Development of Some Multi-Criteria Decision Making Strategies in Uncertain Environment

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DECLARATION

This is to certify that the thesis entitled "**Development of Some Multi-Criteria Decision Making Strategies in Uncertain Environment**", submitted by me to the *Jadavpur University, Kolkata - 700032, West Bengal*, for the award of the degree of Doctor of Philosophy (Science), is an authentic work carried out by me under the supervision of **Prof. Bibhas Chandra Giri** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated

to

Abba, Maa
Asfak and Arju

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14/09/2022

Md Mahatab uddin Molla

MD MAHATAB UDDIN MOLLA

Contents

Acronyms

DM	Decision Making
MCDM	Multi-Criteria Decision Making
MADM	Multi-Attribute Decision Making
MAGDM	Multi-Attribute Group Decision Making
TOPSIS	Techniques for Order Preference by Similarity to Identical Solution
GRA	Grey Relational Analysis
PROMETHEE	Preference Ranking Organization Method for Enrichment Evaluation
DEMATEL	Decision-Making Trial and Evaluation Laboratory
FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
TrFN	Trapezoidal Fuzzy Number
HFS	Hesitant Fuzzy Set
SVNHFS	Single Valued Neutrosophic Hesitant Fuzzy Set
INHFS	Interval Neutrosophic Hesitant Fuzzy Set
IFN	Intuitionistic Fuzzy Number
NS	Neutrosophic Set
SVNS	Single Valued Neutrosophic Set
INS	Interval Neutrosophic Set
SVNTrN	Single Valued Neutrosophic Trapezoidal Number
ITrNN	Interval Trapezoidal Neutrosophic Number
PFS	Pythagorean Fuzzy Set
PFN	Pythagorean Fuzzy Number
TrPFN	Trapezoidal Pythagorean Fuzzy Number

SNS	Spherical Neutrosophic Set
SNN	Spherical Neutrosophic Number
PIS	Positive Ideal Solution
NIS	Negative Ideal Solution
SVNHFPIS	Single Valued Neutrosophic Hesitant Fuzzy Positive Ideal Solution
SVNHFNIS	Single Valued Neutrosophic Hesitant Fuzzy Negative Ideal Solution
INHFPIS	Interval Neutrosophic Hesitant Fuzzy Positive Ideal Solution
INHFNIS	Interval Neutrosophic Hesitant Fuzzy Negative Ideal Solution
PFWA	Pythagorean Fuzzy Weighted Aggregation
SNNWAA	Spherical Neutrosophic Number Weighted Averaging Aggregation

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1
Introduction

“The most difficult thing is the decision to act, the rest is merely tenacity. The fears are paper tigers. You can do anything you decide to do. You can act to change and control your life; and the procedure, the process is its own reward." - Amelia Earhart

"*We are free to choose our paths, but we can’t choose the consequences that come with them.*" — Sean Covey

“Making good decisions is a crucial skill at every level." - Peter F Drucker

In psychology, decision-making is consider as a cognitive process that involves choosing a course of belief or action among the various possible alternatives. It can be logical or irrational. Decision making is inevitable in our personal, professional or social lives on a day-to-day basis depending upon the situation we are in, especially faced with a crisis. There may be situations where decision is forced to be taken instantly. At the time of taking decision, there is no right decision or wrong decision.

Decision making is the study of identifying and choosing options based on its decision values and preferences. Making a decision means that alternatives can be considered, and, in this case, we want to not only identify as many of these options as possible but also choose the most appropriate to our goals, objectives, aspirations, values and so on. It is a process of selecting a single option from a set of available choices in a systematic and logical way.

The steps involved in decision making are referred to as the decision making process. We discuss the decision making process with the following steps (Baker et al., 2001; Fülöp, 2005).

Define the decision problem: Decision makers should be fully aware of the problem of decision making. It is important to identify, understand and define the problem before making a decision. This process must be able to identify the root cause by carefully limiting the assumptions.

Identify the criteria: Identifying and defining criteria that will differentiate between options must be based on goals. Problems with a decision that have a large number of criteria are especially helpful for developing better alternatives. An ideal set of criteria should be effective and meaningful.

Identify alternatives: A major part of decision making involves the analysis of a limited set of alternatives. All available alternatives are compared with the selected aspects and then any aspect that fails to meet is eliminated until only one option remains so that the desired goal can be achieved.

Allocate importance weights to each criteria: The weights of the criteria are allocated accordingly and comparative comparisons are applied.

Score the criteria for each of the alternatives: A matrix is formed by scoring criteria for each option and this matrix is applied to the decision rules.

Apply the decision rules: Criteria must be applied based on the weight and the input of the score of each option decision to determine possible and appropriate options.

Evaluate alternatives against criteria: After evaluation, the decision-making tool can be applied to rank options or to choose more promising options from a set of defined options.

Identify the best alternative: Appropriate alternatives of the final stage of the decision model is identified with the help of evaluation and thus the goal is achieved.

1.1 Multi-criteria decision making

Multi-criteria decision making (MCDM) is very intuitive when considered with single criterion issue, since we only have to choose the option with the highest preferred rating. However, when decision makers evaluate options with multiple criteria, many issues such as the weight of the criteria, the dependence on the choice, and the conflict between the criteria complicate the issues which need to be solved more sophisticatedly.

In order to deal with multiple criteria decision-making problems, the first step is to figure out how many attributes or criteria exist in the problem and how to grasp the way of the problems (i.e., identifying the problems). Next, we need to collect the appropriate data or information in which the preferences of decision maker can be correctly reflected upon and considered (i.e., constructing the preferences). Further work builds a set of possible alternatives or strategies in order to guarantee that the goal will be reached (i.e., evaluating the alternatives). Through these efforts, the next step is to select an appropriate method to help us evaluate and outrank or improve the possible alternatives or strategies (i.e., finding and determining the best alternative).

It is very important to make a distinction between the cases where we have a single criterion or multiple criteria. When a decision problem has a single criterion or a single aggregate measure, the decision can be made implicitly by determining the alternative with the best value of the single criterion or aggregate measure. When a decision problem has a finite number of criteria or multiple criteria, and the number of feasible alternatives is infinite, then the decision problem belongs to the field of multiple criteria optimization. Also, techniques of multiple criteria optimization can be used when there are a finite number of feasible alternatives, but are given only in implicit form. Hwang and Yoon (1981) suggested that MCDM problems can be classified into two main categories based on the different purposes and different data types:

Multi-attribute decision making (MADM): In this problem, the number of criteria (attributes) and alternatives are finite where the alternatives are explicitly given. The decision space of MADM is primarily discrete.

Multiple objective decision making (MODM): In this problem, the number of criteria is finite but the number of feasible alternatives is infinite. The decision space of MODM is often continuous.

1.1.1 Some important multi-criteria decision making techniques

Over the past few years, some well-known MCDM methods are commonly used to analyze the problem and find the desired alternative. Those are Techniques for Order Preference by Similarity to Identical Solution (TOPSIS) (Hwang and Yoon, 1981), Compromise ranking method (VIKOR) (Opricovic and Tzeng, 2004), Grey Relational Analysis (GRA) (Julong et al., 1989), Analytical Hierarchy Process (AHP) (Saaty, 1980), Elimination Et Choice Translating REality(ELECTRE) (Figueira et al., 2016), Preference Ranking Organization Method for Enrichment of Evaluation (PROMETHEE) (Brans et al., 1986), and Decision-Making Trial and Evaluation Laboratory (DEMATEL) (Gabus and Fontela, 1972). Here we discuss some classical techniques for MCDM.

TOPSIS Method:

TOPSIS method is used to determine the best alternative from the concept of compromise solution. The best compromise solution should have the shortest Euclidean distance from the ideal solution and the farthest Euclidean distance from the negative ideal solution. The procedures of TOPSIS can be described as follows:

Let A= $\{A_1, A_2, \dots A_m\}$ be the set of alternatives , C= $\{C_1, C_2, \dots, C_n\}$ be the set of criteria and D= $\{d_{ij}\}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, be the performance ratings with the criteria weight vector $W = \{w_j | j = 1, 2, \dots, n\}$. TOPSIS method is presented with these following steps:

Step 1. Normalization of the decision matrix : The normalized value d_{ij}^N is calculated as follows:

- For benefit criteria (larger the better), d_{ij}^N=$(d_{ij} - d_j^-)/(d_j^+ - d_j^-)$, where d_j^+= $\max_i(d_{ij})$ and d_j^- = $\min_i(d_{ij})$ or setting d_j^+ is the aspired or desired level and d_j^- is the worst level.
- For cost criteria (smaller the better), d_{ij}^N=$(d_j^- - d_{ij})/(d_j^- - d_j^+)$.

Step 2. Calculation of weighted normalized decision matrix : In the weighted normalized decision matrix, the modified ratings are calculated in the following way:

$$v_{ij} = w_j \times d_{ij}^N \text{ for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n. \tag{1.1}$$

where w_j is the weight of the j-th criteria such that $w_j \geq 0$ for $j = 1, 2, \dots, n$ and $\sum_{j=1}^{n} w_j = 1$.

Step 3. Determination of the positive and the negative ideal solutions : The positive ideal solution (PIS) and the negative ideal solution(NIS) are derived as follows:

$$\begin{aligned} \text{PIS} = A^+ &= \left\{v_1^+, v_2^+, \ldots v_n^+, \right\} \\ &= \left\{ \left(\max_j v_{ij} | j \in J_1 \right), \left(\min_j v_{ij} | j \in J_2 \right) | j = 1, 2, \ldots, n \right\} \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \text{NIS} = A^- &= \left\{v_1^-, v_2^-, \ldots v_n^-, \right\} \\ &= \left\{ \left(\min_j v_{ij} | j \in J_1 \right), \left(\max_j v_{ij} | j \in J_2 \right) | j = 1, 2, \ldots, n \right\} \end{aligned} \tag{1.3}$$

where J_1 and J_2 are the benefit and cost type criteria, respectively.

Step 4. Calculate the separation measures for each alternative from the PIS and the NIS : The separation values for the PIS can be measured by using the n-dimensional Euclidean distance which is given as:

$$D_i^+ = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^+\right)^2} \quad i = 1, 2, \ldots, m. \tag{1.4}$$

Similarly, separation values for the NIS is

$$D_i^- = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^-\right)^2} \quad i = 1, 2, \ldots, m. \tag{1.5}$$

Step 5. Calculation of the relative closeness coefficient to the positive ideal solution:

The relative closeness coefficient for the alternative A_i with respect to A^+ is

$$C_i = \frac{D_i^-}{D_i^+ + D_i^-} \text{ for } \quad i = 1, 2, \ldots, m. \tag{1.6}$$

Step 6. Ranking the alternatives : According to relative closeness coefficient to the ideal alternative, larger value of C_i indicates the better alternative A_i.

Grey relational analysis (GRA):

Grey relational analysis (GRA) is an important section of grey system theory which was proposed by Julong et al. (1989). GRA is mainly used to conduct relational analysis of uncertainty of a system having incomplete information. This method is applicable to discrete sequence for co-relational analysis of such sequence with processing uncertainty, multi-variate input and discrete data. GRA method has been successfully applied for MCDM problems. The method can be described as given below (see also Figure1.1):

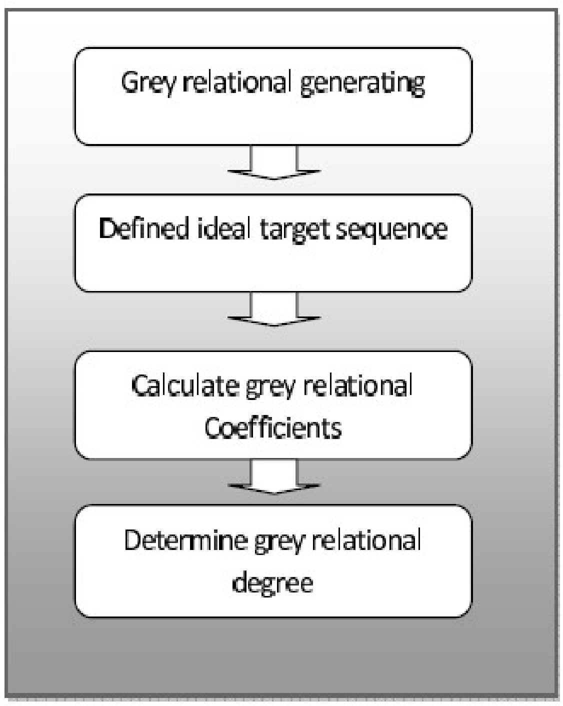

FIGURE 1.1: A schematic diagram of the GRA Method

- **Grey relational generating**
  Translate all the alternatives to comparability sequence. This process is called grey relational generating.
- **Define ideal target sequence**
  Define the ideal target of the sequence of each alternative.
- **Calculate Grey relational coefficient**
  Calculate the grey relational coefficient between ideal target sequence and comparability sequence.

- **Determine Grey relational degree**
  If an alternative achieves the maximum grey relational degree between the ideal target sequence and itself then that alternative is the optimal choice of alternative.

PROMETHEE Method:

PROMETHEE method is an outranking method for MCDM, which was developed by Brans et al. (1986). This method can handle multiple and complex criteria and it depends only on two types of information:

1. Weight of criteria.

2. Information regarding the preference of the decision maker.

The classical PROMETHEE method (Brans et al., 1986) can be described by the following steps:

Step 1. Construct the evaluation matrix
In MCDM problem, if there are m alternatives A and n criteria of each alternative $C = \{C_1, C_2, ...C_n\}$ then the evaluation matrix is of the form $X = (\alpha)_{m\times n}$, where α is the ratting value of the alternatives with respect to the corresponding criteria.

Step 2. Pairwise comparison of the alternatives
In this step, the deviation values are calculated based on pairwise comparison as given below:

$$d_j(\alpha, \beta) = c_j(\alpha) - c_j(\beta)$$

where $d_j(\alpha, \beta)$ denotes the difference between the assessments of the alternatives with respect to the criteria c_j.

Step 3. Calculate the preference function
This step determines the preference function between each pair of alternatives with the function

$$P_j(\alpha, \beta) = f_j[d_j(\alpha, \beta)],\ \forall\, \alpha, \beta \in A.$$

where f_j, the preference function which converts $d_j(\alpha, \beta)$ into a preference degree, lies between 0 and 1. Six types of preference function f_j, proposed by Brans et al. (1986), are as follows:

1. Usual criterion preference function

$$P_k(\alpha,\beta) = \begin{cases} 0, & d_k(\alpha,\beta) \leq 0 \text{ (indifference)} \\ 1, & d_k(\alpha,\beta) > 0 \text{ (strict indifference)} \end{cases}$$

2. U-shape criterion preference function

$$P_k(\alpha,\beta) = \begin{cases} 0, & d_k(\alpha,\beta) \leq p \text{ (indifference)} \\ 1, & d_k(\alpha,\beta) > p \text{ (strict indifference)} \end{cases}$$

3. V- shape criterion preference function

$$P_k(\alpha,\beta) = \begin{cases} 0, & d_k(\alpha,\beta) \leq 0 \text{ (indifference)} \\ \dfrac{d_k(\alpha,\beta)}{p}, & 0 < d_k(\alpha,\beta) \leq p \\ 1, & d_k(\alpha,\beta) > p \text{ (strict indifference)} \end{cases}$$

4. Level criterion preference function

$$P_k(\alpha,\beta) = \begin{cases} 0, & d_k(\alpha,\beta) \leq q \text{ (indifference)} \\ \dfrac{1}{2}, & q < d_k(\alpha,\beta) \leq p \\ 1, & d_k(\alpha,\beta) > p \text{ (strict indifference)} \end{cases}$$

5. V-shape criterion function with indifference area

$$P_k(\alpha,\beta) = \begin{cases} 0, & d_k(\alpha,\beta) \leq q \text{ (indifference)} \\ \dfrac{d_k(\alpha,\beta) - q}{p - q}, & q < d_k(\alpha,\beta) \leq p \\ 1, & d_k(\alpha,\beta) > p \text{ (strict indifference)} \end{cases}$$

6. Gaussian criterion function

$$P_k(\alpha,\beta) = \begin{cases} 0, & d_k(\alpha,\beta) \leq 0 \\ 1 - e^{-\dfrac{d_k^2(\alpha,\beta)}{s^2}}, & d_k(\alpha,\beta) > 0 \end{cases}$$

Step 4. Determine the aggregated preference degree

Let $W = \{w_1, w_2, ..., w_n\}$ be the weight of the criteria where $0 \leq w_j \leq 1$ for $j = 1, 2, ...n$ and $\sum_{j=1}^{n} w_j = 1$. These weights can be considered by the decision maker when the number of criteria is not large. Then the aggregated preference function is defined by

$$\Pi(\alpha, \beta) = \sum_{j=1}^{n} w_j P_j(\alpha, \beta).$$

Step 5. Calculate the positive and the negative outranking flows

In this step, leaving flow or positive outranking flow is calculated to decide the strength of the alternatives, and entering flow or negative outranking flow is calculated to decide the weakness of the alternatives. The positive outranking flow is calculated as

$$\Phi^+ = \frac{1}{m-1} \sum_{\gamma \in A} \Pi(\alpha, \gamma)$$

and the negative outranking flow is calculated as

$$\Phi^- = \frac{1}{m-1} \sum_{\gamma \in A} \Pi(\gamma, \alpha)$$

Step 6. Calculate the net outranking flow

The positive and the negative outranking flows show the best alternative performance. This is demonstrated via PROMETHEE I (partially) and PROMETHEE II (completely). In PROMETHEE I, if an alternative α is better than the alternative β then the following relation holds:

$$\begin{aligned} \alpha P \beta \ \text{ if } \Phi^+(\alpha) > \Phi^+(\beta) \text{ and } \Phi^-(\alpha) < \Phi^-(\beta) \\ \text{or, if } \Phi^+(\alpha) > \Phi^+(\beta) \text{ and } \Phi^-(\alpha) = \Phi^-(\beta) \\ \text{or, if } \Phi^+(\alpha) = \Phi^+(\beta) \text{ and } \Phi^-(\alpha) < \Phi^-(\beta) \end{aligned}$$

If an alternative α is identical to the alternative β, then

$$\alpha I \beta \ \text{ if } \Phi^+(\alpha) = \Phi^+(\beta) \text{ and } \Phi^-(\alpha) = \Phi^-(\beta).$$

Otherwise, the alternatives are incomparable i.e, $\alpha R \beta$.

PROMETHEE II gives a complete ranking and the procedure to compute the net outranking flow can be done from the following:

$$\Phi(\alpha) = \Phi^+(\alpha) - \Phi^-(\alpha)$$

Then a complete rank of the alternatives is obtained according to net outranking flow for each alternative.

DEMATEL Method:

DEMATEL method is framed on the basis of the graph theory (Gabus and Fontela, 1972). This method enables analysis and resolves problems by visualization technique. DEMATEL method investigates and visualizes direct and indirect relationships between components of researching system according to their criteria. By analyzing the relationship level among the whole system's factors, all the elements are categorized into cause group and effect group. DEMATEL method produces a better understanding of relation, and observes the ideal solution of the problem of a complex system. The steps of DEMATEL method are described as follows:

Step 1: Defining the dominant feature in the research methodology, the linguistic measurement scale is set for pairwise comparison among all characteristics. The initial direct relation matrix $D = [d_{ij}]_{n\times n}$ is obtained by pairwise comparison between criteria, in which d_{ij} denotes the degree to which the criterion i affects the criterion j.

Step 2: This step defines the normalization of direct relation matrix. On the basis of direct relation matrix D, the normalized direct relation matrix can be obtained as

$$S = k \times D, \tag{1.7}$$

$$\text{where, } k = \frac{1}{\max\limits_{1\leq i\leq n} \sum_{j=1}^{n} d_{ij}}.$$

Step 3: The total relation matrix is determined as given below:

$$T = S(I-S)^{-1}, \text{where } I \text{ is the } n\times n \text{ identity matrix.} \tag{1.8}$$

Step 4: Construct the DEMATEL map with respect to the total relation matrix. The sum of rows and the sum of columns are denoted by vectors R_j $(j = 1, 2..., n)$ and D_i $(i = 1, 2, ..., n)$, respectively within the total relation matrix $T = [t_{ij}]_{n\times n}$ and are given by

$$R_j = \Big[\sum_{j=1}^{n} t_{ij}\Big]_{1\times n} \tag{1.9}$$

$$D_i = \Big[\sum_{i=1}^{n} t_{ij}\Big]_{n\times 1} \tag{1.10}$$

where $D_i + R_j$ is a horizontal axis vector or 'prominence' which indicates the relative importance of the criterion, and the vertical axis $D_i - R_j$ represents 'relation'. If the value of $D_i - R_j$ is positive then the criterion is formed into the cause group, and if the value of $D_i - R_j$ is negative then the criterion is formed into the effect group.

Step 5: The sum of each column of the total relation matrix is 1 by normalized method, which gives the inner dependency of the matrix.

1.2 Motivation

Classical MCDM methods usually assume that all criteria and their respective weights are expressed in crisp values, and for that reason, the rating and the ranking of the alternatives can be carried out without any problem. In a real-world decision making situation, the application of the classical MCDM method may consider practical constraints from the criteria perhaps containing indeterminacy, or uncertainty in the information. The indeterminacy and uncertainty may come from different sources (Chen and Hwang, 1992).

Unquantifiable information: The price of a new laptop can be easily determined while the quality or look of a laptop is not quantifiable. Quality or look is usually expressed in linguistic terms such as good, fair, poor, etc. known as qualitative data.

Incomplete information: The speed of a fast moving object can be measured by some equipments as "about 50 kmph" but not "exactly 50 kmph." Such data type may be termed as an incomplete information.

Non-obtainable information: Sometimes crisp data are obtainable but the cost is too high, and the decision maker may wish to get an "approximation" of that crisp data. When the data are very sensitive (i.e., government's top secret, an individual's wealth amount, etc.), some "approximated" data or linguistic descriptions are used.

Partial ignorance: Sometimes uncertainty is attributed to partial ignorance of the phenomenon, as a part of the facts can only be known.

Fuzzy set (Zadeh, 1965) is useful and effective for presenting different types of indeterminant or uncertain information, it handles with a kind of uncertainty known as "fuzziness". Each real value of the interval $[0, 1]$ represents the membership degree of an element of a fuzzy set. If $\mu_A(x) \in [0, 1]$ is the membership degree of an element x of a fuzzy set A, then $1 - \mu_A(x)$ is assumed to be the non-membership degree of that element. The fuzzy information is related to three valued logic:true, false and ambiguous, which can be depicted by 1.2.

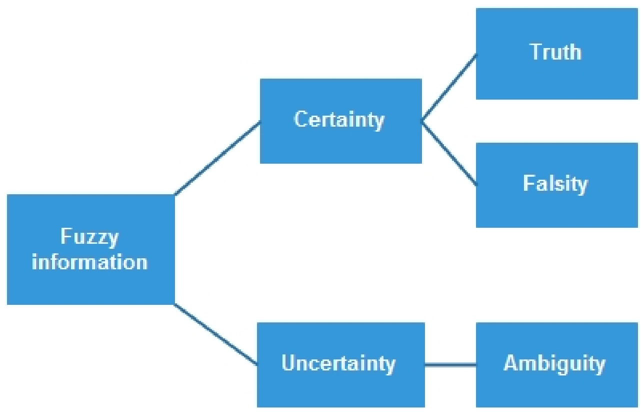

FIGURE 1.2: The structure of the fuzzy information

Generally this does not hold for an element with incomplete information. Atanassov (1986) developed the idea of intuitionistic fuzzy set (IFS). In IFS, two membership functions are expressed by the membership degree and non-membership degree of elements in the universe to the set. If $\mu_A(x) \in [0, 1]$ and $\nu_A(x) \in [0, 1]$ then $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Therefore, it provides a flexible mathematical framework to incomplete and uncertainty information. The intuitionistic fuzzy information is related to a tetra-valued logic where the information could be: true, false, ambiguous and unknown. The information presented by intuitionistic fuzzy sets can be depicted by the Fig. 1.3:

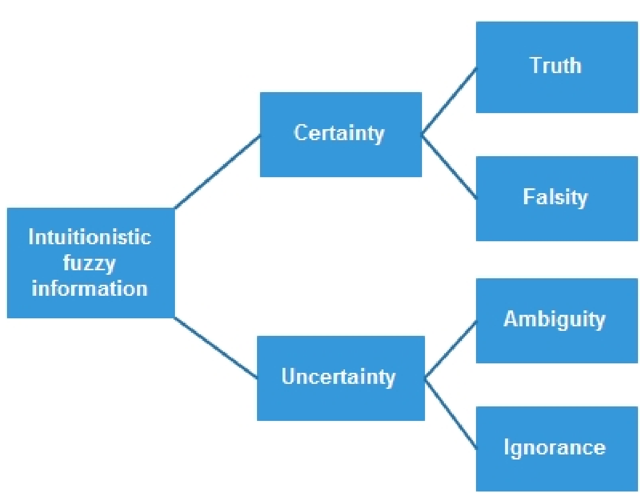

FIGURE 1.3: The structure of the intuitionistic fuzzy information

However, it can only handle incomplete and uncertainty information but not the usual real situation where the indeterminate and inconsistent information can occur. For example, when an expert gives the opinion about a certain statement, he or she may say that the possibility of the statement being true is 0.5, the degree of false statement is 0.6, and the possibility for not sure is 0.2. A further example relates to the field of medicine. Sometimes it is difficult for a doctor to make a certain diagnosis when a patient is suffering from a disease. Therefore, she/he will often give an analysis with a degree of truth and falsity, as well as indeterminacy, such as 60% of "yes", 30% of "no" and 20% of "not sure". These issues are beyond the scope of the FSs and IFSs.

For this purpose, Smarandache (1999b) developed neutrosophic logic and neutrosophic sets (NS). The NS is a set where each element of the universe has a degree of truth, indeterminacy and falsity and which lies in the nonstandard unit interval. Single valued neutrosophic set (SVNs) (Wang et al., 2010) is a special type of neutrosophic set. In neutrosophic set, the membership function value can be greater than 1. If one element of neutrosophic set is appreciated more then the truth membership value in that particular case can be greater than 1. However, in single-valued neutrosophic set, this does not happen because the membership value of single valued neutosophic set lies in $[0, 1]$ and the sum of membership values lies in $[0, 3]$.

Yager (2013)and Yager and Abbasov (2013) introduced Pythagorean fuzzy sets(PFS) which is the extension of IFS. In IFS, the membership function μ and non-membership function ν satisfy the condition $0 \leq \mu + \nu \leq 1$, for $\mu \in [0, 1]$ and $\nu \in [0, 1]$. Note that an element having membership degree $\mu \in [0, 1]$ and non-membership degree $\nu \in [0, 1]$ does not necessarily belong to IFS. For example, if the membership value and non–membership value of an alternative are 0.8 and 0.3 respectively, then the sum of membership and non-membership values of the alternative is greater than 1, which invalidates the criteria for being an IFS. On the other hand, PFS can easily handle this situation because PFS considers the condition $\mu^2 + \nu^2 \leq 1$, which is clearly satisfied as $0.8^2 + 0.3^2 < 1$. This indicates that PFS has an edge over IFS as well as FS in decision-making process under uncertainty. The geometric interpretation of fuzzy set (FS), intuitionistic fuzzy set (IFS), Pythagorean fuzzy set (PFS) and neutrosophic set (NS) are shown in Fig 1.4.

In many cases, it is difficult for decision-makers to definitely express preference in solving MCDM problems with inaccurate, uncertain or incomplete information. Under these circumstances, neutrosophic sets, SVNs and Pythagorean fuzzy sets characterized by their independent membership degree can play an effective role for solving

MCDM problems.

Recently, MCDM under neutrosophic environment is gaining popularity among the researchers. Therefore, there is an opportunity to develop new methods and/or to extend some popular methods in uncertain environment. Development of some methods of MCDM under neutrosophic set and Pythagorean fuzzy set are the main motivation of the thesis.

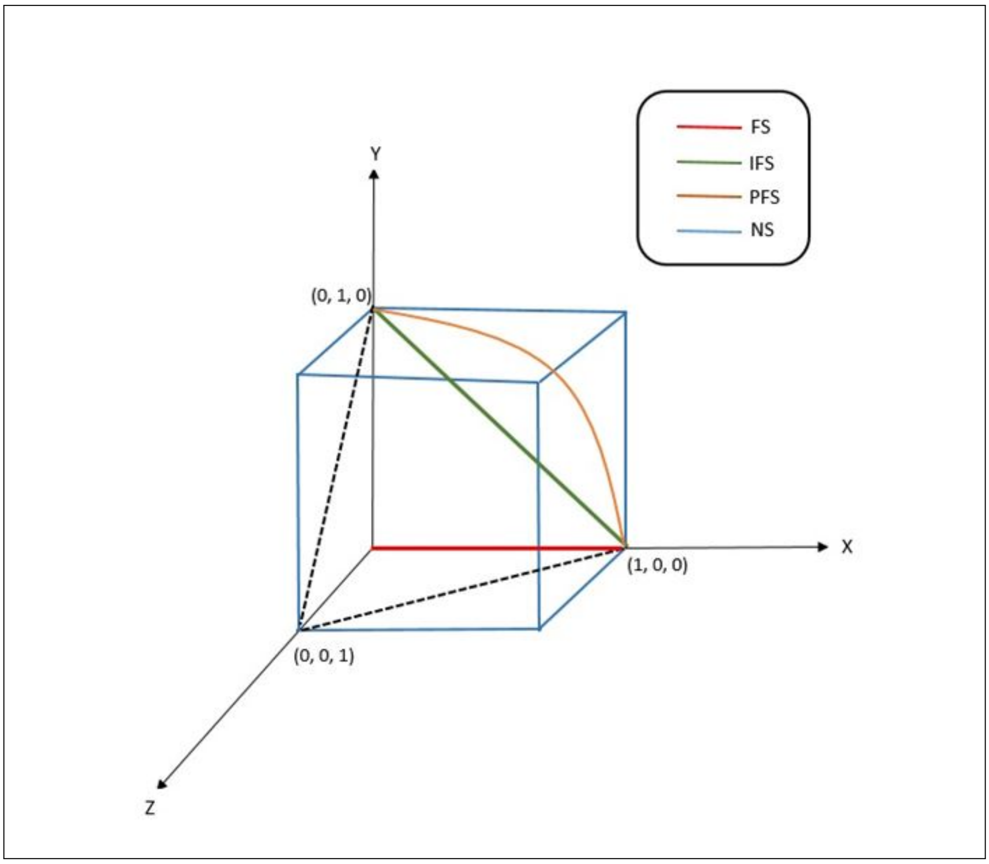

FIGURE 1.4: Geometric representation of FS, IFS, PFS and NS

1.3 Aims and objectives

The primary aim of the thesis is to develop some MCDM models in uncertain environment to deal with real-life decision making problem. The specific objectives of the thesis are as follows:

- To study MADM problem, where the rating values of the attributes are SVTrNNs and weight information is partially known or completely unknown.

- To define a new distance measure of SVTrNN and study some of its properties.
- To develop optimization models to determine the weights of attributes.
- To extend GRA method for solving SVTrNN based MADM problem using a new distance measure.
- To propose TOPSIS method for MADM problem based on interval valued trapezoidal neutrosophic number.
- To develop the model where the rating values of the attributes are ITrNN and weight information is completely known, partially known and completely unknown.
- To formulate SVNHFS based MADM problem, where the weight information is incompletely known and completely unknown.
- To determine the weights of attributes given in incompletely known and completely unknown forms using deviation method.
- To extend TOPSIS method for solving SVNHFS based MADM problem using the proposed optimization model and further extend the proposed approach in INHFS environment.
- To extend the PROMETHEE method for MCDM with Pythagorean fuzzy set.
- To apply the Pythagorean fuzzy PROMETHEE method for a medical diagnosis problem.
- To develop DEMATEL method with Pythagorean fuzzy sets. and solve the proposed method by using trapezoidal Pythagorean fuzzy number (TrPFN).
- To apply the Pythagorean fuzzy DEMATEL method in sustainable supply chain management.
- To define the spherical neutrosophic number weighted averaging aggregation (SNNWAA) operator to solve MCDM problem.
- To calculate the performance of the alternatives with respect to the criteria using SNNWAA operator.

1.4 Preliminaries

In this section, we review some preliminaries regarding fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, Pythagorean fuzzy sets, and spherical neutrosophic sets.

1.4.1 Fuzzy sets

Definition 1.1. (Zadeh, 1965) A fuzzy set $\tilde{A}$ in a universe of discourse X is defined by $\tilde{A}$=$\{\langle x, \mu_{\tilde{A}}(x)\rangle \,|x \in X\}$, where, $\mu_{\tilde{A}}(x)$: $X \to [0,1]$ is called the membership function of $\tilde{A}$ and the value of $\mu_{\tilde{A}}(x)$ is called the degree of membership for $x \in X$.

The $\alpha-$cut of the fuzzy set A is the crisp set A_α given by $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$, $\alpha \in [0,1]$.

Definition 1.2. (Dubois and Prade, 1983; Heilpern, 1992) A fuzzy number $\tilde{A}$ is called a trapezoidal fuzzy number(TrFN), if its membership function is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \dfrac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & otherwise. \end{cases}$$

The TrFN $\tilde{A}$ is denoted by the quadruplet $\tilde{A}$=(a_1, a_2, a_3, a_4) where a_1, a_2, a_3, and a_4 are real numbers and $a_1 \leq a_2 \leq a_3 \leq a_4$. The value of x at $[a_2, a_3]$ gives the maximum of $\mu_{\tilde{A}}(x)$, i.e., $\mu_{\tilde{A}}(x)$ =1; it is the most probable value of the evaluation data. The value of x outside the interval $[a_1, a_4]$ gives the minimum of $\mu_{\tilde{A}}(x)$, i.e., $\mu_{\tilde{A}}(x)$ = 0; it is the least probable value of the evaluation data. Constants a_1 and a_4 are the lower and upper bounds of the available area for the evaluation data. The $\alpha-$cut of TrFN $\tilde{A}$=(a_1, a_2, a_3, a_4) is the closed interval

$$\begin{aligned} A_\alpha &= [L^\alpha(\tilde{A}), R^\alpha(\tilde{A})] \\ &= [(a_2-a_1)\alpha + a_1, -(a_4-a_3)\alpha + a_4], \alpha \in [0,1]. \end{aligned}$$

Definition 1.3. (Dubois and Prade, 1983; Heilpern, 1992) A generalized trapezoidal fuzzy number is an extension of trapezoidal fuzzy number which is denoted by

$A = (a_1, a_2, a_3, a_4; w)$ with membership function μ_A given by

$$\mu_A(x) = \begin{cases} \dfrac{(x-a_1)w}{a_2-a_1}, & a_1 \le x < a_2 \\ w, & a_2 \le x \le a_3 \\ \dfrac{(a_4-x)w}{a_4-a_3}, & a_3 < x \le a_4 \\ 0, & otherwise. \end{cases}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$ and w is called membership degree.

Definition 1.4. (Torra, 2010) Let X be a universe of discourse. A hesitant fuzzy set (HFS) on X is symbolized by

$$A = \{\langle x, h_A(x)\rangle \mid x \in X\}, \tag{1.11}$$

where $h_A(x)$, referred to as the hesitant fuzzy element, is a set of some values in $[0, 1]$ denoting the possible membership degree of the element $x \in X$ to the set A.

From the mathematical point of view, a HFS A can be seen as a FS if there is only one element in $h_A(x)$. For notational convenience, we assume h as hesitant fuzzy element $h_A(x)$ for $x \in X$.

Definition 1.5. (Torra, 2010) Let $h = h_A(x)$ be a hesitant fuzzy element for $x \in X$ to the set A. Then the score function of h is defined as follows:

$$S(h) = \frac{1}{l_h}\sum_{\gamma\in h}\gamma \tag{1.12}$$

where, l_h is the number of the elements in a hesitant fuzzy element h.

Definition 1.6. (Chen et al., 2013) Let X be a non-empty finite set. An interval hesitant fuzzy set on X is represented by

$$E = \big\{\langle x, \tilde{h}_E(x)\rangle | x \in X\big\},$$

where $\tilde{h}_E(x)$ is a set of some different interval values in $[0, 1]$, which denote the possible membership degrees of the element $x \in X$ to the set E. $\tilde{h}_E(x)$ can be represented by an interval hesitant fuzzy element $\tilde{h}$ which is denoted by $\{\tilde{\gamma}|\tilde{\gamma} \in \tilde{h}\}$, where $\tilde{\gamma}$ =$[\gamma^L, \gamma^U]$ is an interval number.

Definition 1.7. (Kutlu Gündoğdu and Kahraman, 2019) Let X be a universe of discourse. A spherical fuzzy set A is an object having the form

$$A = \{(x, (\mu(x), \nu(x), \pi(x))) : x \in X\}$$

where $\mu(x) : X \to [0,1], \nu(x) : X \to [0,1]$ and $\pi(x) : X \to [0,1]$ and satisfy the following relation:

$$0 \leq (\mu(x))^2 + (\nu(x))^2 + (\pi(x))^2 \leq 1$$

1.4.2 Intuitionistic fuzzy sets

Definition 1.8. (Atanassov, 1986) Let a set X be fixed. An intuitionistic fuzzy set A in X is defined as

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \, | x \in X\}$$

which assigns to each x a membership degree $\mu_A(x)$ and a non-membership degree $\nu_A(x)$, where $\mu_A(x)$, $\nu_A(x) \geq 0$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$. In addition $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called hesitancy degree of x to X, which represents the degree of indeterminacy degree of x to X. For simplicity, each pair of $(\mu_A(x), \nu_A(x))$ is called an intuitionistic fuzzy number (IFN).

Definition 1.9. (Atanassov, 2012) Let A=$(\mu_A(x), \nu_A(x))$ and B=$(\mu_A(x), \nu_A(x))$ be two IFNs, then the basic operations of IFNs are presented as follows:

1. $A \oplus B$ = $(\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x))$
2. $A \otimes B$ = $(\mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x))$
3. λA = $\left(1 - (1 - \mu_A(x))^{\lambda}, (\nu_B^{\lambda})\right), \lambda > 0$
4. $(A)^{\lambda}$ = $\left((\nu_B^{\lambda}), 1 - (1 - \mu_A(x))^{\lambda}\right), \lambda > 0$

1.4.3 Neutrosophic sets

A neutrosophic set (Smarandache, 1999b) is characterized by a truth membership degree, an indeterminacy membership degree and a falsity membership degree independently. An important feature of NS is that every element of the universe has not only a certain degree of truth (T) but also a falsity degree (F) and indeterminacy degree (I).

This set is a generalization of crisp, fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, etc. NS is difficult to apply directly in real engineering and scientific applications. In order to deal with difficulties, Wang et al. (2010) introduced a subclass of NS called single-valued neutrosophic set (SVNS) characterized by truth membership degree, an indeterminacy membership degree and a falsity membership degree. SVNS can be applied quite well in real scientific and engineering fields to handle the uncertainty, imprecise, incomplete, and inconsistent information.

Definition 1.10. (Smarandache, 1999b)

Let X be a universe of discourse, with a generic element of X denoted by x. A neutrosophic set $A \subset X$ is characterized by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity-membership function $F_{\mathcal{N}}(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $[0^-, 1^+]$, so that all three neutrosophic components $T_A(x) \to [0^-, 1^+]$, $I_A(x) \to [0^-, 1^+]$ and $F_A(x) \to [0^-, 1^+]$.

The sum of three independent membership degrees $T_A(x)$, $I_A(x)$ and $F_A(x)$ have no restriction such that (Wang et al., 2010)

$$^-0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$$

Definition 1.11. (Wang et al., 2010) Let X be a universe of discourse with a generic element in X denoted by x. A single valued neutrosophic sets A in X is characterized by truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$. The set is denoted by

$$\tilde{A} = \{x, \langle T_A(x), I_A(x), F_A(x)\rangle \,| x \in X\}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real subsets of $[0, 1]$ and $0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3$. For convenience, a SVNS $\tilde{A}$ can be denoted by $\tilde{A}$ =$\langle T_A(x), I_A(x), F_A(x)\rangle$ for all $x \in X$

Definition 1.12. (Wang et al., 2005) Let X be a non empty finite set. Let $D[0, 1]$ be the set of all closed sub intervals of the unit interval $[0, 1]$. An interval neutrosophic set (INS) $\tilde{A}$ in X is an object having the form:

$$\tilde{A} = \{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle | x \in X\} \tag{1.13}$$

where $T_{\tilde{A}} : X \to D[0, 1]$, $I_{\tilde{A}} : X \to D[0, 1]$, $F_{\tilde{A}} : X \to D[0, 1]$ with the condition $0 \leq T_{\tilde{A}}(x) + I_{\tilde{A}}(x) + F_{\tilde{A}}(x) \leq 3$ for any $x \in X$. The intervals $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$

denote respectively, the degree of truth, the indeterminacy and the falsity membership degree of x to $\tilde{A}$. Then for each $x \in X$, the lower and upper limit points of closed intervals $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$ are denoted by $T^L_{\tilde{A}}(x)$, $T^U_{\tilde{A}}(x)$, $I^L_{\tilde{A}}(x)$, $I^U_{\tilde{A}}(x)$, $T^L_{\tilde{A}}(x)$, $T^U_{\tilde{A}}(x)$, respectively. Thus INS $\tilde{A}$ can also be presented in the following form:

$$\tilde{A} = \left\{ \langle x, [T^L_{\tilde{A}}(x), T^U_{\tilde{A}}(x)], [I^L_{\tilde{A}}(x), I^U_{\tilde{A}}(x)], [F^L_{\tilde{A}}(x), F^U_{\tilde{A}}(x)] \rangle | x \in X \right\},$$

where, $0 \leq T^U_{\tilde{A}}(x) + I^U_{\tilde{A}}(x) + F^U_{\tilde{A}}(x) \leq 3$ for any $x \in X$. For convenience of notation, we consider that $\tilde{A} = \langle [T^L_{\tilde{A}}, T^U_{\tilde{A}}], [I^L_{\tilde{A}}, I^U_{\tilde{A}}], [F^L_{\tilde{A}}, F^U_{\tilde{A}}] \rangle$ as an INS, where, $0 \leq T^U_{\tilde{A}} + I^U_{\tilde{A}} + F^U_{\tilde{A}} \leq 3$ for any $x \in X$.

