frac{d^2\Phi(x)}{dx^2} + \left\{[E - V(x)]^2 m(x)^2 - i\frac{dV(x)}{dx}\right\}\Phi(x) = 0 \quad (12)$$

3. Scattering State Solution of Dirac Equation

As we have to find out the solutions for region $x < 0$ and $x > 0$ so we consider a new variable $y = -\frac{\Delta}{\Lambda}e^{\alpha(x+x_0)}$ in Eqn. (11) for region$x < 0$. Then the equation (11) becomes

$$\alpha^2 y^2 \frac{d^2\Theta_L(y)}{dy^2} + \alpha^2 y \frac{d\Theta_L(y)}{dy} + \left\{(E^2 - {m_0}^2) - 2(E-)\left(\frac{Ay^2}{\Delta^2(1-y)^2} - \frac{By}{\Delta(1-y)}\right)\right\}\Theta_L(y) = 0 \quad (13)$$

We consider a trial wave function

$$\Theta_L(y) = y^{\xi}(1-y)^{\lambda} w(y) \quad (14)$$

Then equation (13) reduces to the Gaussian Differential Equation [20] given in following equation

$$y(1-y)\frac{d^2w}{dy^2} + \{1 + 2\xi - (2\xi + 2\lambda + 1)y\}\frac{dw}{dy} - (\xi + \lambda + \eta)(\xi + \lambda - \eta)w = 0 \quad (15)$$

The parametersξ, λ and η are given by

$$\xi = \frac{i\rho}{\alpha}, \text{ where } \rho = \sqrt{E^2 - {m_0}^2}$$
$$\lambda = \frac{1}{2} + \sqrt{\frac{2A(E+m_0)}{\alpha^2\Delta^2} + \frac{1}{4}}$$
$$\eta = \sqrt{-\frac{(E^2 - {m_0}^2)}{\alpha^2} + \frac{2(E+m_0)}{\Delta\alpha^2}\left[\frac{A}{\Delta} + B\right]} \quad (16)$$

The solution of Eqn. (15) is found in the form of hyper-geometric function as

$$w(y) = N_1\ {}_2F_1(\xi + \lambda + \eta, \xi + \lambda - \eta, 1 + 2\xi; y) + N_2 y^{(-2\xi)}\ {}_2F_1(-\xi + \lambda + \eta, -\xi + \lambda - \eta, 1 - 2\xi; y) \quad (17)$$

Then the whole left-hand solution of equation (11) i.e. the solution of equation (13) becomes

$$\Theta_L(y) = N_1 y^{\xi}(1-y)^{\lambda}\ {}_2F_1(\xi + \lambda + \eta, \xi + \lambda - \eta, 1 + 2\xi; y) + N_2 y^{-\xi}(1-y)^{\lambda}\ {}_2F_1(-\xi + \lambda + \eta, -\xi + \lambda - \eta, 1 - 2\xi; y) \quad (18)$$

Now we choose a new variable $z = -\frac{\tilde{\Delta}}{\tilde{\Lambda}}e^{-\beta(x-\tilde{x}_0)}$ for equation (11) for region $x > 0$ and then equation (11) becomes

$$\beta^2 z^2 \frac{d^2\Theta_R(z)}{dz^2} + \beta^2 z\frac{d\Theta_{R(z)}}{dz} + \left\{(E^2 - {m_0}^2) - 2(E + m_0)\left(\frac{Cz^2}{\tilde{\Delta}^2(1-z)^2} - \frac{Dz}{\tilde{\Delta}(1-z)}\right)\right\}\Theta_R(z) = 0 \quad (19)$$

We consider a trial wave function

$$\Theta_R(z) = z^{\tilde{\xi}}(1-z)^{\tilde{\lambda}} v(z) \quad (20)$$

Then equation (19) reduces to the Gaussian Differential Equation given in following equation

$$z(1-z)\frac{d^2v}{dz^2} + \{1 + 2\tilde{\xi} - (2\tilde{\xi} + 2\tilde{\lambda} + 1)z\}\frac{dv}{dz} - (\tilde{\xi} - \tilde{\lambda} + \tilde{\eta})(\tilde{\xi} - \tilde{\lambda} - \tilde{\eta})v = 0 \quad (21)$$

The parameters $\tilde{\xi}$, $\tilde{\lambda}$ *and* $\tilde{\eta}$ are given by

$$\tilde{\xi} = \frac{i\rho}{\beta}, where\ \rho = \sqrt{E^2 - {m_0}^2}$$
$$\tilde{\lambda} = -\frac{1}{2} + \sqrt{\frac{2C(E+m_0)}{\beta^2\tilde{\Delta}^2} + \frac{1}{4}}$$
$$\tilde{\eta} = \sqrt{-\frac{(E^2 - {m_0}^2)}{\beta^2} + \frac{2(E+m_0)}{\beta^2\tilde{\Delta}}\left[\frac{C}{\tilde{\Delta}} - D\right]} \quad (22)$$

The solution of Eqn. (21) is found in the form of hyper-geometric function as

$$v(z) = N_3\ {}_2F_1(\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, \tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 + 2\tilde{\xi}; z) + N_4 z^{(-2\tilde{\xi})}\ {}_2F_1(-\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, -\tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 - 2\tilde{\xi}; z) \quad (23)$$

Then the whole right-hand solution of equation (11) i.e. the solution of equation (19) becomes

$$\Theta_R(z) = N_3 z^{\tilde{\xi}}(1-z)^{-\tilde{\lambda}}\ {}_2F_1(\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, \tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 + 2\tilde{\xi}; z) + N_4 z^{-\tilde{\xi}}(1-z)^{-\tilde{\lambda}}\ {}_2F_1(-\tilde{\xi} - \tilde{\lambda} + \tilde{\eta}, -\tilde{\xi} - \tilde{\lambda} - \tilde{\eta}, 1 - 2\tilde{\xi}; z) \quad (24)$$

