Ref. No.: Ex/ME(M2)/BS/B/MATH/T/211/2023(S)

B.E. MECHANICAL ENGINEERING, SECOND YEAR, FIRST SEMESTER, SUPPLEMENTARY EXAM 2023

Mathematics - III

Full Marks: 100 Time: 3 Hours

Answer any 10 questions

10 x 10 = 100

1. A) In $\mathbb{R}^3$, $\alpha = (4,3,5), \beta = (0,1,3), \gamma = (2,1,1), \delta = (4,2,2)$
   Examine if,
   i) α is a linear combination of β and γ
   ii) β is a linear combination of γ and δ

   B) Prove that the set $S = \{ (0,1,1), (1,0,1), (1,1,0) \}$ is a basis of $\mathbb{R}^3$.

2. A) A mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

   $$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$$

   Show that T is a linear transformation. Find Ker$T$ and dimension of $KerT$.

   B) Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to the vectors $(2,0,0), (0,2,0), (0,0,2)$ respectively. Show that $\dim \ker T + \dim Im\, T = 3$.

3. A) Let V be an inner product space and $u, v \in V; \alpha, \beta \in F$. Then show that,
   i) $\langle \alpha u + \beta v,\ \alpha u + \beta v \rangle =$

   $$|\alpha|^2 \, ||u||^2 + \alpha\bar{\beta} \, \langle u,v \rangle + \bar{\alpha}\beta \, \langle v,u \rangle + |\beta|^2 \, ||v||^2$$

   ii) $||\alpha u|| = |\alpha| \, ||u||$

   B) State and prove Cauchy-Schwartz inequality.

4. A) Write down the definition of unitary operators.
   Show that T in a unitary operator iff $\langle Tv,\ Tv \rangle = \langle v,\ v \rangle \ \forall\ v \in V$.

   B) Write down the definition of normal operator.
   Let T be a normal operator on an inner product space V. Then,
   i) $||T(v)|| = ||T^*(v)|| \ \forall\ v \in V$
   ii) If v is any eigen vector of T, then v is an eigen vector of T^* as well. In fact, if $T(v) = \lambda v$, then $T^*(v) = \bar{\lambda} v$

[Turn over

5. A) Let A be a (7 x 7), matrix over $\mathbb{R}$ with characteristic polynomial $= (t-2)^4\ (t-5)^3$ and minimal polynomial $= (t-2)^2(t-5)^3$. What will be the possible Jordon Canonical form(s) of A?

B) Let V be the vector space over $\mathbb{C}$ of all polynomials in a variable X of degree at most 3. Let $D: V \to V$ be the linear operator given by the differentiation with respect to X. Let A be the matrix of D with respect to some basis for V. Show that the Jordon canonical form of A is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6. Find rational canonical form of the matrix

$$\begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}$$

7. Suppose V be the subspace of $\mathbb{R}^5$ with basis,

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix};\ u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix};\ u_3 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \\ -1 \end{bmatrix};\ u_4 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Apply Gram-Schmidt algorithm to find the orthogonal basis for V.

8. Solve the equation:

$$i.\ xdy - ydx = \sqrt{y^2 + x^2}dx \quad ii.\ \frac{dy}{dx} = \sqrt{y - x}$$

9. Find general solution and singular solution:

$$p = \ln(px - y), \quad where \ \ p = \frac{dy}{dx}$$

10. Find the general solution:

$$(D^2 + D - 6)y = x^2, \quad where \ \ D = \frac{d}{dx}$$

11. Solve the Legendre's differential equation.

12. Define ordinary point and regular singular point of the differential equation

$$P_0(x)y_2 + P_1(x)y_1 + P_2(x)y = 0.$$

Find the series solution near the ordinary point x=0 of the equation

$$y_2 + 3xy_1 + 3y = 0$$

13. Solve the equation using the method of separation of variables.

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

14. Solve the equation using method of variation of parameter.

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

15. For Bessel's function $J_n(x)$ show that

$$i.\ \frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$ii.\ J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\ \sin x$$