Definition 1.13. (Subas, 2018; Ye, 2017) Let α be a single-valued neutrosophic trapezoidal number (SVNTrN). Then its membership functions are given by

$$T_\alpha(x) = \begin{cases} \dfrac{(x-a)t_\alpha}{b-a}, & a \leq x < b \\ t_\alpha, & b \leq x \leq c \\ \dfrac{(d-x)t_\alpha}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

$$I_\alpha(x) = \begin{cases} \dfrac{b-x+(x-a)i_\alpha}{b-a}, & a \leq x < b \\ i_\alpha, & b \leq x \leq c \\ \dfrac{x-c+(d-x)i_\alpha}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

$$F_\alpha(x) = \begin{cases} \dfrac{b-x+(x-a)f_\alpha}{b-a}, & a \leq x < b \\ f_\alpha, & b \leq x \leq c \\ \dfrac{x-c+(d-x)f_\alpha}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

where T_α is truth membership function, I_α is indeterminancy membership function and F_α is falsity membership function, and they all lie between 0 and 1 and satisfy the condition $0 \leq T_\alpha(x) + I_\alpha(x) + F_\alpha(x) \leq 3$ where a, b, c, d are real numbers. Then $\alpha = ([a, b, c, d]; t_\alpha, i_\alpha, f_\alpha)$ is called a neutrosophic trapezoidal number.

Definition 1.14. (Biswas et al., 2018a) Let $\tilde{\alpha}$ be interval trapezoidal neutrosophic number (ITrNN). Then its membership functions are given by

$$T_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{(x-a)t_{\tilde{\alpha}}}{b-a}, & a \le x < b \\ t_{\tilde{\alpha}}, & b \le x \le c \\ \dfrac{(d-x)t_{\tilde{\alpha}}}{d-c}, & c < x \le d \\ 0, & otherwise. \end{cases}$$

$$I_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{b-x+(x-a)i_{\tilde{\alpha}}}{b-a}, & a \le x < b \\ i_{\tilde{\alpha}}, & b \le x \le c \\ \dfrac{x-c+(d-x)i_{\tilde{\alpha}}}{d-c}, & c < x \le d \\ 0, & otherwise. \end{cases}$$

$$F_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{b-x+(x-a)f_{\tilde{\alpha}}}{b-a}, & a \le x < b \\ f_{\tilde{\alpha}}, & b \le x \le c \\ \dfrac{x-c+(d-x)f_{\tilde{\alpha}}}{d-c}, & c < x \le d \\ 0, & otherwise. \end{cases}$$

where $T_{\tilde{\alpha}}$ is truth membership function, $I_{\tilde{\alpha}}$ is indeterminancy membership function and $F_{\tilde{\alpha}}$ is falsity membership function and $t_{\tilde{\alpha}}, i_{\tilde{\alpha}},$ and $f_{\tilde{\alpha}}$ are subsets of [0,1] and $0 \le \sup(t_{\tilde{\alpha}}) + \sup(i_{\tilde{\alpha}}) + \sup(f_{\tilde{\alpha}}) \le 3$. Then α is called an interval trapezoidal neutrosophic number and it is denoted by $\tilde{\alpha} = ([a, b, c, d]; t_{\tilde{\alpha}}, i_{\tilde{\alpha}}, f_{\tilde{\alpha}})$. We take $t_{\tilde{\alpha}} = [\underline{t}, \bar{t}], i_{\tilde{\alpha}} = [\underline{i}, \bar{i}]$ and $f_{\tilde{\alpha}} = [\underline{f}, \bar{f}]$

Definition 1.15. (Biswas et al., 2018a) An interval trapezoidal neutrosophic number (ITrNN) $\tilde{\alpha} = ([a, b, c, d]; [\underline{t}, \bar{t}], [\underline{i}, \bar{i}], [\underline{f}, \overline{f}])$ is said to be positive ITrNN if $a \ge 0$ and one of the four values of a, b, c, d is not equal to zero.

Definition 1.16. Let $\tilde{\alpha} = ([a_1, b_1, c_1, d_1]; [\underline{t}_1, \bar{t}_1], [\underline{i}_1, \bar{i}_1], [\underline{f}_1, \bar{f}_1])$ and $\tilde{\beta} = ([a_2, b_2, c_2, d_2]; [\underline{t}_2, \bar{t}_2], [\underline{i}_2, \bar{i}_2], [\underline{f}_2, \bar{f}_2])$ be two ITrNNs. Then the following operations are valid:

1. $\tilde{\alpha} \bigoplus \tilde{\beta} = \begin{pmatrix} [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \\ [\underline{t}_1 + \underline{t}_2 - \underline{t}_1\underline{t}_2, \bar{t}_1 + \bar{t}_2 - \bar{t}_1\bar{t}_2], [\underline{i}_1\underline{i}_2, \bar{i}_1\bar{i}_2], [\underline{f}_1\underline{f}_2, \bar{f}_1\bar{f}_2] \end{pmatrix};$

2. $\tilde{\alpha} \bigotimes \tilde{\beta} = \begin{pmatrix} ([a_1a_2, b_1b_2, c_1c_2, d_1d_2]; [\underline{t}_1\underline{t}_2, \bar{t}_1\bar{t}_2], \\ [\underline{i}_1 + \underline{i}_2 - \underline{i}_1\underline{i}_2, \bar{i}_1 + \bar{i}_2 - \bar{i}_1\bar{i}_2], \\ [\underline{f}_1 + \underline{f}_2 - \underline{f}_1\underline{f}_2, \bar{f}_1 + \bar{f}_2 - \bar{f}_1\bar{f}_2] \end{pmatrix};$

3. $\lambda\tilde{\alpha} = \left(\begin{array}{c} \left[\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1\right]; \left[1-(1-\underline{t}_1)^\lambda, 1-(1-\bar{t}_1)^\lambda\right] \\ \left[(\underline{i}_1)^\lambda, (\bar{i}_1)^\lambda\right], \left[(\underline{f}_1)^\lambda, (\bar{f}_1)^\lambda\right] \end{array} \right), \lambda \geq 0;$

4. $(\tilde{\alpha})^\lambda = \left(\begin{array}{c} \left[(a_1)^\lambda, (b_1)^\lambda, (c_1)^\lambda, (d_1)^\lambda\right]; \left[(\underline{t}_1)^\lambda, (\bar{t}_1)^\lambda\right], \\ \left[1-(1-\underline{i}_1)^\lambda, 1-(1-\bar{i}_1)^\lambda\right], \\ \left[1-(1-\underline{f}_1)^\lambda, 1-(1-\bar{f}_1)^\lambda\right] \end{array} \right), \lambda \geq 0.$

Definition 1.17. (Ye, 2015c)

Let X be a fixed set. Then a N on X is defined as

$$N = \{\langle x, t(x), i(x), f(x)\rangle \mid x \in X\} \tag{1.14}$$

in which $t(x)$, $i(x)$ and $f(x)$ represent three sets of some values in $[0,1]$, denoting respectively the possible truth, indeterminacy and falsity membership degrees of the element $x \in X$ to the set N. The membership degrees $t(x)$, $i(x)$ and $f(x)$ satisfy the following conditions:

$$0 \leq \delta, \gamma, \eta \leq 1,\ 0 \leq \delta^+ + \gamma^+ + \eta^+ \leq 3$$

where, $\delta \in t(x), \gamma \in i(x), \eta \in f(x), \delta^+ \in t^+(x) = \bigcup_{\delta \in t(x)} \max t(x), \gamma^+ \in i^+(x) = \bigcup_{\gamma \in t(x)} \max i(x)$ and $\eta^+ \in f^+(x) = \bigcup_{\eta \in f(x)} \max f(x)$ for all $x \in X$.

$n(x)$=$\langle t(x), i(x), f(x)\rangle$ is called as single valued neutrosophic hesitant fuzzy element (SVNHFE) denoted by n=$\langle t, i, f\rangle$. The number of values for possible truth, indeterminacy and falsity membership degrees of the element in different SVNHFEs may be different.

Definition 1.18. (Liu and Shi, 2015)

Let X be a non-empty finite set. Then an interval neutrosophic hesitant fuzzy set on X is represented by

$$\tilde{n} = \left\{\langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\rangle | x \in X\right\}$$

where $\tilde{t}(x)$= $\left\{\tilde{\gamma} | \tilde{\gamma} \in \tilde{t}(x)\right\}$, $\tilde{i}(x)$= $\left\{\tilde{\gamma} | \tilde{\gamma} \in \tilde{i}(x)\right\}$ and $\tilde{f}(x)$= $\left\{\tilde{\gamma} | \tilde{\gamma} \in \tilde{f}(x)\right\}$ are three sets of some interval values in real unit interval $[0,1]$, which denotes the possible truth, indeterminacy and falsity membership hesitant degrees of the element $x \in X$ to the set N. These values satisfy the limits:

$$\tilde{\gamma} = [\gamma^L, \gamma^U] \subseteq [0,1], \tilde{\delta} = [\delta^L, \delta^U] \subseteq [0,1], \tilde{\eta} = [\eta^L, \eta^U] \subseteq [0,1]$$

and $0 \leq \tilde{\gamma}^{+} + \tilde{\delta}^{+} + \tilde{\eta}^{+} \leq 3$, where $\tilde{\gamma}^{+} = \bigcup_{\tilde{\gamma} \in \tilde{t}(x)} \sup \tilde{t}(x)$, $\tilde{\delta}^{+} = \bigcup_{\tilde{\delta} \in \tilde{t}(x)} \sup \tilde{i}(x)$ and $\tilde{\eta}^{+} = \bigcup_{\tilde{\eta} \in \tilde{t}(x)} \sup \tilde{f}(x)$. Then $\tilde{n} = \left\{\tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\right\}$ is called an interval neutrosophic hesitant fuzzy element (INHFE) which is the basic unit of the INHFS and is represented by the symbol $\tilde{n} = \left\{\tilde{t}, \tilde{i}, \tilde{f}\right\}$ for convenience.

1.4.4 Pythagorean fuzzy sets

Yager (2013); Yager and Abbasov (2013) proposed Pythagorean fuzzy sets which is generalization of intutionistic fuzzy set.

Definition 1.19. (Yager, 2013; Yager and Abbasov, 2013) Let X be a universe of discourse. Then Pythagorean fuzzy set defined on X is of the form

$$P = \{< x, \mu_p(x), \nu_p(x) > | x \in X\}$$

where $\mu_p : X \to [0, 1]$ and $\nu_p : X \to [0, 1]$ are, respectively, the membership and the non-membership functions which satisfy the condition

$$0 \leq (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1, \quad \forall x \in X$$

and the degree of indeterminacy membership is denoted by $\pi_p(x)$ and is defined by $\pi_p(x) = \sqrt{1 - (\mu_p(x))^2 - (\nu_p(x))^2}$

Zhang and Xu (2014) considered $\beta = < \mu_p, \nu_p >$ as a Pythagorean fuzzy number (PFN) where $\mu_p \in [0, 1]$ and $\nu_p \in [0, 1]$ are membership and non-membership values, respectively and $\pi_p = \sqrt{1 - \mu_p^2 - \nu_p^2}$ and $0 \leq \mu_p^2 + \nu_p^2 \leq 1$.

Definition 1.20. (Zhang and Xu, 2014) Let $\alpha_1 = < \mu_{p_1}, \nu_{p_1} >$ and $\alpha_2 = < \mu_{p_2}, \nu_{p_2} >$ be two PFNs. Then the ordering between these two PFNs is described as follows:

$$\alpha_1 \geq \alpha_2 \Leftrightarrow \mu_{p_1} \geq \mu_{p_2} \text{ and } \nu_{p_1} \leq \nu_{p_2}$$

Definition 1.21. (Xian et al., 2018) A trapezoidal pythagorean fuzzy number (TrPFN) is represented as

$$A = \langle (a_1, a_2, a_3, a_4); \mu, \nu \rangle$$

with the parameters a_1, a_2, a_3, a_4 are such that $a_1 \leq a_2 \leq a_3 \leq a_4$ and the membership and the non-membership degrees μ and ν satisfy the condition $\mu^2 + \nu^2 \leq 1$. Then the membership function μ_A and the non-membership function ν_A are given by

$$\mu_A(x) = \begin{cases} \dfrac{(x-a_1)\mu}{a_2-a_1}, & a_1 \leq x < a_2 \\ \mu, & a_2 \leq x < a_3 \\ \dfrac{(a_4-x)\mu}{a_4-a_3}, & a_3 \leq x < a_4 \\ 0, & otherwise. \end{cases} \tag{1.15}$$

$$\nu_A(x) = \begin{cases} \dfrac{a_2-x+\nu(x-a_1)}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \nu, & a_2 \leq x \leq a_3 \\ \dfrac{x-a_3+\nu(a_4-x)}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 1, & otherwise. \end{cases} \tag{1.16}$$

1.4.5 Spherical neutrosophic sets

Spherical neutrosophic set is an integration of single valued neutrosophic set and Pythagorean fuzzy set. In spherical neutrosophic set, the membership grades are truth membership ($T(x)$), indeterminacy membership ($I(x)$) and falsity membership ($F(x)$), each lies in the standard interval $[0,1]$ and their square sum i.e. $T^2(x)+I^2(x)+F^2(x)$ is less than or equal to 3. Pythagorean fuzzy set has two membership functions and their square sum is less than 1, while in single valued neutrosophic set, the sum of membership grades is less than or equal to 3.

Definition 1.22. (Smarandache, 2017a) Let X be a universe of discourse. A spherical neutrosophic set S is an object having the form

$$S = \{< x, s(T(x), I(x), F(x)) >\}$$

where the function $T(x) : X \to [0,1]$ defines the truth membership, $I(x) : X \to [0,1]$ defines the indeterminant membership and $F(x) : X \to [0,1]$ defines the falsity membership functions, and for any $x \in X$, they satisfy the following relation: $0 \leq (T(x))^2 + (I(x))^2 + (F(x))^2 \leq 3$

Definition 1.23. (Smarandache, 2017a) Let X be a universe of discourse. Then a spherical neutrosophic number (SNN) is denoted by $A = s(T_A, I_A, F_A)$ where $T_A, I_A, F_A \in [0,1]$ and $0 \leq {T_A}^2 + {I_A}^2 + {F_A}^2 \leq 3$.

It is to be noted that zero of spherical neutrosophic number O and unity of spherical neutrosophic number U can be defined as follows:

$$O = s(0, 1, 1), U = s(1, 0, 0)$$

Needless to say that spherical neutrosophic number (SNN) is an extension of single valued neutrosophic number (SVN). In SVN, the sum of truth membership, indeterminant membership and falsity membership lies between 0 and 3 and, in SNN, the sum of their squares lies between 0 and 3 i.e., $0 \leq {T_A}^2 + {I_A}^2 + {F_A}^2 \leq 3$.

1.5 Related literature

MCDM, which identifies the best alternative from a set of available alternatives depends on various criteria. MCDM problem is very common in operations research, management science, medical diagnosis, data mining, etc. In classical MADM methods, such as TOPSIS (Hwang and Yoon, 1981), PROMETHEE (Brans et al., 1986), GRA (Julong et al., 1989), AHP (Saaty, 1980), ELECTRE (Figueira et al., 2016), DEMATEL (Gabus and Fontela, 1972) the weight of each attribute and ratings of alternatives are presented by crisp numbers. Lots of research work have been done on MADM problems, where the ratings of alternatives and/or attribute values are expressed in terms of crisp numbers (Hwang and Yoon, 1981), interval numbers (Zhang and Liu, 2010), fuzzy numbers (Chen, 2000), intuitionistic fuzzy numbers (Boran et al., 2009), interval-valued intuitionistic fuzzy numbers (Nayagam et al., 2011), grey numbers (Wang et al., 2013; Zhang et al., 2005a), etc. However, in realistic situations, due to time pressure, complexity of real world, lack of information processing capabilities, poor knowledge of the public domain and information make the MCDM less and less possible to decision makers to give exact evaluations of decision parameters. In such situations, preference information of alternatives with respect to the attributes provided by the decision makers may be imprecise or incomplete in nature.

1.5.1 Grey relational analysis method for MCDM

Grey relational analysis (GRA)(Julong et al., 1989) method is one of the accepted MCDM methods. Researchers have extended the GRA method for MADM problem in different environments. Wei (2010) introduced GRA method for intuitionistic fuzzy MADM problem with incomplete weight information. Zhang and Liu (2011) proposed GRA method based on intuitionistic fuzzy multi-criteria group decision making problem (MCGDM). Pramanik and Mukhopadhyaya (2011) employed GRA method for intuitionistic fuzzy MCGDM in teacher selection problem. Dey et al. (2015a) applied GRA method for intuitionistic fuzzy MCGDM for weaver selection in Khadi institution.

Biswas et al. (2014) proposed GRA method for MADM under single valued neutrosophic environment using entropy method. Mondal and Pramanik (2015a) developed a neutrosophic MADM model for clay-brick selection in construction field and solved the problem with GRA method. Mondal and Pramanik (2015b) proposed a GRA method for rough neutrosophic MADM. Biswas et al. (2016b) applied GRA method for MADM with single valued neutrosophic hesitant fuzzy set. Biswas et al. (2019) developed NH-MADM strategy in neutrosophic hesitant fuzzy set environment based on extended GRA.

1.5.2 TOPSIS method for MCDM

In classical MCDM methods, the ratings and weights of the criteria are known precisely. TOPSIS (Hwang and Yoon, 1981) is one of the classical methods among many MCDM techniques. Chen (2000) extended the concept of TOPSIS method to develop a methodology for MCDM problem in fuzzy environment. Boran et al. (2009) extended the TOPSIS method for MCDM in intutionistic fuzzy sets. Zhao (2014) proposed TOPSIS method under interval intutionistic fuzzy number. Liu (2014) proposed TOPSIS method for MCDM under trapezoidal intuitionistic fuzzy environment with partial and unknown attribute weight information. Ye (2010) extended the TOPSIS method with interval valued intutionistic fuzzy number. Xu and Zhang (2013) proposed TOPSIS method for MADM under the hesitant fuzzy set with incomplete weight information. Chi and Liu (2013) developed TOPSIS method based on interval neutrosophic set. Ye (2015a) extended the TOPSIS method for single valued linguistic neutrosophic number. Joshi and Kumar (2016) introduced Choquet integral based TOPSIS method for multi-criteria group decision making with interval valued intutionistic hesitant fuzzy set. Fu and Liao (2019) developed TOPSIS method for multi-expert qualitative decision making involving green mine selection under unbalanced double hierarchy linguistic term set. Memari et al. (2019) solved sustainable supplier selection process by fuzzy TOPSIS method. Kutlu Gündoğdu and Kahraman (2019) proposed spherical fuzzy TOPSIS method to solve MCDM problem.

1.5.3 PROMETHEE method for MCDM

MCDM, which identifies the best alternative from a set of available alternatives depends on various criteria. Preference Ranking Organization Method for Enrichment of Evalution (PROMETHEE) (Brans et al., 1986) is a popular method to solve MCDM problem because it not only determines the degree for which an alternative satisfies the criteria but also provides a degree for which the alternative dissatisfies the criteria. PROMETHEE method compares the criteria for each pair of alternatives and preference alternative grade which lies between 0 and 1. PROMETHEE method can be successfully applied in fuzzy (Zadeh, 1965) environment to solve MCDM problem. Goumas and Lygerou (2000) extended the PROMETHEE method for decision making in fuzzy environment for optimal ranking of the alternative in energy exploitation project. Chen et al. (2011) proposed fuzzy PROMETHEE method for information system outsourcing. They used fuzzy number as the rating value of the criteria with respect to alternative. Abedi et al. (2012) developed PROMETHEE II method in fuzzy environment for copper exploration. Gul et al. (2018) developed PROMETHEE method based on fuzzy logic, and used fuzzy number for MCDM problem. However, MCDM process may contain several uncertainties and indeterminate situations. Liao and Xu (2014) proposed the PROMETHEE method in intutionistic fuzzy environment.

1.5.4 DEMATEL method for MCDM

DEMATEL (Gabus and Fontela, 1972) is a method which develops mutual relationships of the criteria and their correlated dependencies. This method provides a casual-effect diagram to describe mutual relationships and influences of the criteria (Wu and Tsai, 2011). It can analyse total relations among sets of variables to obtain logical relationships and direct impact relationships. The method is well suited to situations where it becomes necessary to upgrade the evaluation of one criterion by adding new criterion even if the number of criteria is quite large. The DEMATEL method can also be applied to solve various complex problems (Govindan et al., 2015b; Huang et al., 2007; Ren et al., 2013; Shieh et al., 2010).

Wu and Lee (2007) extended DEMATEL method with fuzzy logic for solving problems of high-tech companies. Lin and Wu (2008) developed DEMATEL method for MCDM (MCDM) problem in fuzzy environment for R&D project selection. Tseng (2009) proposed grey-fuzzy DEMATEL approach for cause and effect groups of service quality expectation. Chang et al. (2011) proposed fuzzy DEMATEL method for supplier selection criteria in supply chain management. Lin (2013) proposed fuzzy DEMATEL method for green supply chain management. Lin et al. (2014) considered DEMATEL method with T_w fuzzy sets and applied in the model for green supply chain. Govindan et al. (2015a) considered DEMATEL method with intuitionistic fuzzy sets for developing green practices and performances in a green supply chain management. Wu et al. (2015) provided a fuzzy DEMATEL method to investigate the influence of green supply chain practices in Vietnamese automobile industry. Lin et al. (2018) developed approximate fuzzy DEMATEL method to find cause and effect relationships among the criteria of sustainable supply chain management.

1.5.5 MCDM methods under Pythagorean fuzzy environment

Atanassov (1986) introduced the concept of intutionistic fuzzy set (IFS) which has both membership and non-membership degrees. The sum of membership and non-membership degrees of an IFS lies between 0 and 1. IFS has been successfully applied in MCDM problem (Atanassov et al., 2005; Xu and Yager, 2008; Yager, 2010). However, in IFS, there are membership function μ and non-membership function ν such that $\mu \in [0, 1]$, $\nu \in [0, 1]$, and $0 \leq \mu + \nu \leq 1$. But, if the rating values of alternative have membership degree 0.9 and non-membership degree 0.3, then the restriction of sum value to be in $[0, 1]$ for membership degree and non-membership degree for IFS is not satisfied. Therefore, IFS cannot handle the situation as $0.9 + 0.3 > 1$. Pythagorean fuzzy set (PFS)(Yager, 2013), an extension of intutionistic fuzzy set (IFS), can easily handle this situation because the set considers the restriction $\mu^2 + \nu^2 \leq 1$ which gives $0.9^2 + 0.3^2 \leq 1$. Yager (2013) introduced Pythagorean membership grades for MCDM problem and solved the MCDM problem using the aggregation operator. Yager and Abbasov (2013) proposed the relationship between Pythagorean membership grade and complex number. They proved that Pythagorean membership grade is one type of complex numbers ($\Pi - i$ numbers) and they solved the MCDM problem with aggregation operator of $\Pi - i$ numbers. Zhang and Xu (2014) extended the TOPSIS method with PFS and considered the Pythagorean fuzzy number (PFN) to sovle the MCDM problem. They defined a distance measure of PFN for developing TOPSIS method to get the optimal result. Recently, many researchers developed the MCDM method with Pythagorean fuzzy information (Zeng et al., 2016; Zhang, 2016). Garg (2016) developed a new generalized Pythagorean fuzzy aggregation operator using Einstein operations and applied to decision making problem. Ren et al. (2016) proposed TODIM method under pythagorean fuzzy environment for decision making. Akram et al. (2020) proposed ELECTREI method in Pythagorean fuzzy information for multi-criteria group decision making problem.

1.5.6 MCDM technique for medical diagnosis problem

Modern medical diagnosis process considers a lot of parameters some of which may contain incomplete and uncertain information. In practice, some diseases have common symptoms. Therefore, these symptoms bear an ambiguous information for detecting the exact disease. This type of medical diagnosis problem could be solved by using MCDM process, where disease and symptom can be set as an alternative and a criteria, respectively. In this problem, the preference values not only gives the degree for which the disease satisfies the symptoms but also provides the degree for which the disease dissatisfies the symptom. Therefore, Pythagorean fuzzy set can be used to handle uncertain and incomplete information. Medical diagnosis process is successfully applied in uncertain environment (Guleria and Bajaj, 2019; Ye, 2011) . Ye (2015b) introduced the cosine similarity measure for simplified neutrosophhic set in decision making for medical diagnosis problem.

1.5.7 MCDM technique for supplier selection problem in supply chain management

Supply chain management (SCM) is the process of managing the movement of a company's supplies, products and services in the most efficient and economic way possible. Sustainable supply chain management considers economic, environmental and social issues all together. Gilbert (2001) studied green supply chain management to establish connection between environment concern and business activity. Noci (1997) was possibly the first who worked on sustainable supply chain management and supplier selection criteria. Zimmer et al. (2016) reviewed the literature on suppler selection. They considered 143 papers published during the period 1997 to 2014, and highlighted the basic criteria of sustainable supplier selection process. Shen et al. (2013) proposed fuzzy multi criteria approach for evaluating green supplier's performance in green supply chain with linguistic preferences. Wu et al. (2015) studied exploring decisive factors in green supply chain practices under uncertainty. Memari et al. (2019) solved sustainable supplier selection process by fuzzy TOPSIS method.

1.5.8 Some other methods for MCDM

Besides the methods of MCDM discussed in the immediate previous subsections, a number of other methods exist in the literature. We briefly review some of them. Zavadskas et al. (1994) proposed a method of multi-criteria complex proportional assessment of projects. Xu (2007) developed some aggregation operators on intuitionistic fuzzy sets such as intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, and intuitionistic fuzzy hybrid aggregation operator. Brauers and Zavadskas (2010) proposed MULTIMOORA method to solve MCDM problem. Wei (2012) introduced hesitant fuzzy prioritized operators for solving MADM problem. Wei (2012) developed approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making. Wei et al. (2013) proposed various types of aggregation operator for MCDM problem with hesitant fuzzy sets. Ye (2014) proposed aggregation operators for simplified neutrosophic set to solve MCDM problem. Garg (2018) developed MCDM method with neutrosophic set using prioritized muirhead mean aggregation operator.

1.6 Scope of the thesis

The goal of the thesis is to study MCDM under different uncertain environments. We develop some models of MCDM, where assessment values of alternatives are considered as different type of neutrosophic sets and Pythagorean fuzzy sets. We solve the medical diagnosis problem, supplier selection problem, personnel selection problem and some real-life decision making problems through the proposed MCDM models. A brief outline of the thesis is given below. The thesis consists of **eight** chapters:

- **Chapter 1** presents an introduction to the MCDM, some preliminaries of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, and Pythagorean fuzzy sets, a brief review of the relevant literature, and aims and objectives of the study.

- **Chapter 2** presents GRA method for MCDM problem where the rating values of the attributes are SVTrNNs and weight information is partially known or completely unknown.

- **Chapter 3** proposes an extended TOPSIS method to solve MADM problem with ITrNNs and introduces new distance measure of ITrNNs. This chapter develops the model where the rating values of the attributes are ITrNN and weight information is completely known, partially known and completely unknown.
- **Chapter 4** develops MCDM method with neutrosophic hesitant fuzzy sets. This chapter formulates SVNHFS and IVNHFS based MCDM problem, where the weight information is incompletely known or completely unknown. TOPSIS method has been used to solve the proposed optimization model.
- **Chapter 5** introduces PROMETHEE method under Pythagorean fuzzy environment, provides some basic operations of Pythagorean fuzzy numbers (PFN) and compares Pythagorean fuzzy sets (PFS) with intuitionistic fuzzy sets (IFS). A medical diagnosis problem is considered as MCDM problem and solved using the proposed method.
- **Chapter 6** proposes DEMATEL method in Pythagorean fuzzy environment. The proposed model utilizes the concept of Pythagorean fuzzy sets and trapezoidal Pythagorean fuzzy number (TrPFN). A supplier selection problem in sustainable supply chain management has been solved using the proposed method.
- **Chapter 7** defines spherical neutrosophic set (SNS), which is a generalized version of FS, IFS, NS and PFS. This chapter introduces SNNWAA operator for aggregating spherical neutrosophic number as criteria value of the alternatives and develops a MCDM method based on SNNWAA operator, which is applied in a real life decision making problem, namely, personnel selection problem.
- **Chapter 8** covers conclusions, main findings, and future scopes of study.

1.7 Chapter summary

This chapter presents an overview of the study carried out in this thesis. It contains a brief introduction of decision-making, MCDM techniques such as TOPSIS, GRA, PROMETHEE, DEMTEL etc, and preliminaries of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, and Pythagorean fuzzy sets. It also includes a literature review on different MCDM models, medical diagnosis problem, supplier selection problem in sustainable supply chain management, and MCDM problem with Pythagorean fuzzy sets.

2

Grey Relational Analysis Method for Single Valued Trapezoidal Neutrosophic Number

2.1 Introduction

Grey relational analysis (GRA) is an important part of grey system theory, which is used to conduct relational analysis of uncertainty of the system. There are many applications of this method in different multi-attribute decision making (MADM) problems (Wei, 2010,1; Zhang et al., 2005b). However, in practice, decision makers face difficulties to collect accurate information of preference values of alternatives in MADM due to imprecise and incomplete data (Xu, 2015). Single valued trapezoidal neutrosophic number (SVTrNN) (Subas, 2018; Ye, 2017) is an extension of trapezoidal fuzzy number. It is presented by a trapezoidal number which has three independent membership functions, the truth membership function, the indeterminate membership function and the falsity membership function. This number can present incomplete or indeterminate information effectively with its three membership degrees. Therefore, it has an advantage over the trapezoidal fuzzy number and the trapezoidal intuitionistic fuzzy number. Deli and Şubaş (2017) developed a ranking method for single valued neutrosophic number and employed the method for solving MADM problem. Biswas et al. (2016b) proposed GRA method for SVTrNN based MADM with value

This chapter is based on the paper published in **Granular Computing** (2020), 5(4), 561–570 and ambiguity based ranking strategy. Biswas et al. (2018b) developed TOPSIS strategy for SVTrNN based MADM with unknown weight information. However, the GRA method has not been studied yet to deal with MADM problems with partially known or completely unknown weight information in the framework of SVTrNN, which can although play an effective role to deal with uncertain and indeterminate information in MADM problem. In view of the above facts, the primary objectives of this chapter are as follows:

- To study MADM problem, where the rating values of the attributes are SVTrNNs and weight information is partially known or completely unknown.
- To define a new distance measure of SVTrNN and study some of its properties.
- To develop optimization models to determine the weights of attributes.
- To extend GRA method for solving SVTrNN based MADM problem using a new distance measure.
- To validate the proposed approach with a numerical example.
- To compare the proposed approach with some existing methods including TOPSIS.

The structure of the chapter is as follows. In Section 2.2, we present some preliminaries of single valued trapezoidal neutrosophic number and define a new distance measure. In Section 2.3, we propose GRA method for SVTrNN based MADM, where the weight information of attributes is partially known or completely unknown. Section 2.4 deals with a numerical example to demonstrate the developed model. In Section 2.5, we conclude the chapter with some remarks.

2.2 Distance measure of SVTrNN

Definition 2.1. (Subas, 2018; Ye, 2017) A single valued trapezoidal neutrosophic number α is a generalization of trapezoidal fuzzy number and its membership functions are given by

$$T_\alpha(x) = \begin{cases} \dfrac{(x-a)t_\alpha}{b-a}, & a \le x < b \\ t_\alpha, & b \le x \le c \\ \dfrac{(d-x)t_\alpha}{d-c}, & c < x \le d \\ 0, & otherwise. \end{cases}$$

$$I_\alpha(x) = \begin{cases} \dfrac{b-x+(x-a)i_\alpha}{b-a}, & a \le x < b \\ i_\alpha, & b \le x \le c \\ \dfrac{x-c+(d-x)i_\alpha}{d-c}, & c < x \le d \\ 0, & otherwise. \end{cases}$$

$$F_\alpha(x) = \begin{cases} \dfrac{b-x+(x-a)f_\alpha}{b-a}, & a \le x < b \\ f_\alpha, & b \le x \le c \\ \dfrac{x-c+(d-x)f_\alpha}{d-c}, & c < x \le d \\ 0, & otherwise. \end{cases}$$

where T_α, I_α and F_α are truth membership function, indeterminacy membership function and falsity membership function, respectively, and they lie between 0 and 1 and their sum lies between 0 and 3 where a, b, c and d are real numbers. Therefore, $\alpha = ([a, b, c, d]; t_\alpha, i_\alpha, f_\alpha)$ is called a single valued trapezoidal neutrosophic number (SVTrNN).

Definition 2.2. Let $\tilde{\alpha} = ([p_1, q_1, r_1, s_1]; t_1, i_1, f_1)$ and $\tilde{\beta} = ([p_2, q_2, r_2, s_2]; t_2, i_2, f_2)$ be two SVTrNNs. Then we define the distance measure between these two numbers as

$$\begin{aligned} d(\tilde{\alpha}, \tilde{\beta}) = \frac{1}{3}\Bigg(&\Big|\Big(1 - \frac{p_1+q_1+r_1+d_1}{4}\Big)t_1 - \Big(1 - \frac{p_2+q_2+r_2+s_2}{4}\Big)t_2\Big| \\ &+\Big|\Big(1 - \frac{p_2+q_2+r_2+s_2}{4}\Big)i_2 - \Big(1 - \frac{p_1+q_1+r_1+s_1}{4}\Big)i_1\Big| \\ &+\Big|\Big(1 - \frac{p_2+q_2+r_2+s_2}{4}\Big)f_2 - \Big(1 - \frac{p_1+q_1+r_1+s_1}{4}\Big)f_1\Big|\Bigg) \end{aligned} \tag{2.1}$$

A real valued function $d : X \times X \longrightarrow [0, 1]$ is said to be distance function if it satisfies the following properties:

1. $d(\tilde{\alpha}, \tilde{\beta}) \ge 0$
2. $d(\tilde{\alpha}, \tilde{\beta}) = d(\tilde{\beta}, \tilde{\alpha})$
3. $d(\tilde{\alpha}, \tilde{\gamma}) \le d(\tilde{\alpha}, \tilde{\beta}) + d(\tilde{\beta}, \tilde{\gamma}) \quad \forall\, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in X$

Proof:

1. The distance measure $d(\tilde{\alpha}, \tilde{\beta})$ is non-negative and $d(\tilde{\alpha}, \tilde{\beta}) = 0$ when $\tilde{\alpha} = \tilde{\beta}$ i.e., $p_1 = p_2, q_1 = q_2, r_1 = r_2, s_1 = s_2, t_1 = t_2, i_1 = i_2$ and $f_1 = f_2$. Therefore, $d(\tilde{\alpha}, \tilde{\beta}) \geq 0$.

2. It is obvious that $d(\tilde{\alpha}, \tilde{\beta}) = d(\tilde{\beta}, \tilde{\alpha})$.

3. $d(\tilde{\alpha}, \tilde{\gamma})$
$=\frac{1}{3}\Big(\big|\big(1-\frac{p_1+q_1+r_1+s_1}{4}\big)t_1-\big(1-\frac{p_3+q_3+r_3+s_3}{4}\big)t_3\big| +\big|\big(1-\frac{p_3+q_3+r_3+s_3}{4}\big)i_3-\big(1-\frac{p_1+q_1+r_1+s_1}{4}\big)i_1\big|$
$+ \big|\big(1 - \frac{p_3+q_3+r_3+s_3}{4}\big)f_3 - \big(1 - \frac{p_1+q_1+r_1+s_1}{4}\big)f_1\big|\Big)$
$= \frac{1}{3}\Big(\big|\big(1-\frac{p_1+q_1+r_1+s_1}{4}\big)t_1-\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)t_2 +\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)t_2-\big(1-\frac{p_3+q_3+r_3+s_3}{4}\big)t_3\big|$
$+\big|\big(1-\frac{p_3+q_3+r_3+s_3}{4}\big)i_3-\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)i_2 +\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)i_2-\big(1-\frac{p_1+q_1+r_1+s_1}{4}\big)i_1\big|$
$+\big|\big(1-\frac{p_3+q_3+r_3+s_3}{4}\big)f_3-\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)f_2 +\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)f_2-\big(1-\frac{p_1+q_1+r_1+s_1}{4}\big)f_1\big|\Big)$

$\leq \dfrac{1}{3}\Bigg(\Big|\big(1 - \frac{p_1+q_1+r_1+s_1}{4}\big)t_1 - \big(1 - \frac{p_2+q_2+r_2+s_2}{4}\big)t_2\Big| + \Big|\big(1 - \frac{p_2+q_2+r_2+s_2}{4}\big)i_2 - \big(1 - \frac{p_1+q_1+r_1+s_1}{4}\big)i_1\Big|$
$+ \Big|\big(1 - \frac{p_2+q_2+r_2+s_2}{4}\big)f_2 - \big(1 - \frac{p_1+q_1+r_1+s_1}{4}\big)f_1\Big|\Bigg) + \dfrac{1}{3}\Bigg(\Big|\big(1 - \frac{p_2+q_2+r_2+s_2}{4}\big)t_2 - \big(1 - \frac{p_3+q_3+r_3+s_3}{4}\big)t_3\Big|$
$+\Big|\big(1-\frac{p_3+q_3+r_3+s_3}{4}\big)i_3-\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)i_2\Big| +\Big|\big(1-\frac{p_3+q_3+r_3+s_3}{4}\big)f_3-\big(1-\frac{p_2+q_2+r_2+s_2}{4}\big)f_2\Big|\Bigg)$
$= d(\tilde{\alpha}, \tilde{\beta}) + d(\tilde{\beta}, \tilde{\gamma})$

   Therefore, $d(\tilde{\alpha}, \tilde{\gamma}) \leq d(\tilde{\alpha}, \tilde{\beta}) + d(\tilde{\beta}, \tilde{\gamma}) \quad \forall\, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in X$.

2.3 GRA method for MADM based on SVTrNN

Grey relational analysis (GRA) is an important section of grey system theory which was proposed by Julong et al. (1989). GRA is mainly used to conduct relational analysis of uncertainty of a system having incomplete information. The method can be described as given below:

Let $A = \{A_1, A_2, ..., A_m\}$ be a finite set of alternatives, $C = \{C_1, C_2, ..., C_n\}$ be the set of attributes, and the rating values of attributes be represented by SVTrNNs. Let $x_{ij} = ([a_{ij}, b_{ij}, c_{ij}, d_{ij}]; t_{ij}, i_{ij}, f_{ij})$ be the rating values of A_i, the i-th alternative over the attribute C_j. Then the decision matrix is given by

$$X = (x_{ij})_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} \end{array} \quad (2.2)$$

Let $W = \{w_1, w_2, ..., w_n\}$ be the weight vector for the attributes and Δ be the set of known weight information which can be constructed in the form as given by Park et al. (2011) and Park (2004); Park et al. (1997):

1. When weak ranking : $\{w_i \geq w_j\}$, $i \neq j$;
2. When strict ranking: $\{w_i - w_j \geq \epsilon_i\ (> 0)\}$, $i \neq j$;
3. The ranking of difference: $\{w_i - w_j \geq w_k - w_p\}$, $i \neq j \neq k \neq p$;
4. The ranking with multiples: $\{w_i \geq \alpha_i w_j\}$, $0 \leq \alpha_i \leq 1, i \neq j$;
5. An interval form: $\{\beta_i \leq w_i \leq \beta_i + \epsilon_i(> 0)\}$, $0 \leq \beta_i \leq \beta_i + \epsilon_i \leq 1$.

We now propose the GRA method for MADM based on SVTrNN with partially known and completely unknown weight information. The steps are as follows:

Step 1: Normalize the decision matrix

This step transforms dimensional attributes into non-dimensional attributes which permit comparison among criteria because different criteria are usually measured in different units. In general, there are two types of attribute. One is benefit type attribute and another one is cost type attribute. Let $X = (x_{ij})_{m\times n}$ be a decision matrix, where SVTrNN $x_{ij} = ([a_{ij}, b_{ij}, c_{ij}, d_{ij}]; t_{ij}, i_{ij}, f_{ij})$ is the rating value of the alternative A_i with respect to the attribute C_j.

In order to eliminate the influence of attribute type, we consider the following technique and obtain the standardize matrix $R = (r_{ij})_{m\times n}$, where $r_{ij} = ([r^1_{ij}, r^2_{ij}, r^3_{ij}, r^4_{ij}]; t_{ij}, i_{ij}, f_{ij})$ is SVTrNN. Then we have

$$r_{ij} = \Big(\Big[\frac{a_{ij}}{u_j^+}, \frac{b_{ij}}{u_j^+}, \frac{c_{ij}}{u_j^+}, \frac{d_{ij}}{u_j^+}\Big]; t_{ij}, i_{ij}, f_{ij}\Big), \quad \text{for benefit type attribute.} \tag{2.3}$$

$$r_{ij} = \Big(\Big[\frac{u_j^-}{d_{ij}}, \frac{u_j^-}{c_{ij}}, \frac{u_j^-}{b_{ij}}, \frac{u_j^-}{a_{ij}}\Big]; t_{ij}, i_{ij}, f_{ij}\Big), \quad \text{for cost type attribute.} \tag{2.4}$$

where $u_j^+ = max\{d_{ij},$ for $i = 1, 2, ..., m\}$ and $u_j^- = min\{a_{ij},$ for $i = 1, 2, ..., m\}$ for $j = 1, 2, ..., n$.

Step 2: Calculate positive and negative ideal solutions

The positive ideal solution and the negative ideal solution of SVTrNN are P^+ and N^-, respectively for the matrix $R = (r_{ij})_{m\times n}$, and those are given below:

- For benefit type attribute,

  $P^+ = \{P_1^+, P_2^+, ..., P_n^+\}$

  where $P_j^+ = \Big([\underset{i}{max}(r^1_{ij}), \underset{i}{max}(r^2_{ij}), \underset{i}{max}(r^3_{ij}), \underset{i}{max}(r^4_{ij})]; \underset{i}{max}(t_{ij}), \underset{i}{min}(i_{ij}), \underset{i}{min}(f_{ij})\Big)$

  and $N^- = \{N_1^-, N_2^-, ..., N_n^-\}$

  where $N_j^- = \Big([\underset{i}{min}(r^1_{ij}), \underset{i}{min}(r^2_{ij}), \underset{i}{min}(r^3_{ij}), \underset{i}{min}(r^4_{ij})]; \underset{i}{min}(t_{ij}), \underset{i}{max}(i_{ij}), \underset{i}{max}(f_{ij})\Big)$

- For cost type attribute,

  $P^+ = \{P_1^+, P_2^+, ..., P_n^+\}$

  where $P_j^+ = \Big([\underset{i}{min}(r^1_{ij}), \underset{i}{min}(r^2_{ij}), \underset{i}{min}(r^3_{ij}), \underset{i}{min}(r^4_{ij})]; \underset{i}{min}(t_{ij}), \underset{i}{max}(i_{ij}), \underset{i}{max}(f_{ij})\Big)$

  and $N^- = \{N_1^-, N_2^-, ..., N_n^-\}$

  where $N_j^- = \Big([\underset{i}{max}(r^2_{ij}), \underset{i}{max}(r^2_{ij}), \underset{i}{max}(r^3_{ij}), \underset{i}{max}(r^4_{ij})]; \underset{i}{max}(t_{ij}), \underset{i}{min}(i_{ij}), \underset{i}{min}(f_{ij})\Big)$

Step 3: Calculate the grey relational coefficient

In this step, we determine the grey relational coefficient of each alternative from positive ideal solution P^+ and negative ideal solution N^-, which can be obtained from the following:

$$\xi_{ij}^+ = \frac{\underset{1\le i\le m}{min}\ \underset{1\le j\le n}{min}\, d(r_{ij}, P_j^+) + \rho\, \underset{1\le i\le m}{max}\ \underset{1\le j\le n}{max}\, d(r_{ij}, P_j^+)}{d(r_{ij}, P_j^+) + \rho\, \underset{1\le i\le m}{max}\ \underset{1\le j\le n}{max}\, d(r_{ij}, P_j^+)} \tag{2.5}$$

$$\xi_{ij}^- = \frac{\underset{1\le i\le m}{min}\ \underset{1\le j\le n}{min}\, d(r_{ij}, N_j^-) + \rho\, \underset{1\le i\le m}{max}\ \underset{1\le j\le n}{max}\, d(r_{ij}, N_j^-)}{d(r_{ij}, N_j^-) + \rho\, \underset{1\le i\le m}{max}\ \underset{1\le j\le n}{max}\, d(r_{ij}, N_j^-)} \tag{2.6}$$

ρ is the identification coefficient and we consider $\rho = 0.5$ in this study.