4. Transmission and Reflection Coefficient for Electric Current Density

By using the asymptotic behaviours as $x \to -\infty$ and as $x \to +\infty$ for the solutions given in equation (18) and equation (24), we get the transmission (T) and reflection(R) coefficients.
As $x \to -\infty$ the left-hand solution i.e., equation (18) becomes

$$\Theta_{\mathrm{L}}(\mathrm{x}) = N_1\left(\frac{\Delta}{\Lambda}\right)^{\frac{i\rho}{\alpha}} e^{\frac{-\pi\rho}{\alpha}} e^{i\rho(x+x_0)} + N_2\left(\frac{\Delta}{\Lambda}\right)^{\frac{-i\rho}{\alpha}} e^{\frac{\pi\rho}{\alpha}} e^{-i\rho(x+x_0)} \quad (25)$$

And, as $x \to +\infty$ the right hand solution i.e., equation (24) becomes,

$$\Theta_{\mathrm{R}}(\mathrm{x}) = N_4\left(\frac{\tilde{\Delta}}{\tilde{\Lambda}}\right)^{\frac{-i\rho}{\beta}} e^{\frac{\pi\rho}{\beta}} e^{i\rho(x-\tilde{x}_0)} \quad (26)$$

So to find the electric current density $J(x)$ for Dirac particle

$$J(x) = \frac{1}{2}[|\Theta(\mathrm{x})|^2 - |\Phi(x)|^2] \quad (27)$$

By putting equations (25) and (26) in equation (6), we have the asymptotic behavior for the left hand solution, $\Phi_L(x)$ become

$$\Phi_L(x) = N_1\left(\frac{\Delta}{\Lambda}\right)^{\frac{i\rho}{\alpha}} e^{\frac{-\pi\rho}{\alpha}} e^{i\rho(x+x_0)} \frac{(E-\rho)}{m(x)} + N_2\left(\frac{\Delta}{\Lambda}\right)^{\frac{-i\rho}{\alpha}} e^{\frac{\pi\rho}{\alpha}} e^{-i\rho(x+x_0)} \frac{(E+\rho)}{m(x)} \quad (28)$$

and the asymptotic behaviors for right hand solution $\Phi_R(x)$ becomes

$$\Phi_R(\mathrm{x}) = N_4\left(\frac{\tilde{\Delta}}{\tilde{\Lambda}}\right)^{\frac{-i\rho}{\beta}} e^{\frac{\pi\rho}{\beta}} e^{i\rho(x-\tilde{x}_0)} \frac{(E-\rho)}{m(x)} \quad (29)$$

Using equations (25), (26), (28) and (29) in equation (27) we get left hand value of $J(x)$ i.e. $J_L(x)$ and the right hand value of $J(x)$ i.e. $J_R(x)$.

From $J_L(x)$ and $J_R(x)$ we can calculate the value of J_{trans}(transmitted current), J_{inc}(incident current) and J_{ref}(reflected current) .Finally the reflection coefficient (R) and the transmission coefficient (T) are found as the following equations, respectively as

$$R = \frac{J_{ref}}{J_{inc}} = \left|\frac{N_2}{N_1}\right|^2 \frac{E+\rho}{E-\rho} e^{\frac{4\pi\rho}{\alpha}} \quad (30)$$

$$T = \frac{J_{trans}}{J_{inc}} = \left|\frac{N_4}{N_1}\right|^2 e^{2\pi\rho\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \quad (31)$$

Where R and T satisfy the condition: $R + T = 1$. For the clear view of the coefficients, we have to use continuity condition

$$\Theta_L(x=0) = \Theta_R(x=0) \quad (32)$$

$$\left.\frac{d\Theta_L}{dx}\right|_{x=0} = \left.\frac{d\Theta_R}{dx}\right|_{x=0} \quad (33)$$

Using the continuity condition we obtain the following result:-

$$\frac{\mathrm{N}_4}{\mathrm{N}_1} = \frac{\mathrm{D}_2\mathrm{K}_2[(\mathrm{R}_1+\mathrm{R}_2)\mathrm{D}_1 + \mathrm{R}_3\mathrm{D}_4] - \mathrm{D}_1\mathrm{K}_1[(\mathrm{R}_4+\mathrm{R}_5)\mathrm{D}_2 + \mathrm{R}_6\mathrm{D}_5]}{\mathrm{D}_2\mathrm{K}_2[(\mathrm{R}_7+\mathrm{R}_8)\mathrm{D}_3 + \mathrm{R}_9\mathrm{D}_6] - \mathrm{D}_3\mathrm{K}_3[(\mathrm{R}_4+\mathrm{R}_5)\mathrm{D}_2 + \mathrm{R}_6\mathrm{D}_5]}$$

$$\frac{\mathrm{N}_2}{\mathrm{N}_1} = \frac{\mathrm{D}_1\mathrm{K}_1[(\mathrm{R}_7+\mathrm{R}_8)\mathrm{D}_3 + \mathrm{R}_9\mathrm{D}_6] - \mathrm{D}_3\mathrm{K}_3[(\mathrm{R}_1+\mathrm{R}_2)\mathrm{D}_1 + \mathrm{R}_3\mathrm{D}_4]}{\mathrm{D}_3\mathrm{K}_3[(\mathrm{R}_4+\mathrm{R}_5)\mathrm{D}_2 + \mathrm{R}_6\mathrm{D}_5] - \mathrm{D}_2\mathrm{K}_2[(\mathrm{R}_7+\mathrm{R}_8)\mathrm{D}_3 + \mathrm{R}_9\mathrm{D}_6]} \quad (34)$$

