Bachelor of Mechanical Engineering Examination, 2023

Second Year, first Semester

Mathematics - III

Full Marks: 100 Time: 3 Hours

(Symbols/Notations have their usual meanings)

Answer any Ten questions

10 x 10 = 100

1\. A) Determine the subspace of $\mathbb{R}^3$ spanned by vectors

$$\alpha = (1,2,3), \beta = (3,1,0)$$

Examine if,

i) $\gamma = (2,1,3)$ is in the subspace

ii) $\delta = (-1,3,6)$ is in the subspace

B) Prove that the set $S = \{ (2,1,1), (1,2,1), (1,1,2) \}$ is a basis of $\mathbb{R}^3$.

2\. A) A mapping $T: \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3)$$

Find ImT and dimension of ImT.

B) Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to the vectors $(2,0,0), (0,2,0), (0,0,2)$ respectively. Show that $\dim \ker T + \dim Im\, T = 3$.

3\. A) Let V be an inner product space and $u, v \in V; \alpha, \beta \in F$. Then show that,

i) $\langle \alpha u + \beta v,\ \alpha u + \beta v \rangle =$

$$|\alpha|^2 \,||u||^2 + \alpha\bar{\beta}\,\langle u,v\rangle + \bar{\alpha}\beta\,\langle v,u\rangle + |\beta|^2\,||v||^2$$

ii) $||\alpha u|| = |\alpha|\,||u||$

B) State and prove Cauchy-Schwartz inequality.

4\. A) Write down the definition of unitary operators.
Show that T in a unitary operator iff $T^*T = I$.

B) Write down the definition of normal operator.
Let T be a normal operator on an inner product space V. Then,

i) $||T(v)|| = ||T^*(v)||\ \forall\, v \in V$

ii) If v_1 and v_2 are eigen vectors of T with respect to distinct eigen values λ_1 and λ_2 respectively then v_1 and v_2 are orthogonal.

5. A) Let A be a (6 x 6), matrix over $\mathbb{R}$ with characteristic polynomial $= (x-3)^2 (x-4)^4$ and minimal polynomial $(x-3)(x-2)^2$. What will be the possible Jordon Canonical form(s) of A?

B) Let V be the vector space over $\mathbb{C}$ of all polynomials in a variable X of degree at most 3. Let $D: V \to V$ be the linear operator given by the differentiation with respect to X. Let A be the matrix of D with respect to some basis for V. Show that A is a nilpotent matrix.

6. Find rational canonical form of the matrix

$$\begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}$$

7. Suppose V be the subspace of $\mathbb{R}^5$ with basis,

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \ u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}; \ u_3 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \\ -1 \end{bmatrix}; \ u_4 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Apply Gram-Schmidt algorithm to find the orthogonal basis for V.

8. Solve the following differential equations:

a) $\left\{ y\left(1+\frac{1}{x}\right) + \cos y \right\} dx + (x + \ln x - x \sin y)\, dy = 0$

b) $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$

9. Solve the following differential equations:

a) $\frac{dy}{dx} + y \cos x = y^2 \sin 2x$

b) $(D^2 - 2D + 1)y = xe^x$, where $D = \frac{d}{dx}$.

10 a) Solve : $x^2D^2 - 3xD + 5)y = x^2 \sin(\ln x)$, where $D = \frac{d}{dx}$.

b) Solve by the method of variation of parameters, of the following differential equation

$(D^2 - 1)y = \frac{2}{1+e^x}$, where $D = \frac{d}{dx}$.

11. a) Derive a partial differential equation (by eliminating the constants) from the equation

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

b) Solve $\frac{y^2 z}{x} p + xzq = y^2$

12. Prove the following relations

a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

b) $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$.

13. Show that $P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2 - 1)^n$, hence find $P_3(x)$

14. Solve the following differential equation by series solution about the point $x = 0$

$$\frac{d^2y}{dx^2} + xy = 0$$ 10