Step 4: Calculate the attribute weight

When the attribute weights are known, calculate the largest degree of grey relation from positive ideal solution and the smallest degree from negative ideal solution and determine the best alternative in GRA method. We develop the following models when the weight information is partially known or completely unknown.

1. Weight information is partially known : If the weight information is partially known then we develop the following optimization model:

$$\textbf{Model–1}\begin{cases} min\ \xi_i^- = \sum\limits_{j=1}^{n} w_j \xi_{ij}^- \\ max\ \xi_i^+ = \sum\limits_{j=1}^{n} w_j \xi_{ij}^+ \\ \text{subject to } w \in \Delta, \ \sum\limits_{i=1}^{n} w_j = 1, w_j \geq 0, \\ \text{for } j = 1, 2, ..., n. \end{cases}$$

Since every alternative is important, therefore, no preference should be given to any alternative. We can aggregate the above multi-objective optimization model into the following single objective model with equal weights.

$$\textbf{Model–2}\begin{cases} min\ \xi = \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} \big(\xi_{ij}^- - \xi_{ij}^+\big) w_j \\ \text{subject to } w \in \Delta, \ \sum\limits_{j=1}^{n} w_j = 1, w_j \geq 0, \\ \text{for } j = 1, 2, ..., n. \end{cases}$$

We find the optimal solution of Model-2 and use it as weight vector.

2. Weight information is completely unknown

In this case, we have the following single objective model:

$$\textbf{Model–3}\begin{cases} min\ \xi = \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} \big(\xi_{ij}^- - \xi_{ij}^+\big) w_j \\ \text{subject to } w \in \Delta, \ \sum\limits_{j=1}^{n} w_j^2 = 1, w_j \geq 0, \\ \text{for } j = 1, 2, ..., n. \end{cases}$$

To solve this model, we construct the Lagrangian function

$$L(w,\theta) = \sum_{i=1}^{m}\sum_{j=1}^{n}(\xi_{ij}^{-} - \xi_{ij}^{+}) + \frac{\theta}{2}\Big(\sum_{j=1}^{n} w_j^2 - 1\Big) \tag{2.7}$$

where $\theta \in \mathbb{R}$ is the Lagrange multiplier. The first order conditions for optimality of L give

$$\frac{\partial L}{\partial w_i} = \sum_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+}) + \theta w_j = 0 \tag{2.8}$$

$$\frac{\partial L}{\partial \theta} = \sum_{j=1}^{n} w_j^2 - 1 = 0 \tag{2.9}$$

From Eq.(2.8), we get the weight vector of the form

$$w_j = \frac{-\sum\limits_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})}{\theta}, \quad j = 1, 2, ...n. \tag{2.10}$$

Putting this value in Eq.(2.9), we get

$$\theta^2 = \sum_{j=1}^{m}\sum_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})^2 \tag{2.11}$$

$$i.e., \quad \theta = -\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})^2} \text{ for } \theta < 0 \tag{2.12}$$

$$\theta = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})^2} \text{ for } \theta > 0 \tag{2.13}$$

From Eqs.(2.10),(2.12) and (2.13), we get the weight vector of the form

$$w_j = \frac{\sum\limits_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})}{\sqrt{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})^2}} \quad \text{for } w_j > 0 \tag{2.14}$$

$$w_j = -\frac{\sum\limits_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})}{\sqrt{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}(\xi_{ij}^{-} - \xi_{ij}^{+})^2}} \quad \text{for } w_j < 0 \tag{2.15}$$

Therefore, the normalized weight vector is given by

$$\bar{w}_j = \frac{w_j}{\sum\limits_{j=1}^{n} w_j} \tag{2.16}$$

Step 5: Determine the degree of grey relational coefficient

The degree of grey relational coefficient of each alternative A_i from the positive ideal solution and that from the negative ideal solution with respect to attribute weight can be obtained, respectively, from the following:

$$\xi_i^+ = \sum_{j=1}^{n} w_j \xi_{ij}^+, \quad i = 1, 2, ...m. \tag{2.17}$$

$$\xi_i^- = \sum_{j=1}^{n} w_j \xi_{ij}^-, \quad i = 1, 2, ...m. \tag{2.18}$$

Step 6: Compute the relative closeness co-efficient

In this step, we determine the relative closeness co-efficient ξ_i of each alternative A_i with respect to the ideal alternative A^+ as

$$\xi_i = \frac{\xi_i^+}{\xi_i^+ + \xi_i^-}, \text{ for } i = 1, 2, ..., m. \tag{2.19}$$

Step 7: Rank the alternatives

We rank each of alternative A_i with respect to ξ_i. The greatest value of $\xi_i (i = 1, 2, ...m)$ of alternative $A_i (i = 1, 2, ...m)$ is the best alternative.

2.4 Numerical example

To demonstrate the proposed GRA method, we consider the following problem: In supply chain management, supplier selection is a major issue. Supplier evaluation is the process to access new or existing suppliers based on their price, production, delivery, quality of service, etc. Evaluation criteria of supplier is uncertain. Purchasing department of an overseas multi-national company intends to pick a suitable supplier to get better development.

To formulate the problem, suppose that there are four suppliers $\{A_1, A_2, A_3, A_4\}$ and each supplier has four attributes such as price, quality, delivery and e-commerce capability. We consider C_1, C_2, C_3, C_4 for price, quality, delivery and e-commerce capability, respectively. The rating values of the attributes are SVTrNN numbers. Then we get the following decision matrix:

	C_1	C_2
A_1	$\left([7,8,9,10]; 0.3, 0.4, 0.5\right)$	$\left([6,7,8,9]; 0.4, 0.5, 0.6\right)$
A_2	$\left([5,6,7,8]; 0.3, 0.4, 0.5\right)$	$\left([3,4,5,6]; 0.3, 0.4, 0.5\right)$
A_3	$\left([3,4,5,6]; 0.5, 0.6, 0.7\right)$	$\left([1,2,3,4]; 0.4, 0.5, 0.6\right)$
A_4	$\left([5,6,7,8]; 0.6, 0.7, 0.8\right)$	$\left([3,4,5,6]; 0.4, 0.5, 0.6\right)$

	C_3	C_4
A_1	$\left([4,5,6,7]; 0.1, 0.2, 0.3\right)$	$\left([4,5,6,7]; 0.3, 0.4, 0.5\right)$
A_2	$\left([6,7,8,9]; 0.4, 0.5, 0.6\right)$	$\left([7,8,9,10]; 0.4, 0.5, 0.6\right)$
A_3	$\left([4,5,6,7]; 0.2, 0.3, 0.4\right)$	$\left([6,7,8,9]; 0.4, 0.5, 0.6\right)$
A_4	$\left([3,4,5,6]; 0.5, 0.6, 0.7\right)$	$\left([5,6,7,8]; 0.3, 0.4, 0.5\right)$

We now determine the best alternative with the help of the proposed GRA method. For this, we adopt the following steps:

Step 1: Normalize the decision matrix.

In the decision matrix, the first column C_1 represents the cost attribute, and second (C_2), third (C_3) and fourth (C_4) columns represent benefit type of attribute. Then, from

Eqs.(2.3) and (2.4), we get the standardize decision matrix as given below.

	C_1	C_2
A_1	$\left([0.30, 0.33, 0.37, 0.42]; 0.3, 0.4, 0.5\right)$	$\left([0.66, 0.77, 0.88, 1.00]; 0.4, 0.5, 0.6\right)$
A_2	$\left([0.37, 0.42, 0.50, 0.60]; 0.3, 0.4, 0.5\right)$	$\left([0.33, 0.44, 0.55, 0.66]; 0.3, 0.4, 0.5\right)$
A_3	$\left([0.50, 0.60, 0.75, 1.00]; 0.5, 0.6, 0.7\right)$	$\left([0.11, 0.22, 0.33, 0.44]; 0.4, 0.5, 0.6\right)$
A_4	$\left([0.37, 0.42, 0.50, 0.60]; 0.6, 0.7, 0.8\right)$	$\left([0.33, 0.44, 0.55, 0.66]; 0.4, 0.5, 0.6\right)$

	C_3	C_4
A_1	$\left([0.44, 0.55, 0.66, 0.77]; 0.1, 0.2, 0.3\right)$	$\left([0.40, 0.50, 0.60, 0.70]; 0.3, 0.4, 0.5\right)$
A_2	$\left([0.66, 0.77, 0.88, 1.00]; 0.4, 0.5, 0.6\right)$	$\left([0.70, 0.80, 0.90, 1.00]; 0.4, 0.5, 0.6\right)$
A_3	$\left([0.44, 0.55, 0.66, 0.77]; 0.2, 0.3, 0.4\right)$	$\left([0.60, 0.70, 0.80, 0.90]; 0.4, 0.5, 0.6\right)$
A_4	$\left([0.33, 0.44, 0.55, 0.66]; 0.5, 0.6, 0.7\right)$	$\left([0.50, 0.60, 0.70, 0.80]; 0.3, 0.4, 0.5\right)$

Step 2: Calculate the positive and negative ideal solutions.
In the decision matrix, the first column C_1 represents the cost attribute, and other columns represent benefit attribute. Therefore, the positive ideal solution
$P^+ = \{P_1^+, P_2^+, P_3^+, P_4^+\}$ is given by

$$\begin{pmatrix} \left([0.30, 0.33, 0.37, 0.42]; 0.3, 0.7, 0.8\right) \\ \left([0.66, 0.77, 0.88, 1.00]; 0.4, 0.4, 0.5\right) \\ \left([0.66, 0.77, 0.88, 1.00]; 0.4, 0.2, 0.3\right) \\ \left([0.70, 0.80, 0.90, 1.00]; 0.4, 0.4, 0.5\right) \end{pmatrix}$$

and the negative ideal solution $N^- = \{N_1^-, N_2^-, N_3^-, N_4^-\}$ is given by

$$\begin{pmatrix} \left([0.50, 0.60, 0.75, 1.00]; 0.6, 0.4, 0.0.5\right) \\ \left([0.11, 0.22, 0.33, 0.44]; 0.3, 0.5, 0.6\right) \\ \left([0.33, 0.44, 0.55, 0.66]; 0.2, 0.6, 0.7\right) \\ \left([0.40, 0.50, 0.60, 0.70]; 0.3, 0.5, 0.6\right) \end{pmatrix}$$

Step 3: Calculate the grey relational coefficient.

Grey relational coefficients of alternatives from ideal solutions (positive, negative) are given by

$$\xi^{+} = (\xi^{+}_{ij})_{4\times 4} = \begin{pmatrix} 0.562 & 1.00 & 0.80 & 0.593 \\ 0.479 & 0.567 & 0.899 & 1.00 \\ 0.428 & 0.355 & 0.629 & 0.753 \\ 0.629 & 0.478 & 0.389 & 0.701 \end{pmatrix}$$

$$\xi^{-} = (\xi^{-}_{ij})_{4\times 4} = \begin{pmatrix} 0.625 & 0.396 & 0.501 & 0.961 \\ 0.742 & 0.572 & 0.513 & 0.574 \\ 0.862 & 1.00 & 0.576 & 0.711 \\ 0.427 & 0.708 & 0.849 & 0.761 \end{pmatrix}$$

Step 4: Calculate the attribute weight.

Here we consider two cases for the attribute weights: (1) when information of the attribute weights is partially known and (2) when information of the attribute weights is completely unknown.

Case 1: When the information of the attribute weights is partially known. Suppose that we have the following weight information:

$$\Delta = \begin{cases} 0.15 \leq w_1 \leq 0.20 \\ 0.20 \leq w_2 \leq 0.40 \\ 0.30 \leq w_3 \leq 0.45 \\ 0.05 \leq w_4 \leq 0.15 \\ \text{and } w_1 + w_2 + w_3 + w_4 = 1 \end{cases}$$

Using Model-2, we construct the single objective programming problem as

$$\begin{cases} min\ \xi(w) = 0.558w_1 + 0.249w_2 - 0.278w_3 - 0.036w_4 \\ \text{subject to } w \in \Delta \text{ and } \sum_{j=1}^{4} w_j = 1,\ w_j > 0, \\ \quad \text{for } j = 1, 2, 3, 4. \end{cases}$$

Solving this problem with the optimization software LINGO 11, we get the optimal weight vector as

$\bar{w} = (0.15, 0.38, 0.45, 0.02).$

Case 2 : In this case, the attribute weights are completely unknown. Using Model-3 and Eqs. (2.14), (2.15) and (2.16), we get the following weight vector

$\bar{w} = (0.498, 0.222, 0.248, 0.032)$.

Step 5: Compute the degree of grey relational coefficient.

Using Eq.(2.17), the degree of grey relational coefficient from positive ideal solution is obtained as

$\xi_i^+ = \{\xi_1^+, \xi_2^+, \xi_3^+, \xi_4^+\}$ which is given in Table 2.1.

TABLE 2.1: Relative closeness co-efficient

ξ_i^+	Case 1	Case 2
ξ_1^+	0.831	0.716
ξ_2^+	0.711	0.619
ξ_3^+	0.497	0.471
ξ_4^+	0.465	0.538

Similarly, using Eq.(2.18), the degree of grey relational coefficient from negative ideal solution is obtained as $\xi_i^- = \{\xi_1^-, \xi_2^-, \xi_3^-, \xi_4^-\}$ which is given in Table 2.2.

TABLE 2.2: Relative closeness co-efficient

ξ_i^-	Case 1	Case 2
ξ_1^-	0.488	0.554
ξ_2^-	0.554	0.642
ξ_3^-	0.782	0.816
ξ_4^-	0.730	0.605

Step 6: Calculate the relative relational degree.

Using Eq.(2.19), the relative closeness co-efficient of each alternative can be obtained as given in Table 2.3. From Table 2.3, we see that, both in Case 1, the relational degrees are in the order $\xi_2 > \xi_1 > \xi_4 > \xi_3$.

Step 7: Rank the alternatives

Considering the relative relational degrees, we determine the ranking of the alternatives as follows:
Case 1: $A_2 \succ A_1 \succ A_4 \succ A_3$
Case 2: $A_2 \succ A_1 \succ A_4 \succ A_3$

TABLE 2.3: Relative relational degree

ξ_i	Case 1	Case 2
ξ_1	0.630	0.563
ξ_2	0.562	0.490
ξ_3	0.388	0.365
ξ_4	0.389	0.470

The above shows that the ranking is same in two cases. However, in both the cases, A_2 emerges as the best alternative. In the following, we compare our proposed approach with the method suggested by Biswas et al. (2018b), because only Biswas et al. (2018b) is suitable for the considered MADM problem where the preference values of alternatives take the form of SVTrNN and attribute weights are partially known or incompletely unknown. We solve the numerical example using Biswas et al. (2018b) method and obtain the similar ranking result which demonstrates the validity of our proposed approach. A comparison of the results is shown in Table 2.4.

TABLE 2.4: A comparison of the results

Methods	Weight information		Ranking result
	Partially known	Unknown	
Biswas et al. (2018b)	$(0.15, 0.40, 0.43, 0.02)$	–	$A_2 \succ A_1 \succ A_4 \succ A_3$
	–	$(0.215, 0.269, 0.241, 0.275)$	$A_2 \succ A_1 \succ A_4 \succ A_3$
The proposed method	$(0.15, 0.38, 0.45, 0.02)$	–	$A_2 \succ A_1 \succ A_4 \succ A_3$
	–	$(0.498, 0.222, 0.248, 0.032)$	$A_2 \succ A_1 \succ A_4 \succ A_3$

The proposed GRA method is flexible to deal with MADM problems with SVTrNNs because the decision maker can analyze solution results by choosing different referential sequences and distinguishing coefficients. On the other hand, Biswas et al. (2018b) method is limited because it depends only on distance measure. Therefore, the proposed approach is better than Biswas et al. (2018b) method to deal with MADM problems. Currently some other methods(Subas, 2018; Ye, 2017) are available for MADM problem with SVTrNNs, where the weight information of attributes is assumed to be completely known. These methods can not deal with SVTrNN based MADM problem with partially known or completely unknown weight information. On the other hand, our proposed method can handle SVTrNN based MADM problem with known weight information, partially known weight and completely unknown weight information. Therefore, our method is better than the existing methods.

The proposed method has the following features:

- The method considers the preference values of the alternatives in terms of SVTrNNs that effectively deal with neutrosophic information in MADM problem.
- The method offers flexible choices for choosing the importance of attribute weights.
- The method only considers relative closeness coefficient obtained from GRA to rank the alternatives. Therefore, the method is simple and understandable.
- The proposed strategy is free from information loss due to use of any complex aggregation operator or transformation of SVTrNN based attribute values into crisp values.
- The method considers a new distance measure for solving MADM problems.

2.5 Conclusion

Single valued trapezoidal neutrosophic number is a well built tool for dealing with indeterminate and incomplete information exist in real MADM problems. In this chapter, we have extended GRA method for MADM problem based on SVTrNN, where the weight information is partially known and completely unknown. We have calculated grey relational degrees between every alternative and positive ideal solution, and between every alternative and negative ideal solution, and then defined relative relational degrees to determine the ranking of the alternatives. In order to determine the attribute weights, we have developed two optimization models under the condition that the attribute weights are partially known or completely unknown. We have provided a numerical example to demonstrate the developed method. The proposed model can be utilized in many practical problems like personnel selection, medical diagnosis, center location selection (Pramanik et al., 2016), weaver selection (Dey et al., 2016), etc. under SVTrNN environment.

3

TOPSIS Method for MADM Based on Interval Trapezoidal Neutrosophic Number

3.1 Introduction

Interval trapezoidal neutrosophic number (ITrNN) (Biswas et al., 2018a) is a generalization of single valued trapezoidal neutrosophic number (SVTrNN) (Subas, 2018; Ye, 2017). Decision makers may face difficulties to express their opinions in terms of single valued truth, indeterminacy and falsity membership degrees. In interval trapezoidal neutrosophic number, truth, indeterminacy and falsity membership degrees are interval valued. Therefore, decision makers can express their opinion throughout this number in a flexible way to face such difficulties. Our objectives in the chapter are as follows:

- To propose TOPSIS method for MADM problem based on interval valued trapezoidal neutrosophic number.
- To develop the model where the rating values of the attributes are ITrNN and weight information is completely known, partially known and completely unknown.

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We organise the chapter as follows: Section 3.2 defines Hamming distance between ITrNNs. Section 3.3 briefly presents classical TOPSIS method. Section 3.4 presents TOPSIS method for MADM based on ITrNN. An illustrative example with comparative analysis is given in Section 3.5. Finally, Section 3.6 presents some concluding remarks.

3.2 Distance between two ITrNNs

Definition 3.1. (Biswas et al., 2018a) Let $\tilde{\alpha} = ([a_1, b_1, c_1, d_1]; [\bar{t}_1, \underline{t}_1], [\bar{i}_1, \underline{i}_1], [\bar{f}_1, \underline{f}_1])$ and $\tilde{\beta} = ([a_2, b_2, c_2, d_2]; [\bar{t}_2, \underline{t}_2], [\bar{i}_2, \underline{i}_2], [\bar{f}_2, \underline{f}_2])$ be two ITrNNs. Then the distance between two numbers is defined as

$$\begin{aligned} d(\tilde{\alpha}, \tilde{\beta}) =& \frac{1}{24}\Bigg(\Big| a_1(2 + \underline{t}_1 - \underline{i}_1 - \underline{f}_1) + a_1(2 + \bar{t}_1 - \bar{i}_1 - \bar{f}_1) \\ & - a_2(2 + \underline{t}_2 - \underline{i}_2 - \underline{f}_2) - a_2(2 + \bar{t}_2 - \bar{i}_2 - \bar{f}_2) \Big| \\ & + \Big| b_1(2 + \underline{t}_1 - \underline{i}_1 - \underline{f}_1) + b_1(2 + \bar{t}_1 - \bar{i}_1 - \bar{f}_1) \\ & - b_2(2 + \underline{t}_2 - \underline{i}_2 - \underline{f}_2) - b_2(2 + \bar{t}_2 - \bar{i}_2 - \bar{f}_2) \Big| \\ & + \Big| c_1(2 + \underline{t}_1 - \underline{i}_1 - \underline{f}_1) + c_1(2 + \bar{t}_1 - \bar{i}_1 - \bar{f}_1) \\ & - c_2(2 + \underline{t}_2 - \underline{i}_2 - \underline{f}_2) - c_2(2 + \bar{t}_2 - \bar{i}_2 - \bar{f}_2) \Big| \\ & + \Big| d_1(2 + \underline{t}_1 - \underline{i}_1 - \underline{f}_1) + d_1(2 + \bar{t}_1 - \bar{i}_1 - \bar{f}_1) \\ & - d_2(2 + \underline{t}_2 - \underline{i}_2 - \underline{f}_2) - d_2(2 + \bar{t}_2 - \bar{i}_2 - \bar{f}_2) \Big| \Bigg) \end{aligned} \tag{3.1}$$

This distance is called normalized Hamming distance. The normalized Hamming distance satisfies following properties:

1. $d(\tilde{\alpha}, \tilde{\beta}) \geq 0$,
2. $d(\tilde{\alpha}, \tilde{\beta}) = d(\tilde{\beta}, \tilde{\alpha})$,
3. $d(\tilde{\alpha}, \tilde{\gamma}) \leq d(\tilde{\alpha}, \tilde{\beta}) + d(\tilde{\beta}, \tilde{\gamma})$, where $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are ITrNNs.

3.3 TOPSIS method for MADM

TOPSIS method is based on the concept that the chosen alternative should have the shortest geometric distance from the positive ideal solution and the longest geometric distance from the negative ideal solution. Let $A=\{A_i|i=1,\ldots,m\}$ be the set of alternatives, $C=\{C_j|j=1,\ldots,n\}$ be the set of criteria and $D=\{d_{ij}|i=1,\ldots,m:j=1,\ldots,n\}$ be the performance ratings with the criteria weight vector $W=\{w_j|j=1,2,\ldots,n\}$. The idea of classical TOPSIS method can be expressed in a series of following steps:

Step 1. Normalize the decision matrix.

The normalized value $\bar{d}_{ij}$ is calculated as follows:

$$\bar{d}_{ij}=\frac{d_{ij}}{\sqrt{\sum_{i=1}^{m}(d_{ij})^2}},\ i=1,2,\ldots,m \text{ and } j=1,2,\ldots,n.$$

Step 2. Calculate the weighted normalized decision matrix.

In the weighted normalized decision matrix, the modified ratings are calculated as given below:

$$v_{ij}=w_j\times\bar{d}_{ij} \text{ for } i=1,2,\ldots,m \text{ and } j=1,2,\ldots,n. \tag{3.2}$$

where w_j is the weight of the j-th attribute such that $w_j\geq 0$ for $j=1,2,\ldots,n$ and $\sum_{j=1}^{n}w_j=1$.

Step 3. Determine the positive and the negative ideal solutions.

The positive ideal solution (PIS) and the negative ideal solution (NIS) are determined as follows:

$$\begin{aligned}\text{PIS}=A^{+}&=\left\{v_1^{+},v_2^{+},\ldots v_n^{+}\right\} \\ &=\left\{\left(\max_j v_{ij}|j\in J_1\right),\left(\min_j v_{ij}|j\in J_2\right)|j=1,2,\ldots,n\right\};\end{aligned} \tag{3.3}$$

$$\begin{aligned}\text{NIS}=A^{-}&=\left\{v_1^{-},v_2^{-},\ldots v_n^{-}\right\} \\ &=\left\{\left(\min_j v_{ij}|j\in J_1\right),\left(\max_j v_{ij}|j\in J_2\right)|j=1,2,\ldots,n\right\},\end{aligned} \tag{3.4}$$

where J_1 and J_2 are the benefit type and the cost type attributes, respectively.

Step 4. Calculate the separation measures for each alternative from the PIS and the NIS.

The separation values for the PIS can be measured using the n-dimensional Euclidean distance measure as follows:

$$D_i^+ = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^+\right)^2}, \quad i = 1, 2, \ldots m. \tag{3.5}$$

Similarly, separation values for the NIS can be measured as

$$D_i^- = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^-\right)^2}, \quad i = 1, 2, \ldots m. \tag{3.6}$$

Step 5. Calculate the relative closeness coefficient to the positive ideal solution.

The relative closeness coefficient for the alternative A_i with respect to A^+ is calculated as

$$C_i = \frac{D_i^-}{D_i^+ + D_i^-} \text{ for } \quad i = 1, 2, \ldots m. \tag{3.7}$$

Step 6. Rank the alternatives.

According to relative closeness coefficient to the ideal alternative, the larger value of C_i reflects the better alternative A_i.

3.4 TOPSIS for multi-attribute decision making based on ITrNN

In this section, we put forward a framework for determining the attribute weights and the ranking orders for all the alternatives with incomplete weight information under neutrosophic environment.

For a multi-attribute decision making problem, let $A = (A_1, A_2, ..., A_n)$ be a discrete set of alternatives and $C = (C_1, C_2, ..., C_n)$ be a discrete set of attributes. Suppose that $D = [\tilde{a}_{ij}]$ is the decision matrix, where $\tilde{a}_{ij} = ([a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4]; \tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$ is ITrNN for alternative A_i with respect to attribute C_j and $\tilde{t}_{ij}, \tilde{i}_{ij}$ and $\tilde{f}_{ij}$ are subsets of $[0, 1]$ and $0 \leq \sup \tilde{t}_{ij} + \sup \tilde{i}_{ij} + \sup \tilde{f}_{ij} \leq 3$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Here $\tilde{t}_{ij}$ denotes interval truth membership function, $\tilde{i}_{ij}$ denotes interval indeterminate membership function, and $\tilde{f}_{ij}$ denotes interval falsity membership function. Then we have the following decision matrix:

$$D = (\tilde{a}_{ij})_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mn} \end{pmatrix} \end{array} \tag{3.8}$$

Now, we develop this method when attribute weights are completely known, partially known and completely unknown. The steps of the ranking are as follows:

Step 1: Standardize the decision matrix.

This step transforms various attribute dimensions into non-dimensional attributes which allow comparison across criteria because various criteria are usually measured in various units. In general, there are two types of attribute. One is benefit type attribute and another one is cost type attribute. Let $D = (a_{ij})_{m\times n}$ be a decision matrix where the ITrNN $\tilde{a}_{ij} = ([a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}]; \tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$ is the rating value of the alternative A_i with respect to the attribute C_j.

In order to eliminate the influence of attribute type, we consider the following technique and obtain the standardize matrix $R = (\tilde{r}_{ij})_{m\times n}$,
where $\tilde{r}_{ij} = ([r^1_{ij}, r^2_{ij}, r^3_{ij}, r^4_{ij}]; [\underline{t}_{ij}, \bar{t}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}])$ is ITrNN. Then we have

$$\tilde{r}_{ij} = \Big(\Big[\frac{a^1_{ij}}{u^+_j}, \frac{a^2_{ij}}{u^+_j}, \frac{a^3_{ij}}{u^+_j}, \frac{a^4_{ij}}{u^+_j}\Big]; [\underline{t}_{ij}, \bar{t}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}]\Big), \text{ for benefit type attribute} \tag{3.9}$$

$$\tilde{r}_{ij} = \Big(\Big[\frac{u^-_j}{a^4_{ij}}, \frac{u^-_j}{a^3_{ij}}, \frac{u^-_j}{a^2_{ij}}, \frac{u^-_j}{a^1_{ij}}\Big]; [\underline{t}_{ij}, \bar{t}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}]\Big), \text{ for cost type attribute} \tag{3.10}$$

where $u^+_j = max\{a^4_{ij} : i = 1, 2, ..., m\}$ and $u^-_j = min\{a^1_{ij} : i = 1, 2, ..., m\}$ for $j = 1, 2, ..., n$.

Step 2: Calculate the attribute weight.

The attribute weights may be completely known, partially known or completely unknown. So we need to determine the attribute weights by maximum deviation method which is proposed by Yingming (1997). If the attributes have larger deviation, smaller deviation and no deviation then we assign larger weight, smaller weight and zero weight, respectively.

For MADM problem, the deviation values of alternative A_i to the other alternatives under the attribute C_j can be defined as follows:

$$d_{ij}(w) = \sum_{k=1}^{m} d(\tilde{a}_{ij}, \tilde{a}_{kj}) w_j,\ i = 1, 2, ..., m;\ \ j = 1, 2, ..., n, \text{where}$$

$$\begin{aligned} d(\tilde{a}_{ij}, \tilde{a}_{kj}) =& \frac{1}{24}\Bigg(\begin{vmatrix} a^1_{ij}(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + a^1_{ij}(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - a^1_{kj}(2 + \underline{t}_{kj} - \underline{i}_{kj} - \underline{f}_{kj}) - a^1_{kj}(2 + \bar{t}_{kj} - \bar{i}_{kj} - \bar{f}_{kj}) \end{vmatrix} \\ &+ \begin{vmatrix} a^2_{ij}(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + a^2_{ij}(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - a^2_{kj}(2 + \underline{t}_{kj} - \underline{i}_{kj} - \underline{f}_{kj}) - a^2_{kj}(2 + \bar{t}_{kj} - \bar{i}_{kj} - \bar{f}_{kj}) \end{vmatrix} \\ &+ \begin{vmatrix} a^3_{ij}(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + a^3_{ij}(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - a^3_{kj}(2 + \underline{t}_{kj} - \underline{i}_{kj} - \underline{f}_{kj}) - a^3_{kj}(2 + \bar{t}_{kj} - \bar{i}_{kj} - \bar{f}_{kj}) \end{vmatrix} \\ &+ \begin{vmatrix} a^4_{ij}(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + a^4_{ij}(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - a^4_{kj}(2 + \underline{t}_{kj} - \underline{i}_{kj} - \underline{f}_{kj}) - a^4_{kj}(2 + \bar{t}_{kj} - \bar{i}_{kj} - \bar{f}_{kj}) \end{vmatrix} \Bigg) \\ =& \frac{1}{24} \sum_{p=1}^{4} \left(\begin{vmatrix} a^p_{ij}(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + a^p_{ij}(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - a^p_{kj}(2 + \underline{t}_{kj} - \underline{i}_{kj} - \underline{f}_{kj}) - a^p_{kj}(2 + \bar{t}_{kj} - \bar{i}_{kj} - \bar{f}_{kj}) \end{vmatrix} \right) \end{aligned}$$

The deviation values of all the alternatives to other alternatives for the attributes C_j can be defined as

$$\begin{aligned} D_j(w) =& \sum_{i=1}^{m} d_{ij}(w) = \sum_{i=1}^{m}\sum_{k=1}^{m} d(\tilde{a}_{ij}, \tilde{a}_{kj}) w_j \\ =& \sum_{i=1}^{m}\sum_{k=1}^{m} \Bigg(\frac{1}{24} \sum_{p=1}^{4} \begin{vmatrix} a^p_{ij}(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + a^p_{ij}(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - a^p_{kj}(2 + \underline{t}_{kj} - \underline{i}_{kj} - \underline{f}_{kj}) - a^p_{kj}(2 + \bar{t}_{kj} - \bar{i}_{kj} - \bar{f}_{kj}) \end{vmatrix} \Bigg) w_j \end{aligned}$$

Therefore, the total deviation value $D(w) = \sum_{j=1}^{n} D_j(w)$.
In the following, we develop three cases:
Case 1. When the attribute weights are completely known.
In this case, the attribute weights $w_1, w_2, ..., w_n$ are known in advance and $\sum_{j=1}^{n} w_j = 1, w_j \geq 0,\ for\ j = 1, 2, ..., n.$

Case 2. When the attributes weights are partially known.
In this case, we assume a non-linear programming model. This model maximizes all

deviation values of the attributes.

$$\textbf{Model 1}\begin{cases} max\ D(w) \\ = \dfrac{1}{24}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{4}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)w_j \\ \text{subject to } w\in\Delta,\ \sum\limits_{j=1}^{n} w=1, w_j\geq 0,\ \text{for } j=1,2,......,n. \end{cases}$$

Here, the incomplete attribute weight information Δ is taken in the following form (Park et al., 2011, 1997):

1. A weak ranking:$\{w_i \geq w_j\}$, $i \neq j$;
2. A strict ranking:$\{w_i - w_j \geq \epsilon_i(>0)\}$, $i \neq j$;
3. A ranking of difference:$\{w_i - w_j \geq w_k - w_p\}$, $i \neq j \neq k \neq p$;
4. A ranking with multiples:$\{w_i \geq \alpha_i w_j\}$, $0 \leq \alpha_i \leq 1, i \neq j$;
5. An interval form:$\{\beta_i \leq w_i \leq \beta_i + \epsilon_i(>0)\}$, $0 \leq \beta_i \leq \beta_i + \epsilon_i \leq 1$.

Solving this model, we get the optimal solution which is to be used as the weight vector.

Case 3. When attribute weights are completely unknown:

In this case, we can establish the following programming model:

$$\textbf{Model 2}\begin{cases} max\ D(w) \\ = \dfrac{1}{24}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{4}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)w_j \\ \text{subject to } w\in\Delta,\ \sum\limits_{j=1}^{n} w_j^2=1, w_j\geq 0,\ \text{for } j=1,2,......,n. \end{cases}$$

To solve this model, we construct the Lagrangian function:

$$L(w,\xi)=\frac{1}{24}\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{4}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)w_j + \frac{\xi}{48}\Big(\sum_{j=1}^{n} w_j^2 - 1\Big) \quad (3.11)$$

where $\xi \in \mathbb{R}$ is Lagrange multiplier.

Now, we calculate the partial derivatives of L with respect to $w_j(j = 1, 2, ...n)$ and ξ:

$$\frac{\partial L}{\partial w_j} = \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{4}\left(\left| \begin{array}{l} a_{ij}^p(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^p(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -\,a_{kj}^p(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^p(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)+\xi w_j = 0 \tag{3.12}$$

$$\frac{\partial L}{\partial \xi} = \sum_{j=1}^{n} w_j^2 - 1 = 0 \tag{3.13}$$

From Eq. (3.12), we get

$$w_j = \frac{-\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{4}\left(\left| \begin{array}{l} a_{ij}^p(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^p(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -\,a_{kj}^p(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^p(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)}{\xi},$$

$$j = 1, 2, ..., n \tag{3.14}$$

Putting this value in Eq.(3.13), we get

$$\xi^2 = \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left| \begin{array}{l} a_{ij}^p(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^p(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -\,a_{kj}^p(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^p(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)\right)^2 \tag{3.15}$$

$$\xi = -\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left| \begin{array}{l} a_{ij}^p(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^p(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -\,a_{kj}^p(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^p(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)\right)^2}\text{, for } \xi < 0 \tag{3.16}$$

From Eq.(3.14) and Eq. (3.16), we get the formula for determining attribute weights for $C_j(j = 1, 2, ..., n)$:

$$w_j = \frac{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{m}\left(\left| \begin{array}{l} a_{ij}^p(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^p(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -\,a_{kj}^p(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^p(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)}{\sqrt{\sum\limits_{j=1}^{n}\left(\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{m}\left(\left| \begin{array}{l} a_{ij}^p(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^p(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -\,a_{kj}^p(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^p(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)\right)^2}} \tag{3.17}$$

Now, we can get the normalized attribute weight as

$$\bar{w}_j = \frac{w_j}{\sum_{j=1}^{n} w_j} = \frac{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{m}\left(\begin{vmatrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij}) + a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ - a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj}) - a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{vmatrix}\right)}{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{m}\left(\begin{vmatrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij}) + a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ - a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj}) - a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{vmatrix}\right)} \tag{3.18}$$

Therefore, we get the normalized weight vector $\bar{w} = \{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n\}$.

Step 3: Determine the positive and negative ideal solutions.

The normalized decision matrix $R = (\tilde{r}_{ij})_{m\times n}$ in the interval trapezoidal neutroshopic number, the positive and negative ideal solutions are defined as follows:

$\tilde{r}^{+} = (\tilde{r}_1^{+}, \tilde{r}_2^{+}, \ldots, \tilde{r}_n^{+})$ and $\tilde{r}^{-} = (\tilde{r}_1^{-}, \tilde{r}_2^{-}, \ldots, \tilde{r}_n^{-})$ where,

$$\begin{aligned}\tilde{r}_j^{+} =& ([r_j^{1+}, r_j^{2+}, r_j^{3+}, r_j^{4+}]; [\underline{t}_j^{+}, \bar{t}_j^{+}], [\underline{i}_j^{+}, \bar{i}_j^{+}], [\underline{f}_j^{+}, \bar{f}_j^{+}]) \\ =& \Big([\max_i(r_{ij}^{1}), \max_i(r_{ij}^{2}), \max_i(r_{ij}^{3}), \max_i(r_{ij}^{4})]; \\ & [\max_i(\underline{t}_{ij}), \max_i(\bar{t}_{ij})][\min_i(\underline{i}_{ij}), \min_i(\bar{i}_{ij})], [\min_i(\underline{f}_{ij}), \min_i(\bar{f}_{ij})] \Big)\end{aligned} \tag{3.19}$$

$$\begin{aligned}\tilde{r}_j^{-} =& ([r_j^{1-}, r_j^{2-}, r_j^{3-}, r_j^{4-}]; [\underline{t}_j^{-}, \bar{t}_j^{-}], [\underline{i}_j^{-}, \bar{i}_j^{-}], [\underline{f}_j^{-}, \bar{f}_j^{-}]) \\ =& \Big([\min_i(r_{ij}^{1}), \min_i(r_{ij}^{2}), \min_i(r_{ij}^{3}), \min_i(r_{ij}^{4})]; \\ & [\min_i(\underline{t}_{ij}), \min_i(\bar{t}_{ij})][\max_i(\underline{i}_{ij}), \max_i(\bar{i}_{ij})], [\max_i(\underline{f}_{ij}), \max_i(\bar{f}_{ij})] \Big)\end{aligned} \tag{3.20}$$

The global positive and negative ideal solutions for ITrNN can be considered as

$$\tilde{r}_j^{+} = ([1,1,1,1]; [1,1], [0,0], [0,0])$$

and

$$\tilde{r}_j^{-} = ([0,0,0,0]; [0,0], [1,1], [1,1]).$$

Step 4: Calculate the separation measure from ideal solutions.
Now, using Eqs.(3.2), (3.19) and (3.20), we calculate separation measure d_i^+ from positive ideal solution and d_i^- from negative ideal solution as

$$\begin{aligned} d_i^+ &= \sum_{j=1}^{n} w_j d(\tilde{r}_{ij}, \tilde{r}_j^+) \\ &= \frac{1}{24} \sum_{j=1}^{n} w_j \sum_{p=1}^{4} \left(\left| \begin{array}{l} r_{ij}^p(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + r_{ij}^p(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - r_j^{p+}(2 + \underline{t}_j^+ - \underline{i}_j^+ - \underline{f}_j^+) - r_j^{p+}(2 + \bar{t}_j^+ - \bar{i}_j^+ - \bar{f}_j^+) \end{array} \right| \right), \quad i = 1, 2, ..., m. \end{aligned} \tag{3.21}$$

$$\begin{aligned} d_i^- &= \sum_{j=1}^{n} w_j d(\tilde{r}_{ij}, \tilde{r}_j^-) \\ &= \frac{1}{24} \sum_{j=1}^{n} w_j \sum_{p=1}^{4} \left(\left| \begin{array}{l} r_{ij}^p(2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + r_{ij}^p(2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - r_j^{p-}(2 + \underline{t}_j^- - \underline{i}_j^- - \underline{f}_j^-) - r_j^{p-}(2 + \bar{t}_j^- - \bar{i}_j^- - \bar{f}_j^-) \end{array} \right| \right), \quad i = 1, 2, ..., m. \end{aligned} \tag{3.22}$$

Step 5: Calculate the relative closeness co-efficient.
We calculate the relative closeness co-efficient of an alternative A_i with respect to the ideal alternative A^+ as

$$RCC(A_i) = \frac{d_i^-}{d_i^+ + d_i^-}, \; for \; i = 1, 2, ..., n, \tag{3.23}$$

where $0 \leq RCC(A_i) \leq 1$. We then rank the best alternative according to RCC.
Step 6: End.

3.5 An illustrative example

In order to demonstrate the proposed method, we consider the following MADM problem. Suppose that a person wants to buy a laptop. Let there be four companies A_1, A_2, A_3, A_4 and laptop of each company has three attributes such as cost, warranty, and quality. We consider C_1 for cost, C_2 for warranty and C_3 for quality type of attribute.