Where, the abbreviations are given below

$$\sigma = -\frac{\Delta}{\Lambda}$$
$$\tilde{\sigma} = -\frac{\tilde{\Delta}}{\tilde{\Lambda}}$$
$$K_1 = \sigma^{\xi} e^{\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}$$
$$K_2 = \sigma^{-\xi} e^{-\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}$$
$$K_3 = \tilde{\sigma}^{-\tilde{\xi}} e^{-\beta\tilde{\xi}\tilde{x}_0}(1-\tilde{\sigma}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}}$$
$$D_1 = {}_2F_1(\xi+\lambda+\eta, \xi+\lambda-\eta, 1+2\xi; \sigma e^{\alpha x_0})$$
$$D_2 = {}_2F_1(-\xi+\lambda+\eta, -\xi+\lambda-\eta, 1-2\xi; \sigma e^{\alpha x_0})$$
$$D_3 = {}_2F_1\left(-\tilde{\xi}-\tilde{\lambda}+\tilde{\eta}, -\tilde{\xi}-\tilde{\lambda}-\tilde{\eta}, 1-2\tilde{\xi}; \tilde{\sigma}e^{\beta\tilde{x}_0}\right)$$
$$D_4 = {}_2F_1(\xi+\lambda+\eta+1, \xi+\lambda-\eta+1, 2+2\xi; \sigma e^{\alpha x_0})$$
$$D_5 = {}_2F_1(-\xi+\lambda+\eta+1, -\xi+\lambda-\eta+1, 2-2\xi; \sigma e^{\alpha x_0})$$
$$D_6 = {}_2F_1\left(-\tilde{\xi}-\tilde{\lambda}+\tilde{\eta}+1, -\tilde{\xi}-\tilde{\lambda}-\tilde{\eta}+1, 2-2\tilde{\xi}; \tilde{\sigma}e^{\beta\tilde{x}_0}\right)$$
$$R_1 = \sigma^{\xi}(\alpha\xi)e^{\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}$$
$$R_2 = \sigma^{(\xi+1)}(-\lambda\alpha)e^{\alpha(\xi+1)x_0}(1-\sigma e^{\alpha x_0})^{\lambda-1}$$
$$R_3 = \sigma^{\xi} e^{\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}\frac{(\xi+\lambda+\eta)(\xi+\lambda-\eta)}{(1+2\xi)}$$
$$R_4 = \sigma^{-\xi}(-\alpha\xi)e^{-\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}$$
$$R_5 = \sigma^{(-\xi+1)}(-\lambda\alpha)e^{-\alpha(1-\xi)x_0}(1-\sigma e^{\alpha x_0})^{\lambda-1}$$
$$R_6 = \sigma^{-\xi} e^{-\alpha\xi x_0}(1-\sigma e^{\alpha x_0})^{\lambda}\frac{(-\xi+\lambda+\eta)(-\xi+\lambda-\eta)}{(1-2\xi)}$$
$$R_7 = \tilde{\sigma}^{-\tilde{\xi}}\left(\beta\tilde{\xi}\right)e^{-\beta\tilde{\xi}\tilde{x}_0}(1-\tilde{\sigma}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}}$$
$$R_8 = \tilde{\sigma}^{(-\tilde{\xi}+1)}\left(\beta\tilde{\lambda}\right)e^{-\beta(\tilde{\xi}-1)\tilde{x}_0}(1-\tilde{\sigma}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}}$$
$$R_9 = \tilde{\sigma}^{-\tilde{\xi}} e^{-\beta\tilde{\xi}\tilde{x}_0}(1-\tilde{\sigma}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}}\frac{\left(-\tilde{\xi}-\tilde{\lambda}+\tilde{\eta}\right)(-\tilde{\xi}-\tilde{\lambda}-\tilde{\eta})}{(1-2\tilde{\xi})}$$

5. Bound State and Ground State Solution for Energy Eigen Value

Case I. Solution for Negative Region($x < 0$)

We put $V(x) \to -V(x)$ in equation (1) to obtain the shape of the potential as a potential well and for the potential well we get the bound states solution. Thus we

consider a new variable $y = -\frac{\Delta}{\Lambda}e^{\alpha(x+x_0)}$ in (11) for region$x < 0$. Then the equation (11) becomes

$$\alpha^2 y^2 \frac{d^2\Theta_L(y)}{dy^2} + \alpha^2 y \frac{d\Theta_L(y)}{dy} + \left\{(E^2 - {m_0}^2) - 2(E - m_0)\left(\frac{Ay^2}{\Delta^2(1-y)^2} - \frac{By}{\Delta(1-y)}\right)\right\}\Theta_L(y) = 0 \quad (35)$$

We consider a trial wave function,

$$\Theta_L(y) = y^{\xi_1}(1-y)^{\lambda_1}P(y) \quad (36)$$

Then equation (35) reduces to the Gaussian Differential Equation given in following equation

$$y(1-y)\frac{d^2P}{dy^2} + \{1 + 2\xi_1 - (2\xi_1 + 2\lambda_1 + 1)y\}\frac{dP}{dy} - (\xi_1 + \lambda_1 + \eta_1)(\xi_1 + \lambda_1 - \eta_1)P = 0 \quad (37)$$

The parametersξ_1, λ_1 and η_1 are given by

$$\xi_1 = \frac{i\rho}{\alpha}, \text{ where } \rho = \sqrt{E^2 - {m_0}^2}$$
$$\lambda_1 = \frac{1}{2} + \sqrt{\frac{-2A(E+m_0)}{\alpha^2\Delta^2} + \frac{1}{4}}$$
$$\eta_1 = \sqrt{-\frac{(E^2 - {m_0}^2)}{\alpha^2} + \frac{2(E+m_0)}{\Delta\alpha^2}\left[\frac{-A}{\Delta} - B\right]} \quad (38)$$

