b) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, where C is the circle $|z| = 3$. 4+6

6. a) Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.

Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

b) Find the Laplace transform of $\left(1 - e^t\right)/t$. 6+4

Bachelor of Engineering In Chemical Engineering Examination, 2022

(1st Year, 2nd Semester, Supplementary)

Mathematics - II

Time : Three hours Full Marks : 100

(50 Marks for each Part)

(Use separate answer script for each Part)

(Unexplained Symbols / Notations have their usual meanings)

Special credit will be given for precise answer.

PART – I (50 Marks)

Answer any ***Five*** questions. 10×5=50

1. a) Give an example (with proper explanation) of a bounded sequence which is not convergent. 2

b) Check if the sequence $x_n = 1 - \frac{1}{n}$ is monotone increasing or decreasing or oscillatory and hence discuss its convergence. 2

c) Apply sandwich theorem to show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n^2 + k} = 0$$ 3

d) Using Cauchy's General Principles of Convergence show that the sequence $(x_n)_{n\geq 1}$ can not converge, where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. 3

2. a) Determine the limits of the double integral $\iint_R f(x,y)\,dxdy$ where R is the region bounded by $y^2 = x$ and $x^2 = y$ considering

   i) x as the first variable

   ii) y as the first variable. 2+2

   b) Change the order of integration of the integral

   $\int_0^{\frac{12}{7}}\int_0^{x} f(x,y)\,dxdy + \int_{\frac{12}{7}}^{4}\int_0^{\frac{12-3x}{4}} f(x,y)\,dxdy$. 3

   c) Evaluate $\iint_R y\,dxdy$ where R is the region in the xy-plane bounded by the lines $y = x$ and the parabola $y = 4x - x^2$. 3

3. a) Why are the following integrals improper? When do they converge (no proof is required)? What are their names when they converge?

   i) $\int_0^{\infty} e^{-x}x^{n-1}dx$, n is a real number

   ii) $\int_0^{1} x^{m-1}(1-x)^{n-1}dx$, m, n are real numbers.

   2+2

   b) Define geometric series and discuss its convergence.

   3

   b) If $\omega = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function ϕ and the function ω in terms of z. 5+5

3. a) Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$, along (i) the line $y = \frac{x}{2}$, (ii) the real axis to 2 and then vertically to $2+i$.

   b) If $f(z)$ is a regular function of z, prove that

   $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. 6+4

4. State Laurent's theorem.

   Expand $\frac{1}{z^2 - 3z + 2}$ for the regions (i) $0 < |z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$, (iv) $0 < |z-1| < 1$. 2+8

5. a) If $F(\zeta) = \int_C \frac{4z^2 + z + 5}{z - \zeta}dz$, where C is the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, find the value of (i) $F(3{\cdot}5)$, (ii) $F(i)$, (iii) $F'(-1)$ and (iv) $F''(-i)$.

[Turn over

b) Find the Jacobian of the transformation from Cartesian coordinates to spherical coordinates. 3

c) Evaluate: $\iiint_R \sqrt{1-x^2-y^2-z^2}\,dxdydz$, where R is the region given by $x^2+y^2+z^2 \le 1$. 4

PART – II (50 Marks)

Answer any ***Five*** questions.

All questions carry equal marks.

1. State the necessary and sufficient conditions for the derivative of the funtion $f(z)$ to exist for all values of $z(=x+iy)$ in a region *R*.

   Prove that the function $f(z)$ defined by

   $$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}(z\neq 0),\ f(0)=0$$

   is continuous and Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. 10

2. a) State and prove Cauchy's Theorem and extend it to a multiply connected region.

c) If the series $\sum_{n=1}^{\infty} x_n$ converges then prove that $\lim_{n\to\infty} x_n = 0$. Is the converse true? Justify. 2+1

4. a) Compute the triple integral $\iiint_R xyz\ dxdydz$ over the domain bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=1$. 5

   b) What is *p*-series? When does it converge (no proof is required)? 2

   c) Discuss absolute convergence and conditional convergence of a series of real numbers. Give an example (with proper explanation) of a series which is convergent but not absolutely convergent. 2+1

5. a) State Leibniz's rule and Generalized Leibniz's rule (no proof is required) related with differentiation under the sign of integration. 2

   b) Test the convergence of

   (i) $\int_0^1 \frac{dx}{\sqrt{1-x}}$ and (ii) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^3}dx$. 4

   c) Write the relation between Beta function and Gamma function and use it to evaluate $\Gamma\left(\frac{1}{2}\right)$. 4

6. a) Find the radius of convergence (R) of the following power series and check the behaviour at *R*.

   $$\sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

[Turn over