The person evaluates the rating values of the alternatives A_i $(i = 1, 2, 3, 4)$ with respect to attributes $C_j (j = 1, 2, 3)$. Then we get the neutrosophic decision matrix

$D = (\tilde{a}_{ij})_{4\times 3} =$

	C_1
A_1	$([50, 60, 70, 80]; [0.1, 0.2], [0.2, 0.3], [0.4, 0.5])$
A_2	$([30, 40, 50, 60]; [0.3, 0.4], [0.2, 0.3], [0.1, 0.2])$
A_3	$([70, 80, 90, 100]; [0.6, 0.7], [0.2, 0.3], [0.4, 0.5])$
A_4	$([40, 50, 60, 70]; [0.4, 0.5], [0.6, 0.7], [0.2, 0.3])$
	C_2
A_1	$([30, 40, 50, 60]; [0.2, 0.3], [0.4, 0.5], [0.6, 0.7])$
A_2	$([10, 20, 30, 40]; [0.1, .2], [0.3, 0.4], [0.6, 0.7])$
A_3	$([50, 60, 70, 80]; [0.1, 0.2], [0.3, 0.4], [0.6, 0.7])$
A_4	$([70, 80, 90, 100]; [0.2, 0.3], [0.4, 0.5], [0.6, 0.8])$
	C_3
A_1	$([40, 50, 60, 70]; [0.4, 0.5], [0.6, 0.7], [0.7, 0.8])$
A_2	$([20, 30, 40, 50]; [0.1, 0.2], [0.3, 0.4], [0.8, 0.9])$
A_3	$([70, 80, 90, 100]; [0.3, 0.5], [0.4, 0.6], [0.7, 0.8])$
A_4	$([30, 40, 50, 60]; [0.4, 0.5], [0.6, 0.7], [0.7, 0.8])$

Now, with the help of the proposed method, we find the best alternative following the steps given below:

Step 1: Standardize the decision matrix.

In the decision matrix, the first column represents the cost type attribute, and the second and the third columns represent benefit type attribute. Then, using Eqs. (3.9) and (3.10), we get the following standardize decision matrix R_{ij} =

	C_1
A_1	$([0.38, 0.43, 0.50, 0.60]; [0.1, 0.2], [0.2, 0.3], [0.4, 0.5])$
A_2	$([0.50, 0.60, 0.75, 1.0]; [0.3, 0.4], [0.2, 0.3], [0.1, 0.2])$
A_3	$([0.30, 0.33, 0.38, 0.43]; [0.6, 0.7], [0.2, 0.3], [0.4, 0.5])$
A_4	$([0.43, 0.50, 0.60, 0.75]; [0.4, 0.5], [0.6, 0.7], [0.2, 0.3])$
	C_2
A_1	$([0.30, 0.40, 0.50, 0.60]; [0.2, .3], [0.4, 0.5], [0.6, 0.7])$
A_2	$([0.10, 0.20, 0.30, 0.40]; [0.1, 0.2], [0.3, 0.4], [0.6, 0.7])$
A_3	$([0.50, 0.60, 0.70, 0.80]; [0.1, 0.2], [0.3, 0.4], [0.6, 0.7])$
A_4	$([0.70, 0.80, 0.90, 1.0]; [0.2, 0.3], [0.4, .5], [0.6, 0.8])$
	C_3
A_1	$([0.40, 0.50, 0.60, 0.70]; [0.4, 0.5], [0.6, 0.7], [0.7, 0.8])$
A_2	$([0.20, 0.30, 0.40, 0.50]; [0.1, 0.2], [0.3, 0.4], [0.8, 0.9])$
A_3	$([0.70, 0.80, 0.90, 1.0]; [0.3, 0.5], [0.4, 0.6], [0.7, 0.8])$
A_4	$([0.30, 0.40, 0.50, 0.60]; [0.4, .5], [0.6, 0.7], [0.7, 0.8])$

Step 2: Calculate the attribute weight.

Here we assume three cases for the attribute weight.

Case 1 : When the attribute weights are completely known, let the weight vector be $\bar{w} = (0.25, 0.55, 0.20)$.

Case 2 : When the attribute weights are partially known, we select the weight information as follows:

$$\Delta = \begin{cases} 0.35 \leq w_1 \leq 0.75 \\ 0.25 \leq w_2 \leq 0.60 \\ 0.30 \leq w_3 \leq 0.45 \\ \text{and } w_1 + w_2 + w_3 = 1 \end{cases}$$

Using Model 1, we develop the single objective programming problem as

$$\begin{cases} max(D) = 45.92w_1 + 109.56w_2 + 98.20w_3 \\ \text{subject to } w \in \Delta \text{ and } \sum_{j=1}^{3} w_j = 1,\ w_j > 0 \ for\ j = 1, 2, 3. \end{cases}$$

Solving this problem with optimization software LINGO 11, we get the optimal weight vector as

$\bar{w} = (0.35, 0.35, 0.30)$.

Case 3 : When the attribute weights are completely unknown, we use Model 2 and Eqn. (3.18) and obtain the following weight vector:

$\bar{w} = (0.18, 0.43, 0.39)$.

Step 3: Determine the positive and negative ideal solutions.

Since the cost of the laptop is cost type attribute, and warranty and quality are benefit type attributes, therefore, using Eqs. (3.19) and (3.20), we get the following neutrosophic positive and negative ideal solutions:

$$A^+ = \begin{pmatrix} \left([0.30,0.33,0.38,0.43];[0.10,0.20],[0.20,0.30],[0.20,0.30]\right) \\ \left([0.70,0.80,0.90,1.0];[0.20,0.30],[0.40,0.50],[0.60,0.70]\right) \\ \left([0.70,0.80,0.90,1.0];[0.40,0.50],[0.60,0.70],[0.80,0.90]\right) \end{pmatrix}$$

$$A^- = \begin{pmatrix} \left([0.50,0.60,0.75,1.0];[0.60,0.70],[0.40,0.50],[0.40,0.50]\right) \\ \left([0.10,0.20,0.30,0.40];[0.10,0.20],[0.30,0.40],[0.60,0.70]\right) \\ \left([0.20,0.30,0.40,0.50];[0.10,0.20],[0.30,0.40],[0.70,0.80]\right) \end{pmatrix}$$

Step 4 : Calculate the separation measure from ideal solutions.

Case 1 : From Eq. (3.21), we get the separation measure d_i^+ of each A_i from A^+:

$d_1^+ = d(A_1, A^+) = 0.179, d_2^+ = d(A_2, A^+) = 0.425$
$d_3^+ = d(A_3, A^+) = 0.106, d_4^+ = d(A_4, A^+) = 0.325$

From Eq. (3.22), we get the separation measure d_i^- of each A_i from A^-:

$d_1^- = d(A_1, A^-) = 0.304, d_2^- = d(A_2, A^-) = 0.083$
$d_3^- = d(A_3, A^-) = 0.485, d_4^- = d(A_4, A^)) = 0.503$

Case 2 : From Eq. (3.21), we get the separation measure d_i^+ of each A_i from A^+:

$d_1^+ = d(A_1, A^+) = 0.185, d_2^+ = d(A_2, A^+) = 0.434$
$d_3^+ = d(A_3, A^+) = 0.141, d_4^+ = d(A_4, A^+) = 0.335$

From Eq. (3.22), we get the separation measure d_i^- of each A_i from A^-:

$d_1^- = d(A_1, A^-) = 0.299, d_2^- = d(A_2, A^-) = 0.084$
$d_3^- = d(A_3, A^-) = 0.479, d_4^- = d(A_4, A^)) = 0.381$

Case 3 : From Eq. (3.21), we get the separation measure d_i^+ of each A_i from A^+:

$d_1^+ = d(A_1, A^+) = 0.167, d_2^+ = d(A_2, A^+) = 0.429$
$d_3^+ = d(A_3, A^+) = 0.126, d_4^+ = d(A_4, A^+) = 0.307$

From Eq.(3.22), we get the separation measure d_i^- of each A_i from A^-:

$d_1^- = d(A_1, A^-) = 0.604, d_2^- = d(A_2, A^-) = 0.094$
$d_3^- = d(A_3, A^-) = 0.554, d_4^- = d(A_4, A^)) = 0.467$

Step 5: Calculate the relative closeness co-efficient.

In this step, using Eq.(3.23), we calculate the relative closeness coefficient of the alternatives A_1, A_2, A_3, A_4 and obtain the following results (see Table 3.1):

TABLE 3.1: Relative closeness co-efficient

$RCC(A_i)$	Case 1	Case 2	Case 3
$RCC(A_1)$	0.629	0.618	0.783
$RCC(A_2)$	0.163	0.162	0.180
$RCC(A_3)$	0.819	0.773	0.814
$RCC(A_4)$	0.607	0.532	0.603

From the above table, we see that $RCC(A_3) \geq RCC(A_1) \geq RCC(A_4) \geq RCC(A_2)$ in all cases. Therefore, we conclude that

$$A_3 \succ A_1 \succ A_4 \succ A_2$$

where A_3 is the best alternative.

Step 6: End.

3.5.1 Comparative analysis

The study made by Liu (2014) presents TOPSIS method for MADM based on trapezoidal intuitionistic fuzzy number and does not include indeterminate type information in the decision making process. The preference value considered in this chapter is interval trapezoidal neutrosophic number, which deals with indeterminate type information effectively along with truth and falsity type information. The method presented by Ye (2017) and Subas (2018) discusses some aggregation operators of trapezoidal neutrosophic number and the decision making method proposed by Biswas et al. (2018b) presents trapezoidal neutrosophic number based TOPSIS method for MADM with partially known, and completely unknown weight information. We know that interval trapezoidal neutrosophic number is a generalization of trapezoidal neutrosophic number. The approach provided by Biswas et al. (2018a) discusses ITrNN based MADM with known weight information, whereas our proposed model develops ITrNN based MADM model with known, partially known, and completely unknown weight information. Furthermore, the methods suggested by Biswas et al. (2018b) , Ye (2017), and Subas (2018) are not suitable for the decision making problem discussed in this chapter. We compare our results with the method Biswas et al. (2018a) given in Table 3.2

Therefore, our proposed method is more general than the existing methods because the existing methods cannot deal with ITrNN based MADM with partially known, and completely unknown weight information.

TABLE 3.2: A comparison of the results

Method	Type of weight information	Ranking result
Biswas et al. (2018a) method	Partially known	Not Applicable
	Completely unknown	Not Applicable
Proposed method	Partially known	$A_3 \succ A_1 \succ A_4 \succ A_2$
	Completely unknown	$A_3 \succ A_1 \succ A_4 \succ A_2$

3.6 Conclusion

TOPSIS method is a very popular method for MADM problem and this method has been extended under different environments like fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets. In this chapter, we have extended TOPSIS method based on ITrNN. First, we have developed an optimization model to calculate the attribute weight with the help of maximum deviation strategy when the weight information is partially known. We have also developed another model by using Lagrangian function to determine attributes' weights for unknown weight information case. With these weights we have solved MADM problem by TOPSIS method. Finally, we have provided a numerical example of MADM problem and compared with existing methods. The proposed strategy can be extended to multi-attribute group decision making problem with ITrNN. This model can be used in various selection problem like weaver selection problem (Dey et al., 2015a,1), data mining (Mondal et al., 2016), teacher selection (Mondal and Pramanik, 2014), brick field selection problem (Mondal and Pramanik, 2015a), center location selection (Pramanik et al., 2016), etc. under ITrNN environment.

4

TOPSIS Method for Neutrosophic Hesitant Fuzzy Multi-Criteria Decision Making

4.1 Introduction

In decision making problem, decision makers may sometime hesitate to assign a single value for rating the alternatives due to doubt or incomplete information. Instead, they prefer to assign a set of possible values to represent the membership degree for any element to the set. To deal with the issue, Torra (2010) coined the idea of hesitant fuzzy set, which is a generalization of fuzzy set and intutionsistic fuzzy set. Till then, hesitant fuzzy set has been successfully applied in decision making problems (Rodriguez et al., 2011; Xia and Xu, 2011; Xu and Zhang, 2013). Xu and Xia (2011) proposed a variety of distance measures for hesitant fuzzy set. Wei (2012) introduced hesitant fuzzy prioritized operators for solving MADM problem. Beg and Rashid (2013) proposed TOPSIS method for MADM with hesitant fuzzy linguistic term set. Liao and Xu (2015) developed approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making. Joshi and Kumar (2016) introduced Choquet integral based TOPSIS method for multi-criteria group decision making with interval valued intutionistic hesitant fuzzy set.

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However, hesitant fuzzy set can not present inconsistent, imprecise, inappropriate and incomplete information because the set has only truth hesitant membership degree to express any element to the set. To handle this problem, Ye (2015a) introduced single valued neutrosophic hesitant fuzzy sets (SVNHFS). Şahin and Liu (2017) defined correlation coefficient of SVNHFS and applied in decision making problem. Biswas et al. (2016b) proposed GRA method for MADM with SVNHFS for known attribute weight. Ji et al. (2018) proposed a projection–based TODIM approach under multi-valued neutrosophic environments for personnel selection problem. Biswas et al. (2019) further extended the GRA method for solving MADM with SVNHFS and INHFS for partially known or unknown attribute weight.

We have the following objectives in this study:

- To formulate SVNHFS based MADM problem, where the weight information is incompletely known and completely unknown.
- To determine the weights of attributes given in incompletely known and completely unknown forms using deviation method.
- To extend TOPSIS method for solving SVNHFS based MADM problem using the proposed optimization model.
- To further extend the proposed approach in INHFS environment
- To validate the proposed approach with two numerical examples.
- To compare the proposed method with some existing methods.

The remainder of this chapter is organized as follows. Section 4.2 gives preliminaries for hesitant fuzzy set, SVNHFS, INHFS and also represents score function, accuracy function and distance function of SVNHFS and INHFS. Section 4.3 and Section 4.4 develop TOPSIS method for MADM under SVNHFS and INHFS, respectively. Section 4.5 presents two numerical examples to validate the proposed method and provides a comparative study between the proposed method and existing methods. Finally, conclusion and future research directions are given in Section 4.6 .

4.2 Preliminaries

Definition 4.1. (Torra, 2010) Let X be a universe of discourse. A hesitant fuzzy set on X is symbolized by

$$A = \{\langle x, h_A(x)\rangle \mid x \in X\}, \tag{4.1}$$

where $h_A(x)$, referred to as the hesitant fuzzy element, is a set of some values in $[0, 1]$ denoting the possible membership degree of the element $x \in X$ to the set A.

From mathematical point of view, a HFS A can be seen as a FS if there is only one element in $h_A(x)$. For notational convenience, we assume h as hesitant fuzzy element $h_A(x)$ for $x \in X$.

Definition 4.2. (Chen et al., 2013) Let X be a non-empty finite set. An interval hesitant fuzzy set on X is represented by

$$E = \left\{\langle x, \tilde{h}_E(x)\rangle | x \in X\right\},$$

where $\tilde{h}_E(x)$ is a set of some different interval values in $[0, 1]$, which denote the possible membership degrees of the element $x \in X$ to the set E. $\tilde{h}_E(x)$ can be represented by an interval hesitant fuzzy element $\tilde{h}$ which is denoted by $\{\tilde{\gamma} | \tilde{\gamma} \in \tilde{h}\}$, where $\tilde{\gamma}$ =$[\gamma^L, \gamma^U]$ is an interval number.

Definition 4.3. (Ye, 2015a) Let X be a fixed set. Then a N on X is defined as

$$N = \{\langle x, t(x), i(x), f(x)\rangle \mid x \in X\} \tag{4.2}$$

in which $t(x)$, $i(x)$ and $f(x)$ represent three sets of some values in $[0, 1]$, denoting respectively the possible truth, indeterminacy and falsity membership degrees of the element $x \in X$ to the set N. The membership degrees $t(x)$, $i(x)$ and $f(x)$ satisfy the following conditions:

$$0 \leq \delta, \gamma, \eta \leq 1,\ 0 \leq \delta^+ + \gamma^+ + \eta^+ \leq 3$$

where, $\delta \in t(x)$, $\gamma \in i(x)$, $\eta \in f(x)$, $\delta^+ \in t^+(x) = \bigcup_{\delta \in t(x)} \max t(x)$, $\gamma^+ \in i^+(x) = \bigcup_{\gamma \in t(x)} \max i(x)$ and $\eta^+ \in f^+(x) = \bigcup_{\eta \in f(x)} \max f(x)$ for all $x \in X$.

$n(x)$=$\langle t(x), i(x), f(x)\rangle$ is called as single valued neutrosophic hesitant fuzzy element (SVNHFE) denoted by n=$\langle t, i, f\rangle$. The number of values for possible truth, indeterminacy and falsity membership degrees of the element in different SVNHFEs may be different.

Definition 4.4. (Liu and Shi, 2015) Let X be a non-empty finite set. Then an interval neutrosophic hesitant fuzzy set on X is represented by

$$\tilde{n} = \left\{\langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\rangle | x \in X\right\}$$

where $\tilde{t}(x)$= $\left\{\tilde{\gamma}|\tilde{\gamma} \in \tilde{t}(x)\right\}$, $\tilde{i}(x)$= $\left\{\tilde{\gamma}|\tilde{\gamma} \in \tilde{i}(x)\right\}$ and $\tilde{f}(x)$= $\left\{\tilde{\gamma}|\tilde{\gamma} \in \tilde{f}(x)\right\}$ are three sets of some interval values in real unit interval $[0, 1]$, which denotes the possible truth, indeterminacy and falsity membership hesitant degrees of the element $x \in X$ to the set N. These values satisfy the limits:

$$\tilde{\gamma} = [\gamma^L, \gamma^U] \subseteq [0, 1], \tilde{\delta} = [\delta^L, \delta^U] \subseteq [0, 1], \tilde{\eta} = [\eta^L, \eta^U] \subseteq [0, 1]$$

and $0 \leq \tilde{\gamma}^+$+$\tilde{\delta}^+$+$\tilde{\eta}^+ \leq 3$, where $\tilde{\gamma}^+$= $\bigcup\limits_{\tilde{\gamma}\in\tilde{t}(x)}$ sup $\tilde{t}(x)$, $\tilde{\delta}^+$= $\bigcup\limits_{\tilde{\delta}\in\tilde{t}(x)}$ sup $\tilde{i}(x)$ and $\tilde{\eta}^+$= $\bigcup\limits_{\tilde{\eta}\in\tilde{t}(x)}$ sup $\tilde{f}(x)$. Then $\tilde{n}$= $\left\{\tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\right\}$ is called an interval neutrosophic hesitant fuzzy element (INHFE) which is the basic unit of the INHFS and is represented by the symbol $\tilde{n}$= $\left\{\tilde{t}, \tilde{i}, \tilde{f}\right\}$ for convenience.

4.2.1 Score function, accuracy function and distance function of SVNHFEs and INHFEs

Definition 4.5. (Biswas et al., 2016b) Let n_i=$\langle t_i, i_i, f_i\rangle (i = 1, 2, \ldots, n)$ be a collection of SVNHFEs. Then the score function $S(n_i)$, the accuracy function $A(n_i)$ and the certainty function $C(n_i)$ of n_i $(i = 1, 2, \ldots, n)$ can be defined as follows:

1. $S(n_i)$=$\frac{1}{3}\left[2 + \frac{1}{l_t}\sum\limits_{\gamma\in t}\gamma - \frac{1}{l_i}\sum\limits_{\delta\in i}\delta - \frac{1}{l_f}\sum\limits_{\eta\in f}\eta\right]$;
2. $A(n_i)$=$\frac{1}{l_t}\sum\limits_{\gamma\in t}\gamma - \frac{1}{l_f}\sum\limits_{\eta\in f}\eta$;
3. $C(n_i)$=$\frac{1}{l_t}\sum\limits_{\gamma\in t}\gamma$.

Example 4.1. *Let* n_1= $\langle\{0.3, 0.4, 0.5\}, \{0.1\}, \{0.3, 0.4\}\rangle$ *be an SVNHFE, and then by Definition 4.5, we have*

1. $S(n_1) = \frac{1}{3}\Big[2 + \frac{1.2}{3} - 0.1 - \frac{0.7}{2}] = 0.65$;
2. $A(n_1) = \frac{1.2}{3} - \frac{0.7}{2} = 0.05$;
3. $C(n_1) = \frac{1.2}{3} = 0.4$.

Definition 4.6. (Biswas et al., 2016b) Let n_1=$\langle t_1, i_1, f_1\rangle$ and n_2=$\langle t_2, i_2, f_2\rangle$ be two SVN-HFEs. Then the following rules can be defined for comparison purpose:

1. If $s(n_1) > s(n_2)$, then n_1 is greater than n_2, i.e., n_1 is superior to n_2, denoted by $n_1 \succ n_2$.
2. If $s(n_1) = s(n_2)$ and $A(n_1) > A(n_2)$, then n_1 is greater than n_2, i.e., n_1 is superior to n_2, denoted by $n_1 \succ n_2$.
3. If $s(n_1) = s(n_2)$ and $A(n_1) = A(n_2)$, and $C(n_1) > C(n_2)$, then n_1 is greater than n_2, i.e., n_1 is superior to n_2, denoted by $n_1 \succ n_2$.
4. If $s(n_1) = s(n_2)$ and $A(n_1) = A(n_2)$, and $C(n_1) = C(n_2)$, then n_1 is equal to n_2, i.e., n_1 is indifferent to n_2, denoted by $n_1 \sim n_2$.

Example 4.2. *Let* n_1= $\langle\{0.3, 0.4, 0.5\}, \{0.1\}, \{0.3, 0.4\}\rangle$ *and* n_2= $\langle\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\}\rangle$ *be two SVNHFEs, and then by Definition 4.5, we have*

$$S(n_1) = 0.65, A(n_1) = 0.05, C(n_1) = 0.40$$
$$S(n_2) = 0.75, A(n_2) = 0.40, C(n_2) = 0.65.$$

Since $S(n_2) > S(n_1)$, *therefore, we have* $n_2 \succ n_1$ *from Definition 4.6. We take another example to compare SVNHFEs.*

Example 4.3. *Let* n_1= $\langle\{0.5, 0.6\}, \{0.2\}, \{0.2, 0.3\}\rangle$ *and* n_2= $\langle\{0.7, 0.8\}, \{0.3\}, \{0.3, 0.4\}\rangle$ *be two SVNHFEs. Then by Definition 4.5, we have*

$$S(n_1) = 0.70, A(n_1) = 0.30, C(n_1) = 0.55$$
$$S(n_2) = 0.70, A(n_2) = 0.40, C(n_2) = 0.75.$$

Since $S(n_2) = S(n_1)$ *and* $A(n_2) > A(n_1)$ *we have* $n_2 \succ n_1$ *from Definition 4.6.*

Definition 4.7. (Biswas et al., 2016b) Let $\tilde{n}_i$ = $\langle\tilde{t}_i, \tilde{i}_i, \tilde{f}_i\rangle (i = 1, 2, \ldots, n)$ be a collection of INHFEs. Then the score function $S(\tilde{n}_i)$, the accuracy function $A(\tilde{n}_i)$ and the certainty function $C(\tilde{n}_i)$ of $\tilde{n}_i (i = 1, 2, \ldots, n)$ can be defined as follows:

1. $S(\tilde{n}_i)=\frac{1}{6}\Big[4+\frac{1}{l_t}\sum\limits_{\gamma\in t}(\gamma^L+\gamma^U)-\frac{1}{l_i}\sum\limits_{\delta\in i}(\delta^L+\delta^U)-\frac{1}{l_f}\sum\limits_{\eta\in f}(\eta^L+\eta^U)\Big]$;
2. $A(\tilde{n}_i)=\frac{1}{2}\Big[\frac{1}{l_t}\sum\limits_{\gamma\in t}(\gamma^L+\gamma^U)-\frac{1}{l_f}\sum\limits_{\eta\in f}(\eta^L+\eta^U)\Big]$;
3. $C(\tilde{n}_i)=\frac{1}{2}\Big[\frac{1}{l_t}\sum\limits_{\gamma\in t}(\gamma^L+\gamma^U)\Big]$.

Example 4.4. *Let* $\tilde{n}_1 = \langle\{[0.3,0.4],[0.4,0.5]\},\{[0.1,0.2]\},\{[0.3,0.4]\}\rangle$ *be an INHFE, and then by the above definition, we have*

1. $S(\tilde{n}_1)=\frac{1}{6}\Big[4+\frac{1}{2}(0.7+0.9)-(0.1+0.2)-(0.3+0.4)\Big]=0.63$;
2. $A(\tilde{n}_1)=\frac{1}{2}\Big[\frac{1}{2}(0.7+0.9)-(0.3+0.4)\Big]=0.05$;
3. $C(\tilde{n}_1)=\frac{1}{2}\Big[\frac{1}{2}(0.7+0.9)\Big]=0.4$.

Definition 4.8. Let $n_1=\langle t_1,i_1,f_1\rangle$ and $n_2=\langle t_2,i_2,f_2\rangle$ be two INHFEs. Then the following rules can be defined to compare INHFEs:

1. If $s(\tilde{n}_1)>s(\tilde{n}_2)$, then $\tilde{n}_1$ is greater than $\tilde{n}_2$, denoted by $\tilde{n}_1\succ\tilde{n}_2$.
2. If $s(\tilde{n}_1)=s(\tilde{n}_2)$ and $A(\tilde{n}_1)>A(\tilde{n}_2)$, then $\tilde{n}_1$ is greater than $\tilde{n}_2$, denoted by $\tilde{n}_1\succ\tilde{n}_2$.
3. If $s(\tilde{n}_1)=s(\tilde{n}_2)$ and $A(\tilde{n}_1)=A(\tilde{n}_2)$, and $C(\tilde{n}_1)>C(\tilde{n}_2)$, then $\tilde{n}_1$ is greater than $\tilde{n}_2$, denoted by $\tilde{n}_1\succ\tilde{n}_2$.
4. If $s(\tilde{n}_1)=s(\tilde{n}_2)$ and $A(\tilde{n}_1)=A(\tilde{n}_2)$, and $C(\tilde{n}_1)=C(\tilde{n}_2)$, then $\tilde{n}_1$ is equal to $\tilde{n}_2$, denoted by $\tilde{n}_1\sim\tilde{n}_2$.

Example 4.5. *Let* $\tilde{n}_1 = \langle\{[0.3,0.4],[0.4,0.5]\},\{[0.1,0.2]\},\{[0.3,0.4]\}\rangle$ *and* $\tilde{n}_2 = \langle\{[0.5,0.6]\},\{[0.1,0.2],[0.2,0.3]\},\{[0.2,0.3]\}\rangle$ *be two INHFEs, and then by Definition 4.7, we have*

$$S(\tilde{n}_1)=0.63, A(\tilde{n}_1)=0.05, C(\tilde{n}_1)=0.40;$$

$$S(\tilde{n}_2)=0.70, A(\tilde{n}_2)=0.30, C(\tilde{n}_2)=0.55.$$

Following Definition 4.8, and the relation $S(\tilde{n}_2)>S(\tilde{n}_1)$, *we can say* $n_2\succ n_1$.

Definition 4.9. (Biswas et al., 2018b) Let $n_1 = \langle t_1, i_1, f_1\rangle$ and $n_2 = \langle t_2, i_2, f_2\rangle$ be two SVNHFEs. Then the normalized hamming distance between n_1 and n_2 is defined as follows:

$$D(n_1,n_2) = \frac{1}{3}\left(\left|\frac{1}{l_{t_1}}\sum_{\gamma_1\in t_1}\gamma_1 - \frac{1}{l_{t_2}}\sum_{\gamma_2\in t_2}\gamma_2\right| + \left|\frac{1}{l_{i_1}}\sum_{\delta_1\in i_1}\delta_1 - \frac{1}{l_{i_2}}\sum_{\delta_2\in i_2}\delta_2\right| + \left|\frac{1}{l_{f_1}}\sum_{\eta_1\in f_1}\eta_1 - \frac{1}{l_{f_2}}\sum_{\eta_2\in f_2}\eta_2\right|\right) \tag{4.3}$$

where, l_{t_k}, l_{i_k} and l_{f_k} are numbers of possible membership values in n_k for $k = 1, 2$.

Example 4.6. *Let* n_1= $\langle\{0.3, 0.4, 0.5\}, \{0.1\}, \{0.3, 0.4\}\rangle$ *and* n_2= $\langle\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\}\rangle$ *be two SVNHFEs, and then by the above definition, the distance measure between* n_1 *and* n_2 *is given by*

$$\begin{aligned} D(n_1,n_2) =& \frac{1}{3}\left(\left|\frac{1}{3}(0.3+0.4+0.5) - \frac{1}{2}(0.6+0.7)\right| + \left|0.1 - \frac{1}{2}(0.1+0.2)\right| \right.\\ &\left. + \left|\frac{1}{2}(0.3+0.4) - \frac{1}{2}(0.2+0.3)\right|\right) \\ =& 0.1333 \end{aligned}$$

Definition 4.10. (Biswas et al., 2018b) Let $\tilde{n}_1$=$\langle\tilde{t}_1, \tilde{i}_1, \tilde{f}_1\rangle$ and $\tilde{n}_2$=$\langle\tilde{t}_2, \tilde{i}_2, \tilde{f}_2\rangle$ be two INHFEs. Then the normalized hamming distance between $\tilde{n}_1$ and $\tilde{n}_2$ is defined as follows:

$$\tilde{D}(\tilde{n}_1,\tilde{n}_2) = \frac{1}{6}\left(\begin{array}{l} \left|\frac{1}{l_{\tilde{t}_1}}\sum_{\gamma_1\in\tilde{t}_1}\gamma_1^L - \frac{1}{l_{\tilde{t}_2}}\sum_{\gamma_2\in\tilde{t}_2}\gamma_2^L\right| + \left|\frac{1}{l_{\tilde{t}_1}}\sum_{\gamma_1\in\tilde{t}_1}\gamma_1^U - \frac{1}{l_{\tilde{t}_2}}\sum_{\gamma_2\in\tilde{t}_2}\gamma_2^U\right| \\ +\left|\frac{1}{l_{\tilde{i}_1}}\sum_{\delta_1\in\tilde{i}_1}\delta_1^L - \frac{1}{l_{\tilde{i}_2}}\sum_{\delta_2\in\tilde{i}_2}\delta_2^L\right| + \left|\frac{1}{l_{\tilde{i}_1}}\sum_{\delta_1\in\tilde{i}_1}\delta_1^U - \frac{1}{l_{\tilde{i}_2}}\sum_{\delta_2\in\tilde{i}_2}\delta_2^U\right| \\ +\left|\frac{1}{l_{\tilde{f}_1}}\sum_{\eta_1\in\tilde{f}_1}\eta_1^L - \frac{1}{l_{\tilde{f}_2}}\sum_{\eta_2\in\tilde{f}_2}\eta_2^L\right| + \left|\frac{1}{l_{\tilde{f}_1}}\sum_{\eta_1\in\tilde{f}_1}\eta_1^U - \frac{1}{l_{\tilde{f}_2}}\sum_{\eta_2\in\tilde{f}_2}\eta_2^U\right| \end{array}\right) \tag{4.4}$$

where, $l_{\tilde{t}_k}$, $l_{\tilde{i}_k}$ and $l_{\tilde{f}_k}$ are numbers of possible membership values in n_k for $k = 1, 2$.

Example 4.7. *Let* $\tilde{n}_1 = \langle\{[0.3, 0.4], [0.4, 0.5]\}, \{[0.1, 0.2]\}, \{[0.3, 0.4]\}\rangle$ *and* $\tilde{n}_2 = \langle\{[0.5, 0.6]\}, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.2, 0.3]\}\rangle$ *be two INHFEs. Using the above definition, the distance measure between* $\tilde{n}_1$ *and* $\tilde{n}_2$ *is given by*

$$\tilde{D}(\tilde{n}_1,\tilde{n}_2) = \frac{1}{6}\,(0.15+0.15+0.05+0.05+0.10+0.10) = 0.10.$$

4.3 TOPSIS method for MADM with SVNHFS information

In this section, we propose TOPSIS method to find out the best alternative in MADM with SVNHFSs. Suppose that $A=\{A_1, A_2, \ldots, A_m\}$ be the discrete set of m alternatives and $C=\{C_1, C_2, \ldots, C_n\}$ be the set of n attributes for a SVNHFSs based multi-attribute decision making problem. Also, assume that the rating value of the i-th alternative $A_i(i = 1, 2, \ldots, m)$ over the attribute $C_j(j = 1, 2, \ldots, n)$ is considered with SVNHFSs $x_{ij}=(t_{ij}, i_{ij}, f_{ij})$, where $t_{ij}=\{\gamma_{ij} \mid \gamma_{ij} \in t_{ij}, 0 \leq \gamma_{ij} \leq 1\}$, $i_{ij}=\{\delta_{ij} \mid \delta_{ij} \in i_{ij}, 0 \leq \delta_{ij} \leq 1\}$ and $f_{ij}=\{\eta_{ij} \mid \eta_{ij} \in f_{ij}, 0 \leq \eta_{ij} \leq 1\}$ indicate the possible truth, indeterminacy and falsity membership degrees of the i-th alternative A_i over the j-th attribute C_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then we can construct a SVNHFS based decision matrix $X=(x_{ij})_{m\times n}$ which has entries as the SVNHFSs and can be written as

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \tag{4.5}$$

Now, we extend the TOPSIS method for MADM in single-valued neutrosophic hesitant fuzzy environment. Before going to discuss in details, we briefly mention some important steps of the proposed model. First, we consider the weights of attributes which may be known, incompletely known or completely unknown. We develop optimization models to determine the exact weights of attributes using maximum deviation method (Yingming, 1997). Following TOPSIS method, we then determine the Hamming distance measure of each alternative from the positive and negative ideal solutions. Finally, we obtain the relative closeness co-efficient of each alternative to determine the most preferred alternative.

We elaborate the following steps used in the proposed model.

Step 1. Determine the weights of attributes.

Case 1a. If the information of attribute weights is completely known and is given as $w = (w_1, w_2, \ldots, w_n)^T$, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then go to Step 2.

However, in case of real decision making, due to time pressure, lack of knowledge or decision makers' limited expertise in the public domain, the information about the attribute weights is often incompletely known or completely unknown. In this situation when the attribute weights are partially known or completely unknown, we use the maximizing deviation method proposed by Yingming (1997) to deal with MADM problems. For a MADM problem, Yingming suggested that when the attribute has larger deviation among the alternatives, a larger weight should be assigned and when the attribute has smaller deviation among the alternatives, a smaller weight should be assigned, and when attribute has no deviation, zero weight should be assigned.

Now, we develop an optimization model based on maximizing deviation method to determine the optimal relative weights of attributes under SVNHF environment. For the attribute $C_j \in C$, the deviation of alternative A_i to all the other alternatives can be defined as

$$D_{ij}(w) = \sum_{k=1}^{m} w_j D(x_{ij}, x_{kj}), \quad \text{for } i = 1, 2, \ldots, m; j = 1, 2, \ldots, n. \tag{4.6}$$

In Eq.(4.3), the Hamming distance $D(x_{ij}, x_{kj})$ is defined as

$$\begin{aligned} D(x_{ij}, x_{kj}) &= \frac{1}{3}\begin{pmatrix} \left|\frac{1}{l_{t_{ij}}}\sum_{\gamma_{ij}\in t_{ij}}\gamma_{ij} - \frac{1}{l_{t_{kj}}}\sum_{\gamma_{kj}\in t_{kj}}\gamma_{kj}\right| \\ +\left|\frac{1}{l_{i_{ij}}}\sum_{\delta_{ij}\in i_{ij}}\delta_{ij} - \frac{1}{l_{i_{kj}}}\sum_{\delta_{kj}\in i_{kj}}\delta_{kj}\right| \\ +\left|\frac{1}{l_{f_{ij}}}\sum_{\eta_{ij}\in f_{ij}}\eta_{ij} - \frac{1}{l_{f_{kj}}}\sum_{\eta_{kj}\in f_{kj}}\eta_{kj}\right| \end{pmatrix} \\ &= \frac{1}{3}\left(\Delta T(x_{ij}, x_{kj}) + \Delta I(x_{ij}, x_{kj}) + \Delta F(x_{ij}, x_{kj})\right) \end{aligned} \tag{4.7}$$

where,

$$\begin{aligned} \Delta T(x_{ij}, x_{kj}) &= \left|\frac{1}{l_{t_{ij}}}\sum_{\gamma_{ij}\in t_{ij}}\gamma_{ij} - \frac{1}{l_{t_{kj}}}\sum_{\gamma_{kj}\in t_{kj}}\gamma_{kj}\right|; \\ \Delta I(x_{ij}, x_{kj}) &= \left|\frac{1}{l_{f_{ij}}}\sum_{\eta_{ij}\in f_{ij}}\eta_{ij} - \frac{1}{l_{f_{kj}}}\sum_{\eta_{kj}\in f_{kj}}\eta_{kj}\right|; \\ \Delta F(x_{ij}, x_{kj}) &= \left|\frac{1}{l_{f_{ij}}}\sum_{\eta_{ij}\in f_{ij}}\eta_{ij} - \frac{1}{l_{f_{kj}}}\sum_{\eta_{kj}\in f_{kj}}\eta_{kj}\right|; \end{aligned}$$

and $l_{t_{ij}}$, $l_{i_{ij}}$ and $l_{f_{ij}}$ denote the numbers of possible membership values in x_{il} for $l = j, k$.

We now consider the deviation values of all alternatives to other alternatives for the attribute $x_j \in X(j = 1, 2, \ldots, n)$:

$$\begin{aligned} D_j(w) &= \sum_{i=1}^{m} D_{ij}(w_j) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{m} \frac{w_j}{3} \left(\Delta T(x_{ij}, x_{kj}) + \Delta I(x_{ij}, x_{kj}) + \Delta F(x_{ij}, x_{kj}) \right). \end{aligned} \tag{4.8}$$

Case 2a: The information about the attribute weights is incomplete.

In this case, we develop some model to determine the attribute weights. Suppose that the attribute's incomplete weight information H is given by

1. A weak ranking: $\{w_i \geq w_j\}$, $i \neq j$;
2. A strict ranking: $\{w_i - w_j \geq \epsilon_i(> 0)\}$, $i \neq j$;
3. A ranking of difference: $\{w_i - w_j \geq w_k - w_p\}$, $i \neq j \neq k \neq p$;
4. A ranking with multiples: $\{w_i \geq \alpha_i w_j\}$, $0 \leq \alpha_i \leq 1, i \neq j$;
5. An interval form: $\{\beta_i \leq w_i \leq \beta_i + \epsilon_i(> 0)\}$, $0 \leq \beta_i \leq \beta_i + \epsilon_i \leq 1$.

For these cases, we construct the following constrained optimization model based on the set of known weight information H:

$$\text{M–1.} \quad \begin{cases} \max D(w) = \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} \sum\limits_{k=1}^{m} \frac{w_j}{3} \begin{pmatrix} \Delta T(x_{ij}, x_{kj}) + \Delta I(x_{ij}, x_{kj}) \\ + \Delta F(x_{ij}, x_{kj}) \end{pmatrix} \\ \text{subject to } \; w \in H, w_j \geq 0, \sum\limits_{j=1}^{n} w_j = 1, j = 1, 2, \ldots, n. \end{cases} \tag{4.9}$$

Solving Model-1, we can obtain the optimal solution $w = (w_1, w_2, \ldots, w_n)^T$ which can be used as the weight vector of the attributes to proceed to Step 2.

Case 3a: The information about the attribute weights is completely unknown.

In this case, we develop the following non-linear programming model to select the weight vector W, which maximizes all deviation values for all the attributes:

$$\text{M–2.}\quad \begin{cases} \max D(w) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m} \frac{w_j}{3}\left(\begin{array}{r}\Delta T(x_{ij},x_{kj}) + \Delta I(x_{ij},x_{kj}) \\ + \Delta F(x_{ij},x_{kj})\end{array}\right) \\ s.t. \quad w_j \geq 0, j=1,2,\ldots,n; \sum\limits_{j=1}^{n} w_j^2 = 1. \end{cases} \tag{4.10}$$

The Lagrange function corresponding to the above constrained optimization problem is given by

$$L(w,\lambda) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} \frac{w_j}{3}\left(\begin{array}{r}\Delta T(x_{ij},x_{kj}) + \Delta I(x_{ij},x_{kj}) \\ + \Delta F(x_{ij},x_{kj})\end{array}\right) + \frac{\lambda}{6}\left(\sum_{j=1}^{n} w_j^2 - 1\right), \tag{4.11}$$

where λ is a real number denoting the Lagrange multiplier. The partial derivatives of L with respect to w_j and λ are given by

$$\frac{\partial L}{\partial w_j} = \frac{1}{3}\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\begin{array}{rr}\Delta T(x_{ij},x_{kj}) & +\Delta I(x_{ij},x_{kj}) \\ & +\Delta F(x_{ij},x_{kj})\end{array}\right) + \frac{\lambda}{3} w_j = 0 \tag{4.12}$$

$$\frac{\partial L}{\partial \lambda} = \frac{1}{6}\big(\sum_{j=1}^{n} w_j^2 - 1\big) = 0. \tag{4.13}$$

It follows from Eq. (4.12) that

$$w_j = -\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\Delta T(x_{ij},x_{kj}) + \Delta I(x_{ij},x_{kj}) + \Delta F(x_{ij},x_{kj})\right)\right)/\lambda, \tag{4.14}$$

for $i = 1, 2, \ldots, m$.

Putting this value of w_j in Eq. (4.13), we get

$$\lambda^2 = \sum_{j=1}^{n}\big(\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\Delta T(x_{ij},x_{kj}) + \Delta I(x_{ij},x_{kj}) + \Delta F(x_{ij},x_{kj})\right)\big)^2 \tag{4.15}$$

$$or, \quad \lambda = -\sqrt{\sum_{j=1}^{n}\big(\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\Delta T(x_{ij},x_{kj}) + \Delta I(x_{ij},x_{kj}) + \Delta F(x_{ij},x_{kj})\right)\big)^2} \tag{4.16}$$

where $\lambda < 0$ and $\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\left(\Delta T(x_{ij},x_{kj}) + \Delta I(x_{ij},x_{kj}) + \Delta F(x_{ij},x_{kj})\right)$, represents the sum of deviations of all the attributes with respect to the j-th attribute and $\sum\limits_{j=1}^{n}\big(\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\left(\Delta T(x_{ij},x_{kj}) + \Delta I(x_{ij},x_{kj}) + \Delta F(x_{ij},x_{kj})\right)\big)^2$, represents the sum of deviations of all the alternatives with respect to all the attributes.

Then combining equations (4.14) and (4.16), we obtain weight w_j for $j = 1, 2, \ldots, n$ as

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\Delta T(x_{ij}, x_{kj}) + \Delta I(x_{ij}, x_{kj}) + \Delta F(x_{ij}, x_{kj})\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\Delta T(x_{ij}, x_{kj}) + \Delta I(x_{ij}, x_{kj}) + \Delta F(x_{ij}, x_{kj})\right)\right)^2}}. \tag{4.17}$$

We make the sum of w_j $(j = 1, 2, \ldots, n)$ into a unit to normalize the weight of the j-th attribute:

$$w_j^N = \frac{w_j}{\sum_{j=1}^{n} w_j}, j = 1, 2, \ldots, n; \tag{4.18}$$

and consequently, we obtain the weight vector of the attribute as

$$W = (w_1^N, w_2^N, \ldots, w_n^N)$$

for proceeding to Step-2.

Step 2. Determine the positive ideal alternative and negative ideal alternative.