The solution of (37) is found in the form of hyper-geometric function as

$$P(y) = A_1\,{}_2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; y) + A_2 y^{-2\xi_1}\,{}_2F_1(-\xi_1 + \lambda_1 + \eta_1, -\xi_1 + \lambda_1 - \eta_1, 1 - 2\xi_1; y) \quad (39)$$

Then the left-hand solution of equation (11) i.e., the solution of equation (36) becomes

$$\Theta_L(y) = A_1 y^{\xi_1}(1-y)^{\lambda_1}\,{}_2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; y) + A_2 y^{-\xi_1}(1-y)^{\lambda_1}\,{}_2F_1(-\xi_1 + \lambda_1 + \eta_1, -\xi_1 + \lambda_1 - \eta_1, 1 - 2\xi_1; y) \quad (40)$$

Case II. Solution for Positive Region($x > 0$)

Now we choose a new variable $z = -\frac{\bar{\Delta}}{\bar{\Lambda}}e^{-\beta(x-\bar{x}_0)}$ for equation (11) for region $x > 0$ and then equation (11) becomes

$$\beta^2 z^2 \frac{d^2\,\Theta_R(z)}{dz^2} + \beta^2 z \frac{d\Theta_{R(z)}}{dz} + \left\{(E^2 - {m_0}^2) - 2(E + m_0)\left(\frac{Cz^2}{\bar{\Delta}^2(1-z)^2} - \frac{Dz}{\bar{\Delta}(1-z)}\right)\right\}\Theta_R(z) = 0 \quad (41)$$

We consider a trial wave function

$$\Theta_R(z) = z^{\tilde{\xi}_1}(1-z)^{\tilde{\lambda}_1}Q(z) \quad (42)$$

Then equation (41) reduces to the Gaussian Differential Equation given in following equation

$$z(1-z)\frac{d^2Q}{dz^2} + \{1 + 2\tilde{\xi}_1 - (2\tilde{\xi}_1 + 2\tilde{\lambda}_1 + 1)z\}\frac{dQ}{dz} - (\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1)(\tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1)Q = 0 \quad (43)$$

The parameters $\tilde{\xi}_1$, $\tilde{\lambda}_1$ *and* $\tilde{\eta}_1$ are given by

$$\tilde{\xi}_1 = \frac{i\rho}{\beta}\ , where\ \rho = \sqrt{E^2 - {m_0}^2}$$
$$\tilde{\lambda}_1 = -\frac{1}{2} + \sqrt{\frac{-2C(E+m_0)}{\beta^2\tilde{\Delta}^2} + \frac{1}{4}}$$
$$\tilde{\eta}_1 = \sqrt{-\frac{(E^2 - {m_0}^2)}{\beta^2} + \frac{2(E+m_0)}{\beta^2\bar{\Delta}}\left[\frac{-C}{\bar{\Delta}} - D\right]} \quad (44)$$

The solution of (43) is found in the form of hyper-geometric function as

$$Q(z) = A_3\ {}_2F_1(\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1, \tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1, 1 + 2\tilde{\xi}_1; z) + A_4 z^{(-2\tilde{\xi}_1)}\,{}_2F_1(-\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1, -\tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1, 1 - 2\tilde{\xi}_1; z) \quad (45)$$

Then the whole right-hand solution of equation (11) i.e. the solution of equation (42) becomes

$$\Theta_R(z) = A_3 z^{\tilde{\xi}_1}(1-z)^{-\tilde{\lambda}_1}\,{}_2F_1(\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1, \tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1, 1 + 2\tilde{\xi}_1; z) + A_4 z^{-\tilde{\xi}_1}(1-z)^{-\tilde{\lambda}_1}\,{}_2F_1(-\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1, -\tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1, 1 - 2\tilde{\xi}_1; z) \quad (46)$$

By the boundary condition i.e. as $x \to \mp\infty, y \to 0$ and $z \to 0$ and the wave functions go to zero at infinity we get from equation (40) and (46)$A_2 = A_4 = 0$.

Then equation (40) becomes

$$\Theta_L(y) = A_1 y^{\xi_1}(1-y)^{\lambda_1}\,{}_2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; y) \quad (47)$$

And, the equation (46) becomes

$$\Theta_R(z) = A_3 z^{\tilde{\xi}_1}(1-z)^{-\tilde{\lambda}_1}\,{}_2F_1(\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1, \tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1, 1 + 2\tilde{\xi}_1; z) \quad (48)$$

Case III. Solution for Ground state($x = 0$)

Then by using the continuity condition given by

$$\Theta_L(x=0) = \Theta_R(x=0)$$

$$\left.\frac{d\Theta_L}{dx}\right|_{x=0} = \left.\frac{d\Theta_R}{dx}\right|_{x=0}$$

We obtain now from equations (47) and (48), the expression for the energy eigen-values given by

$$[(C_5 - C_6)C_1 + C_7C_3]C_{11}C_1 - [(C_8 - C_9)C_2 + C_{10}C_4]C_{12}C_2 = 0 \quad (49)$$

