Ref. No.: Ex/ME(M2)/BS/B/MATH/T/111/2023

Bachelor of Engineering Examination, 2023
Mechanical Engineering
First Year, First Semester
Mathematics-I

Time : Three hours **Full Marks:100**

Use Separate Answer script for each Group.
Notations and symbols have their usual meanings.

GROUP A
Answer any *five* questions from the following.

1. Test the convergence of the following series.

   i) $\Sigma_{n=1}^{\infty}\frac{n!2^n}{n^n}$

   ii) $\Sigma_{n=1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1})$

   iii) $\Sigma_{n=1}^{\infty}\frac{\sqrt{n}}{a.n^{\frac{3}{2}}+b}, a>0.$

3 + 3 + 4

2. (i) Prove that the sequence $\{x_n\}$ where $x_{n+1}=\frac{6(1+x_n)}{7+x_n}$, $x_1=c>0$ is monotone increasing or decreasing according as $c<2$ or $c>2$ and that in either case the sequence converges to 2. Discuss the case when $c=2$.

   (ii) Use Cauchy's criterion to prove that $\{x_n\}$ converges when $x_n=1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}$.

6 + 4

3. (i)Find Lagrange's form of remainder after n terms in the expansion of $log(1+x)$ in Taylor's series, $0<x<1$ and show that this remainder tends to zero as $n\to\infty$.

   (ii)Show that the function $f(x,y)$ defined by

$$f(x,y)=x^2tan^{-1}\frac{y}{x}-y^2tan^{-1}\frac{x}{y},(x,y)\neq(0,0)$$

$$f(0,y)=0=f(x,0)$$

   satisfies $f_{xy}=f_{yx}$ for all points except $(0,0)$.

5 + 5

4. (i) Suppose a function $f(x,y)$ defined by

$$f(x,y)=\frac{x^3y^3}{x^2+y^2},(x,y)\neq(0,0)$$

$$f(x,y)=0,(x,y)=(0,0).$$

   Is $f(x,y)$ continuous at $(0,0)$?

   (ii)State and prove Cauchy's Mean value theorem.

4 + 6

[Turn over

5. i) Using Mean Value Theorem prove that $\frac{x}{1+x^2} < tan^{-1}x < x$ for $0 < x < \frac{\pi}{2}$.

ii) Show that the functions $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$ and $w = \frac{z}{x-y}$ are not independent. Find the relation among them.

5 + 5

6. i)If $y = \frac{sin^{-1}x}{\sqrt{1-x^2}}, | x |< 1$, show that a) $(1-x^2)y_2 - 3xy_1 - y = 0$.
b)$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0$.
(ii) Using Lagrange's method of undetermined multiplier find the extreme value of $x^2 + y^2 + z^2$ subject to the condition $2x + 3y + 5z = 30$.

4 + 6

7. i) If u be a homogeneous function of x and y of degree n having continuous partial derivatives then show that
$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

ii) If $u = t^n e^{-\frac{r^2}{4t}}$, find n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = (\partial u/\partial t)$.

6 + 4

GROUP B

Answer Q.8 and any three from the remaining.

8. Evaluate:

$$\int_0^a x^4\sqrt{a^2 - x^2}\, dx. \qquad (5)$$

9. (a) State the "Fundamental theorem of integral calculus". Discuss the applicability of the fundamental theorem for the function $f(x) = x[x]$, $\forall x \in [0,3]$. (1+6)

(b) Define primitive/anti-derivative of a function.
Let, $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ x & \text{if } x \in [1,2] \end{cases}$. Find the anti-derivative of this function which is defined by $F(x) = \int_0^x f(t)dt$, $x \in [0,2]$. Show that $F'(x) = f(x)$ on $[0,2]$. (1+7)

10. Examine convergence of the following improper integrals: (5+5+5)

$$(a) \int_0^2 \frac{dx}{\sqrt{x(2-x)}}, \quad (b) \int_a^\infty \frac{\sin^2 x}{x^p}dx \text{ for } a > 0, p > 1, \quad (c) \int_0^\infty \left(\frac{1}{1+x} - \frac{1}{e^x}\right)\frac{1}{x}dx$$

11. (a) Define Beta-function. Prove that

$$\int_a^b (x-a)^m(b-x)^n = (b-a)^{m+n+1}B(m+1, n+1) \qquad (1+6)$$

(b) Define Gamma-function. Prove that

$$\int_0^{\frac{\pi}{2}} \sin^p x dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x dx = \frac{\pi}{2(p+1)} \qquad (1+7)$$

12. (a) Find the perimeter of the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad (7)$$

(b) Find the volume of revolutions of the ellipse $\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$ when rotated about x-axis and y-axis. Show that the ratio of volumes is $\frac{b}{a}$. (8)

13. (a) Evaluate

$$\iint_R [2a^2 - 2a(x+y) - (x^2+y^2)]\,dx\,dy,$$

where the region R is bounded by the circle $x^2 + y^2 + 2a(x+y) = 2a^2$. (7)

(b) Evaluate the following integral by changing the order of the integral

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{dy}{(1+e^y)\sqrt{1-x^2-y^2}}. \qquad (8)$$