From decision matrix X=$(x_{ij})_{m\times n}$, we can determine the single valued neutrosophic hesitant fuzzy positive ideal solution(SVNHFPIS) A^+ and single valued neutrosophic hesitant fuzzy negative ideal solution (SVNHFNIS) A^- of alternatives as follows:

$$\begin{aligned} A^+ &= \left(A_1^+, A_2^+, \ldots, A_n^+\right) \\ &= \left\{\left\langle \max_i\{\gamma_{ij}^{\sigma(p)}\}, \min_i\{\delta_{ij}^{\sigma(q)}\}, \min_i\{\eta_{ij}^{\sigma(r)}\}\right\rangle | i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\right\} \end{aligned} \tag{4.19}$$

$$\begin{aligned} A^- &= \left(A_1^-, A_2^-, \ldots, A_n^-\right) \\ &= \left\{\left\langle \min_i\{\gamma_{ij}^{\sigma(p)}\}, \max_i\{\delta_{ij}^{\sigma(q)}\}, \max_i\{\eta_{ij}^{\sigma(r)}\}\right\rangle | i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\right\} \end{aligned} \tag{4.20}$$

Here we compare the attribute values x_{ij} by using score, accuracy and certainty values of SVNHFEs defined in Definition 4.5.

Step 3. Determine the distance measure from the ideal alternatives to each alternative.

We determine the distance measure between positive ideal alternative A^+ and alternative A_i, as given in the following

$$D_i^+ = \sum_{j=1}^{n} w_j D(x_{ij}, x_j^+)$$
$$= \frac{w_j}{3} \left(\begin{array}{l} \left| \frac{1}{l_{t_{ij}}} \sum_{\gamma_{ij} \in t_{ij}} \gamma_{ij} - \frac{1}{l_{t_j^+}} \sum_{\gamma_j^+ \in t_j^+} \gamma_j^+ \right| \\ \quad + \left| \frac{1}{l_{i_{ij}}} \sum_{\delta_{ij} \in i_{ij}} \delta_{ij} - \frac{1}{l_{i_{kj}}} \sum_{\delta_j^- \in i_j^+} \delta_j^+ \right| \\ \quad + \left| \frac{1}{l_{f_{ij}}} \sum_{\eta_{ij} \in f_{ij}} \eta_{ij} - \frac{1}{l_{f_{kj}}} \sum_{\eta_j^- \in f_j^+} \eta_j^+ \right| \end{array} \right) \tag{4.21}$$

for $i = 1, 2, \ldots, m$.

Similarly, we can determine the distance measure between negative ideal alternative A^- and alternative $A_i (i = 1, 2, \ldots, m)$ as given in the following:

$$D_i^- = \sum_{j=1}^{n} w_j D(x_{ij}, x_j^-)$$
$$= \frac{w_j}{3} \left(\begin{array}{l} \left| \frac{1}{l_{t_{ij}}} \sum_{\gamma_{ij} \in t_{ij}} \gamma_{ij} - \frac{1}{l_{t_j^-}} \sum_{\gamma_j^- \in t_j^-} \gamma_j^- \right| \\ \quad + \left| \frac{1}{l_{i_{ij}}} \sum_{\delta_{ij} \in i_{ij}} \delta_{ij} - \frac{1}{l_{i_{kj}}} \sum_{\delta_j^+ \in i_j^-} \delta_j^- \right| \\ \quad + \left| \frac{1}{l_{f_{ij}}} \sum_{\eta_{ij} \in f_{ij}} \eta_{ij} - \frac{1}{l_{f_{kj}}} \sum_{\eta_j^- \in f_j^+} \eta_j^- \right| \end{array} \right) \tag{4.22}$$

for $i = 1, 2, \ldots, m$.

Step 4. Determine the relative closeness coefficient.

We determine closeness coefficient C_i for each alternative $A_i (i = 1, 2, \ldots, m)$ with respect to SVNHFPIS A^+ as given in the following:

$$RC_i = \frac{D_i^-}{D_i^+ + D_i^-} \text{ for } i = 1, 2, \ldots, m. \tag{4.23}$$

where $0 \leq C_i \leq 1$ $(i = 1, 2, \ldots, m)$. We observe that an alternative A_i is closer to the SVNHFPIS A^+ and farther to the SVNHFNIS A^- as C_i approaches unity.

Step 5. Rank the alternatives.

We can rank the alternatives according to the descending order of relative closeness coefficient values of alternatives to determine the best alternative from a set of feasible alternatives.

Step 6. End.

4.4 TOPSIS method for MADM with INHFS information

In this section, we further extend the proposed model into interval neutrosophic hesitant fuzzy environment.

For a MADM problem, let A=$(A_1, A_2, \ldots, A_m)$ be a set of alternatives, C=$(C_1, C_2, \ldots, C_n)$ be a set of attributes, and $\tilde{W}$=$(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^T$ be the weight vector of the attributes such that $\tilde{w}_j \in [0, 1]$ and $\sum\limits_{j=1}^{n} \tilde{w}_j = 1$.

Suppose that $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$ be the decision matrix where $\tilde{x}_{ij}$ be the INHFS for the alternative A_i with respect to the attribute C_j and $\tilde{x}_{ij} = (\tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$ where $\tilde{t}_{ij}, \tilde{i}_{ij},$ and $\tilde{f}_{ij}$ are truth, indeterminacy and falsity membership degrees, respectively. The decision matrix is given by

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{bmatrix} \tag{4.24}$$

Now, we develop TOPSIS method based on INHFS when the attribute weights are completely known, partially known or completely unknown.

Step 1. Determine the weights of the attributes.

We suppose that attribute weights are completely known, partially known or completely unknown. We use maximum deviation method when the attribute weights are partially known or completely unknown.

Case 1b. The information of attribute weights is completely known

Assume the attribute weights as $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)^T$ with $\tilde{w}_j \in [0, 1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$ and then go to Step 2.

For partially known or completely unknown attribute weights, we calculate the deviation values of the alternative A_i to other alternatives under the attribute C_j defined as follows:

$$\tilde{D}_{ij}(\tilde{w}) = \sum_{k=1}^{m} \tilde{w}_j D(\tilde{x}_{ij}, \tilde{x}_{kj}), \text{ for } i = 1, 2, \ldots, m; j = 1, 2, \ldots, n. \tag{4.25}$$

Using Eq. (4.4), the Hamming distance $\tilde{D}(\tilde{x}_{ij}, \tilde{x}_{kj})$ is obtained as

$$\begin{aligned}
&\tilde{D}(\tilde{x}_{ij}, \tilde{x}_{kj}) \\
&= \frac{1}{6}\left(\begin{array}{c}
\left|\frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in\tilde{t}_{ij}}{\gamma_{ij}}^L - \frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in\tilde{t}_{kj}}{\gamma_{ij}}^L\right| + \left|\frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in t_{ij}}{\gamma_{ij}}^U - \frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in\tilde{t}_{kj}}{\gamma_{ij}}^U\right| \\
+\left|\frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in i_{ij}}{\delta_{ij}}^L - \frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in\tilde{i}_{kj}}{\delta_{ij}}^L\right| + \left|\frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in i_{ij}}{\delta_{ij}}^U - \frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in\tilde{i}_{kj}}{\delta_{ij}}^U\right| \\
+\left|\frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in f_{ij}}{\eta_{ij}}^L - \frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in\tilde{f}_{kj}}{\eta_{ij}}^L\right| + \left|\frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in f_{ij}}{\eta_{ij}}^U - \frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in\tilde{f}_{kj}}{\eta_{ij}}^U\right|
\end{array}\right) \\
&= \frac{1}{6}\left(\Delta\tilde{T}(\tilde{x}_{ij}, \tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij}, \tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij}, \tilde{x}_{kj})\right)
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
\Delta\tilde{T}(\tilde{x}_{ij}, \tilde{x}_{kj}) &= \left|\frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in\tilde{t}_{ij}}{\gamma_{ij}}^L - \frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in\tilde{t}_{kj}}{\gamma_{ij}}^L\right| + \left|\frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in t_{ij}}{\gamma_{ij}}^U - \frac{1}{l_{\tilde{t}_{ij}}}\sum_{\tilde{\gamma}_{ij}\in\tilde{t}_{kj}}{\gamma_{ij}}^U\right|; \\
\Delta I(x_{ij}, x_{kj}) &= \left|\frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in i_{ij}}{\delta_{ij}}^L - \frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in\tilde{i}_{kj}}{\delta_{ij}}^L\right| + \left|\frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in i_{ij}}{\delta_{ij}}^U - \frac{1}{l_{\tilde{i}_{ij}}}\sum_{\tilde{\delta}_{ij}\in\tilde{i}_{kj}}{\delta_{ij}}^U\right|; \\
\Delta\tilde{F}(\tilde{x}_{ij}, \tilde{x}_{kj}) &= \left|\frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in f_{ij}}{\eta_{ij}}^L - \frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in\tilde{f}_{kj}}{\eta_{ij}}^L\right| + \left|\frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in f_{ij}}{\eta_{ij}}^U - \frac{1}{l_{\tilde{f}_{ij}}}\sum_{\tilde{\eta}_{ij}\in\tilde{f}_{kj}}{\eta_{ij}}^U\right|;
\end{aligned}$$

and $l_{\tilde{t}_{ij}}$, $l_{\tilde{i}_{ij}}$ and $l_{\tilde{f}_{ij}}$ are numbers of possible membership values in x_{il} for $l = j, k$.

The deviation values of all the alternatives to the other alternatives for the attribute $C_j (j = 1, 2, \ldots, n)$ can be obtained from the following:

$$\begin{aligned}
\tilde{D}_j(\tilde{w}) &= \sum_{i=1}^{m} \tilde{D}_{ij}(\tilde{w}_j) \\
&= \sum_{i=1}^{m}\sum_{k=1}^{m} \frac{\tilde{w}_j}{6}\left(\Delta\tilde{T}(\tilde{x}_{ij}, \tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij}, \tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij}, \tilde{x}_{kj})\right).
\end{aligned} \tag{4.27}$$

Case 2b. The information of attribute weights is partially known

In this case, we assume a non-linear programming model to calculate attribute weights.

$$\text{M–3.} \quad \begin{cases} \max \tilde{D}(w) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} \frac{\tilde{w}_j}{6}\begin{pmatrix}\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) \\ + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\end{pmatrix} \\ \text{subject to } \tilde{w} \in \tilde{H}, \tilde{w}_j \geq 0, \sum_{j=1}^{n}\tilde{w}_j = 1, j = 1,2,\ldots,n. \end{cases} \tag{4.28}$$

where $\tilde{H}$ is the set of partially known weight information.

Solving Model-3, we can get the optimal attribute weight vector.

Case 3b. The information of attribute weights is completely known

In this case, we consider the following model:

$$\text{M–4.} \quad \begin{cases} \max D(w) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} \frac{w_j}{6}\begin{pmatrix}\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) \\ + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\end{pmatrix} \\ s.t. \quad \tilde{w}_j \geq 0, \sum_{j=1}^{n}\tilde{w}_j^2 = 1, j = 1,2,\ldots,n. \end{cases} \tag{4.29}$$

The Lagrangian function corresponding to the above nonlinear programming problem is given by

$$\tilde{L}(\tilde{w},\tilde{\lambda}) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\frac{\tilde{w}_j}{6}\begin{pmatrix}\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) & +\Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) \\ & +\Delta\tilde{F}(\tilde{x}_{ij},\xi x_{kj})\end{pmatrix} + \frac{\tilde{\lambda}}{12}\left(\sum_{j=1}^{n}\tilde{w}_j^2 - 1\right), \tag{4.30}$$

where $\tilde{\lambda}$ is the Lagrange multiplier. Then the partial derivatives of $\tilde{L}$ are computed as

$$\frac{\partial\tilde{L}}{\partial\tilde{w}_j} = \frac{1}{6}\sum_{i=1}^{m}\sum_{k=1}^{m}\begin{pmatrix}\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) & +\Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) \\ & +\Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\end{pmatrix} + \frac{\lambda}{6}\tilde{w}_j = 0 \tag{4.31}$$

$$\frac{\partial\tilde{L}}{\partial\tilde{\lambda}} = \frac{1}{12}\Big(\sum_{j=1}^{n}\tilde{w}_j^2 - 1\Big) = 0. \tag{4.32}$$

It follows from Eq.(4.31) that the weight $\tilde{w}_j$ for $i = 1,2,\ldots,m$ is

$$\tilde{w}_j = -\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta I(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\Big)\right)/\tilde{\lambda}, \tag{4.33}$$

Putting w_j in Eq. (4.32), we get

$$\tilde{\lambda}^2 = \sum_{j=1}^{n}(\sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\Big))^2 \tag{4.34}$$

$$or,\ \ \tilde{\lambda} = -\sqrt{\sum_{j=1}^{n}(\sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\Big))^2} \tag{4.35}$$

where $\tilde{\lambda} < 0$ and $\sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\tilde{\Delta}T(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\Big)$ represents the sum of deviations of all the attributes with respect to the j-th attribute and $\sum_{j=1}^{n}(\sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\tilde{\Delta}T(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\Big))^2$ represents the sum of deviations of all the alternatives with respect to all the attributes.

Then combining Eqs. (4.33) and (4.35), we obtain the weight $\tilde{w}_j(j = 1, 2, \ldots, n)$ as

$$\tilde{w}_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\Delta\tilde{T}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\Big)}{\sqrt{\sum_{j=1}^{n}(\sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\tilde{\Delta}T(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{I}(\tilde{x}_{ij},\tilde{x}_{kj}) + \Delta\tilde{F}(\tilde{x}_{ij},\tilde{x}_{kj})\Big))^2}}. \tag{4.36}$$

We make the sum of w_j $(j = 1, 2, \ldots, n)$ into a unit to normalize the weight of the j-th attribute:

$$\tilde{w}_j^N = \frac{\tilde{w}_j}{\sum_{j=1}^{n}\tilde{w}_j}, j = 1, 2, \ldots, n; \tag{4.37}$$

and consequently, we obtain the weight vector of the attribute as

$$\tilde{W} = (\tilde{w}_1^N, \tilde{w}_2^N, \ldots, \tilde{w}_n^N)$$

for proceeding to Step-2.

Step 2. Determine the positive ideal alternative and negative ideal alternative.

From decision matrix $\tilde{X}$=$(\tilde{x}_{ij})_{m\times n}$, we determine the interval neutrosophic hesitant fuzzy positive ideal solution(INHFPIS) A^+ and interval neutrosophic hesitant fuzzy negative ideal solution (INHFNIS) A^- of alternatives as follows:

$$\begin{aligned}\tilde{A}^+ &= (\tilde{A}_1^+, \tilde{A}_2^+, \ldots, \tilde{A}_n^+) \\ &= \left\{ \left\langle \max_i \{\tilde{\gamma}_{ij}^{\sigma(p)}\}, \min_i \{\tilde{\delta}_{ij}^{\sigma(q)}\}, \min_i \{\tilde{\eta}_{ij}^{\sigma(r)}\} \right\rangle | i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \right\}\end{aligned} \tag{4.38}$$

$$\begin{aligned}\tilde{A}^- &= (\tilde{A}_1^-, \tilde{A}_2^-, \tilde{\ldots}, A_n^-) \\ &= \left\{ \left\langle \min_i \{\tilde{\gamma}_{ij}^{\sigma(p)}\}, \max_i \{\tilde{\delta}_{ij}^{\sigma(q)}\}, \max_i \{\tilde{\eta}_{ij}^{\sigma(r)}\} \right\rangle | i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \right\}\end{aligned} \tag{4.39}$$

Here, we compare the attribute values $\tilde{x}_{ij}$ by using score, accuracy and certainty values of INHFSs defined in Definition 4.7.

Step 3. Determine the distance measure from the ideal alternatives to each alternative.

We determine the distance measure between positive ideal alternative A^+ and alternative $A_i (i = 1, 2, \ldots, m)$ as follows:

$$\begin{aligned}\tilde{D}_i^+ &= \sum_{j=1}^{n} \tilde{w}_j \tilde{D}(\tilde{x}_{ij}, \tilde{x}_j^+) \\ &= \frac{\tilde{w}_j}{6} \begin{pmatrix} \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\gamma}_{ij} \in \tilde{t}_{ij}} \gamma_{ij}^L - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\gamma}_j^+ \in \tilde{t}_j^+} \gamma_j^{L+} \right| + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\gamma}_{ij} \in \tilde{t}_{ij}} \gamma_{ij}^U - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\gamma}_j^+ \in \tilde{t}_j^+} \gamma_j^{U+} \right| \\ + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\delta}_{ij} \in \tilde{t}_{ij}} \delta_{ij}^L - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\delta}_j^+ \in \tilde{t}_j^+} \delta_j^{L+} \right| + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\delta}_{ij} \in \tilde{t}_{ij}} \delta_{ij}^U - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\delta}_j^+ \in \tilde{t}_j^+} \delta_j^{U+} \right| \\ + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\eta}_{ij} \in \tilde{t}_{ij}} \eta_{ij}^L - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\eta}_j^+ \in \tilde{t}_j^+} \eta_j^{L+} \right| + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\eta}_{ij} \in \tilde{t}_{ij}} \eta_{ij}^U - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\eta}_j^+ \in \tilde{t}_j^+} \eta_j^{U+} \right| \end{pmatrix}\end{aligned} \tag{4.40}$$

for $i = 1, 2, \ldots, m$. Similarly, we determine the distance measure between negative ideal alternative A^- and alternative $A_i (i = 1, 2, \ldots, m)$ as follows:

$$\tilde{D}_i^- = \sum_{j=1}^{n} \tilde{w}_j \tilde{D}(\tilde{x}_{ij}, \tilde{x}_j^-)$$

$$= \frac{\tilde{w}_j}{6} \left(\begin{array}{l} \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\gamma}_{ij} \in \tilde{t}_{ij}} \gamma_{ij}^L - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\gamma}_j^+ \in \tilde{t}_j^+} \gamma_j^{L+} \right| + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\gamma}_{ij} \in \tilde{t}_{ij}} \gamma_{ij}^U - \frac{1}{l_{\tilde{t}_j^+}} \sum_{\tilde{\gamma}_j^- \in \tilde{t}_j^+} \gamma_j^{U-} \right| \\ + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\delta}_{ij} \in \tilde{t}_{ij}} \delta_{ij}^L - \frac{1}{l_{\tilde{t}_j^-}} \sum_{\tilde{\delta}_j^- \in \tilde{t}_j^-} \delta_j^{L-} \right| + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\delta}_{ij} \in \tilde{t}_{ij}} \delta_{ij}^U - \frac{1}{l_{\tilde{t}_j^-}} \sum_{\tilde{\delta}_j^- \in \tilde{t}_j^-} \delta_j^{U-} \right| \\ + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\eta}_{ij} \in \tilde{t}_{ij}} \eta_{ij}^L - \frac{1}{l_{\tilde{t}_j^-}} \sum_{\tilde{\eta}_j^- \in \tilde{t}_j^-} \eta_j^{L-} \right| + \left| \frac{1}{l_{\tilde{t}_{ij}}} \sum_{\tilde{\eta}_{ij} \in \tilde{t}_{ij}} \eta_{ij}^U - \frac{1}{l_{\tilde{t}_j^-}} \sum_{\tilde{\eta}_j^- \in \tilde{t}_j^-} \eta_j^{U-} \right| \end{array} \right) \tag{4.41}$$

Step 4. Determine the closeness coefficient.

In this step, we calculate closeness coefficient C_i for each alternative $A_i (i = 1, 2, \ldots, m)$ with respect to INHFPIS $\tilde{A}^+$ as given below:

$$\tilde{R}C_i = \frac{\tilde{D}_i^-}{\tilde{D}_i^+ + \tilde{D}_i^-} \text{ for } i = 1, 2, \ldots, m. \tag{4.42}$$

where $0 \leq \tilde{C}_i \leq 1$ $(i = 1, 2, \ldots, m)$. We observe that an alternative A_i is closer to the INHFPIS $\tilde{A}^+$ and farther to the INHFNIS A^- as $\tilde{C}_i$ approaches unity.

Step 5. Rank the alternatives.

Finally, we can rank the alternatives according to the descending order of relative closeness coefficient values of alternatives to choose the best alternative from a set of feasible alternatives.

Step 6. End.

4.5 Numerical examples

In this section, we consider two examples to illustrate the utility of the proposed method for single valued neutrosophic hesitant fuzzy set (SVNHFS) and interval hesitant fuzzy set (INHFS).

4.5.1 Example for SVNHFS

Suppose that an investment company wants to invest a sum of money in the following four alternatives:

- car company (A_1)
- food company (A_2)
- computer company (A_3)
- arms company (A_4)

The company considers the following three attributes to make the decision:

- risk analysis (C_1)
- growth analysis (C_2)
- environment impact analysis (C_3)

We assume the ratting values of the alternatives $A_i,\ i = 1,2,3,4$ with respect to attributes $C_j,\ j = 1,2,3$ and get the SVNHFS matrix presented in Table 4.1. The steps to

TABLE 4.1: Single valued neutrosophic hesitant fuzzy decision matrix

	C_1	C_2	C_3
A_1	$\langle\{0.3,0.4,0.5\},\{0.1\},\{0.3,0.4\}\rangle$	$\langle\{0.5,0.6\},\{0.2,0.3\},\{0.3,0.4\}\rangle$	$\langle\{0.2,0.3\},\{0.1,0.2\},\{0.5,0.6\}\rangle$
A_2	$\langle\{0.6,0.7\},\{0.1,0.2\},\{0.2,0.3\}\rangle$	$\langle\{0.6,0.7\},\{0.1\},\{0.3\}\rangle$	$\langle\{0.6,0.7\},\{0.1,0.2\},\{0.1,0.2\}\rangle$
A_3	$\langle\{0.5,0.6\},\{0.4\},\{0.2,0.3\}\rangle$	$\langle\{0.6\},\{0.3\},\{0.4\}\rangle$	$\langle\{0.5,06\},\{0.1\},\{0.3\}\rangle$
A_4	$\langle\{0.7,0.8\},\{0.1\},\{0.1,0.2\}\rangle$	$\langle\{0.6,0.7\},\{0.1\},\{0.2\}\rangle$	$\langle\{0.3,0.5\},\{0.2\},\{0.1,0.2,0.3\}\rangle$

get the best alternative are as follows:

Step 1: Determine the weights of attributes.

There are three cases for attribute weights:

Case 1 : When the attribute weights are completely known, let the weight vector be $w^N = (0.35, 0.25, 0.40)$.

Case 2 : When the attribute weights are partially known, the weight information is as follows:

$$H = \begin{cases} 0.30 \leq w_1 \leq 0.40 \\ 0.20 \leq w_2 \leq 0.30 \\ 0.35 \leq w_3 \leq 0.45 \\ \text{and } w_1 + w_2 + w_3 = 1 \end{cases}$$

Using Model-1, we get the single objective programming problem as

$$\begin{cases} max(D) = 1.796w_1 + 1.164w_2 + 1.962w_3 \\ \text{subject to } w \in H \text{ and } \sum_{j=1}^{3} w_j = 1,\ w_j > 0\ for\ j = 1, 2, 3. \end{cases}$$

Solving this problem with optimization software LINGO 11, we get the optimal weight vector as $w^N = (0.35, 0.20, 0.45)$.

Case 3 : When the attribute weights are completely unknown, using Model-2 and Eqs. (4.17) and (4.18), we obtain the following weight vector:

$w^N = (0.351, 0.265, 0.384)$.

Step 2: Determine the positive ideal alternative and negative ideal alternative.

In this step, we calculate the positive and the negative ideal solutions from Eqs. (4.19) and (4.20), respectively.

$$\begin{aligned} A^+ &= (A_1^+, A_2^+, A_3^+) \\ &= \left(\begin{array}{c} \left\langle \{0.7, 0.8\}, \{0.1\}, \{0.1, 0.2\} \right\rangle, \left\langle \{0.6, 0.7\}, \{0.1\}, \{0.2\} \right\rangle, \\ \left\langle \{0.6, 0.7\}, \{0.1, 0.2\}, \{0.1, 0.2\} \right\rangle \end{array} \right) \end{aligned} \tag{4.43}$$

$$\begin{aligned} A^- &= (A_1^-, A_2^-, A_3^-) \\ &= \left(\begin{array}{c} \left\langle \{0.5, 0.6\}, \{0.4\}, \{0.2, 0.3\} \right\rangle, \left\langle \{0.6\}, \{0.3\}, \{0.4\} \right\rangle, \\ \left\langle \{0.2, 0.3\}, \{0.1, 0.2\}, \{0.5, 0.6\} \right\rangle \end{array} \right) \end{aligned} \tag{4.44}$$

Step 3: Determine the distance measure from the ideal alternatives to each alternative.

In this step, we determine the distance measure from positive and negative ideal solutions from Eqs. (4.21) and (4.22) as given in Tables 4.2 and 4.3.

Step 4: Determine the relative closeness coefficient.

TABLE 4.2: Distance measure from positive ideal solution

$D^+(A_i)$	Case 1	Case 2	Case 3
D_1^+	0.210	0.210	0.201
D_2^+	0.037	0.035	0.037
D_3^+	0.140	0.145	0.148
D_4^+	0.046	0.052	0.044

TABLE 4.3: Distance measure from negative ideal solution

$D^-(A_i)$	Case 1	Case 2	Case 3
D_1^-	0.180	0.164	0.198
D_2^-	0.176	0.183	0.173
D_3^-	0.120	0.115	0.102
D_4^-	0.181	0.182	0.180

We now calculate the relative closeness coefficients from Eq. (4.23) and the results are shown in Table 4.4.

TABLE 4.4: Relative closeness coefficient

$RC(A_i)$	Case 1	Case 2	Case 3
$RC(A_1)$	0.461	0.438	0.496
$RC(A_2)$	0.826	0.839	0.823
$RC(A_3)$	0.462	0.451	0.408
$RC(A_4)$	0.796	0.778	0.800

Step 5: Rank the alternatives.

From Table 4.4, ranks of the alternatives are determined as follows:

$$\textbf{Case 1:}\ A_2 \succ A_4 \succ A_3 \succ A_1$$
$$\textbf{Case 2:}\ A_2 \succ A_4 \succ A_3 \succ A_1$$
$$\textbf{Case 3:}\ A_2 \succ A_4 \succ A_1 \succ A_3$$

The above shows that A_2 is the best alternative for all cases.

Step 6: End.

4.5.2 Example for INHFS

In order to demonstrate the proposed method for INHFS, we consider the same numerical example for SVNHFS but the ratting values of the attributes are INHFS. The INHFS based decision matrix is presented in Table 4.5.

TABLE 4.5: Interval neutrosophic hesitant fuzzy decision matrix

	C_1	C_2	C_3
A_1	$\{[0.3,0.4],[0.4,0.5]\}$ $\{[0.1,0.2]\}$ $\{[0.3,0.4]\}$	$\{[0.4,0.5],[0.5,0.6]\}$ $\{[0.2,0.3]\}$ $\{[0.3,0.3],[0.3,0.4]\}$	$\{[0.3,0.5]\}$ $\{[0.2,0.3]\}$ $\{[0.1,0.2],[0.3,0.3]\}$
A_2	$\{[0.6,0.7]\}$ $\{[0.1,0.2]\}$ $\{[0.1,0.2],[0.2,0.3]\}$	$\{[0.6,0.7]\}$ $\{[0.1,0.2]\}$ $\{[0.2,0.3]\}$	$\{[0.6,0.7]\}$ $\{[0.1,0.2]\}$ $\{[0.1,0.2]\}$
A_3	$\{[0.3,0.4],[0.5,0.6]\}$ $\{[0.2,0.4]\}$ $\{[0.2,0.3]\}$	$\{[0.6,0.7]\}$ $\{[0.0,0.1]\}$ $\{[0.2,0.3]\}$	$\{[0.5,06]\}$ $\{[0.1,0.2],[0.2,0.3]\}$ $\{[0.2,0.3]\}$
A_4	$\{[0.7,0.8]\}$ $\{[0.0,0.1]\}$ $\{[0.1,0.2]\}$	$\{[0.5,0.6]\}$ $\{[0.2,0.3]\}$ $\{[0.3,0.4]\}$	$\{[0.2,0.3]\}$ $\{[0.1,0.2]\}$ $\{[0.4,0.5],[0.5,0.6]\}$

Step 1: Determine the weights of attributes.

Here we consider completely known, partially known and completely unknown attribute weights in three cases.

Case 1 : When the attribute weights are known in advance, let the weight vector be

$$\bar{w}^N = (0.30, 0.25, 0.45).$$

Case 2 : When the attribute weights are partially known, the weight information is as follows:

$$\tilde{H} = \begin{cases} 0.30 \leq \tilde{w}_1 \leq 0.40 \\ 0.20 \leq \tilde{w}_2 \leq 0.30 \\ 0.35 \leq \tilde{w}_3 \leq 0.45 \\ \text{and } \tilde{w}_1 + \tilde{w}_2 + \tilde{w}_3 = 1 \end{cases}$$

Now, with the help of Model-3, we consider the following optimization problem:

$$\begin{cases} max(D) = 1.7626\tilde{w}_1 + 1.526\tilde{w}_2 + 1.848\tilde{w}_3 \\ \text{subject to } \tilde{w} \in \tilde{H} \text{ and } \sum_{j=1}^{3} \tilde{w}_j = 1,\ \tilde{w}_j > 0\ for\ j = 1,2,3. \end{cases}$$

Solving this problem with optimization software LINGO 11, we get the optimal weight vector as

$$\bar{w}^N = (0.35, 0.20, 0.45)$$

Case 3 : When the attribute weights are completely unknown, using Model-2 and Eqs. (4.36) and (4.37), we obtain the following weight vector:

$$\bar{w}^N = (0.343, 0.297, 0.360).$$

Step 2: Determine the positive ideal alternative and negative ideal alternative.

In this step, we calculate the positive and the negative ideal solutions, where the positive ideal solution is the best solution and negative ideal solution is the worst solution. From Eqs. (4.19) and (4.20), we get

$$\begin{aligned} \tilde{A}^+ &= (\tilde{A}_1^+, \tilde{A}_2^+, \tilde{A}_3^+) \\ &= \begin{pmatrix} \Big\langle \{[0.7, 0.8]\}, \{[0.0, 0.1]\}, \{[0.1, 0.2]\} \Big\rangle, \\ \Big\langle \{[0.6, 0.7]\}, \{[0.0, 0.1]\}, \{[0.2, 0.3]\} \Big\rangle, \\ \Big\langle \{[0.6, 0.7]\}, \{[0.1, 0.2]\}, \{[0.1, 0.2]\} \Big\rangle \end{pmatrix} \end{aligned} \tag{4.45}$$

$$\begin{aligned} \tilde{A}^- &= (\tilde{A}_1^-, \tilde{A}_2^-, \tilde{A}_3^-) \\ &= \begin{pmatrix} \Big\langle \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.1, 0.2]\}, \{[0.3, 0.4]\} \Big\rangle, \\ \Big\langle \{[0.4, 0.5], [0.5, 0.6]\}, \{[0.2, 0.3]\}, \{[0.3, 0.3], [0.3, 0.4]\} \Big\rangle, \\ \Big\langle \{[0.2, 0.3]\}, \{[0.1, 0.2]\}, \{[0.4, 0.5], [0.5, 0.6]\} \Big\rangle \end{pmatrix} \end{aligned} \tag{4.46}$$

Step 3: Determine the distance measure from the ideal alternatives to each alternative.

In this step, using Eqs. (4.40) and (4.41), we determine the distance measure from positive ideal solution and negative ideal solution as given in Tables 4.6 and 4.7, respectively.

Step 4: Determine the relative closeness coefficient.

We now calculate the relative closeness coefficient from Eq. (4.42). The results are shown in Table 4.8.

Step 5: Rank the alternatives.

TABLE 4.6: Distance measure from positive ideal solution

$\tilde{D}^+(A_i)$	Case 1	Case 2	Case 3
$\tilde{D}_1^+$	0.164	0.168	0.167
$\tilde{D}_2^+$	0.032	0.035	0.037
$\tilde{D}_3^+$	0.102	0.113	0.104
$\tilde{D}_4^+$	0.146	0.139	0.129

TABLE 4.7: Distance measure from negative ideal solution

$\tilde{D}^-(A_i)$	Case 1	Case 2	Case 3
$\tilde{D}_1^-$	0.078	0.079	0.063
$\tilde{D}_2^-$	0.179	0.180	0.168
$\tilde{D}_3^-$	0.155	0.153	0.148
$\tilde{D}_4^-$	0.071	0.080	0.082

TABLE 4.8: Relative closeness coefficient

$\tilde{RC}(A_i)$	Case 1	Case 2	Case 3
$\tilde{RC}(A_1)$	0.322	0.312	0.273
$\tilde{RC}(A_2)$	0.848	0.837	0.819
$\tilde{RC}(A_3)$	0.603	0.576	0.587
$\tilde{RC}(A_4)$	0.327	0.365	0.389

From Table 4.8, we obtain the ranks of the alternatives as follows:

Case 1 :$A_2 \succ A_3 \succ A_4 \succ A_1$

Case 2 :$A_2 \succ A_3 \succ A_4 \succ A_1$

Case 3 :$A_2 \succ A_3 \succ A_4 \succ A_1$

The above shows that A_2 is best alternative for all cases.

Step 6: End.

4.5.3 Comparative analysis and discussion:

We divide this section into two parts. Firstly, we compare our proposed method with the existing methods for multi-attribute decision making under SVNHFS and then for INHFS.

4.5.3.1 Comparative analysis of SVNHFS

Ye (2015c) developed the method to find out the best alternative under single valued neutrosophic hesitant fuzzy environment and Şahin and Liu (2017) proposed correlation coefficient of single valued neutrosophic hesitant fuzzy set for MADM. Rankings of the alternatives of the above existing method and our proposed method are shown in Table 4.9. When the attribute weights are known in advance, three methods result in the same ranking. However, when the attribute weights are partially known or completely unknown, the above two methods are not applicable.

TABLE 4.9: A comparison of the results under SVNHFS

Methods	Type of weight information	Ranking result
Ye (2015c) method	Completely known	$A_2 \succ A_4 \succ A_3 \succ A_1$
Şahin and Liu (2017) method	Completely known	$A_2 \succ A_4 \succ A_3 \succ A_1$
Proposed method	Completely known	$A_2 \succ A_4 \succ A_3 \succ A_1$
Ye (2015c) method	Partially known	Not Applicable
Şahin and Liu (2017) method	Partially known	Not Applicable
Proposed method	Partially known	$A_2 \succ A_4 \succ A_3 \succ A_1$
Ye (2015c) method	Completely unknown	Not Applicable
Şahin and Liu (2017) method	Completely unknown	Not Applicable
Proposed method	Completely unknown	$A_2 \succ A_4 \succ A_1 \succ A_3$

4.5.3.2 Comparative analysis of INHFS

Liu and Shi (2015) proposed MADM method for the best alternative under interval neutrosophic hesitant fuzzy environment. Table 4.10 shows a comparison between Liu and Shi (2015) method and our proposed method.

The advantages of the proposed method for SVNHFS and INHFS are as follows:

- The existing methods are developed based on aggregation operator, correlation coefficient and hybrid weighted operator, but our proposed method is developed on the basis of deviation method.

TABLE 4.10: A comparison of the results under INHFS

Methods	Type of weight information	Ranking result
Liu and Shi (2015)	Completely known	$A_2 \succ A_3 \succ A_4 \succ A_1$
Proposed method	Completely known	$A_2 \succ A_3 \succ A_4 \succ A_1$
Liu and Shi (2015)	Partially known	Not Applicable
Proposed method	Partially known	$A_2 \succ A_3 \succ A_4 \succ A_1$
Liu and Shi (2015)	Completely unknown	Not Applicable
Proposed method	Completely unknown	$A_2 \succ A_3 \succ A_4 \succ A_1$

- The proposed method offers more flexible choice of weight information because it is also applicable to partially known and unknown weight information.

4.6 Conclusion

Neutrosophic hesitant fuzzy set encompasses single valued neutrosophic set, interval neutrosophic set, hesitant fuzzy set, intuitionistic fuzzy set and fuzzy set. The neutrosophic set has three components: truth membership, falsity membership and indeterminacy membership functions. Therefore, neutrosophic hesitant fuzzy set is flexible to deal with imprecise, indeterminate and incomplete information for MADM problem. In this chapter, we have extended TOPSIS method for solving MADM problem under SVNHFS and INHFS environments. We have considered three types of weight information of attributes, completely known, partially known and completely unknown weight information. We have developed optimization models for calculating attribute weights for partially known, and completely unknown weight information with the help of maximizing deviation method. Finally, numerical examples have been given to support and illustrate the validation and efficiency of the proposed method. The proposed strategy can be extended to multi-attribute group decision making problem as well as the case when weight information is unknown. The developed model can be applied to many real decision making problems such as pattern recognition, supply chain management, data mining, etc. For future research, the proposed method can be extended in MADM problem with plithogenic set (Smarandache, 2017b).

5

PROMETHEE Method with Pythagorean Fuzzy Sets for Medical Diagnosis Problems

5.1 Introduction

Modern medical diagnosis process considers a lot of parameters some of which may contain incomplete and uncertain information. In practice, some diseases have common symptoms. Therefore, these symptoms bear an ambiguous information for detecting the exact disease. This type of medical diagnosis problems could be solved by using MCDM process, where disease and symptom can be set as an alternative and a criterion, respectively. In this study, the preference value not only gives the degree for which the disease satisfies the symptoms but also provides the degree for which the disease dissatisfies the symptom. Ye (2015b) introduced the cosine similarity measure for simplified neutrosophic sets (Smarandache, 1999a) in decision making for medical diagnosis problem. Xiao (2018) solved medical diagnosis problem as a decision making problem with hybrid fuzzy soft sets. Preference Ranking Organization Method for Enrichment of Evalution (PROMETHEE) (Brans et al., 1986) is a popular method to solve MCDM problem. PROMETHEE method compares the criteria for each pair of alternatives and preference alternative grade which lies between 0 and 1. Pythagorean fuzzy sets (PFS)(Yager, 2013), an extension of intuitionistic fuzzy sets

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(IFS), can easily handle incomplete and indeterminant situation of MCDM problem. Yager (2013) introduced Pythagorean membership grades for MCDM problem and solved the MCDM problem using the aggregation operator. Yager (2013) proposed the relationship between Pythagorean membership grade and complex number. They proved that Pythagorean membership grade is one type of complex numbers ($\Pi - i$ numbers) and they solved the MCDM problem with aggregation operator of $\Pi - i$ numbers. Zhang and Xu (2014) extended the TOPSIS method with PFS and considered the Pythagorean fuzzy number (PFN) to solve the MCDM problem. They defined a distance measure of PFN for developing TOPSIS method to get the optimal result. Many researchers developed the MCDM method with Pythagorean fuzzy information (Zeng et al., 2016; Zhang, 2016). In this chapter, we propose PROMETHEE method for MCDM problem under Pythagorean fuzzy environment and its application to medical diagnosis problem. The objectives of our study are as follows:

- To extend the PROMETHEE method for MCDM with Pythagorean fuzzy sets.
- To validate the proposed method by comparing with two existing methods.
- To apply the proposed method for a medical diagnosis problem.

The structure of the chapter is as follows. In Section 5.2, we briefly analyze the basic concept of Pythagorean fuzzy sets and compare the Pythagorean fuzzy sets with intuitionistic fuzzy sets. In Section 5.3, we explain the PROMETHEE method for Pythagorean fuzzy sets. In Section 5.4, we provide a numerical example and perform comparative analysis between the proposed Pythagorean PROMETHEE method and the existing MCDM method under Pythagorean fuzzy information. In Section 5.5, a medical diagnosis problem is solved by using the proposed Pythagorean fuzzy PROMETHEE method. Finally, some concluding remarks are drawn in Section 5.6.

5.2 Basic operations on Pythagorean fuzzy number(PFN)

Definition 5.1. (Yager, 2013) Let X be a universe of discourse. Then a Pythagorean fuzzy set defined on X is of the from

$$P = \{< x, \mu_p(x), \nu_p(x) > | x \in X\}$$

where $\mu_p : X \to [0,1]$ and $\nu_p : X \to [0,1]$ are membership and non-membership functions, respectively and satisfy the following relation:

$$0 \leq (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1 \qquad \forall x \in X$$

Then the degree of indeterminacy membership $\pi_p(x)$ is defined as
$\pi_p(x) = \sqrt{1 - (\mu_p(x))^2 - (\nu_p(x))^2}$

Zhang and Xu (2014) considered $\beta =< \mu_p, \nu_p >$ as a Pythagorean fuzzy number (PFN) where $\mu_p \in [0,1]$ and $\nu_p \in [0,1]$ are membership and non-membership values, respectively and $\pi_p = \sqrt{1 - \mu_p^2 - \nu_p^2}$ and $0 \leq \mu_p^2 + \nu_p^2 \leq 1$.

Let $\alpha_1 =< \mu_{p_1}, \nu_{p_1} >$, and $\alpha_2 =< \mu_{p_2}, \nu_{p_2} >$ be two PFNs. Zhang and Xu (2014) defined the following operations :

1. $\alpha_1 \cup \alpha_2 =< \max\{\mu_{p_1}, \mu_{p_2}\}, \min\{\nu_{p_1}, \nu_{p_2}\} >$
2. $\alpha_1 \cap \alpha_2 =< \min\{\mu_{p_1}, \mu_{p_2}\}, \max\{\nu_{p_1}, \nu_{p_2}\} >$
3. $\alpha_1^c =< \nu_{p_1}, \mu_{p_1} >$
4. $\alpha_1 \oplus \alpha_2 =< \sqrt{\mu_{p_1}^2 + \mu_{p_2}^2 - \mu_{p_1}^2 \mu_{p_2}^2}, \nu_{p_1}\nu_{p_2} >$
5. $\alpha_1 \otimes \alpha_2 =< \mu_{p_1}\mu_{p_2}, \sqrt{\nu_{p_1}^2 + \nu_{p_2}^2 - \nu_{p_1}^2 \nu_{p_2}^2} >$
6. $\lambda\alpha_1 =< \sqrt{1 - (1 - \mu_{p_1}^2)^\lambda}, (\nu_{p_1})^\lambda >, \ \lambda > 0$
7. $\alpha_1^\lambda =< (\mu_{p_1})^\lambda, \sqrt{1 - (1 - \nu_{p_1}^2)^\lambda} >, \ \lambda > 0$

Using the above rules of operation, it can be easily shown that the followings are valid.