Where, the abbreviations are

$$S = -\frac{\Delta}{\Lambda}$$
$$\tilde{S} = -\frac{\tilde{\Delta}}{\tilde{\Lambda}}$$
$$C_1 = {}_2F_1(\xi_1 + \lambda_1 + \eta_1, \xi_1 + \lambda_1 - \eta_1, 1 + 2\xi_1; Se^{\alpha x_0})$$
$$C_2 = {}_2F_1\left(\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1, \tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1, 1 + 2\tilde{\xi}_1; \tilde{S}e^{\beta \tilde{x}_0}\right)$$
$$C_3 = {}_2F_1(\xi_1 + \lambda_1 + \eta_1 + 1, \xi_1 + \lambda_1 - \eta_1 + 1, 2 + 2\xi_1; Se^{\alpha x_0})$$
$$C_4 = {}_2F_1\left(\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1 + 1, \tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1 + 1, 2 + 2\tilde{\xi}_1; \tilde{S}e^{\beta \tilde{x}_0}\right)$$
$$C_5 = S^{\xi_1}(\alpha\xi_1)e^{\alpha\xi_1 x_0}(1 - Se^{\alpha x_0})^{\lambda_1}$$
$$C_6 = S^{(\xi_1+1)}(\lambda_1\alpha)e^{\alpha(\xi_1+1)x_0}(1 - \sigma e^{\alpha x_0})^{\lambda_1 - 1}$$
$$C_7 = S^{\xi_1}e^{\alpha\xi_1 x_0}(1 - Se^{\alpha x_0})^{\lambda_1}\frac{(\xi_1 + \lambda_1 + \eta_1)(\xi_1 + \lambda_1 - \eta_1)}{(1 + 2\xi_1)}$$
$$C_8 = -\beta\tilde{\xi}_1\tilde{S}^{\tilde{\xi}_1}e^{\beta\tilde{\xi}_1\tilde{x}_0}(1 - \tilde{S}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}_1}$$
$$C_9 = \beta\tilde{\lambda}_1\tilde{S}^{\tilde{\xi}_1}e^{\beta(\tilde{\xi}_1+1)\tilde{x}_0}(1 - \tilde{S}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}_1 - 1}$$
$$C_{10} = \tilde{S}^{\tilde{\xi}_1}e^{\beta\tilde{\xi}_1\tilde{x}_0}(1 - \tilde{S}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}_1}\frac{(\tilde{\xi}_1 - \tilde{\lambda}_1 + \tilde{\eta}_1)(\tilde{\xi}_1 - \tilde{\lambda}_1 - \tilde{\eta}_1)}{(1 + 2\tilde{\xi}_1)}$$
$$C_{11} = S^{\xi_1}e^{\alpha\xi_1 x_0}(1 - Se^{\alpha x_0})^{\lambda_1}$$
$$C_{12} = \tilde{S}^{\tilde{\xi}_1}e^{\beta\tilde{\xi}_1\tilde{x}_0}(1 - \tilde{S}e^{\beta\tilde{x}_0})^{-\tilde{\lambda}_1}$$

6. Conclusion

The scattering state and the bound state solutions for the one-dimensional Dirac equation with position dependent mass for GAMR potential is obtained. Solving the equation for the positive and negative region, we get the wave function in terms of the hyper-geometric functions. Bound state and ground state solutions are also obtained. For the solution at $x = 0$ we use continuity condition and finally an expression for energy eigen-value is obtained. Transmission (T) and reflection (R) coefficients for electric current density for this equation are found and the unitary condition $R + T = 1$ is preserved for PDM case. For bound states, we get the discrete spectrum for GAMR potential well with effective PDM, finding an exact condition for the energy eigen-values. Here, we gave a schematic graphical representation of the GAMR potential and the mass distribution.

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Klein-Gordon Equation with Double Ring Shaped Coulomb Potential via AIM

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Abstract. In this article Klein-Gordon equation is studied for Double Ring Shaped Coulomb Potential by Asymptotic Iteration Method. The bound state solution is obtained for inverse square potential from Radial part in terms of confluent hypergeometric function. Energy eigen value for isotropic harmonic oscillator and ring shaped oscillator with its solution in terms of Gauss hypergeometric function are also obtained from the angular part.

Keywords: Klein-Gordon (KG) equation · Double ring shaped Coulomb Potential · Asymptotic Iteration Method (AIM)

1 Introduction

It is well known that the relativistic part of the Schrödinger equation describing free particles is the Klein-Gordan (KG) equation. In relativistic quantum mechanics solution of KG-equation plays an important role for some physical potential. It has been created a line of great interest to solve KG-equation by equating scalar and vector potential for some typical central and non-central potential such as Hulthen [1], Morse [2], Ring shaped Hartmann [3], Kratzer Fuez [4], Pösch-Teller [5], Coulomb [6], Harmonic Oscillator [7] etc.

Particularly the Coulombic ring shaped Potential is introduced by Hartmann et al. [8] in Quantum Chemistry. The double ring shaped Coulomb potential is a 3-dimensional Coulomb potential surrounded by a double ring shaped inverse square potential. Mainly the double ring shaped Coulomb potential is a non-central potential in spherical coordinate and can be written as:-

$$V(r,\theta) = -\frac{A}{r} + \frac{B}{r^2 sin^2\Theta} + \frac{C}{r^2 cos^2\Theta} \tag{1}$$

where, $A = \eta\sigma^2 e^2$, $B = \frac{\hbar^2\eta^2\sigma^2}{2\mu}$, $C = \frac{\hbar^2}{2\mu}a$ where, $a \geq 0$ μ is the mass of the particle, η and σ are positive real parameters which range from 1 to 10 and r and Θ are the spherical coordinates. In Eq. (1) when we put $B = 0$ and $C = 0$ then V(r) reduces to coulomb potential and when $B = 0$ then V(r) reduces to ring shaped Hartmann potential. Specially the ring shaped non central potentials

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are used to describe the molecular structure [9] and the interaction between the deformed nuclie [10] and specially the molecular structure of benzene [11].

