Bachelors of Computer Science and Engineering 2023

(2nd Year, 2nd Semester)

Mathematics IV

Full Marks: 100

USE SEPARATE ANSWER SCRIPTS FOR GROUP A AND GROUP B

Group A

FULL MARKS: 50

Answer question 1 any SIX from the rest:

1. Prove that a set A is infinite if A has a proper subset B such that $|A| = |B|$. 2

2. Let A, B, C be three subsets of a set X. Show that $A\Delta(B\Delta C) = (A\Delta B)\Delta C$. 8

3. Define an equivalence relation on a nonempty set. Prove that ρ is an equivalence relation on A if and only if the following conditions hold:

   (a) $\triangle_A \subseteq \rho$, where $\triangle_A = \{(a, a) : a \in A\}$;

   (b) $\rho = \rho^{-1}$;

   (c) $\rho \circ \rho \subseteq \rho$. 8

4. Define the cardinal number $|A|$ of a set $|A|$. Let $\mathcal{P}(A)$ be the set of all subsets of A. Prove that $|A| < |\mathcal{P}(A)|$ for any set A. 8

5. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural numbers and real numbers respectively. Let $\aleph_0 = |\mathbb{N}|$ and $c = |\mathbb{R}|$. Prove that $\aleph_0 c = c$. 8

6. Define a countable set. Prove that the set of all algebraic numbers is countable. 8

7. What is an archimedean order property. Show that both the set of rational numbers and the set of real numbers satisfy this property. 8

8. Find the truth table of $(p \to q) \wedge (q \to r)$. 8

9. Let A, B, C be three nonempty sets and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions. Prove the following:

   (a) If f and g are both injective, then $g \circ f$ is injective;

   (b) If $g \circ f$ is injective, then f is injective;

   (c) If f and g are both surjective, then $g \circ f$ is surjective. 8

CSE/BS/B/Math/T/221/ 2023

Group B

Probability and Stochastic Processes

Symbols/notations used have their usual meaning

FULL MARKS: 50

Answer question numbers 1 and any FOUR from the rest:

1. (i)State Chevyshev's inequality.

(ii) A coin is flipped twice. Assuming that all four points in the sample space are equally likely, what is the probability that both flips land on same heads, given that

(a) the first flip lands on heads.

Given that (b) at least one flip lands on heads.

(iii)In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $(1-p)$ be the probability that the student guesses. Assume that a student who guesses at theanswer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he or she answered it correctly? 2+3+5

2. (i) Let X denote the tangent of an angle (measured in radians) chosen at random from $(-\pi/2, \pi/2)$. Find the distributiuon of X.

(ii) Show that the function $| x |$ in $(-1, 1)$ and zero elsewhere is a possible density function and find the corresponding distribution function. 5 + 5

3. (i) Some airlines find that each passenger who reserves a seat fails to turn up with probability 0.1 independently of other passsengers of these airlines. Airline A always sells 10 tickets for their 9 seat aeroplane while airline B always sells 20 tickets for their 18 seat aeroplane. Using Poissson approximtion to binomial distribution find which one of A and B is more overbooked.

(ii) Derive the mean and standard deviation of a Binomial distribution with parameters n and p. 4 + 6

4. (i) The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

compute:

(a) $P(X > 1, Y < 1)$

(b) $P(X < Y)$

(c) $P(X < a)$

(ii) A point P is chosen at random on a line segment AB of length $2a$. Find the probability that area of the rectangle AP, PB will exceed $\frac{1}{2}a^2$ 4+6

5. (i) I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

(a) If the probability of rain is p, what is the probability that I get wet?

(b) Current estimates show that $p = 0.6$ in Edinburgh. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

(ii) State and prove Chainpman-Kolmogorov equation 6+4

6. A two-server queueing system is in a steady-state condition and the steady state probabilities are $p_0 = \frac{1}{16}$, $p_1 = \frac{4}{16}$, $p_2 = \frac{6}{16}$, $p_3 = \frac{4}{16}$, $p_4 = \frac{1}{16}$, $p_n = 0$ if $n > 4$. Calculate

(a) L (the expected number of customers in the system) and L_q (the expected number of customers in the queue).

(b) The expected number of customers being served.

Suppose the arrival rate is 2 customers per hour. Calculate

(c) W (the mean waiting time in the system) and W_q (the mean waiting time in the queue).

(d) $\frac{1}{\mu}$ (the mean service time per customer). 10