1. $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$
2. $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$
3. $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2, \ \lambda > 0$
4. $\lambda_1\alpha_1 \oplus \lambda_2\alpha_1 = (\lambda_1 \oplus \lambda_2)\alpha_1, \ \lambda_1, \lambda_2 > 0$
5. $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda, \ \lambda > 0$
6. $\alpha_1^{\lambda_1} \otimes \alpha_1^{\lambda_2} = \alpha_1^{(\lambda_1+\lambda_2)}, \ \lambda_1, \lambda_2 > 0$

Definition 5.2. (Zhang and Xu, 2014) Let $\alpha_1 =< \mu_{p_1}, \nu_{p_1} >$ and $\alpha_2 =< \mu_{p_2}, \nu_{p_2} >$ be two PFNs. Then the ordering between these two PFNs is described as follows:

$$\alpha_1 \geq \alpha_2 \Leftrightarrow \mu_{p_1} \geq \mu_{p_2} \text{ and } \nu_{p_1} \leq \nu_{p_2}$$

Definition 5.3. (Zhang and Xu, 2014) If $\alpha =< \mu, \nu >$ be a PFN then the score function of $\alpha =< \mu, \nu >$ is denoted by $s(\alpha)$ and is defined by

$$s(\alpha) = \mu^2 - \nu^2$$

Now, the following propositions hold:

1. $s(\alpha) \in [-1, 1]$
2. For two PFNs $\alpha_1 =< \mu_{p_1}, \nu_{p_1} >$ and $\alpha_2 =< \mu_{p_2}, \nu_{p_2} >$, if $s(\alpha_1) > s(\alpha_2)$ then $\alpha_1 > \alpha_2$.

Definition 5.4. (Zhang and Xu, 2014) Let $\alpha_1 =< \mu_{p_1}, \nu_{p_1} >$ and $\alpha_2 =< \mu_{p_2}, \nu_{p_2} >$ be two PFNs and $s(\alpha_1)$ and $s(\alpha_2)$ be the score values of α_1 and α_2, respectively. Then the following relations hold for the two PFNs:

1. $s(\alpha_1) < s(\alpha_2) \Rightarrow \alpha_1 \prec \alpha_2$
2. $s(\alpha_1) > s(\alpha_2) \Rightarrow \alpha_1 \succ \alpha_2$
3. $s(\alpha_1) = s(\alpha_2) \Rightarrow \alpha_1 \sim \alpha_2$, where $\alpha_1 \sim \alpha_2$ means that α_1 and α_2 are not comparable.

For example, let us consider $\alpha_1 =< 0.9, 0.3 >$ and $\alpha_2 =< 0.7, 0.4 >$ be two PFNs. Then we have, $s(\alpha_1) = (0.9)^2 - (0.3)^2 = 0.72,\ s(\alpha_2) = (0.7)^2 - (0.4)^2 = 0.33.$ Therefore, $s(\alpha_1) > s(\alpha_2) \Rightarrow \alpha_1 \succ \alpha_2$.

The membership value of IFN satisfies $0 \leq \mu_I + \nu_I \leq 1$, whereas the membership value of PFN satisfies $0 \leq \mu_P^2 + \nu_P^2 \leq 1$. Yager (2013) showed that the space of intuitionistic fuzzy membership grade is a subspace of the space of Pythagorean membership grade, which is shown in figure 5.1 . Therefore, every IFN is PFN but converse is not true. With this advantage, the decision maker can express preference values of alternatives in a more flexible way with PFN than IFN.

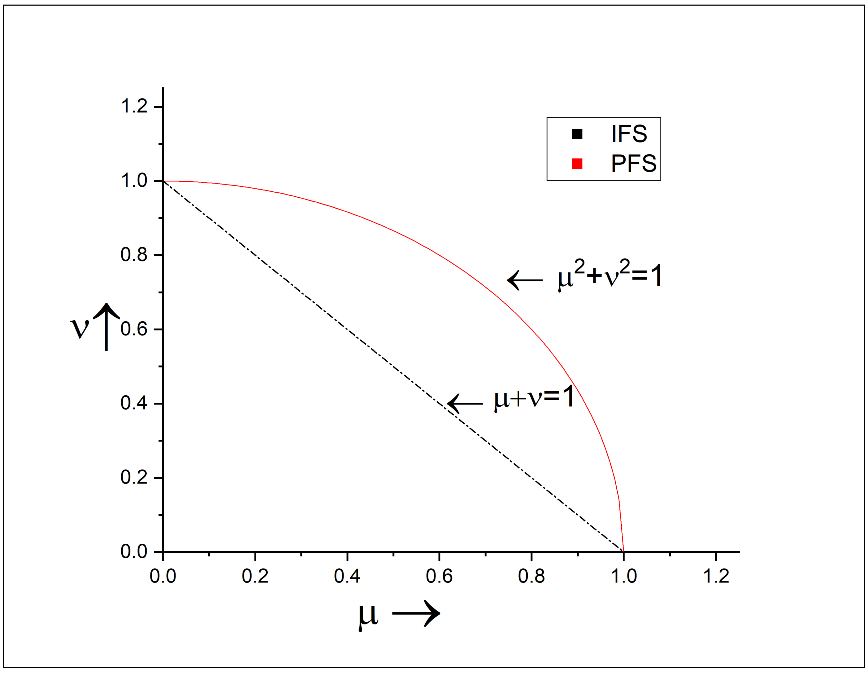

FIGURE 5.1: Comparison of spaces between IFS and PFS

5.3 Pythagorean fuzzy PROMETHEE method

In this section, we develop PROMETHEE method under Pythagorean fuzzy environment. To develop this model, we use Pythagorean fuzzy number (PFN). In a MCDM problem, let us consider m alternatives $A = \{A_1, A_2, ..., A_m\}$ and n criteria $C = \{C_1, C_2, ..., C_n\}$. Then we obtain the Pythagorean fuzzy decision matrix in the following form:

$$X = (x_{ij})_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix} \end{array} \tag{5.1}$$

where $\alpha_{ij} =< \mu_{ij}, \nu_{ij} >$ is a PFN presenting the rating value of the alternative A_i with respect to criteria C_j for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$. We consider the weight information of the criteria as $w = \{w_1, w_2, ..., w_n\}$ which is the normalized weight vector and it satisfies $0 \leq w_j \leq 1$ for $j = 1, 2, ..., n$ and $\sum_{j=1}^{n} w_j = 1$.

Now, we discuss the following steps to organize the proposed model.

Step 1: Determine the performance differences

In this step, we compare each pair of alternatives. Since the rating values of the alternatives are PFN, we compare the deviations between any two alternatives over each criterion with score function of PFN (see Definition 5.3). The difference value of i-th alternative to other alternatives is given by

$$d_k(\alpha_{ik}, \alpha_{jk}) = s(\alpha_{ik}) - s(\alpha_{jk}) = \mu_{ik}^2 - \nu_{ik}^2 - (\mu_{jk}^2 - \nu_{jk}^2),\ k = 1, 2, .., n \tag{5.2}$$

where $s(\alpha_{ik})$ is the score value of α_{ik}.

Step 2: Construct the preference function

We consider no preference if difference is negligible, and large preference for larger difference value. We consider V-shape criterion function with indifference area, which is defined as

$$P_k(\alpha_{ik}, \alpha_{jk}) = \begin{cases} 0, & d_k(\alpha_{ik}, \alpha_{jk}) \leq q \\ \dfrac{d_k(\alpha_{ik}, \alpha_{jk}) - q}{p - q}, & q < d_k(\alpha_{ik}, \alpha_{jk}) \leq p \\ 1, & d_k(\alpha_{ik}, \alpha_{jk}) > p \end{cases} \tag{5.3}$$

where p and q are parameters, p is the value of strict preference threshold and q is the value of indifference threshold. This preference function is shown in Figure 5.2.

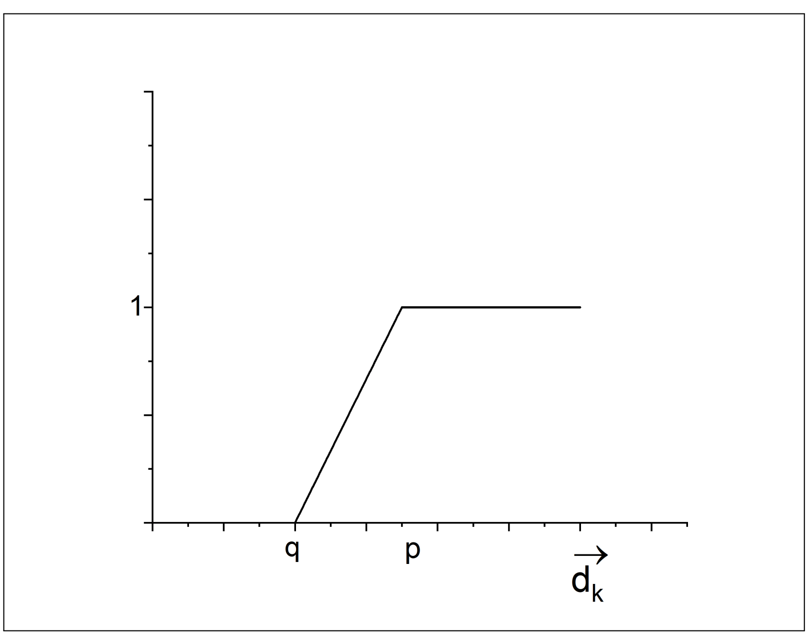

FIGURE 5.2: V-shape criterion function with indifference area

Step 3: Calculate aggregated preference value and construct preference matrix

In this step, we calculate the total preference index for an alternative A_i with respect to an alternative A_j over all the criteria, which is defined by

$$\pi(A_i, A_j) = \sum_{k=1}^{n} w_k P_k(\alpha_{ik}, \alpha_{jk}), \quad \text{for all} \quad A_i, A_j \in A \tag{5.4}$$

where $\{w_1, w_2, ..., w_n\}$ are the weights of the corresponding criteria $\{C_1, C_2, ..., C_n\}$. Therefore, we get the following preference matrix whose entries are aggregated preference values.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \dots & A_m \end{array} \\ \begin{pmatrix} - & \pi(A_1, A_2) & \dots & \pi(A_1, A_m) \\ \pi(A_2, A_1) & - & \dots & \pi(A_2, A_m) \\ \vdots & \vdots & \ddots & \vdots \\ \pi(A_m, A_1) & \pi(A_m, A_2) & \dots & - \end{pmatrix} \end{array} \tag{5.5}$$

The diagonal elements of the above matrix assign no value. Therefore, when we compare the alternative A_i with the same alternative A_i then no preference value is assigned.

Step 4: Determine the leaving and the entering outranking flows

From the preference matrix, we see that each alternative A_i is compared with other $(m-1)$ alternatives in $A = \{A_1, A_2, ..., A_m\}$. Then leaving outranking flow (positive outranking flow) is determined as follows:

$$\phi^+(A_i) = \frac{1}{m-1} \sum_{X \in A} \pi(A_i, X), \quad \text{for each } A_i \tag{5.6}$$

i.e, positive outranking flow of the i-th alternative is the average value of the i-th row of the preference matrix. The entering outranking flow (negative outranking flow) is determined as follows:

$$\phi^-(A_i) = \frac{1}{m-1} \sum_{X \in A} \pi(X, A_i), \quad \text{for each } A_i \tag{5.7}$$

and this is the average value of the i-th column for the alternative A_i.

Positive outranking flow points out the magnitude in which one alternative dominates other alternatives. Likewise, negative outranking flow points out the magnitude in which one alternative is dominated by other alternatives.

Step 5: Calculate net outranking flows

In **PROMETHEE I**, the net outranking flows are not required for ranking the alternatives. In this method, we compare one alternative with another one at a time. Therefore, **PROMETHEE I** gives a partial order ranking with the help of positive outranking flow $\phi^+(A_i)$ and negative outranking flow $\phi^-(A_i)$. The preference relation and partial pre-orders P, I, R are derived as follows:

$$\begin{aligned} A_iPA_j \ (A_i \text{ outranks } A_j) \text{ if } &\phi^+(A_i) > \phi^+(A_j) \text{ and } \phi^-(A_i) < \phi^-(A_j) \\ \text{or, if } &\phi^+(A_i) > \phi^+(A_j) \text{ and } \phi^-(A_i) = \phi^-(A_j) \\ \text{or, if } &\phi^+(A_i) = \phi^+(A_j) \text{ and } \phi^-(A_i) < \phi^-(A_j) \end{aligned}$$

If an alternative A_i is identical to the alternative A_j, Then

$$A_iIA_j \ (A_i \text{ is indifferent with } A_j) \text{ if } \phi^+(A_i) = \phi^+(A_j) \text{ and } \phi^-(A_i) = \phi^-(A_j)$$

Otherwise, the alternatives are incomparable i.e, A_iRA_j.

PROMETHEE II gives complete ranking of the alternatives. In our proposed model, we mainly focus on this method. To build **PROMETHEE II** method with PFN, we calculate the net outranking flow as follows:

$$\phi(A_i) = \phi^+(A_i) - \phi^-(A_i) \quad \text{for } i = 1, 2, ..., m. \tag{5.8}$$

Then the complete orders P and I are derived as:

$$\begin{aligned} A_iPA_j \ (A_i \text{ outranks } A_j) \text{ iff } \ &\phi(A_i) > \phi(A_j) \\ A_iIA_j \ (A_i \text{ is indifferent with } A_j) \text{ iff } \ &\phi(A_i) = \phi(A_j). \end{aligned}$$

Step 6: Rank of the alternatives

In this step, we can rank the alternatives according to the descending order of net outranking flow of the alternatives using **PROMETHEE II** method and choose the best alternative from the set of all alternatives $A = \{A_1, A_2, ..., A_m\}$.

5.4 Comparative analysis using various Pythagorean MCDM methods

5.4.1 Airlines selection problem

In the following, we consider a numerical example adapted from Zhang and Xu (2014) for decision making problem to evaluate the best alternative. We solve the problem by the proposed method and perform a comparative analysis.

The civil aviation administration of Taiwan (CAAT) wants to improve the service quality of domestic airlines. In order to do so, CAAT builds a committee to establish the best domestic airline from the four leading airlines, which are UNI AIR (A_1), Transasia (A_2), Mandarin (A_3), and Daily Air (A_4). Each alternative choice has four essential criteria:

1. Booking and ticketing service (C_1). This criteria require convenience of buying or booking ticket, and courtesy of buying or booking ticket.
2. Check-in and boarding procedure (C_2). This procedure consists of convenience check-in, courtesy of employee, etc.
3. Cabin service (C_3). Cabin service considers the felicities of cabin in flight.
4. Responsiveness (C_4). This criteria consider fair waiting list call, handing of delayed flight and missing baggage, etc.

The weight information of the criteria is given by $W = (0.15, 0.25, 0.35, 0.25)$. Assume that the rating values of the alternative provided by the committee are expressed in terms of PFNs. Then we obtain the following Pythagorean fuzzy decision matrix:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad C_1 \quad & \quad C_2 \quad & \quad C_3 \quad & \quad C_4 \quad \end{array} \\ \begin{pmatrix} \langle 0.9, 0.3 \rangle & \langle 0.7, 0.6 \rangle & \langle 0.5, 0.8 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.4, 0.7 \rangle & \langle 0.9, 0.2 \rangle & \langle 0.8, 0.1 \rangle & \langle 0.5, 0.3 \rangle \\ \langle 0.8, 0.4 \rangle & \langle 0.7, 0.5 \rangle & \langle 0.6, 0.2 \rangle & \langle 0.7, 0.4 \rangle \\ \langle 0.7, 0.2 \rangle & \langle 0.8, 0.2 \rangle & \langle 0.8, 0.4 \rangle & \langle 0.6, 0.6 \rangle \end{pmatrix} \end{array} \tag{5.9}$$

Now, we employ our proposed model to determine the best alternative.

First, we calculate the difference value between two alternatives using Eq.(5.2). We consider the V-shape criterion function with indifference area given in Eq.(5.3) and take $q = 0.1$ and $p = 0.8$. Then we get the preference matrix whose entries are the aggregated preference values calculated by Eq.(5.4). The preference matrix is then obtained as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} - & 0.15 & 0.03 & 0.11 \\ 0.55 & - & 0.27 & 0.09 \\ 0.51 & 0.15 & - & 0.09 \\ 0.48 & 0.14 & 0.12 & - \end{pmatrix} \end{array} \tag{5.10}$$

We now calculate the leaving outranking flows from the matrix (see Eq. 5.10) and Eq. (5.6). The results are given in Table 5.1. Similarly, we calculate the entering outranking flows from the matrix (see Eq. 5.10) and Eq. (5.7). The results are given in Table 5.2.

TABLE 5.1: Leaving outranking flow

$\phi^+(A_i)_i$	Value
$\phi^+(A_1)$	0.096
$\phi^+(A_2)$	0.303
$\phi^+(A_3)$	0.250
$\phi^+(A_4)$	0.246

TABLE 5.2: Entering outranking flow

$\phi^-(A_i)_i$	Value
$\phi^-(A_1)$	0.513
$\phi^-(A_2)$	0.146
$\phi^-(A_3)$	0.140
$\phi^-(A_4)$	0.097

Now, we determine net outranking flow by Eq. (5.8) for PROMETHEE II and ranking alternatives according to the net outranking flow as given in Table 5.3

From Table 5.3, we see that the optimal ranking of the domestic airline is $A_2 \succ A_4 \succ A_3 \succ A_1$. Therefore, the best alternative is A_2 i.e., Transasia.

TABLE 5.3: Net outranking flow

$\phi(A_i)_i$	Value	Ranking
$\phi(A_1)$	-0.417	4
$\phi(A_2)$	0.157	1
$\phi(A_3)$	0.110	3
$\phi(A_4)$	0.149	2

5.4.2 Comparative analysis

In this subsection, we compare our proposed method with Yager (2013) method and Zhang and Xu (2014) method. Yager (2013) proposed a useful method for MCDM problem based on the PFWA aggregation operator under Pythagorean fuzzy information. Zhang and Xu (2014) extended the TOPSIS method to MCDM with Pythagorean fuzzy sets. To compare the proposed method with the above two methods, we solve the MCDM problem mentioned above. Rankings of the alternatives of the proposed method, and Yager (2013) and Zhang and Xu (2014) methods are shown in Table 5.4.

TABLE 5.4: A comparison of the results

Methods	Ranking
Yager (2013)	$A_2 \succ A_4 \succ A_3 \succ A_1$
Zhang and Xu (2014)	$A_2 \succ A_3 \succ A_4 \succ A_1$
Proposed method	$A_2 \succ A_4 \succ A_3 \succ A_1$

It can be easily seen from Table 5.4 that the ranking of the four potential alternatives found in the proposed method is similar to that of Yager (2013) method. Moreover, Transasia (A_2) is the best alternative in all the three methods. While comparing the result of the proposed method with that of Zhang and Xu (2014) method, it is to be noted that, in TOPSIS method, the criteria for the alternatives are compared with positive and negative ideal alternatives but in PROMETHEE method, the criteria for the alternatives are compared between two alternatives and then the ranking of the alternatives is made.

5.5 Medical diagnosis using Pythagorean PROMETHEE method

Modern medical diagnosis accommodates a lot of incomplete, inconsistent and uncertain information because the information available to physicians from medical technologies is increasing day by day. Pythagorean fuzzy sets can handle inconsistent and uncertain information than fuzzy sets and intuitionistic fuzzy sets.
In some medical diagnosis situations, there are some symptoms which may occur or may not occur for a particular disease. Hence PFN is an appropriate tool to assign medical diagnosis problem. In this section, we solve a medical diagnosis problem with the proposed Pythagorean fuzzy PROMETHEE method.
We now consider the medical diagnosis problem which is adapted from Ye (2015). Let there be m diseases $\{D_1, D_2, ..., D_m\} = D$ and a set of symptoms for each disease be $S = \{S_1, S_2, ..., S_n\}$. If a patient P has all symptoms then, to determine the appropriate disease of that patient, we consider the MCDM problem for which we assume the characteristic information as PFN. To emphasize this diagnosis problem, we provide an example and solve by the Pythagorean fuzzy PROMETHEE method.

5.5.1 Numerical example: Medical diagnosis problem

Now-a-days two detrimental fevers are very common in India. One is due to Dengu and another one is due to Nipa-virus infection. It is seen that the symptoms of this fever are similar to the symptoms of usual fevers like Malaria and typhoid. To decide the actual fever of a patient P, we consider the medical diagnosis problem as the MCDM problem where the fevers indicate the alternatives, and symptoms indicate the criteria of the considered MCDM problem. We take a sample for the patient P with symptoms as Pythagorean fuzzy information. We consider the following alternatives (diseases):

- Nipa-virus infection (D_1)
- Dengue (D_2)
- Malaria (D_3)
- Typhoid (D_4)

and consider the following criteria:

- fever (S_1)
- headache (S_2)
- vomiting (S_3)
- nausea (S_4)

The characteristic values of the symptoms are PFN. We have the following decision matrix for the MCDM problem:

	S_1(fever)	S_2(headache)	S_3(vomiting)	S_4(nausea)
D_1(Malaria)	$\langle 0.4, 0.6\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.2, 0.9\rangle$	$\langle 0.1, 0.6\rangle$
D_2(Typhoid)	$\langle 0.9, 0.3\rangle$	$\langle 0.2, 0.9\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.5, 0.6\rangle$
D_3(Dengue)	$\langle 0.7, 0.4\rangle$	$\langle 0.5, 0.6\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.5, 0.8\rangle$
D_4(Nipa-virus infection)	$\langle 0.2, 0.3\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.3, 0.4\rangle$

We now determine the optimal alternative with the help of our proposed method. The weight information of the criteria is given by $w = (0.40, 0.15, 0.25, 0.20)$. We solve the medical diagnosis problem by employing the Pythagorean fuzzy PROMETHEE method with the following steps:

Step 1: Determine the performance difference.
In this step, we determine the performance difference of one alternative to other alternatives using Eq. (5.2). The results are given in Table 5.5.

TABLE 5.5: Performance difference value

	S_1	S_2	S_3	S_4
$d(D_1, D_2)$	0.13	0.46	-1.04	0.04
$d(D_1, D_3)$	-0.52	1.12	-0.37	-0.24
$d(D_1, D_4)$	0.15	-0.20	-0.70	0.20
$d(D_2, D_1)$	-0.13	-0.46	1.04	-0.04
$d(D_2, D_3)$	-0.39	-0.88	0.67	-0.28
$d(D_2, D_4)$	0.20	-0.66	0.34	-0.32
$d(D_3, D_1)$	0.52	-1.12	0.37	0.24
$d(D_3, D_2)$	0.39	0.88	-0.67	0.28
$d(D_3, D_4)$	0.67	-1.32	-0.33	-0.04
$d(D_4, D_1)$	-0.15	0.20	0.70	-0.20
$d(D_4, D_2)$	-0.20	0.66	-0.34	0.32
$d(D_4, D_3)$	-0.67	1.32	0.33	0.04

Step 2: Construct the preference function.
There are six types of preference function. In our proposed model, we use the V-shape criterion function with indifference area. Now, we convert the difference value with the preference function given in Eq. (5.3) and assume $q = 0.1$ and $p = 0.8$. The performance values lie in $[0, 1]$ as shown in Table 5.6.

TABLE 5.6: Preference function value

	S_1	S_2	S_3	S_4
$d(D_1, D_2)$	0.04	0.51	0.00	0.00
$d(D_1, D_3)$	0.00	1.00	0.00	0.00
$d(D_1, D_4)$	0.07	0.00	0.00	0.14
$d(D_2, D_1)$	0.00	0.00	1.00	0.00
$d(D_2, D_3)$	0.00	0.00	0.81	0.00
$d(D_2, D_4)$	0.14	0.00	0.34	0.00
$d(D_3, D_1)$	0.60	0.00	0.38	0.20
$d(D_3, D_2)$	0.41	1.00	0.00	0.25
$d(D_3, D_4)$	0.81	0.00	0.00	0.00
$d(D_4, D_1)$	0.00	0.14	0.85	0.00
$d(D_4, D_2)$	0.00	0.80	0.00	0.31
$d(D_4, D_3)$	0.00	1.00	0.32	0.00

Step 3: Calculate the aggregated preference value.
In this step, we calculate the total preference index using the weight information of the symptoms $w = (0.40, 0.15, 0.25, 0.20)$. Using Eq. (5.4), we get the following preference matrix:

$$\begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{pmatrix} - & 0.092 & 0.150 & 0.056 \\ 0.250 & - & 0.20 & 0.14 \\ 0.375 & 0.364 & - & 0.324 \\ 0.233 & 0.182 & 0.230 & - \end{pmatrix} \end{array} \tag{5.11}$$

Step 4: Determine the positive and the negative outranking flows.
Using Eqs. (5.6) and (5.7), we determine the positive outranking flows $\phi^+(D_i)$ and the negative outranking flows $\phi^-(D_i)$ for the alternatives as given in Table 5.7.

Step 5: Calculate net outranking flows.
In this step, we calculate the net outranking values by using Eq. (5.8). The results are shown in Table 5.8. From Table 5.8, we see that the ranking of the alternatives

TABLE 5.7: Positive and negative outranking values

Alternative	Positive outranking flows	Negative outranking flows
D_1	$\phi^+(D_1) = 0.099$	$\phi^-(D_1) = 0.286$
D_2	$\phi^+(D_2) = 0.197$	$\phi^-(D_2) = 0.212$
D_3	$\phi^+(D_3) = 0.354$	$\phi^-(D_3) = 0.193$
D_4	$\phi^+(D_4) = 0.215$	$\phi^-(D_4) = 0.173$

TABLE 5.8: Net outranking values and ranking

Alternative	Net outranking flows	Ranking
D_1	$\phi^(D_1) = -0.187$	4
D_2	$\phi^(D_2) = -0.015$	3
D_3	$\phi^(D_3) = 0.161$	1
D_4	$\phi^(D_4) = 0.042$	2

(diseases) is $D_3 \succ D_4 \succ D_3 \succ D_1$. Thus, the diagnosis result indicates that the patient P has a Dengue(D_3) fever.

5.6 Conclusion

PROMETHEE is one of the classical methods for solving MCDM problem with crisp number as well as fuzzy and intuitionistic fuzzy numbers. In this chapter, we have extended the PROMETHEE method under Pythagorean fuzzy environment and solved a real MCDM problem. We have discussed some basic operations of Pythagorean fuzzy numbers (PFN) and compared the Pythagorean fuzzy sets (PFS) with intuitionistic fuzzy sets(IFS). We have derived classical PROMETHEE method for MCDM problem. We have proposed PROMETHEE method for PFN and calculated the performance value of the alternatives using the score function of PFN. We have used V-shape criterion function to determine the preference value of the alternative and solved a numerical example to compare the existing methods under Pythagorean fuzzy environment to ensure the validity of our proposed method. Finally, we have introduced a medical diagnosis problem as an MCDM problem and solved the problem using our proposed method.

6

Pythagorean Fuzzy DEMATEL Method for Supplier Selection in Sustainable Supply Chain Management

6.1 Introduction

Decision-making trial and evaluation laboratory (DEMATEL) (Gabus and Fontela, 1972) is a method which develops mutual relationships of the criteria and their correlated dependencies. This method provides a casual-effect diagram to describe mutual relationships and influences of the criteria (Wu and Tsai, 2011). It can analyse total relations among sets of variables to obtain logical relationships and direct impact relationships. The method is well suited to situations where it becomes necessary to upgrade the evaluation of one criterion by adding new one even if the number of criteria is quite large. It is well known that if the number of evaluation criteria is not restricted, then the decision difficulty increases, and the decision quality is degraded for some decision-making methods such as AHP, TOPSIS, etc. But in the DEMATEL method, such a situation will not occur as it divides the entire criteria, however large it is, into two groups cause and effect, and displays casual relationships between criteria visually.

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It is undeniable that gain or loss of a business organization largely depends on supplier selection in its supply chain management system. Many organizations choose suitable suppliers around the world to make collaborative commerce, and to increase trade and productivity. Besides supply chain management, they also take a look on environmental management for special concern in industry. In fact, green image, social concern and economic policy together are forcing companies to integrate sustainable supply chain management. Therefore, it is a challenging task to select the proper criteria of the supplier in sustainable supply chain management, because choosing of various criteria involves assessment and selection of the ideal supplier. Moreover, the number of criteria for supplier selection may be large and some criteria may contain incomplete and inconsistent information. In uncertain environment, exact determination of criteria is quite difficult. However, Pythagorean fuzzy set can effectively deal with uncertain environment in the decision making process. It is well known that a fuzzy number can be expressed as a fuzzy set defining a fuzzy interval in real number. Since the boundary of this interval is ambiguous, the interval is also a fuzzy set. Generally, a fuzzy interval is represented by two end points and some intermediary peak points. Among various shapes of fuzzy number, triangular and trapezoidal fuzzy numbers (Dubois and Prade, 1983) are widely used by researchers. Like trapezoidal fuzzy number (TrFN), we have Pythagorean trapezoidal fuzzy number (PTrFN) in Pythagorean fuzzy set. Therefore, Pythagorean fuzzy set based DEMATEL method can be considered as a suitable tool to handle supplier selection problem in sustainable supply chain management. The main objectives of our study are as follows:

- To develop DEMATEL method with Pythagorean fuzzy sets.
- To solve the proposed method by using trapezoidal Pythagorean fuzzy number (TrPFN).
- To apply the proposed method in sustainable supply chain management.

The remainder of the chapter is organized as follows. In Section 6.2, we discuss preliminaries of classical DEMATEL method and some basics of Pythagorean fuzzy sets. Section 6.3 is devoted to the proposed Pythagorean fuzzy DEMATEL method. Section 6.4 presents numerical results of a supplier selection problem which is solved using the proposed method. Section 6.5 analyzes the numerical results. Finally, the chapter ends with concluding remarks in section 6.6.

6.2 Preliminaries

In this section, we briefly discuss the classical DEMATEL method and provide some preliminaries of Pythagorean fuzzy sets.

6.2.1 Classical DEMATEL method

The steps of DEMATEL method are described as follows.

Step 1: Defining the dominant feature in the research methodology, the linguistic measurement scale is set for pairwise comparison among all characteristics. The initial direct relation matrix $D = [d_{ij}]_{n\times n}$ is obtained by pairwise comparison between criteria, in which d_{ij} denotes the degree to which the criteria i affects the criteria j.

Step 2: This step defines the normalization of direct relation matrix. On the basis of direct relation matrix D, the normalized direct relation matrix can be obtained as

$$S = k \times D, \tag{6.1}$$

$$\text{where, } k = \frac{1}{\max\limits_{1\leq i\leq n} \sum_{j=1}^{n} d_{ij}}.$$

Step 3: The total relation matrix is determined as given below:

$$T = S(I-S)^{-1}, \text{where } I \text{ is the } n\times n \text{ identity matrix.} \tag{6.2}$$

Step 4: Construct the DEMATEL map with respect to the total relation matrix. The sum of rows and the sum of columns are denoted by vectors R_j $(j = 1, 2..., n)$ and D_i $(i = 1, 2, ..., n)$, respectively within the total relation matrix $T = [t_{ij}]_{n\times n}$ and are given by

$$R_j = \Big[\sum_{j=1}^{n} t_{ij}\Big]_{1\times n} \tag{6.3}$$

$$D_i = \Big[\sum_{i=1}^{n} t_{ij}\Big]_{n\times 1} \tag{6.4}$$

where $D_i + R_j$ is a horizontal axis vector or 'prominence' which indicates the relative importance of the criterion, and the vertical axis $D_i - R_j$ represents 'relation'. If the value of $D_i - R_j$ is positive then the criterion is formed into the cause group, and if the value of $D_i - R_j$ is negative then the criterion is formed into the effect group.

Step 5: The sum of each column of the total relation matrix is 1 by normalized method, which gives the inner dependency of the matrix.

6.2.2 The basics of Pythagorean fuzzy arithmetic

Definition 6.1. (Yager, 2013; Yager and Abbasov, 2013) Let X be a universe of discourse. Then Pythagorean fuzzy set defined on X is of the form

$$P = \{< x, \mu_p(x), \nu_p(x) > | x \in X\}$$

where $\mu_p : X \to [0,1]$ and $\nu_p : X \to [0,1]$ are, respectively, the membership and the non-membership functions which satisfy the condition

$$0 \leq (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1, \quad \forall x \in X$$

and the degree of indeterminacy membership is denoted by $\pi_p(x)$ and is defined by $\pi_p(x) = \sqrt{1 - (\mu_p(x))^2 - (\nu_p(x))^2}$

The membership value and the non-membership value of intuitionistic fuzzy set I satisfy the condition $0 \leq \mu_I + \nu_I \leq 1$, whereas the membership value and the non-membership value of Pythagorean fuzzy set P satisfy the condition $0 \leq \mu_P^2 + \nu_P^2 \leq 1$.

Definition 6.2. (Dubois and Prade, 1983) A generalized trapezoidal fuzzy number is an extension of trapezoidal fuzzy number, which is denoted by

$$A = (a, b, c, d; \mu)$$

and is described by a fuzzy subset of a real number $\mathbb{R}$ with membership function μ_A given by

$$\mu_A(x) = \begin{cases} \dfrac{(x-a)\mu}{b-a}, & a \leq x < b \\ \mu, & b \leq x \leq c \\ \dfrac{(d-x)\mu}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases} \tag{6.5}$$

where $a, b, c, d \in \mathbb{R}$ and μ is a membership degree.

Definition 6.3. (Xian et al., 2018) A trapezoidal pythagorean fuzzy number(TrPFN) is represented as

$$A = \langle (a_1, a_2, a_3, a_4); \mu, \nu \rangle$$

with the parameters $a_1, a_2, a_3,$ and a_4 such that $a_1 \leq a_2 \leq a_3 \leq a_4$ and the membership and the non-membership degrees μ and ν satisfy the condition $\mu^2 + \nu^2 \leq 1$. Then the membership function μ_A and the non-membership function ν_A are given by

$$\mu_A(x) = \begin{cases} \dfrac{(x-a_1)\mu}{a_2-a_1}, & a_1 \leq x < a_2 \\ \mu, & a_2 \leq x < a_3 \\ \dfrac{(a_4-x)\mu}{a_4-a_3}, & a_3 \leq x < a_4 \\ 0, & otherwise. \end{cases} \tag{6.6}$$

$$\nu_A(x) = \begin{cases} \dfrac{a_2 - x + \nu(x-a_1)}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \nu, & a_2 \leq x \leq a_3 \\ \dfrac{x - a_3 + \nu(a_4-x)}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 1, & otherwise. \end{cases} \tag{6.7}$$

Let $A_1 = \langle (a_{p_1}, a_{p_2}, a_{p_3}, a_{p_4}); \mu_{p_1}, \nu_{p_1} \rangle$, $A_2 = \langle (b_{p_1}, b_{p_2}, b_{p_3}, b_{p_4}); \mu_{p_2}, \nu_{p_2} \rangle$ be two TrPFNs. Then the following operations hold good:

1. $A_1 \oplus A_2 = \langle (a_{p_1} + b_{p_1}, a_{p_2} + b_{p_2}, a_{p_3} + b_{p_3}, a_{p_4} + b_{p_4}); \sqrt{\mu_{p_1}^2 + \mu_{p_2}^2 - \mu_{p_1}^2 \mu_{p_2}^2}, \nu_{p_1} \nu_{p_2} \rangle$
2. $A_1 \otimes A_2 = \langle (a_{p_1} b_{p_1}, a_{p_2} b_{p_2}, a_{p_3} b_{p_3}, a_{p_4} b_{p_4}); \mu_{p_1} \mu_{p_2}, \sqrt{\nu_{p_1}^2 + \nu_{p_2}^2 - \nu_{p_1}^2 \nu_{p_2}^2} \rangle$
3. $\lambda A_1 = \langle (\lambda a_{p_1}, \lambda a_{p_2}, \lambda a_{p_3}, \lambda a_{p_4}); \sqrt{1 - (1 - \mu_{p_1}^2)^\lambda}, (\nu_{p_1})^\lambda \rangle, \ \lambda > 0$
4. $A_1{}^\lambda = \langle ({a_{p_1}}^\lambda, {a_{p_2}}^\lambda, {a_{p_3}}^\lambda, {a_{p_4}}^\lambda); (\mu_{p_1})^\lambda, \sqrt{1 - (1 - \nu_{p_1}^2)^\lambda} \rangle, \ \lambda > 0$

Using the above operations, it can be shown that the following relations are valid.

1. $A_1 \oplus A_2 = A_2 \oplus A_1$
2. $A_1 \otimes A_2 = A_2 \otimes A_1$
3. $\lambda(A_1 \oplus A_2) = \lambda A_1 \oplus \lambda A_2, \ \lambda > 0$

4. $\lambda_1 A_1 \otimes \lambda_2 A_1 = (\lambda_1 + \lambda_2) A_1,\ \lambda_1, \lambda_2 > 0$

5. $(A_1 \otimes A_2)^\lambda = A_1^\lambda \otimes A_2^\lambda,\ \lambda > 0$

6. $A_1^{\lambda_1} \otimes A_1^{\lambda_2} = A_1^{(\lambda_1+\lambda_2)},\ \lambda_1, \lambda_2 > 0$

Definition 6.4. (Grzegrorzewski, 2003) Let $A = \langle (a_1, a_2, a_3, a_4); \mu, \nu \rangle$ be a TrPFN where a_1, a_2, a_3 and a_4 are real numbers. Then the expected value of A is given by

$$E(A) = \frac{(a_1 + a_2 + a_3 + a_4)}{4} \sqrt{\mu^2 + \nu^2} \tag{6.8}$$

It can be easily shown that if TrPFN is $A = \langle (1, 1, 1, 1); 1, 0 \rangle$ then $E(A) = 1$, and if TrPFN is $B = \langle (0, 0, 0, 0); 0, 1 \rangle$ then $E(B) = 0$.

6.3 Pythagorean fuzzy DEMATEL method

In decision making, decision makers usually make judgement according to their experience and expertise. Exact evaluation of criteria for DEMATEL method or any other decision-making method is quite difficult in uncertain environment. Pythagorean fuzzy set effectively deals with uncertain environment in decision making process. The proposed method is discussed in the following.

Step 1: Extracting the Pythagorean fuzzy direct relation

We consider Pythagorean fuzzy linguistic scale which is assigned to the corresponding TrPFN with the view point of the expert to deal with ambiguities of human assessment. We construct Pythagorean fuzzy direct relation matrix D for the criteria $C_1, C_2, ..., C_n$ as

$$D = [d_{ij}]_{n \times n} \tag{6.9}$$

where d_{ij}'s are TrPFNs. Govindan et al. (2015a) calculated the expected value of trapezoidal intuitionistic fuzzy number(TrIFN). Here we determine the expected value of each d_{ij} and obtain the expected Pythagorean fuzzy direct relation matrix $\tilde{D}$ using equation (6.8) (Grzegrorzewski, 2003) as

$$\tilde{D} = [\tilde{d_{ij}}]_{n \times n} \tag{6.10}$$

where $\tilde{d_{ij}}$ is the expected value of TrPFN d_{ij} .

Step 2: Normalize the expected Pythagorean fuzzy direct relation matrix

This step transforms various criteria into non-dimensional criteria. This allows comparison across criteria because various criteria are usually measured in different units. Hence the normalized expected Pythagorean fuzzy direct relation matrix is obtained as follows:

$$\begin{aligned} X &= k \times \tilde{D} \\ &= k \times [\tilde{d_{ij}}]_{n\times n} \end{aligned} \tag{6.11}$$

where

$$k = \frac{1}{\max\limits_{1\leq i\leq n} \sum_{j=1}^{n} \tilde{d_{ij}}}$$

Then X can be written as

$$X = [n_{ij}]_{n\times n} \tag{6.12}$$

Step 3: Construction of Pythagorean fuzzy total relation matrix

The Pythagorean fuzzy total relation matrix is calculated as

$$T = X(I - X)^{-1} \tag{6.13}$$

where T is an $(n\times n)$ Pythagorean fuzzy total relation matrix and I is an $(n\times n)$ identity matrix. Therefore,

$$T = [t_{ij}]_{n\times n}. \tag{6.14}$$

Step 4: Generating casual diagram

Calculate D and R that denote respectively the sum of rows and the sum of columns of the Pythagorean fuzzy total relation matrix $T = [t_{ij}]_{n\times n}$:

$$R = \Big[\sum_{j=1}^{n} t_{ij}\Big]_{1\times n} \tag{6.15}$$

$$D = \Big[\sum_{i=1}^{n} t_{ij}\Big]_{n\times 1} \tag{6.16}$$

Here $D + R$ denotes the impact strength index and $D - R$ represents the importance factor index. The important relation map can be drawn in cause and effect groups by putting the value in the form of $(D - R, D + R)$. The vertical axis, $D - R$ represents 'relation'. If the value of $D - R$ is positive then the criterion is grouped into the cause group and if the value of $D - R$ is negative then the criterion is grouped into the effect group. A schematic diagram of the proposed DEMATEL method in Pythagorean fuzzy environment is depicted in Figure 6.1.

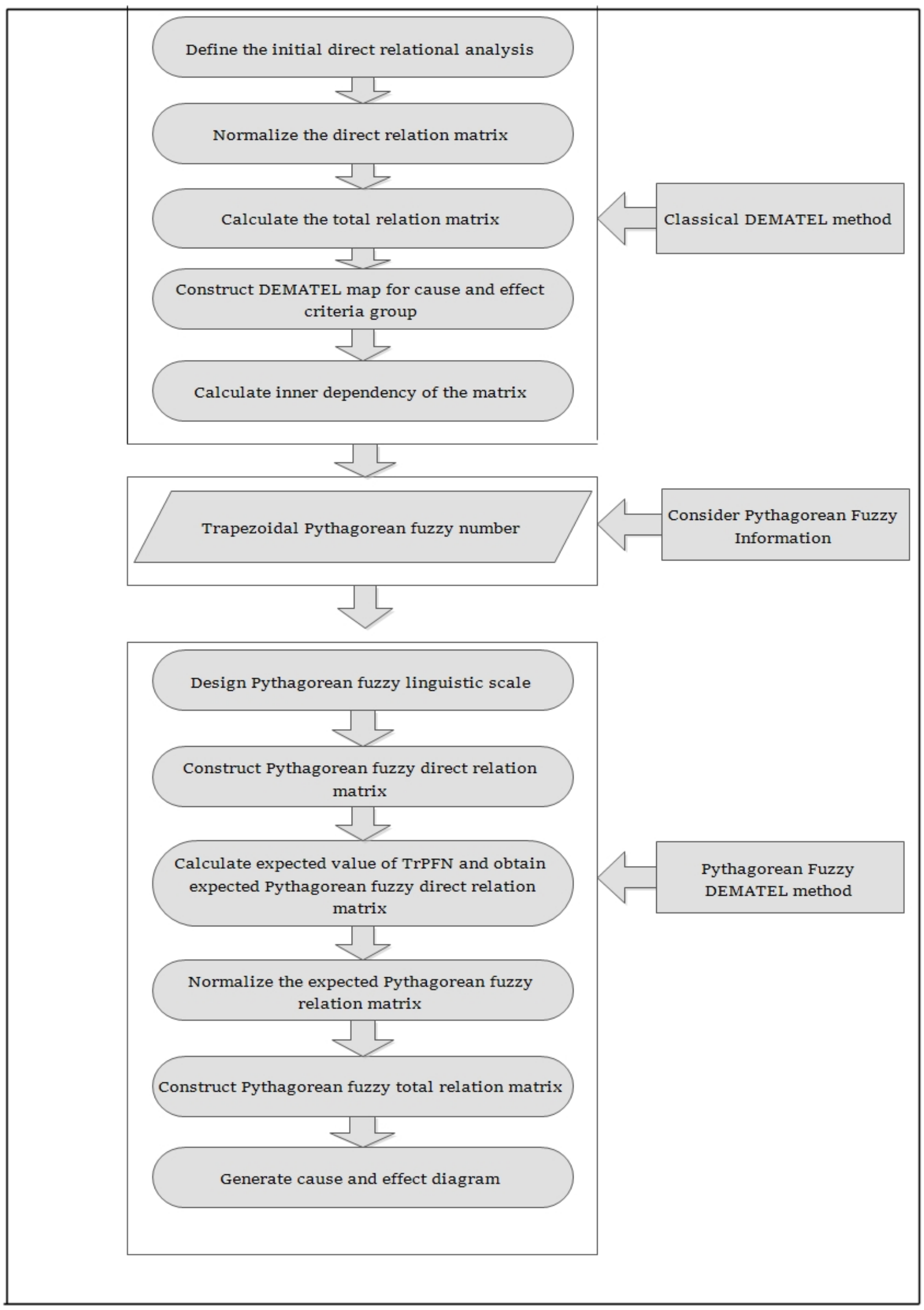

FIGURE 6.1: A schematic diagram of research workflow.