The methods considered to solve KG-equation are Super Symmetric Approximation Method (SUSY) [12], Nikiforov Uvarov Method (NU) [13], WKB Approximation Method [14], Variational, Functional Analysis etc. The Asymptotic Iteration Method (AIM) [15], proposed by Ciffce et al. gives more accurate and efficient results for the spectrum of many particles in relativistic and non-relativistic Quantum Mechanics compared to the other techniques.

In this paper we solve the 3-dimensional KG-equation for the double ring shaped Coulomb potential considering equal scalar and vector potential by using AIM. This paper is as follows: In Sect. 2, we give a brief description about AIM method. Section 3 gives the stationary radial and angle dependent part of the KG equation. In Sect. 4 we found the solution of the Radial part of the KG equation. Section 5 gives the solution of the Angular part of the Kg equation. Section 6 is left for the conclusion.

2 Overview of Asymptotic Iteration Method

The AIM method is based on solving a second order differential equation of the form:

$$f_n''(x) = \lambda_0(x) f_n'(x) + s_0(x) f_n(x) \tag{2}$$

Where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to x. The variables, $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To find a general solution to this equation, we differentiate (1) with respect to x and find

$$f_n'''(x) = \lambda_1(x) f_n'(x) + s_1(x) f_n(x) \tag{3}$$

Where $\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x)$,

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x). \tag{4}$$

Similarly, the second derivative of (1) yields

$$f_n^4(x) = \lambda_2(x) f_n'(x) + s_2(x) f_n(x), \tag{5}$$

Where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{6}$$

Equation (1) can be easily iterated up to $(k+1)th$ and $(k+2)th$ derivatives, $k = 1, 2, 3, \ldots$ Therefore, we have the recurrence relations

$$\begin{aligned} f_n^{(k+1)}(x) &= \lambda_{k-1}(x) f_n'(x) + s_{k-1}(x) f_n(x), \\ f_n^{(k+2)}(x) &= \lambda_k(x) f_n'(x) + s_k(x) f_n(x), \end{aligned} \tag{7}$$

Where

$$\lambda_k(x) = \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x),$$
$$s_k(x) = s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x). \tag{8}$$

From the ratio of the (k + 2)th and (k + 1)th derivatives, we have

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{f_n^{(k+2)}(x)}{f_n^{(k+1)}(x)} = \frac{\lambda_k(x)[f'_n(x) + \frac{s_k(x)}{\lambda_k(x)}f_n(x)]}{\lambda_{k-1}(x)[f'_n(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)}f_n(x)]}. \tag{9}$$

For sufficiently large k, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x) \tag{10}$$

which is the "asymptotic" aspect of the method, then, (8) reduces to

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{11}$$

which yields

$$f_n^{(k+1)}(x) = C_1 exp(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)}dx) = C_1\lambda_{k-1}(x)exp(\int[\alpha(x) + \lambda_0(x)]dx), \tag{12}$$

where C_1 is the integration constant and the right hand side of (11) is obtained by using (9) and (10). By inserting (11) into (6), the first-order differential equation is obtained as

$$f'_n(x) + \alpha(x)f_n(x) = C_1 exp(\int[\alpha(x) + \lambda_0(x)]dx). \tag{13}$$

This first-order differential equation can easily be solved and the general solution of (1) can be obtained as:

$$f_n(x) = exp(-\int^x \alpha(x_1)dx_1)[C_2 + C_1\int^x exp(\int^{x_1}[\lambda_0(x_2) + 2\alpha(x_2)]dx_2)dx_1] \tag{14}$$

For a given potential, the radial Klein-Gordon equation is converted to the form of (1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by (7). The termination condition of the method in (9) can be arranged as

$$\Delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \tag{15}$$

where k shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of (15) and the radial quantum number n is equal to the iteration number k for this case. For nontrivial potentials that have no exact solutions, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of

the asymptotic wave function or the minimum value of the potential and the approximate energy eigenvalues are obtained from the roots of (15) for sufficiently great values of k with iteration for which k is always greater than n in these numerical solutions.

The general solution of (1) is given by (13). The first part of (13) gives us the polynomial solutions that are convergent and physical, whereas the second part of (13) gives us non-physical solutions that are divergent. Although (13) is the general solution of (1), we take the coefficient of the second part (C_1) as zero, in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wave function generator for exactly solvable potentials:

$$f_n(x) = C_2 exp(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)} dx_1), \tag{16}$$

where n represents the principal quantum number.

3 Stationary Radial and Angle-Dependent Klein-Gordon Equation with Equal Scalar and Vector Potential

The stationary 3D K-G equation with the coupling of a vector potential V(r) and a scalar potential S(r) for a particle of rest mass m_0 in the natural units $\hbar = c = 1$ can be expressed as

$$\nabla^2\Psi(r,\theta,\phi) + [(E - V(r,\theta,\phi))^2 - (m_0 + S(r,\theta,\phi)^2)]\Psi(r,\theta,\phi) = 0 \tag{17}$$

where, E, V(r) and S(r) are the relativistic energy of the particle, vector and scaler potentials, respectively. Assuming V(r) = S(r) we get from Eq. (17),

$$\nabla^2\Psi(r,\theta,\phi) + [(E^2 - {m_0}^2) - 2(E + m_0)V(r,\theta)]\Psi(r,\theta,\phi) = 0 \tag{18}$$

Now considering the double ring shaped Coulomb potential the KG-equation reduces to,

$$\nabla^2\Psi(r,\theta,\phi) + [(E^2 - {m_0}^2) - 2(E + m_0)\{-\frac{A}{r} + \frac{B}{r^2 sin^2\Theta} + \frac{C}{r^2 cos^2\Theta}\}]\Psi(r,\theta,\phi) = 0 \tag{19}$$

To separate the variables for the stationary wave function we assume,

$$\Psi(r,\theta,\phi) = \frac{R(r)}{r}\frac{\Theta(\theta)}{sin^{\frac{1}{2}}\theta}\Phi(\phi) \tag{20}$$

By following the standard procedure of separation of variables we get the component equations as follows:-

$$\frac{d^2R}{dr^2} + (\delta^2 + \frac{\lambda A}{r} - \frac{\alpha^2}{r^2})R(r) = 0 \tag{21}$$

$$\frac{d^2\Theta}{d\theta^2} - [\frac{\lambda B}{sin^2\theta} + \frac{\lambda C}{cos^2\theta} - \frac{1}{4} - \frac{1}{4sin^2\theta} - \alpha^2 + \frac{\beta^2}{sin^2\theta}]\Theta(\theta) = 0 \tag{22}$$

$$\frac{d^2\Phi}{d\phi^2} = -\beta^2\Phi(\phi) \tag{23}$$

where, $\lambda = 2(E + m_0)$, $\delta^2 = (E^2 - {m_0}^2)$, represents the relativistic energy of a particle and α^2 and β^2 are separation constants. Putting $\alpha^2 = l(l+1)$, which we often encounter in various Schrödinger quantum systems, with the orbital angular momentum $l = 0, 1, 2,...$ and the magnetic quantum number $\beta = 0, \pm1, \pm2,...$.

The solution of Eq. (23) is the azimuthal angle solution and it is,

$$\Phi(\phi) = De^{i\beta\phi} \tag{24}$$

The Eq. (21) is radial equation and Eq. (22) is angle-dependent equation for the KG equation. For these two equations we use AIM in our next parts.

4 Solution for Radial Part of KG Equation

To solve the Eq. (21) with AIM for $l \neq 0$, we should transform Eq. (21) to the form of Eq. (2). For bound state solution of the Eq. (21) we consider $R(0) = 0$ and $R(\infty) = 0$. Therefore for the physically acceptable radial solution we consider the radial wave function as follows:

$$R(r) = r^{(l+1)}e^{-i\delta}f(r) \tag{25}$$

Thus by substituting $y = -2i\delta r$ and taking $R(r)$ as in Eq. (21) the wave function reduces to,

$$\frac{d^2f}{dy^2} - 2(\frac{1}{2} - \frac{l+1}{y})\frac{df}{dy} - (\frac{l+1}{y} - \frac{i\xi A}{y\delta})f(y) = 0 \tag{26}$$

where, $\xi = \frac{\lambda}{2}$ and $\lambda_0(y) = 2(\frac{1}{2} - \frac{l+1}{y})$ and $S_0(y) = \frac{l+1}{y} - \frac{i\xi A}{y\delta}$. After calculating $\lambda_n(y)$ and $S_n(y)$ we get,

$$\begin{aligned}
\lambda_0(y) &= 2(\frac{1}{2} - \frac{l+1}{y}) \\
S_0(y) &= \frac{l+1}{y} - \frac{i\xi A}{y\delta} \\
\lambda_1(y) &= \frac{2(l+1)}{y^2} + \frac{(l+1)-a}{y} - \frac{4(l+1)}{y} + \frac{4(l+1)^2}{y^2} \\
S_1(y) &= \frac{(l+1-a)(y-2l-3)}{y^2} \\
\lambda_2(y) &= 1 - \frac{2(a-2l+2)}{y} + \frac{(a+3l+3)-2(l+1)(l+a)}{y^2} - \frac{4(l+1)(2l^2+9l+8)}{y^3} \\
S_2(y) &= \frac{1}{y^3}[(l+1-a)\{y^2 - y(a+3l+4) + (4l^2+16l+15)\}]...etc.
\end{aligned} \tag{27}$$

where, $a = \frac{i\xi A}{\delta}$ combining these results with the condition given in Eq. (10) yields:

$$\frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} \Rightarrow \quad \left[\frac{i\xi A}{\delta}\right]_{n=0} = (l+1) \tag{28}$$

$$\frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} \Rightarrow \quad \left[\frac{i\xi A}{\delta}\right]_{n=1} = 2(l+1) \tag{29}$$

. . .

$$\frac{S_n}{\lambda_n} = \frac{S_{(n+1)}}{\lambda_{(n+1)}} \Rightarrow \left[\frac{i\xi A}{\delta}\right]_{n} = (n+1)(l+1) \tag{30}$$

Thus the generalised term in Eq. (30) gives the energy spectrum of the KG-equation with double ring shaped Coulomb potential, where n is radial quantum number $(n = 0, 1, 2, ..)$.

For $A = 0$ in Eq. (21) we get a singular solution which corresponds to the inverse square potential which gives bound state only if the separation constant is negative specially less than $-\frac{1}{4}$. The bound states are determined by potential well type and hence the quantum number n is limited by the study of potential well.

Equation (26) satisfies the confluent hypergeometric function and the solution of the differential equation given in (26) can be written as,

$$f(r) = 1F_1(l+1-\frac{i\xi A}{\delta}, 2l+2; -2i\delta r) \tag{31}$$

Thus the radial part of the wave function can be written as,

$$R(r) = N_1 r^{(l+1)} e^{-i\delta} 1F_1(l+1-\frac{i\xi A}{\delta}, 2l+2; -2i\delta r) \tag{32}$$

where N_1 is the normalizing constant.