6.4 Numerical results

In this section, we consider the supplier selection problem in sustainable supply chain management for numerical illustration of the proposed method. The effective factors of supplier selection problem in sustainable supply chain management are generally complex. We use the Pythagorean fuzzy DEMATEL method to simplify the cause and effect criteria for the supplier. The proposed method is discussed in the following steps.

Step 1: Based on literature review, the following fifteen criteria of supplier are identified for sustainable supply chain management: Environmental efficiency (C_1), green image (C_2), pollution depletion (C_3), green design (C_4), safety and health (C_5), employment practices (C_6), supplier/customer collaboration (C_7), stakeholder relations (C_8), quality (C_9), flexibility (C_{10}), cost (C_{11}), technical capability (C_{12}), logistics cost (C_{13}), rejection ratio (C_{14}), and e-commerce capability (C_{15}). These influential criteria for the entire system are chosen for interrelation comparison. In Table 6.1, we list the major criteria and sub-criteria involved in this supplier selection problem.
Human assessments for interrelation comparison between chosen criteria are generally given by crisp values. However, assessments with preferences are often vague and difficult to estimate by crisp values. In this case, linguistic assessment is the reasonable approach for deciding the relationship between two criteria. We use 7-point linguistic rating scale which is described by linguistic term and it's corresponding TrPFN (Govindan et al., 2015a) (See Table 6.2). The corresponding expected value of each TrPFN is calculated using equation (6.8) and is shown in Table 6.2.

We consider an expert team of ten members which includes two professors, four research scholars and four students who work in the relevant field of our study. To obtain the relationships among the evaluation criteria, we consult with the experts using a survey instrument with 7-point linguistic rating scale. The linguistic data are obtained from each individual expert's assessment. The maximum linguistic rating value of the corresponding criterion is selected and the initial direct relation matrix is then obtained, which is shown in Table 6.3.

Step 2: Substituting the TrPFN with the corresponding expected value, we obtain the expected Pythagorean fuzzy direct relation matrix shown in Table 6.4. The normalized Pythagorean fuzzy direct relation matrix is performed by using equations (6.1) and

TABLE 6.1: Criteria and sub-criteria for the supplier selection in sustainable supply chain

Criteria	Sub-criteria
Environmental criteria	environmental efficiency (C_1) green image (C_2) pollution depletion (C_3) green design (C_4)
Social criteria	safety and health (C_5) employment practices (C_6) supplier/customer collaboration (C_7) stakeholder relations (C_8)
Economic criteria	quality (C_9) flexibility (C_{10}) cost (C_{11}) technical capability (C_{12}) logistics cost (C_{13}) rejection ratio (C_{14}) e-commerce capability (C_{15})

TABLE 6.2: Pythagorean fuzzy linguistic scale

Linguistic variable	Influence score	Corresponding TrPFN	Expected value
Absolutely low (AL)	0	$\langle(0, 0, 0, 0); 0, 1\rangle$	0
Low (L)	1	$\langle(0, 0.1, 0.2, 0.3); 0.1, 0.8\rangle$	0.121
Fairly low (FL)	2	$\langle(0.1, 0.2, 0.3, 0.4); 0.3, 0.7\rangle$	0.190
Medium Low (ML)	3	$\langle(0.3, 0.4, 0.5, 0.6); 0.5, 0.5\rangle$	0.318
Fairly high (FH)	4	$\langle(0.5, 0.6, 0.7, 0.8); 0.6, 0.4\rangle$	0.469
High (H)	5	$\langle(0.7, 0.8, 0.9, 1); 0.8, 0.2\rangle$	0.701
Absolutely high (AH)	6	$\langle(1, 1, 1, 1); 1, 0\rangle$	1

TABLE 6.3: Initial direct relation matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_1	-	AH	H	H	FH	ML	ML	L	AH	ML	FH	H	L	L	FL
C_2	H	-	H	AH	H	L	H	ML	AH	H	H	FH	L	L	ML
C_3	H	AH	-	H	H	L	H	L	H	H	H	H	L	L	L
C_4	AH	H	FH	-	AH	H	H	H	H	FH	FH	ML	ML	ML	H
C_5	H	H	AH	FH	-	ML	L	L	H	H	H	L	L	ML	L
C_6	L	L	FL	L	L	-	H	ML	FH	ML	ML	ML	H	L	L
C_7	H	FH	ML	ML	L	L	-	ML	H	H	H	H	H	H	FH
C_8	ML	L	L	L	ML	H	H	-	ML	ML	H	H	L	H	H
C_9	H	H	H	H	L	H	ML	ML	-	AH	AH	H	L	L	H
C_{10}	ML	H	ML	FH	H	L	ML	L	H	-	H	FH	ML	L	FH
C_{11}	H	AH	H	FH	H	L	ML	L	H	H	-	FH	H	L	FH
C_{12}	ML	ML	L	L	FH	L	ML	ML	H	FH	H	-	H	L	H
C_{13}	L	L	L	FL	FL	FL	L	ML	FL	L	H	H	-	L	H
C_{14}	FL	FL	L	L	FL	L	H	H	FH	FH	H	H	L	-	AH
C_{15}	H	H	FH	L	ML	L	H	H	AH	AH	H	H	H	L	-

TABLE 6.4: Expected Pythagorean fuzzy direct relation matrix (D)

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_1	0.00	1.00	0.701	0.701	0.469	0.318	0.318	0.121	1.00	0.318	0.469	0.701	0.121	0.121	0.190
C_2	0.701	0.00	0.701	1.00	0.701	0.121	0.701	0.318	1.00	0.701	0.701	0.469	0.121	0.121	0.318
C_3	0.701	1.00	0.00	0.701	0.701	0.121	0.701	0.121	0.701	0.701	0.701	0.701	0.121	0.121	0.121
C_4	1.00	0.701	0.469	0.00	0.469	0.701	0.701	0.701	0.701	0.469	0.318	0.318	0.318	0.318	0.701
C_5	0.701	0.701	1.00	0.469	0.00	0.318	0.121	0.121	0.701	0.701	0.701	0.121	0.121	0.318	0.121
C_6	0.121	0.121	0.190	0.121	0.121	0.00	0.701	0.318	0.469	0.318	0.318	0.318	0.701	0.121	0.121
C_7	0.701	0.469	0.318	0.318	0.121	0.121	0.00	0.318	0.701	0.701	0.701	0.701	0.701	0.701	0.469
C_8	0.318	0.121	0.121	0.121	0.318	0.701	0.701	0.00	0.318	0.318	0.701	0.701	0.121	0.701	0.701
C_9	0.701	0.701	0.701	0.701	0.121	0.701	0.318	0.318	0.00	1.00	1.00	0.701	0.121	0.121	0.701
C_{10}	0.318	0.701	0.318	0.469	0.701	0.121	0.318	0.121	0.701	0.00	0.701	0.469	0.318	0.121	0.469
C_{11}	0.701	1.00	0.701	0.469	0.701	0.121	0.469	0.121	0.701	0.701	0.00	0.469	0.701	0.121	0.469
C_{12}	0.318	0.318	0.121	0.121	0.469	0.121	0.318	0.318	0.701	0.469	0.701	0.00	0.701	0.121	0.701
C_{13}	0.121	0.121	0.121	0.190	0.190	0.190	0.121	0.318	0.190	0.121	0.701	0.701	0.00	0.121	0.701
C_{14}	0.190	0.190	0.121	0.121	0.190	0.121	0.701	0.701	0.469	0.469	0.701	0.701	0.121	0.00	1.00
C_{15}	0.701	0.701	0.469	0.121	0.318	0.121	0.701	0.701	1.00	1.00	0.701	0.701	0.701	0.121	0.00

(6.12). The normalized Pythagorean fuzzy direct relation matrix is shown in Table 6.5. Table 6.6 represents Pythagorean fuzzy total relation matrix.

TABLE 6.5: Normalized Pythagorean fuzzy direct relation matrix (Y)

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_1	0.00	0.124	0.087	0.087	0.058	0.039	0.039	0.015	0 .124	0.039	0.058	0.087	0.015	0.015	0.024
C_2	0.087	0.00	0.087	0.124	0.087	0.015	0.087	0.039	0.124	0.087	0.087	0.058	0.015	0.015	0.039
C_3	0.087	0.124	0.00	0.087	0.087	0.015	0.087	0.015	0.087	0.087	0.087	0.087	0.015	0.015	0.015
C_4	0.124	0.087	0.058	0.00	0.058	0.087	0.087	0.087	0.087	0.058	0.039	0.039	0.039	0.039	0.087
C_5	0.087	0.087	0.124	0.058	0.00	0.039	0.015	0.015	0.087	0.087	0.087	0.015	0.015	0.039	0.015
C_6	0.015	0.015	0.024	0.015	0.015	0.00	0.087	0.039	0.058	0.039	0.039	0.039	0.087	0.015	0.015
C_7	0.087	0.058	0.039	0.039	0.015	0.015	0.00	0.039	0.087	0.087	0.087	0.087	0.087	0.087	0.058
C_8	0.039	0.015	0.015	0.015	0.039	0.087	0.087	0.00	0.039	0.039	0.087	0.087	0.015	0.087	0.087
C_9	0.087	0.087	0.087	0.087	0.015	0.087	0.039	0.039	0.00	0.124	0.124	0.087	0.015	0.015	0.087
C_{10}	0.039	0.087	0.039	0.058	0.087	0.015	0.039	0.015	0.087	0.00	0.087	0.058	0.039	0.015	0.058
C_{11}	0.087	0.124	0.087	0.058	0.087	0.015	0.058	0.015	0.087	0.087	0.00	0.058	0.087	0.015	0.058
C_{12}	0.039	0.039	0.015	0.015	0.058	0.015	0.039	0.039	0.087	0.058	0.087	0.00	0.087	0.015	0.087
C_{13}	0.015	0.015	0.015	0.024	0.024	0.024	0.015	0.039	0.024	0.015	0.087	0.087	0.00	0.015	0.087
C_{14}	0.024	0.024	0.015	0.015	0.024	0.015	0.087	0.087	0.058	0.058	0.087	0.087	0.015	0.00	0.124
C_{15}	0.087	0.087	0.058	0.015	0.039	0.015	0.087	0.087	0.124	0.124	0.087	0.087	0.087	0.015	0.00

TABLE 6.6: Pythagorean fuzzy total relation matrix (T)

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_1	0.324	0.468	0.359	0.350	0.306	0.202	0.310	0.192	0.511	0.385	0.428	0.392	0.219	0.138	0.290
C_2	0.459	0.417	0.405	0.423	0.372	0.209	0.395	0.244	0.575	0.483	0.515	0.418	0.254	0.163	0.349
C_3	0.434	0.503	0.305	0.374	0.356	0.192	0.374	0.206	0.516	0.457	0.487	0.419	0.239	0.152	0.306
C_4	0.475	0.478	0.365	0.296	0.334	0.269	0.393	0.286	0.532	0.445	0.464	0.399	0.275	0.184	0.386
C_5	0.386	0.422	0.379	0.312	0.241	0.191	0.274	0.179	0.457	0.407	0.433	0.312	0.205	0.152	0.264
C_6	0.192	0.209	0.173	0.158	0.152	0.092	0.233	0.140	0.274	0.231	0.254	0.221	0.205	0.090	0.174
C_7	0.398	0.408	0.310	0.298	0.266	0.177	0.273	0.221	0.479	0.427	0.462	0.404	0.294	0.207	0.336
C_8	0.301	0.305	0.240	0.224	0.242	0.215	0.313	0.155	0.370	0.329	0.398	0.349	0.202	0.191	0.314
C_9	0.450	0.492	0.397	0.384	0.310	0.267	0.356	0.244	0.462	0.512	0.544	0.444	0.262	0.158	0.390
C_{10}	0.326	0.397	0.287	0.291	0.304	0.160	0.277	0.173	0.434	0.309	0.414	0.332	0.222	0.125	0.293
C_{11}	0.434	0.504	0.387	0.349	0.358	0.192	0.350	0.209	0.518	0.459	0.413	0.400	0.305	0.151	0.347
C_{12}	0.293	0.321	0.237	0.223	0.254	0.145	0.253	0.182	0.398	0.334	0.386	0.256	0.252	0.115	0.301
C_{13}	0.198	0.217	0.171	0.166	0.167	0.115	0.172	0.143	0.252	0.215	0.298	0.264	0.128	0.087	0.241
C_{14}	0.295	0.322	0.245	0.230	0.234	0.153	0.316	0.239	0.395	0.355	0.407	0.357	0.204	0.113	0.354
C_{15}	0.448	0.488	0.371	0.320	0.328	0.204	0.392	0.285	0.572	0.513	0.521	0.451	0.323	0.162	0.316

Step 3: We calculate the total relation matrix from normalized Pythagorean fuzzy direct relation by using equation (6.13).

To obtain the numerical values as indicated in Steps 2-3, we use Microsoft excel software. The instructions for the computation are summarized as follows:

- Find the sum of all elements of each row of the expected Pythagorean fuzzy direct relation matrix D.
- Find the maximum element from the sums by using ‘max function’.

- Get the normalized Pythagorean fuzzy direct relation matrix (Y) by dividing every element of D by the maximum element.
- Define the identity matrix I.
- Calculate $I - Y$ by using 'subtraction function'.
- Use 'inverse function' to determine $(I - Y)^{-1}$.
- Determine Pythagorean fuzzy total relation matrix (T) i.e. $Y(I - Y)^{-1}$ using 'm multiplication function'.

Step 4: The sum of all rows of Pythagorean fuzzy total relation matrix is denoted by D and the sum of all columns is denoted by R. Then the values of $(D + R)$ and $(D - R)$ are determined. A criterion is treated as cause category if $(D - R)$ is positive, and effect category if $(D - R)$ is negative. $(D + R)$ presents horizontal axis of the vector which is called prominence due to importance of the criteria. Table 6.7 represents the values of $(D + R)$ and $(D - R)$ for all criteria. The casual diagram of the set $(D + R, D - R)$ is shown in Figure 6.2. This diagram indicates the discernment about the recognition of the whole complex system and recognizes significance of supplier in sustainable supply chain management.

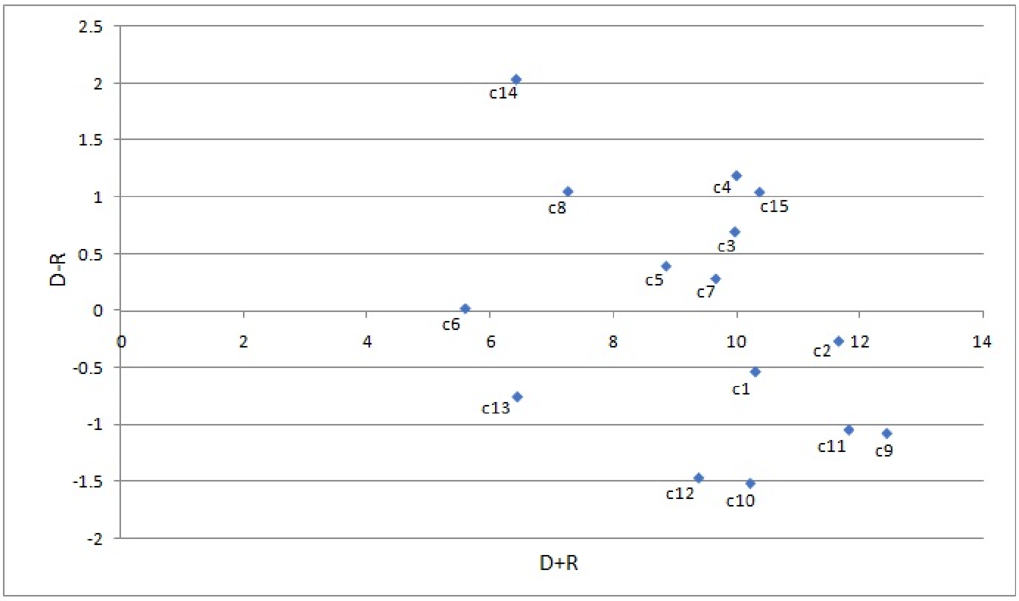

FIGURE 6.2: Cause and effect diagram.

TABLE 6.7: Values of $(D+R)$ and $(D-R)$ for different criteria

Criteria	D	R	$D+R$	$D-R$
Environmental efficiency(C_1)	4.881	5.418	10.300	-0.537
Green image (C_2)	5.687	5.956	11.65	-0.269
Pollution depletion (C_3)	5.328	4.638	9.966	0.689
Green design (C_4)	5.587	4.405	9.992	1.182
Safety and health (C_5)	4.620	4.230	8.851	0.39
Employment practices (C_6)	2.804	2.789	5.594	0.016
Supplier/ Customer collaboration (C_7)	4.968	4.688	9.656	0.279
Stakeholder relations (C_8)	4.155	3.105	7.261	1.045
Quality (C_9)	5.678	6.754	12.433	-1.076
Flexibility (C_{10})	4.351	5.866	10.218	-1.515
Cost (C_{11})	5.384	6.429	11.814	-1.045
Technical capability (C_{12})	3.957	5.425	9.382	-1.468
Logistics cost (C_{13})	2.840	3. 597	6.438	-0.757
Rejection ratio (C_{14})	4.224	2.196	6.42	2.028
E-commerce capability (C_{15})	5.700	4.667	10.367	1.039

6.5 Result analysis

From Figure 6.2 and Table 6.7, we see that the criteria for supplier selection are divided into two groups according to positive and negative values of $(D-R)$. If $(D-R)$ is positive, then the criterion belongs to cause group and if the value of $(D-R)$ is negative then the criteria belongs to effective group. Here, the cause group includes the criterion $\{C_3, C_4, C_5, C_6, C_7, C_8, C_{14}, C_{15}\}$ and the effective group includes the criteria $\{C_1, C_2, C_9, C_{10}, C_{11}, C_{12}, C_{13}\}$. There are many other hints that can be obtained from Figure 6.2. We perform detailed analysis on the cause group and the effect group separately in the following two sub-sections.

6.5.1 Cause criteria analysis

The cause criteria group can have net impact criteria of a supplier in a sustainable supply chain and its performance can influence much on the whole system. Therefore, the criteria which belong to cause group can get more attention for supplier selection in sustainable supply chain management. In our numerical study, out of the fifteen criteria, eight criteria are identified as cause group. Those are pollution depletion (C_3), green design (C_4), safety and health (C_5), employment practices (C_6), supplier/customer collaboration (C_7), stakeholder relation (C_8), rejection ratio (C_{14}), and e-commerce capability (C_{15}). In the cause group, the 'rejection ratio' of the supplier i.e. C_{14} has the highest value of $(D - R)$ which means that C_{14} has more impact on the whole system. In addition, Table 6.7 shows that the value of $(D + R)$ of C_{14} is 6.4 which indicates that C_{14} is a remarkable criterion of the supplier in sustainable supply chain management. The rejection ratio is the percentage of items with unsatisfactory quality from the considered delivery. The items supplied by a supplier may not be 100% perfect due to inadequate internal control at supplier's company, damage due to packing and/or storage, damage during loading and unloading and transport, etc. For this reason, rejection ratio is considered to be one of the key indicators of supplier evaluation.

The criterion for which the value of $(D - R)$ is second highest is 'green design' and it's corresponding value of $(D+R)$ is also relatively high. Thus the criterion 'green design' can dispatch the other criteria in the whole system and therefore, it should attract more attention of the decision maker while selecting a supplier. A supplier should not only provide an organization with the adequate raw materials or products at a competitive price, but also help to improve environmental performance while avoiding hazardous substances and considering green design. With increasing government regulations and public awareness in environmental protection, firms should give much attention to environmental issues for sustainability of their supply chains when selecting suppliers.

The $(D - R)$ values of the criteria 'stakeholder relations' (C_8) and 'e-commerce capability' (C_{15}) are almost same but the importance degree of $(D + R)$ is relatively high for C_{15} than (C_8). Hence the criterion 'e-commerce capability' is more important than the criterion 'stakeholder relation'. A strong relationship between supplier and buyer can improve communication, responsiveness and overall value of the supply chain. So stakeholder relation plays an important role in supplier selection. On the other hand, e-business strategy appears to be a critical component in today's supply chain. Technology is being used to enhance communication and move an organization and its entire supplier network toward paperless transactions. This can help improve efficiency in data transformation and information flow without unnecessary costs. E-commerce strategy can bring flexibility, effectiveness and efficiency in conducting business activities, and hence the supplier's e-commerce capability emerges as a key indicator of its quality.

The criterion 'pollution depletion' (C_3) has positive but relatively low value in comparison to the other criteria in the cause group's other criteria stated above. It's importance degree of $(D + R)$ is 9.966 which is as high as the whole system. So, there is no doubt that the criterion 'pollution depletion' will enhance the effective criteria of the supplier in any sustainable supply chain system. Hence it can be suggested that 'pollution depletion' is also an important criterion of the supplier.

The criterion 'employment practices' (C_6) has the lowest value of $(D - R)$ among the cause group criteria and the value of $(D + R)$ is also relatively low. Therefore, the criterion 'employment practices' does not have enough power to improve the system and hence it cannot be an importance criterion of the supplier. For similar reasons, safety and health (C_5) and supplier/ customer collaboration(C_7) cannot be identified as vital criteria for supplier selection in sustainable supply chain management.

6.5.2 Effect criteria analysis

In general, the criteria in the effect group are smoothly impacted by other criteria which make effect criteria unsuitable to a supplier of sustainable supply chain. It is therefore important to discuss effect group criteria more precisely. Out of the fifteen criteria considered in the numerical example, seven criteria, namely $C_1, C_2, C_9, C_{10}, C_{11}$, $C_{12},$ and C_{13} belong to the effective group. The criterion 'quality' (C_9) has the highest $(D + R)$ value 12.433, which indicates that quality is the most important criterion of supplier in the effect group. Quality level of procured items should be an important factor in supplier selection as it can directly affect quality of finished product and customers' satisfaction. Figure 6.2 shows that its $(D-R)$ value is less than zero. However, it's influential impact index and degree of influenced impact are 5.678 and 6.754, respectively which are relatively high in comparison to all other criteria of this group. This suggests that 'quality' has significant impact on the other criteria of supplier.

Supplier's environmental performance assessment is one of the most important items for supplier selection. From Figure 6.2, we observe that the criterion 'green image' (C_2) is an effective criterion with $(D-R)$ value very close to zero and its $(D+R)$ value is 11.65. This suggests that 'green image' is net receiver and it has an impact on the other criteria. For the same reason, the criterion environmental efficiency(C_1) can be considered as an important criterion of the supplier too. Both the criteria (green image and environmental efficiency) are expected to help enterprises to reduce environmental risks and impacts for sustainable supply chain.

Another effect criterion 'logistics cost' (C_{13}) has negative $(D - R)$ value and it's $(D + R)$ value is not enough to label as a criterion of the supplier of sustainable supply chain management. From Figure 6.2, we also see that the effect group criterion 'cost' (C_{11}) has high $(D + R)$ value 11.814 and relatively low $(D - R)$ value. Therefore, it suggests that C_{11} has low impact on the whole system but it is susceptible to other factors. So 'cost' cannot be an important criterion of the supplier. There are some common features of the criteria 'flexibility' (C_{10}) and 'technical capability' (C_{12}) whose $(D + R)$ values are relatively low, and $(D - R)$ values are very low. Therefore, these two criteria cannot have significant impact on the whole system to select the supplier of a sustainable supply chain.

6.5.3 Discussion

Supplier selection is one of the critical decisions for any organization due to its direct impact on profitability and organizational competitive position. For a sustainable supply chain, it is difficult to choose important criteria of supplier. Our numerical study reveals that out of the fifteen assumed criteria of supplier, rejection ratio (C_{14}), green design (C_4), stakeholder relations (C_8), e-commerce capability (C_{15}), pollution depletion (C_3), quality (C_9), green image (C_2), and environmental efficiency (C_1) are more important than others. Among the above eight important criteria, cause criteria group $\{C_{14}, C_4, C_8, C_{15}, C_3\}$ can be improved, while the effective criteria group $\{C_9, C_2, C_1\}$ can impact the whole system significantly. The rejection ratio of delivered items of supplier and supplier's e-commerce capability are emphasized. Besides, green design and stakeholder relation are found to be of great significance. Pollution depletion is also recognised as a reasonable criterion of supplier. Moreover, quality from effect group is observed as a major criterion. The criterion 'green image' is found to be almost equal importance with the criterion 'environmental efficiency'. In summary, to be considered for selection in a sustainable supply chain, a supplier is required to pay attention to quality of delivered items, its e-commerce capability, establishment of strong relationship with the company, environmental benefit performance and environmental specification requirements. The proposed Pythagorean fuzzy DEMATEL method successfully identifies all these important qualities of supplier.

6.6 Conclusion

This study extended the DEMATEL method in Pythagorean fuzzy environment and applied it to solve the supplier selection problem in sustainable supply chain management. The proposed method utilized the concept of Pythagorean fuzzy sets and trapezoidal Pythagorean fuzzy number (TrPFN). It used Pythagorean fuzzy set to effectively deal with uncertainty, and linguistic variables defined by TrPFN to rate the criteria values of supplier. The proposed method provided a cause-effect diagram to divide the criteria of supplier into two groups and then examined each criterion in each group for its impact on the whole sustainable supply chain system.

7

Solving A Multi-Criteria Decision Making Problem with Spherical Neutrosophic Sets

7.1 Introduction

Single valued neutrosophic set (Wang et al., 2010) is a special type of neutrosophic set. In neutrosophic set, the membership function value can be greater than 1. If one element of neutrosophic set is appreciated more then the truth membership value of that particular case can be greater than 1. However, in single-valued neutrosophic set, this does not happen because the membership value of single valued neutosophic set lies in $[0, 1]$ and the sum of membership values lies in $[0, 3]$. In this chapter, we consider spherical neutrosophic set which is an integration of single valued neutrosophic set and Pythagorean fuzzy set. In spherical neutrosophic set, the membership grades are truth membership ($T(x)$), indeterminacy membership ($I(x)$) and falsity membership ($F(x)$), each lies in the standard interval $[0, 1]$ and their square sum i.e. $T^2(x) + I^2(x) + F^2(x)$ is less than or equal to 3. Pythagorean fuzzy set has two membership functions and their square sum is less than 1 while, in single valued neutrosophic set, the sum of membership grades is less than or equal to 3.

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The objectives of our study are as follows:

- To define the spherical neutrosophic number weighted averaging aggregation (SNNWAA) operator to solve MCDM problem.
- To calculate the performance of the alternatives with respect to the criteria using SNNWAA operator.
- To show that the proposed method is advantageous in the sense that the spherical neutrosophic set integrates the theory scientifically, accepts the characteristics of Pythagorean fuzzy set and neutrosophic set by excluding the criticism of neutrosophic set.

The remainder of the chapter is organized as follows. Section 7.2 gives preliminaries of spherical neutrosophic theory. Section 7.3 presents the spherical neutrosophic aggregation operator. Section 7.4 deals with MCDM method with aggregation operator. Section 7.5 provides a numerical example of personnel selection problem for validation of the proposed method. Finally, in section 7.6, the chapter is concluded with some remarks.

7.2 Precursory

In this section we give some basic of spherical neutrosophic sets.

Definition 7.1. (Yager, 2013) Let X be a universe of discourse. Then a Pythagorean fuzzy set (PFS) defined on X is of the form

$$P = \{< x, \mu_p(x), \nu_p(x) > |x \in X\}$$

where $\mu_p : X \to [0, 1]$ and $\nu_p : X \to [0, 1]$ are membership and non-membership functions, respectively and satisfy the following relation:

$$0 \leq (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1 \quad \forall x \in X$$

Then the degree of indeterminacy membership $\pi_p(x)$ is defined as
$\pi_p(x) = \sqrt{1 - (\mu_p(x))^2 - (\nu_p(x))^2}$.

Definition 7.2. (Smarandache, 1998) Let X be a universe of discourse. Then the neutrosophic set (NS) defined on X is of the form

$$N = \{< x, T(x), I(x), F(x) > | x \in X\}$$

where $T(x) : X \to [0^-, 1^+]$, $I(x) : X \to [0^-, 1^+]$ and $F(x) : X \to [0^-, 1^+]$ are truth membership, indeterminacy membership and falsity membership functions, respectively and

$$0^- \leq T(x) + I(x) + F(x) \leq 3^+$$

Definition 7.3. (Wang et al., 2010) Let X be a universe of discourse. Then a single valued neutrosophic set (SVNS) defined on X is of the form

$$\bar{N} = \{< x, T(x), I(x), F(x) > | x \in X\}$$

where $T(x) : X \to [0, 1]$, $I(x) : X \to [0, 1]$ and $F(x) : X \to [0, 1]$ are truth membership, indeterminacy membership and falsity membership functions, respectively and

$$0 \leq T(x) + I(x) + F(x) \leq 3$$

Definition 7.4. (Kutlu Gündoğdu and Kahraman, 2019) Let X be a universe of discourse. A spherical fuzzy set A is an object having the form

$$A = \{(x, (\mu(x), \nu(x), \pi(x))) : x \in X\}$$

where $\mu(x) : X \to [0, 1], \nu(x) : X \to [0, 1]$ and $\pi(x) : X \to [0, 1]$ and satisfy the following relation:

$$0 \leq (\mu(x))^2 + (\nu(x))^2 + (\pi(x))^2 \leq 1$$

Definition 7.5. (Smarandache, 2019) Let X be a universe of discourse. A spherical neutrosophic set S is an object having the form

$$S = \{< x, s(T(x), I(x), F(x)) >\}$$

where the function $T(x) : X \to [0, 1]$ defines the truth membership, $I(x) : X \to [0, 1]$ defines the indeterminant membership and $F(x) : X \to [0, 1]$ defines the falsity membership functions, and for any $x \in X$, they satisfy the following relation:

$$0 \leq (T(x))^2 + (I(x))^2 + (F(x))^2 \leq 3$$

A comparison between neutrosophic sets and spherical neutrosophic sets is reflected in Fig 7.1

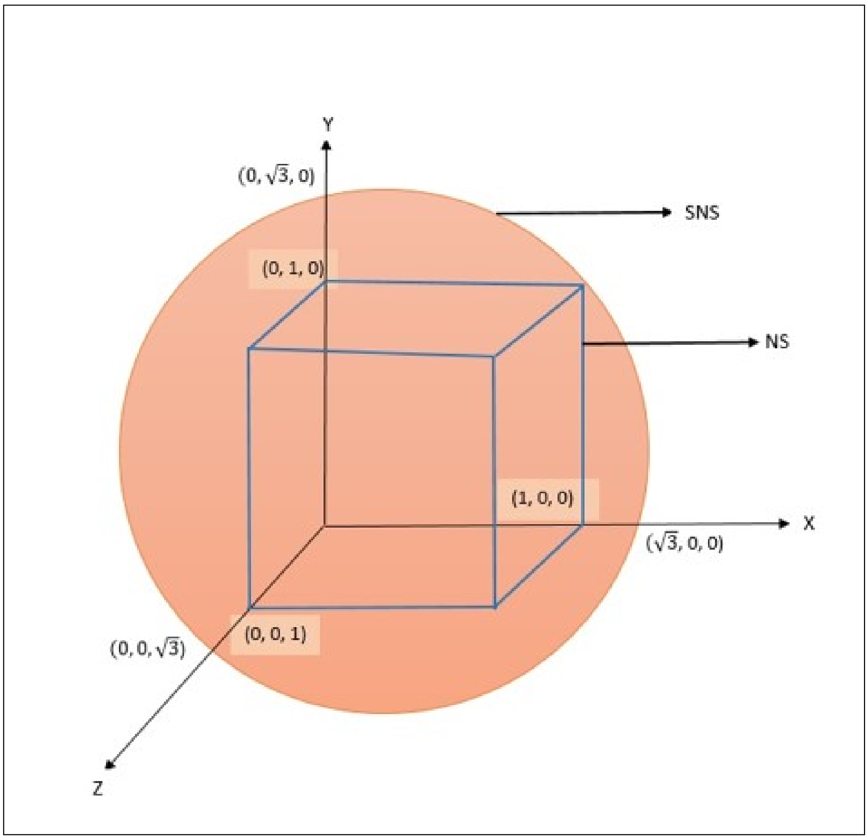

FIGURE 7.1: A comparison between NS and SNS

Definition 7.6. (Smarandache, 2019) Let X be a universe of discourse. Then a spherical neutrosophic number (SNN) is denoted by

$$A = s(T_A, I_A, F_A)$$

where $T_A, I_A, F_A \in [0, 1]$ and $0 \leq {T_A}^2 + {I_A}^2 + {F_A}^2 \leq 3$.

It is to be noted that zero of spherical neutrosophic number O and unity of spherical neutrosophic number U can be defined as follows:

$$O = s(0, 1, 1), U = s(1, 0, 0)$$

Needless to say that spherical neutrosophic number (SNN) is an extension of single valued neutrosophic number (SVN). In SVN, the sum of truth membership, indeterminant membership and falsity membership lies between 0 and 3 and, in SNN, the sum of their squares lies between 0 and 3 i.e., $0 \leq {T_A}^2 + {I_A}^2 + {F_A}^2 \leq 3$.

Definition 7.7. (Smarandache, 2019) Let $A = s(T_A, I_A, F_A)$ and $B = s(T_B, I_B, F_B)$ be two spherical neutrosophic numbers (SNNs). Then the following properties hold:

1. $A \subseteq B$ iff $T_A \leq T_B, I_A \geq I_B$ and $F_A \geq F_B$.
2. Two spherical neutrosophic numbers A and B are equal i.e. $A = B \iff A \subseteq B \, and \, B \subseteq A.$
3. $A \cup B = s(max\{T_A, T_B\}, min\{I_A, I_B\}, min\{F_A, F_B\})$
4. $A \cap B = s(min\{T_A, T_B\}, max\{I_A, I_B\}, max\{F_A, F_B\})$
5. $A^c = s(F_A, \sqrt{1 - {I_A}^2}, T_A)$

Definition 7.8. (Smarandache, 2019) Let $A = s(T_A, I_A, F_A)$ and $B = s(T_B, I_B, F_B)$ be two spherical neutrosophic numbers (SNNs). Then the Hamming distance between two numbers is defined as follows:

$$d(A, B) = \frac{1}{6}\big(|T_A - T_B| + |I_A - I_B| + |F_A - F_B|\big)$$

7.2.1 Some basic operations of SNNS

Let $A = s(T_A, I_A, F_A)$ and $B = s(T_B, I_B, F_B)$ be two spherical neutrosophic numbers (SNNs) and λ be a real number. Then we can define

1. $A + B = s(\sqrt{{T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2}, I_A I_B, F_A F_B)$
2. $A \otimes B = s(T_A T_B, \sqrt{{I_A}^2 + {I_B}^2 - {I_A}^2{I_B}^2}, \sqrt{{F_A}^2 + {F_B}^2 - {F_A}^2{F_B}^2})$
3. $\lambda A = s(\sqrt{1 - (1 - {T_A}^2)^\lambda}, (I_A)^\lambda, (F_A)^\lambda)$
4. $A^\lambda = s((T_A)^\lambda, \sqrt{1 - (1 - {I_A}^2)^\lambda}, \sqrt{1 - (1 - {F_A}^2)^\lambda})$

Theorem 7.9. *Let $A = s(T_A, I_A, F_A), B = s(T_B, I_B, F_B)$ and $C = s(T_C, I_C, F_C)$ be any three spherical neutrosophic numbers (SNNs). Then the followings properties hold:*
(i) $A + B = B + A$
(ii) $A \otimes B = B \otimes A$
(iii) $(A + B) + C = A + (B + C)$
(iv) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
(v) $\lambda(A + B) = \lambda A + \lambda B,$ *where* $\lambda \in \mathbb{R}$ *and* $\lambda > 0.$

(vi) $(A \otimes B)^{\lambda} = A^{\lambda} \otimes B^{\lambda}$

(vii) $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$ *where* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $\lambda_1, \lambda_2 > 0$

(viii) $A^{\lambda_1} \otimes A^{\lambda_2} = A^{\lambda_1 + \lambda_2}$

Proof: (i)

$$\begin{aligned}
L.H.S &= A + B \\
&= s(\sqrt{{T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2}, I_A I_B, F_A F_B) \\
&= s(\sqrt{{T_B}^2 + {T_A}^2 - {T_B}^2{T_A}^2}, I_B I_A, F_B F_A) \\
&= B + A \\
&= R.H.S
\end{aligned}$$

(ii)

$$\begin{aligned}
L.H.S &= A \otimes B \\
&= s(T_A T_B, \sqrt{{I_A}^2 + {I_B}^2 - {I_A}^2{I_B}^2}, \sqrt{{F_A}^2 + {F_B}^2 - {F_A}^2{F_B}^2}) \\
&= s(T_B T_A, \sqrt{{I_B}^2 + {I_A}^2 - {I_B}^2{I_A}^2}, \sqrt{{F_A}^2 + {F_B}^2 - {F_A}^2{F_B}^2}) \\
&= B \otimes A \\
&= R.H.S
\end{aligned}$$

(iii)

$$\begin{aligned}
L.H.S &= (A + B) + C \\
&= s(\sqrt{{T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2}, I_A I_B, F_A F_B) + s(T_C, I_C, F_C) \\
&= s(\sqrt{({T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2) + {T_C}^2 - ({T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2){T_C}^2}, \\
&\quad I_A I_B I_C, F_A F_B F_C) \\
&= s(\sqrt{{T_A}^2 + {T_B}^2 + {T_C}^2 - {T_A}^2{T_B}^2 - {T_A}^2{T_C}^2 - {T_B}^2{T_C}^2 + {T_A}^2{T_B}^2{T_C}^2}, \\
&\quad I_A I_B I_C, F_A F_B F_C) \\
R.H.S &= A + (B + C) \\
&= s(T_A, I_A, F_A) + s(\sqrt{{T_B}^2 + {T_C}^2 - {T_B}^2{T_C}^2}, I_B I_C, F_B F_C) \\
&= s(\sqrt{({T_A}^2 + {T_B}^2 + {T_C}^2 - {T_B}^2{T_C}^2) - {T_A}^2({T_B}^2 + {T_C}^2 - {T_B}^2{T_C}^2)}, \\
&\quad I_A I_B I_C, F_A F_B F_C) \\
&= s(\sqrt{{T_A}^2 + {T_B}^2 + {T_C}^2 - {T_A}^2{T_B}^2 - {T_A}^2{T_C}^2 - {T_B}^2{T_C}^2 + {T_A}^2{T_B}^2{T_C}^2}, \\
&\quad I_A I_B I_C, F_A F_B F_C) \\
\text{Therefore, } L.H.S &= R.H.S
\end{aligned}$$

(iv)

$$\begin{aligned}
L.H.S &= (A \otimes B) \otimes C \\
&= s(T_AT_B, \sqrt{{I_A}^2 + {I_B}^2 - {I_A}^2{I_B}^2}, \sqrt{{F_A}^2 + {F_B}^2 - {F_A}^2{F_B}^2}) \otimes s(T_C, I_C, F_C) \\
&= s(T_AT_BT_C, \sqrt{({I_A}^2 + {I_B}^2 - {I_A}^2{I_B}^2) + {I_C}^2 - ({I_A}^2 + {I_B}^2 - {I_A}^2{I_B}^2){I_C}^2}, \\
&\quad \sqrt{({F_A}^2 + {F_B}^2 - {F_A}^2{F_B}^2) + {F_C}^2 - ({F_A}^2 + {F_B}^2 - {F_A}^2{F_B}^2){F_C}^2}) \\
&= s(T_AT_BT_C, \sqrt{{I_A}^2 + {I_B}^2 + {I_C}^2 - {I_A}^2{I_B}^2 - {I_A}^2{I_C}^2 - {I_B}^2{I_C}^2 + {I_A}^2{I_B}^2{I_C}^2}, \\
&\quad \sqrt{{F_A}^2 + {F_B}^2 + {F_C}^2 - {F_A}^2{F_B}^2 - {F_A}^2{F_C}^2 - {F_B}^2{F_C}^2 + {F_A}^2{F_B}^2{F_C}^2}) \\
R.H.S &= A \otimes (B \otimes C) \\
&= s(T_A, I_A, F_A) \otimes s(T_BT_C, \sqrt{{I_B}^2 + {I_C}^2 - {I_B}^2{I_C}^2}, \sqrt{{F_B}^2 + {F_C}^2 - {F_B}^2{F_C}^2}) \\
&= s(T_AT_BT_C, \sqrt{{I_A}^2 + ({I_B}^2 + {I_C}^2 - {I_B}^2{I_C}^2) - {I_A}^2({I_B}^2 + {I_C}^2 - {I_B}^2{I_C}^2)}, \\
&\quad \sqrt{{F_A}^2 + ({F_B}^2 + {F_C}^2 - {F_B}^2{F_C}^2) - {F_A}^2({F_B}^2 + {F_C}^2 - {F_B}^2{F_C}^2)}) \\
&= s(T_AT_BT_C, \sqrt{{I_A}^2 + {I_B}^2 + {I_C}^2 - {I_A}^2{I_B}^2 - {I_A}^2{I_C}^2 - {I_B}^2{I_C}^2 + {I_A}^2{I_B}^2{I_C}^2}, \\
&\quad \sqrt{{F_A}^2 + {F_B}^2 + {F_C}^2 - {F_A}^2{F_B}^2 - {F_A}^2{F_C}^2 - {F_B}^2{F_C}^2 + {F_A}^2{F_B}^2{F_C}^2})
\end{aligned}$$

Therefore, L.H.S = R.H.S

(v) Let us consider a positive real number λ. Then

$$\begin{aligned}
L.H.S &= \lambda(A + B) \\
&= \lambda\, s(\sqrt{{T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2}, I_AI_B, F_AF_B) \\
&= s(\sqrt{1 - (1 - ({T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2))^\lambda}, {I_A}^\lambda{I_B}^\lambda, {F_A}^\lambda{F_B}^\lambda) \\
R.H.S &= \lambda A + \lambda B \\
&= \lambda s(T_A, I_A, F_A) + \lambda s(T_B, I_B, F_B) \\
&= s(\sqrt{1 - (1 - {T_A}^2)^\lambda}, {I_A}^\lambda, {F_A}^\lambda) + s(\sqrt{1 - (1 - {T_B}^2)^\lambda}, {I_B}^\lambda, {F_B}^\lambda) \\
&= s(\sqrt{1 - (1 - {T_A}^2)^\lambda + 1 - (1 - {T_B}^2)^\lambda - (1 - (1 - {T_A}^2)^\lambda)(1 - (1 - {T_B}^2)^\lambda)}, \\
&\quad {I_A}^\lambda{I_B}^\lambda, {F_A}^\lambda{F_B}^\lambda) \\
&= s(\sqrt{1 - (1 - ({T_A}^2 + {T_B}^2 - {T_A}^2{T_B}^2))^\lambda}, {I_A}^\lambda{I_B}^\lambda, {F_A}^\lambda{F_B}^\lambda) \\
\text{Therefore, } L.H.S &= R.H.S
\end{aligned}$$

(vi) Let us consider a positive real number λ. Then

$$
\begin{aligned}
L.H.S &= (A \otimes B)^{\lambda} \\
&= s((T_A T_B)^{\lambda}, \sqrt{1-(1-({I_A}^2+{I_B}^2-I_A I_A)^{\lambda})}, \\
&\quad \sqrt{1-(1-({F_A}^2+{F_B}^2-F_A F_A)^{\lambda})}) \\
&\quad R.H.S = A^{\lambda} \otimes B^{\lambda} \\
&= s((T_A)^{\lambda}, \sqrt{1-(1-{I_A}^2)^{\lambda}}, \sqrt{1-(1-{F_A}^2)^{\lambda}}) \otimes s((T_B)^{\lambda}, \\
&\quad \sqrt{1-(1-{I_B}^2)^{\lambda}}, \sqrt{1-(1-{F_B}^2)^{\lambda}}) \\
&= s((T_A T_B)^{\lambda}, \sqrt{1-(1-({I_A}^2+{I_B}^2-I_A I_A)^{\lambda})}, \\
&\quad \sqrt{1-(1-({F_A}^2+{F_B}^2-F_A F_A)^{\lambda})}) \\
\text{Therefore, } L.H.S &= R.H.S
\end{aligned}
$$

(vii) Let λ_1 and λ_2 be two positive real numbers and A be a SNN. Then

$$
\begin{aligned}
L.H.S &= (\lambda_1+\lambda_2)A \\
&= s(\sqrt{1-(1-{T_A}^2)^{\lambda_1+\lambda_2}}, {I_A}^{\lambda_1+\lambda_2}, {F_A}^{\lambda_1+\lambda_2}) \\
R.H.S &= \lambda_1 A + \lambda_2 A \\
&= s(\sqrt{1-(1-{T_A}^2)^{\lambda}_1}, {I_A}^{\lambda}_1, {F_A}^{\lambda}_1) + s(\sqrt{1-(1-{T_A}^2)^{\lambda}_2}, {I_A}^{\lambda}_2, {F_A}^{\lambda}_2) \\
&= s(\sqrt{1-(1-{T_A}^2)^{\lambda}_1+1-(1-{T_A}^2)^{\lambda}_2-(1-(1-{T_A}^2)^{\lambda}_1)(1-(1-{T_A}^2)^{\lambda}_2)}, \\
&\quad {I_A}^{\lambda_1+\lambda_2}, {F_A}^{\lambda_1+\lambda_2}) \\
&= s(\sqrt{1-(1-{T_A}^2)^{\lambda_1+\lambda_2}}, {I_A}^{\lambda_1+\lambda_2}, {F_A}^{\lambda_1+\lambda_2}) \\
\text{Therefore, } L.H.S &= R.H.S
\end{aligned}
$$

(viii) Let λ_1 and λ_2 be two positive real numbers and A be a SNN. Then

$$
\begin{aligned}
L.H.S &= A^{\lambda_1} \otimes A^{\lambda_2} \\
&= s((T_A)^{\lambda_1}, \sqrt{1-(1-{I_A}^2)^{\lambda_1}}, \sqrt{1-(1-{F_A}^2)^{\lambda_1}}) \otimes s((T_A)^{\lambda_2}, \\
&\quad \sqrt{1-(1-{I_A}^2)^{\lambda_2}}, \sqrt{1-(1-{F_A}^2)^{\lambda_2}}) \\
&= s((T_A)^{\lambda_1+\lambda_2}, \sqrt{1-(1-{I_A}^2)^{\lambda_1+\lambda_2}}, \sqrt{1-(1-{F_A}^2)^{\lambda_1+\lambda_2}}) \\
&= A^{\lambda_1+\lambda_2} \\
&= R.H.S
\end{aligned}
$$

Hence the theorem is proved.

7.2.2 Score, accuracy and certainty functions of SNN

In this sub-section, we define the score function, accuracy function and certainty function of SNN, and provide a comparison rule of SNN.

Definition 7.10. Let $A_k = s(T_{A_k}, F_{A_k}, I_{A_k})$ be a spherical neutrosophic number. Then the score function, accuracy function and certainty function of A_k can be defined as follows:
Score function, $sc(A_k) = \frac{1}{3}\sqrt{{T_{A_k}}^2 + 1 - {I_{A_k}}^2 + 1 - {F_{A_k}}^2}$
Accuracy function, $ac(A_k) = T_{A_k} - F_{A_k}$
Certainty function, $cr(A_k) = T_{A_k}$.

Let $A_1 = s(T_{A_1}, I_{A_1}, F_{A_1})$ and $A_2 = s(T_{A_2}, I_{A_2}, F_{A_2})$ be two spherical neutrosophic numbers. Then we have the following results:

(i) If $sc(A_1) > sc(A_2)$ then $A_1 > A_2$.

(ii) If $sc(A_1) = sc(A_2)$ and $ac(A_1) > ac(A_2)$ then $A_1 > A_2$.

(iii) If $sc(A_1) = sc(A_2)$, $ac(A_1) = ac(A_2)$ and $cr(A_1) > cr(A_2)$ then $A_1 > A_2$.

(iv) If $sc(A_1) = sc(A_2)$, $ac(A_1) = ac(A_2)$ and $cr(A_1) = cr(A_2)$ then $A_1 = A_2$.

7.3 Spherical neutrosophic aggregation operator

In this section, we discuss spherical neutrosophic number weighted averaging aggregation (SNNWAA) operator of spherical neutrosophic number.

Definition 7.11. Let $A_1, A_2, ..., A_n$ be n numbers of spherical neutrosophic number(SNN) and each A_i be of the from $A_i = s(T_{A_i}, I_{A_i}, F_{A_i})$. Then $SNNWAA : (SNN)^n \rightarrow SNN$ is defined by

$$SNNWAA(A_1, A_2, ..., A_n) = \sum_{i=1}^{n} w_i A_i,$$

where $w_1, w_2, ..., w_n$ are the weights of corresponding SNNs of $A_1, A_2, ..., A_n$ and each $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

Theorem 7.12. *Let* $A_i = s(T_{A_i}, I_{A_i}, F_{A_i})$, $i = 1, 2, ..., n$ *be the collection of SNNs. Then*

$$SNNWAA(A_1, A_2, ..., A_n) = s\Big(\sqrt{1 - \prod_{i=1}^{n}(1 - T_{A_i}^2)^{w_i}}, \prod_{i=1}^{n}(I_{A_i})^{w_i}, \prod_{i=1}^{n}(F_{A_i})^{w_i}\Big)$$

where $\{w_1, w_2, ..., w_n\}$ *is the weight vector of* $A_1, A_2, ..., A_n$ *and* $w_i \geq 0$ *and* $\sum_{i=1}^{n} w_i = 1$.

Proof. We prove this theorem by the principle of mathematical induction. For $n = 2$, we have

$$\begin{aligned} w_1 A_1 &= s(\sqrt{1 - (1 - {T_{A_1}}^2)^{w_1}}, (I_{A_1})^{w_1}, (F_{A_1})^{w_1}) \\ w_2 A_2 &= s(\sqrt{1 - (1 - {T_{A_2}}^2)^{w_2}}, (I_{A_2})^{w_1}, (F_{A_2})^{w_1}) \end{aligned}$$

Then

$$\begin{aligned} &SNNWAA(A_1, A_2) = w_1A_1 + w_2A_2 \\ &= s(\sqrt{1 - (1 - {T_{A_1}}^2)^{w_1}}, (I_{A_1})^{w_1}, (F_{A_1})^{w_1}) + s(\sqrt{1 - (1 - {T_{A_2}}^2)^{w_2}}, (I_{A_2})^{w_1}, (F_{A_2})^{w_1}) \\ &= s(\sqrt{(1 - (1 - {T_{A_1}}^2)^{w_1}) + (1 - (1 - {T_{A_2}}^2)^{w_2}) - (1 - (1 - {T_{A_1}}^2)^{w_1})(1 - (1 - {T_{A_2}}^2)^{w_2})}, \\ &\quad (I_{A_1})^{w_1} \cdot (I_{A_2})^{w_1}, (F_{A_1})^{w_1} \cdot (F_{A_2})^{w_1}) \\ &= s\Big(\sqrt{1 - \prod_{i=1}^{2}(1 - T_{A_i}^2)^{w_i}}, \prod_{i=1}^{2}(I_{A_i})^{w_i}, \prod_{i=1}^{2}(F_{A_i})^{w_i}\Big) \end{aligned}$$

which implies,

$$SNNWAA(A_1, A_2) = s\Big(\sqrt{1 - \prod_{i=1}^{2}(1 - T_{A_i}^2)^{w_i}}, \prod_{i=1}^{2}(I_{A_i})^{w_i}, \prod_{i=1}^{2}(F_{A_i})^{w_i}\Big)$$

Therefore, the theorem is true for $n = 2$.

Let us assume that the theorem is true for $n = m$. Then we have

$$SNNWAA(A_1, A_2, ..., A_m) = s\Big(\sqrt{1 - \prod_{i=1}^{m}(1 - T_{A_i}^2)^{w_i}}, \prod_{i=1}^{m}(I_{A_i})^{w_i}, \prod_{i=1}^{m}(F_{A_i})^{w_i}\Big)$$

Now, we will show that this result is true for $n = m + 1$. We have

$$
\begin{aligned}
& SNNWAA(A_1, A_2, ..., A_m, A_{m+1}) = \sum_{i=1}^{m} w_i A_i + w_{m+1} A_{m+1} \\
= \; & s(\sqrt{1 - \prod_{i=1}^{m}(1 - T_{A_i}^2)^{w_i}}, \prod_{i=1}^{m}(I_{A_i})^{w_i}, \prod_{i=1}^{m}(F_{A_i})^{w_i}) \\
& + s(\sqrt{1 - (1 - T_{A_{m+1}}^2)^{w_{m+1}}}, (I_{A_{m+1}})^{w_{m+1}}, (F_{A_{m+1}})^{w_{m+1}}) \\
= \; & s(\sqrt{(1 - \prod_{i=1}^{m}(1 - T_{A_i}^2)^{w_i}) + (1 - (1 - T_{A_{m+1}}^2)^{w_{m+1}}) - (1 - \prod_{i=1}^{m}(1 - T_{A_i}^2)^{w_i})(1 - (1 - T_{A_{m+1}}^2)^{w_{m+1}})}, \\
& \prod_{i=1}^{m}(I_{A_i})^{w_i} \cdot (I_{A_{m+1}})^{w_{m+1}}, \prod_{i=1}^{m}(F_{A_i})^{w_i} \cdot (F_{A_{m+1}})^{w_{m+1}}) \\
= \; & s(\sqrt{1 - \prod_{i=1}^{m+1}(1 - T_{A_i}^2)^{w_i}}, \prod_{i=1}^{m+1}(I_{A_i})^{w_i}, \prod_{i=1}^{m+1}(F_{A_i})^{w_i})
\end{aligned}
$$

This shows that the result given in the theorem is true for $n = m+1$. Thus the theorem which is assumed to be true for $n = m$ is found to be true for $n = m + 1$. Hence the theorem is true for any natural number n. □

Properties:

SNNWAA operator satisfies the following properties:

1. Idempotency : Let $A_k = s(T_{A_k}, I_{A_k}, F_{A_k})$ where $k = 1, 2, ..., n$ be n numbers of SNNs. If all A_k's are identical i.e. $A_k = A = s(T_A, I_A, F_A)$ for $k = 1, 2, ..., n$ then

$$SNNWAA(A_1, A_2, ..., A_k) = A$$

2. Boundedness : Let $A_k = s(T_{A_k}, I_{A_k}, F_{A_k})$ be any collection of SNNs assuming that $A_k^+ = s(\max_k T_{A_k}, \min_k I_{A_k}, \min_k F_{A_k})$, $A_-^+ = s(\min_k T_{A_k}, \max_k I_{A_k}, \max_k F_{A_k})$. Then,

$$A_k^- \subseteq SNNWAA(A_1, A_2, ..., A_k) \subseteq A_k^+$$

3. Monotonicity : Let $\bar{A}_k = s(T_{\bar{A}_k}, I_{\bar{A}_k}, F_{\bar{A}_k})$ be any collection of SNNs. If it satisfies that $\bar{A}_k \subseteq A_K$ for all $k \in N$ then

$$SNNWAA(\bar{A}_1, \bar{A}_2, ..., \bar{A}_n) \leq SNNWAA(A_1, A_2, ..., A_n)$$

7.4 Multi-criteria decision making using SNNWAA operator

Suppose that a MCDM problem contains uncertainty, inconsistent and incomplete information. To handle the problem, we propose a MCDM method with spherical neutrosophic aggregation operator. Let $A = \{X_1, X_2, ..., X_m\}$ be a finite set of alternatives and $C = \{C_1, C_2, ..., C_n\}$ be the set of criteria. We assume that the criterion value of the corresponding alternative is a spherical neutrosophic number. Let $w = \{w_1, w_2, ..., w_n\}$ be the weight vector and $\sum_{j=1}^{n} w_j = 1$. Then we can get spherical neutrosophic decision matrix $D = (\alpha_{ij})_{m\times n}$ of the from

$$D = (\alpha_{ij})_{m\times n} = \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix} \end{array} \tag{7.1}$$

where $\alpha_{ij} = s(T_{ij}, I_{ij}, F_{ij})$ is a spherical neutrosophic number. Now, we solve the MCDM problem following the steps given below.

Step 1: In MCDM problem, there are two types of criterion $-$ one is cost type criterion and other one is benefit type criterion. Therefore, decision matrix $D = (\alpha_{ij})_{m\times n}$ needs to be normalized and it can be converted into the normalized spherical neutrosophic decision matrix $R = (\tilde{\alpha_{ij}})_{m\times n}$ where

$$\tilde{\alpha}_{ij} = \begin{cases} s\Big(\dfrac{T_{ij}}{P_i}, \dfrac{I_{ij}}{P_i}, \dfrac{F_{ij}}{P_i}\Big), & \text{for benefit type criterion.} \\ \\ s\Big(\dfrac{P_i}{T_{ij}}, \dfrac{P_i}{I_{ij}}, \dfrac{P_i}{F_{ij}}\Big), & \text{for cost type criterion} \end{cases} \tag{7.2}$$

where $P_i = \underset{ij}{max}\{T_{ij}, I_{ij}, F_{ij}\}$ for $j = 1, 2, ..., n$.

Step 2: We calculate the aggregated value of SNNs and use SNNWAA operator discussed in Theorem 7.12 .

Step 3: After calculating aggregation value, we determine the values of score, accuracy and certainty functions to compare the alternatives.

Step 4: Finally, we get the ranking order of the alternatives in ascending order and choose the best alternative.

7.5 Numerical example

In this section, we provide a numerical example to illustrate our proposed approach. Let us consider a personnel selection problem in which selection to be made so that the selected person's skills mostly match for a particular position. This problem can be solved as a MCDM problem. Suppose that candidates are evaluated by alternatives and every alternative has several criteria like communication skill, working experience, general aptitude, etc. In indeterminate and uncertain environments, the selection of proper employees for the particular positions becomes difficult because the selection process involves subjective assessment. Fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets are some excellent tools for dealing with these environments. Moreover, we see that the decision maker faces many difficulties to select the criteria value. So, it would be better to introduce spherical neutrosophic sets and aggregation operator to determine the optimal solution.

Suppose that a company has to select an employee for a particular position. Four candidates $\{X_1, X_2, X_3, X_4\}$ enter the final round of interview after preliminary elimination process. The expert interviews and evaluates the candidates under the following criteria:

1. Communication skill (C_1)
2. Working experience (C_2)
3. General aptitude (C_3)

The weight vector of criteria $w = \{0.35, 0.25, 0.40\}$ is known. Now, the spherical neutrosophic decision matrix is given by

	C_1(Communication skill)	C_2(Working experience)	C_3(General aptitude)
X_1	$s(0.9, 0.3, 0.2)$	$s(1.0, 0.7, 0.5)$	$s(0.8, 0.3, 0.4)$
X_2	$s(0.4, 0.8, 0.9)$	$s(0.7, 0.8, 0.1)$	$s(0.9, 1.0, 0.3)$
X_3	$s(0.7, 0.3, 0.5)$	$s(0.7, 0.1, 0.5)$	$s(0.9, 0.5, 0.2)$
X_4	$s(0.3, 0.7, 0.9)$	$s(1.0, 0.3, 0.4)$	$s(0.7, 0.3, 0.5)$

The steps for finding the solution by SNNWAA operator are as follows:

Step 1: In this problem, all the criteria are of benefit type. Then, using Eq. (7.2), we get the following normalized decision matrix:

	C_1	C_2	C_3
X_1	$s(0.9, 0.3, 0.2)$	$s(1.0, 0.7, 0.5)$	$s(0.8, 0.3, 0.4)$
X_2	$s(0.4, 0.8, 0.9)$	$s(0.7, 0.8, 0.1)$	$s(0.9, 1.0, 0.3)$
X_3	$s(0.7, 0.3, 0.5)$	$s(0.7, 0.1, 0.5)$	$s(0.9, 0.5, 0.2)$
X_4	$s(0.3, 0.7, 0.9)$	$s(1.0, 0.3, 0.4)$	$s(0.7, 0.3, 0.5)$

Step 2: We aggregate the criteria values of the alternatives by SNNWAA operator:

$$SNNWAA(A_1, A_2, ..., A_n) = s\Big(\sqrt{1 - \prod_{i=1}^{n}(1 - T_{A_i}^2)^{w_i}}, \prod_{i=1}^{n}(I_{A_i})^{w_i}, \prod_{i=1}^{n}(F_{A_i})^{w_i}\Big)$$

Then we get

$$\begin{aligned} SNNWAA(X_1) &= s(1.0, 0.37, 0.33) \\ SNNWAA(X_2) &= s(0.76, 0.87, 0.33) \\ SNNWAA(X_3) &= s(0.81, 0.28, 0.34) \\ SNNWAA(X_4) &= s(1.0, 0.40, 0.58) \end{aligned}$$

Step 3: We calculate in Table 7.1 the score, accuracy and certainty values of the alternatives, which are discussed in subsection 7.2.2.

TABLE 7.1: Score, accuracy and certainty values

Score value	Accuracy value	Certainty value
$sc(X_1) = 0.92$	0.63	1.0
$sc(X_2) = 0.57$	-0.11	0.76
$sc(X_3) = 0.82$	0.53	0.81
$sc(X_4) = 0.83$	0.60	1.0

Step 4: Since $sc(X_1) > sc(X_4) > sc(X_3) > sc(X_2)$, the ranking order of the four candidates is obtained as

$$X_1 \succ X_4 \succ X_3 \succ X_2$$

Therefore, the best candidate is X_1.

7.5.1 Discussion

The above numerical study considers three criteria viz. communication skill, working experience and general aptitude for selecting the appropriate candidate for the position. More importance is given to general aptitude and then communication skill and lastly working experience. Aggregation operator is chosen as it is a useful tool in order to summarize different criteria from different sources. Here, the SNNWAA operator aggregates all criteria values by different alternatives and selects the best alternative. X_1 is found to be the best candidate for that particular position because SNNWAA value of X_1 is $s(1.0, 0.37, 0.33)$ and the score value of X_1 is greater than other candidates' score values. Based on the score values of the candidates, the rank of the alternatives is obtained as

$$X_1 \succ X_4 \succ X_3 \succ X_2$$

7.6 Conclusion

Spherical neutrosophic set (SNS) is a generalized version of FS, IFS, NS and PFS applied in real life decision making problems. It is a suitable tool to handle uncertain, incomplete and indeterminant information in MCDM. In this study, we have proposed SNNWAA operator for aggregating spherical neutrosophic number as criteria value of the alternatives and developed a MCDM method based on SNNWAA operator. Finally, we have solved a numerical example of MCDM problem, namely, personnel selection problem to examine the efficiency of the proposed method. There are many avenues for further research on our work. Alternative score, accuracy function and aggregation operator can be developed in SNS. Other MCDM methods such as TOPSIS, VIKOR, PROMETHEE, and AHP can be extended in SNS environment.

8 Conclusions

This chapter presents a summary of the most important contributions of the thesis together with suggestions for future scope of work. The proposed research work has concentrated on developing some methods for solving MCDM problems in uncertain environment.

8.1 Major contributions

- In **Chapter 2**, we have solved GRA method for MCDM problem where the rating values of the attributes are SVTrNNs and weight information is partially known or completely unknown.
- In **Chapter 3**, we have extended TOPSIS method to solve MADM problem with ITrNNs. We have introduced a new distance measure of ITrNNs. We have developed the model where the rating values of the attributes are ITrNN and weight information is completely known, partially known and completely unknown.
- In **Chapter 4**, we have studied MCDM method with neutrosophic hesitant fuzzy sets. We have formulated SVNHFS and IVNHFS based MCDM problem, where the weight information is incompletely known and completely unknown. TOPSIS method been used to solve the proposed optimization model.
- In **Chapter 5**, we have extended the PROMETHEE method under Pythagorean fuzzy environment and solved a real MCDM problem. We have discussed some

basic operations of Pythagorean fuzzy numbers (PFN) and compared Pythagorean fuzzy sets (PFS) with intuitionistic fuzzy sets (IFS). We have introduced a medical diagnosis problem as an MCDM problem and solved the problem using our proposed method.

- In **Chapter 6**, we have proposed DEMATEL method in Pythagorean fuzzy environment. Here the concept of Pythagorean fuzzy sets and trapezoidal Pythagorean fuzzy number (TrPFN) is utilized. The proposed method has been applied to solve supplier selection problem in sustainable supply chain management.

- In **Chapter 7**, we have introduced spherical neutrosophic set (SNS) as a generalized version of FS, IFS, NS and PFS and applied it in real life decision making problems. We have proposed SNNWAA operator for aggregating spherical neutrosophic number as criteria value of the alternatives and developed a MCDM method based on SNNWAA operator. We have solved numerically a MCDM problem, namely, personnel selection problem to examine the efficiency of the proposed method.

The primary contributions of the study conducted in this thesis are to find solutions of various real-world decision-making problems, such as supplier selection, project selection, medical diagnosis, pattern identification, employee selection, data mining, clustering analysis, etc.

8.2 Future scopes of work

MCDM problems generally take place in a complex environment and usually connected with imprecise data and uncertainty. Therefore, MCDM/MADM with fuzzy sets, intutionistic fuzzy sets, hesitant fuzzy sets, and Pythagorean fuzzy sets, neutrosophic sets, and spherical neutrosophic sets have received much attention to the researchers.

Based on the works carried out in this thesis, some observations for potential future research directions are indicated below.

- GRA method of MCDM can be extended to multi-criteria group decision making problem in SVTrNN environment and also weight vector of criteria can be considered as incompletely known or completely unknown.

- GRA, AHP, VIKOR, PROMETHEE and other MCDM methods can be developed with ITrNNs.
- The developed models based on SVNHFS and IVNHFS can be applied to many real-life decision making problems such as pattern recognition, supply chain management, data mining, etc. The proposed method can be extended in MADM problem with plithogenic set (Smarandache, 2017b).
- TOPSIS, GRA, PROMETHEE, and DEMATEL methods can be developed using the following sets:
  - spherical neutrosophic set (Smarandache, 2019)
  - neutrosophic soft set (Maji, 2013)
  - interval valued neutrosophic hesitant fuzzy set (Liu and Shi, 2015)
  - hesitant Pythagorean fuzzy sets (Liang and Xu, 2017)
  - triangular fuzzy neutrosophic set (Biswas et al., 2016a)
- The proposed DEMATEL method would be useful to solve MCDM problem containing a large number of criteria as well as vague, incomplete and inconsistent information.
- The supplier selection problem of sustainable supply chain management can be solved by other MCDM techniques besides DEMATEL method.
- Medical diagnosis problem can be considered as MCDM problem and solved with uncertain environment.
- Alternative score, accuracy function and aggregation operator can be developed in SNS. Other MCDM methods such as TOPSIS, VIKOR, PROMETHEE, and AHP can be extended in SNS environment.

We hope that the concept presented in this thesis will open up new avenue of research in practical problems involving personal selection in academia, project evaluation, data mining, portfolio selection, risk analysis, and many other areas of management systems.

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- TOPSIS method for MADM based on interval trapezoidal neutrosophic number. **Neutrosophic Sets and Systems**, 22, 151-167 (2018).

- TOPSIS method for neutrosophic hesitant fuzzy multi-attribute decision making. **Informatica**, 31(1), 35-63 (2020).

- Grey relational analysis method for SVTrNN based multi-attribute decision making with partially known or completely unknown weight information. **Granular Computing**, 5(4), 561-570 (2020).

- Extended PROMETHEE method with Pythagorean fuzzy sets for medical diagnosis problems. **Soft Computing**, 25(6), 4503-4512 (2021).

- Pythagorean fuzzy DEMATEL method for supplier selection in sustainable supply chain management. **Expert Systems with Applications**, 116396 (2022) .

- Multi-criteria decision making problem with spherical neutrosophic sets. **OPSEARCH**, 1-18 (2022).

Front Pages of Published Papers

TOPSIS Method for MADM based on Interval Trapezoidal Neutrosophic Number

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Abstract: TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) is a very common method for Multiple Attribute Decision Making (MADM) problem in crisp as well as uncertain environment. The interval trapezoidal neutrosophic number can handle incomplete, indeterminate and inconsistent information which are generally occurred in uncertain environment. In this paper, we propose TOPSIS method for MADM, where the rating values of the attributes are interval trapezoidal neutrosophic numbers and the weight information of the attributes are known or partially known or completely unknown. We develop optimization models to obtain weights of the attributes with the help of maximum deviation strategy for partially known and completely unknown cases. Finally, we provide a numerical example to illustrate the proposed approach and make a comparative analysis.

Keywords: Interval trapezoidal neutrosophic number, Multi-attribute decision making, TOPSIS, Unknown weight information.

1 Introduction

Multi-attribute decision making (MADM) is a popular field of study in decision analysis. MADM refers to making choice of the best alternative from a finite set of decision alternatives in terms of multiple, usually conflicting criteria. The decision maker uses the rating value of the attribute in terms of fuzzy sets [1], intuitionistic fuzzy sets [2], hesitant fuzzy sets [3], and neutrosophic sets [4].

In classical MADM methods, the ratings and weights of the criteria are known precisely. TOPSIS [5] is one of the classical methods among many MADM techniques like Preference Ranking Organization METHod for Enrichment of Evaluations (PROMETHEE) [6], Vlse Kriterijuska OptimizacijaI Komoromisno Resenje (VIKOR) [7], ELimination Et Choix Traduisant la REalit (ELECTRE) [8], Analytic Hierarchy Process (AHP) [9], etc. MADM problem has also been studied in fuzzy environment [10–14] and intuitionistic fuzzy environment [15–18]. Researchers have extended the TOPSIS method to deal with MADM problems in different environment. Chen [19] extended the concept of TOPSIS method to develop a methodology for MADM problem in fuzzy environment. Boran et al. [20] extended the TOPSIS method for MADM in intutionistic fuzzy sets. Zhao [21] proposed TOPSIS method under interval intutionistic fuzzy number. Liu [22] proposed TOPSIS method for MADM under trapezoidal intuitionistic fuzzy environment with partial and unknown attribute weight information.

Compared to fuzzy set and intutionistic fuzzy set, neutrosophic set [4] has the potential to deal with MADM problem because it can effectively handle indeterminate and incomplete information. Hybrids of

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ORIGINAL PAPER

Grey relational analysis method for SVTrNN based multi-attribute decision making with partially known or completely unknown weight information

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Abstract

Single-valued trapezoidal neutrosophic number (SVTrNN), an extension of single-valued neutrosophic set, effectively deals with indeterminate and incomplete information in multi-attribute decision making (MADM) problem. In this paper, we extend the grey relational analysis (GRA) method for solving SVTrNN based MADM problem, where the weight information of attributes is partially known or completely unknown. Following the classical GRA method, we define grey relational co-efficient using a new distance measure. We develop two optimization models to determine the weights of the attributes. We calculate grey positive and negative relational degrees and define the relative closeness co-efficient of each alternative to determine the best alternative. We take a numerical example to validate the proposed approach and compare the proposed method with other exiting methods. It is observed from the numerical study that the proposed GRA method has an advantage over the existing methods for solving SVTrNN based MADM problem with partially known or completely unknown attribute weight information.

Keywords Multi-attribute decision making · Single-valued trapezoidal neutrosophic number · Grey relational analysis · Unknown weight information

1 Introduction

Grey relational analysis (GRA) is an important part of grey system theory, which is used to conduct relational analysis of uncertainty of the system. There are many applications of this method in different multi-attribute decision making (MADM) problems (Zhang et al. 2005; Wei 2011; Wei et al. 2011). However, in practice, decision makers face difficulties to collect accurate information of preference values of alternatives in MADM due to imprecise and incomplete data (Xu 2015).

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During the past several years, fuzzy sets (Zadeh 1965), intuitionistic fuzzy sets (Atanasso 1986), and neutrosophic sets (Smarandache 1999) have gained much attention from the researchers to deal with uncertain information in decision making problems. Fuzzy sets is used in various optimization techniques (Chen and Wang 1995; Chen and Tanuwijaya 2011; Chen and Chang 2011; Cheng et al. 2016; Lee and Chen 2008; Chen and Huang 2003. Intuitionistic fuzzy set is useful to handle various MCDM problems (Chen and Chang 2015; Chen et al. 2016a, b, Liu and Chen 2018a; Liu et al. 2017). Recently, MADM method is being developed under hesitant fuzzy sets and type-2 fuzzy sets (Mishra et al. 2018; Qin 2017). GRA method is one of the accepted MADM methods among TOPSIS (Hwang and Yoon 1981), VIKOR (Opricovic and Tzeng 2004), PROMETHEE (Brans et al. 1986), AHP (Wind and Saaty 1980), etc. Researchers have extended the GRA method for MADM problem in different environments. Wei (2010) introduced GRA method for intuitionistic fuzzy MADM problem with incomplete weight information. Zhang and Liu (2011) proposed GRA method

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TOPSIS Method for Neutrosophic Hesitant Fuzzy Multi-Attribute Decision Making

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Abstract. Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) is a very common and useful method for solving multi-criteria decision making problems in certain and uncertain environments. Single valued neutrosophic hesitant fuzzy set (SVNHFS) and interval neutrosophic hesitant fuzzy set (INHFS) are developed on the integration of neutrosophic set and hesitant fuzzy set. In this paper, we extend TOPSIS method for multi-attribute decision making based on single valued neutrosophic hesitant fuzzy set and interval neutrosophic hesitant fuzzy set. Furthermore, we assume that the attribute weights are known, incompletely known or completely unknown. We establish two optimization models for SVNHFS and INHFS with the help of maximum deviation method. Finally, we provide two numerical examples to validate the proposed approach.

Key words: hesitant fuzzy set, neutrosophic set, single valued neutrosophic hesitant fuzzy set, interval neutrosophic hesitant fuzzy set, multi-attribute decision making, TOPSIS.

1. Introduction

Decision making is a popular field of study in the areas of Operations Research, Management Science, Medical Science, Data Mining, etc. Multi-attribute decision making (MADM) refers to making choice of an alternative from a finite set of alternatives. For solving MADM problem, there exist many well-known methods such as TOPSIS (Hwang and Yoon, 1981), VIKOR (Opricovic and Tzeng, 2004), PROMETHEE (Brans *et al.*, 1986), ELECTRE (Roy, 1990), AHP (Satty, 1980), DEMATEL (Gabus and Fontela, 1972), MULTIMOORA (Brauers and Zavadskas, 2006, 2010), TODIM (Gomes and Lima, 1992a, 1992b), WASPAS (Zavadskas *et al.*, 2014), COPRAS (Zavadskas *et al.*, 1994), EDAS (Keshavarz Ghorabaee *et al.*, 2015), MAMVA (Kanapeckiene *et al.*, 2011), DNMA (Liao and Wu, 2019), etc. Wu and Liao (2019) developed consensus-based probabilistic linguistic gained and lost dominance score method for multi-criteria group decision making problem. Hafezalkotob *et al.* (2019) proposed an overview of MULTIMOORA for multi-criteria decision making for theory, developments, applications, and challenges. Mi *et al.* (2019) surveyed on integrations and applications of the best worst method in decision making. Among those methods, TOPSIS method has gained a lot of attention in the past

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METHODOLOGIES AND APPLICATION

Extended PROMETHEE method with Pythagorean fuzzy sets for medical diagnosis problems

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Abstract

Pythagorean fuzzy sets, a generalization of intuitionistic fuzzy sets, can effectively handle uncertain, incomplete and inconsistent information involved in real-life multi-criteria decision making (MCDM) problems. Preference ranking organization method for enrichment of evaluation (PROMETHEE) is one of the popular methods for solving MCDM problem. In this paper, we extend the PROMETHEE method with Pythagorean fuzzy sets. We illustrate the proposed model with a numerical example and compare the method with some existing Pythagorean fuzzy sets-based methods. We also solve a medical diagnosis problem using the proposed Pythagorean fuzzy PROMETHEE method and highlight some advantages of the proposed method.

Keywords Pythagorean fuzzy sets · Multi-criteria decision making · PROMETHEE · Medical diagnosis

1 Introduction

Multi-criteria decision making (MCDM) which identifies the best alternative from a set of available alternatives depends on various criteria. MCDM problem is very common in operation research, management science, medical diagnosis, data mining, etc. Preference ranking organization method for enrichment of evaluation (PROMETHEE) (Brans et al. 1986) is a popular method among TOPSIS (Hwang and Yoon 1981), VIKOR (Opricovic and Tzeng 2004), AHP (Satty 1980), ELECTRE (Roy 1990) and MULTIMOORA (Brauers and Zavadskas 2010) to solve MCDM problem. PROMETHEE method compares the criteria for each pair of alternatives and preference alternative grade which lies between 0 and 1. PROMETHEE method can be successfully applied in fuzzy environment (Zadeh 1965) to solve MCDM problem. Goumas and Lygerou (2000) extended the PROMETHEE method for decision making in fuzzy environment for optimal ranking of the alternative in energy exploitation project. Chen et al. (2011) proposed fuzzy PROMETHEE method for information system outsourcing. They used fuzzy number as the rating value of the criteria with respect to alternative. Abedi et al. (2012) developed PROMETHEE II method in fuzzy environment for copper exploration. Gul et al. (2018) developed PROMETHEE method based on fuzzy logic and used fuzzy number for MCDM problem. Feng et al. (2020) developed an extension of PROMETHEE method with fuzzy soft sets. However, MCDM process may contain several uncertainties and indeterminate situations. It not only determines the degree for which an alternative satisfies the criteria but also provides a degree for which the alternative dissatisfies the criteria. Atanassov (1986) introduced the concept of intuitionistic fuzzy sets (IFS) which has both membership and non-membership degrees. The sum of membership and non-membership degrees of an IFS lies between 0 and 1. IFS has been successfully applied in MCDM problem (Yager 2010; Xu and Yager 2008; Atanassov et al. 2005). Ali et al. (2019) developed graphical method for ranking intuitionistic fuzzy value with uncertainty index and entropy. Feng et al. (2019a, b) proposed lexicographic orders of intuitionistic fuzzy values and their relationships for decision making problem. Feng et al. (2019a, b) proposed another view on generalized intuitionistic fuzzy soft sets and related multi-attribute decision making methods. Liao and Xu (2014)

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Pythagorean fuzzy DEMATEL method for supplier selection in sustainable supply chain management

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Keywords:
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Supplier selection
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ABSTRACT

The fuzzy DEMATEL (Decision Making Trial and Evaluation Laboratory) method is one of the accepted decision-making methods in uncertain environment. The Pythagorean fuzzy set is a generalized concept of fuzzy set and intuitionistic fuzzy set. In this paper, we develop the Pythagorean fuzzy set based DEMATEL method and apply it to solve the supplier selection problem in sustainable supply chain management. We consider the Pythagorean fuzzy set for handling uncertain and incomplete information while selecting the criteria of a supplier and dealing with ambiguity of human decisions. Considering the independence of the criteria, we put forward the proposed method which gives mutual relationships among the criteria, and identifies cause–effect components of the system. The proposed method is illustrated numerically based on data collected from a group of professional personnel.

1. Introduction

Fuzzy sets (FS) (Zadeh, 1965) and intuitionistic fuzzy sets (IFS) (Atanassov, 1986) are some excellent tools for dealing with uncertainties, including indeterminate and inconsistent information. An extension of intuitionistic fuzzy set is Pythagorean fuzzy set (PFS) (Yager, 2013a, 2013b). In IFS, the membership function μ and non-membership function ν satisfy the condition $0 \leq \mu + \nu \leq 1$ for $\mu \in [0,1]$ and $\nu \in [0,1]$. Note that an element having membership degree $\mu \in [0,1]$ and non-membership degree $\nu \in [0,1]$ does not necessarily belong to IFS. For example, if the membership value and non-membership value of an alternative are 0.8 and 0.3, respectively, then the sum of membership and non-membership values of the alternative is greater than 1, which invalidates the criteria for being an IFS. On the other hand, PFS can easily handle this situation because PFS considers the condition $\mu^2 + \nu^2 \leq 1$ which is clearly satisfied as $0.8^2 + 0.3^2 < 1$. This indicates that PFS has an edge over IFS as well as FS in decision-making process under uncertainty.

Yager (2013a) introduced Pythagorean membership grades and solved multi-criteria decision making (MCDM) problem using the aggregation operator. Zhang and Xu (2014) extended the TOPSIS method with PFS and considered Pythagorean fuzzy number (PFN) to solve an MCDM problem. They defined a distance measure of PFN for the TOPSIS method to get the optimal result. Recently, many researchers have developed different MCDM models with PFS. Garg (2016) developed a generalized Pythagorean fuzzy aggregation operator using Einstein operations, and applied that operator to decision making problem. Ren et al. (2016) proposed TODIM method for decision making in Pythagorean fuzzy environment. Perez-Dominguez et al. (2018) introduced MOORA method for decision making under Pythagorean fuzzy set.

Decision-making trial and evaluation laboratory (DEMATEL) (Gabus & Fontela, 1972) is a method which develops mutual relationships of the criteria and their correlated dependencies. This method provides a casual-effect diagram to describe mutual relationships and influences of the criteria (Wu & Tsai, 2011). It can analyze total relations among sets of variables to obtain logical relationships and direct impact relationships. The method is well suited to situations where it becomes necessary to upgrade the evaluation of one criterion by adding new one even if the number of criteria is quite large. It is well known that if the number of evaluation criteria is not restricted, then the decision difficulty increases, and the decision quality is degraded for some decision-making methods such as AHP, TOPSIS, etc. But in the DEMATEL method, such a situation will not occur as it divides the entire criteria, however large it is, into two groups cause and effect, and displays casual relationships between criteria visually. We often need a large number of criteria for supplier selection in sustainable supply chain management. These criteria may contain incomplete and inconsistent information too. Such a problem can therefore be easily handled by the DEMATEL method in Pythagorean fuzzy environment. The DEMATEL method can also be applied to solve various complex problems (Govindan, Khodaverdi et al., 2015; Govindan, Rajendran et al., 2015; Huang et al., 2007; Ren et al., 2013; Shieh et al., 2010).

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APPLICATION ARTICLE

Multi-criteria decision making problem with spherical neutrosophic sets

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Abstract
This paper considers spherical neutrosophic sets (SNS) which can be defined as an integration of Pythagorean fuzzy sets and single valued neutrosophic sets. In single valued neutrosophic sets, the sum of the three membership degrees (truth, indeterminacy and falsity) lies between 0 and 3 whereas in spherical neutrosophic sets, the sum of the squares of three membership degrees lies between 0 and 3. The basic operations of SNS are established and some aggregation operators based on spherical neutrosophic sets are defined. A solution approach for a multi-criteria decision making problem is developed with the help of an aggregation operator of SNS. A numerical example is provided for validation of the proposed approach.

Keywords Spherical neutrosophic sets · Aggregation operator · Multi-criterion decision making

1 Introduction

The concept of fuzzy set introduced by Zadeh [1] in 1965 has been found to be very much applicable in many branches of science and engineering today. Researchers have extended the ordinary fuzzy set to some of those discussed in the following. In 1986, Attansov [2] proposed intuitionistic fuzzy set in which there are membership function and non-membership function, and each membership degree lies between 0 and 1, and the sum of membership and non-membership degrees also lies between 0 and 1. Intuitionistic fuzzy set has also been successfully applied in many branches of science including decision making [3–5]. Later, in 2013, Yarger [6] introduced Pythagorean fuzzy set which is further extension of intuitionistic fuzzy set. The membership degrees of Pythagorean fuzzy set also lie between 0 and 1 but their sum

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