5 Solution for Angular Part of KG Equation

The angular part of the KG equation for double ring shaped coulomb potential is given by Eq. (22). We consider $\tilde{l} = l + \frac{1}{2}$ and $l(l+1) = \tilde{l}^2 - \frac{1}{4}$ and the Eq. (22) becomes,

$$\frac{d^2\Theta}{d\theta^2} - \left[\frac{\lambda C}{cos^2\theta} + \frac{\lambda B - \frac{1}{4} + \beta^2}{sin^2\theta}\right]\Theta = -\tilde{l}^2\Theta(\theta) \tag{33}$$

Defining P and Q as follows,

$$P = -\frac{1}{2} \pm \sqrt{\lambda B + \beta^2}\, and \tag{34}$$

$$Q = -\frac{1}{2} \pm \sqrt{\lambda C + \frac{1}{4}} \tag{35}$$

Equation (33) reduces to,

$$\frac{d^2\Theta}{d\theta^2} - \left[\frac{Q(Q+1)}{cos^2\theta} + \frac{P(P+1)}{sin^2\theta}\right]\Theta = -\tilde{l}^2\Theta(\theta) \tag{36}$$

To solve this equation by AIM with boundary conditions i.e. $\theta(0)$ and $\theta(\pi)$ are finite we consider the following wave function,

$$\Theta(\theta) = sin^{P+1}\theta cos^{Q+1}\theta f(\theta) \tag{37}$$

By using this wave function in Eq. (36) we get the second order homogeneous differential equation as,

$$\frac{d^2f}{d\theta^2} = 2\{(Q+1)tan\theta - (P+1)cot\theta\}\}\frac{df}{d\theta} + [(P+Q+2)^2 - (l+\frac{1}{2})^2]f \tag{38}$$

Now by using AIM method we have,

$$s_0 = [(P+Q+2)^2 - (l+\frac{1}{2})^2] \tag{39}$$

$$\lambda_0 = 2\{(Q+1)tan\theta - (P+1)cot\theta\} \tag{40}$$

$$s_1 = 2[(P+Q+2)^2 - (l+\frac{1}{2})^2]\{(Q+1)tan\theta - (P+1)cot\theta\} \tag{41}$$

$$\lambda_1 = 2\{(Q+1)sec^2\theta + (P+1)cosec^2\theta\} + [(P+Q+2)^2 - (l+\frac{1}{2})^2] + 4[\{(Q+1)tan\theta - (P+1)cot\theta\}]^2 \tag{42}$$

and so on. Thus combining these results with the condition given by Eq. (10) we have,

$$\frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} \Rightarrow \quad \tilde{l}^2 = (P+Q+2)^2 \tag{43}$$

$$\frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} \Rightarrow \quad \tilde{l}^2 = (P+Q+4)^2 \tag{44}$$

$$....etc. \tag{45}$$

and finally we get the generalised form as,

$$\tilde{l}^2 = (P+Q+2n+2)^2 \quad for \quad n = 0, 1, 2, 3, ... \tag{46}$$

By putting the value of P , Q and $\tilde{l}^2$ into the Eq. (46) we get value of l,

$$l = \sqrt{\lambda B + \beta^2} + \sqrt{\lambda C + \frac{1}{4}} + 2n + \frac{1}{2} \tag{47}$$

now by inserting the value of l into the generalised form of energy eigen value for the radial part of the KG-equation with double ring shaped coulomb potential given by Eq. (30), we get the relativistic energy spectrum for a bound electron from the following equation,

$$\left[\frac{i\xi A}{\delta}\right]_n = (n+1)(\sqrt{\lambda B + \beta^2} + \sqrt{\lambda C + \frac{1}{4}} + 2n + \frac{1}{2} + 1) \tag{48}$$

By putting $B = C = 0$ into the above equation we get the energy eigen value for isotropic harmonic oscillator[] and by putting $B = 0$ and $C \neq 0$ into the above equation we get the eigen values for the ring shaped oscillator.

For the eigen function of the angular part we substitute $cos^2(\theta) = x$ in the Eq. (38) and it reduces to the form:

$$x(1-x)\frac{d^2f}{dx^2} + [(Q + \frac{3}{2}) - (P + Q + 3)x]\frac{df}{dx} + \frac{1}{4}[(P + Q + 2)^2 - (l + \frac{1}{2})^2]f = 0 \tag{49}$$

The solution of the above differential equation is of the form of Gauss hypergeometric function given by,

$$f(\theta) = 2F_1(-n, P + Q + 2 + n, Q + \frac{3}{2}; cos^2(\theta) \tag{50}$$

Therefore, using the value of $f(\theta)$ in the Eq. (37) and we get the eigen function for the angular part as follows:

$$\Theta(\theta) = N_2 sin^{P+1}\theta cos^{Q+1}\theta 2F_1(-n, P + Q + 2 + n, Q + \frac{3}{2}; cos^2(\theta)) \tag{51}$$

where N_2 is the normalizing constant.

6 Conclusion

In this paper we have solved the KG-equation for the Double Ring Shaped coulomb Potential via Asymptotic Iteration Method. Using this method we get the general expression of the energy spectrums and the corresponding wave function in terms of confluent hypergeometric function multi-dimensional space. From the solution of the radial part we get the bound state solution of the inverse square potential by equating $A = 0$ in Eq. (21) and we also get a solution of the bound state of potential well type. From the solution of angular part we get the energy eigen values for isotropic harmonic oscillator and for the ring shaped oscillator by equating the coefficients in Eq. (48) with appropriate values. The Double Ring Shaped Potentials have many applications in the field of nuclear

physics and quantum chemistry which are mainly used to describe the interaction between the deformed pair of nuclei in Physics and to describe the molecular structure of benzene in chemistry. Thus the non-central potential named double Ring Shaped Coulomb Potential is used to find the quantum information in chemical and Molecular Physics.

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