6. a) Compute $\ln(x+i\in)-\ln(x-i\in)$, where $\in$ is an infinitesimal positive real number, and x can be positive or negative.

b) Suppose one wants to integrate a function, $f(x)=\ln\sin z$ over a rectangular contour whose vertices are located at 0, π, $\pi+iR$, iR (in an anti-clockwise manner), with appropriate indentations of the contour. Identify the locations of these identations and evaluate them. Hence show that $\int_0^{\pi} dx \ln\sin x = -\pi\ln 2$.

c) Evaluate the following integral using Cauchy's formula,

$$\int_{-\infty}^{\infty}\frac{dx}{\left(x^2+a^2\right)^2\left(x^2+b^2\right)} \quad \left(\text{Re}(a)>0,\ \text{Re}(b)<0\right)$$

1.5+5+3.5

Ex/SC/PHY/PG/CORE/TH/102/2023

M. Sc. Physics Examination, 2023

(1st Year, 1st Semester)

Paper – PHY/PG/CORE/TH/102

[Mathematical Methods]

Time : Two hours | Full Marks : 40

Use separate script for each Part.

Part – A

Answer ***any two*** questions.

1. a) Define an inner product space. Show that the linearity in ket essentially implies anti-linearity in bra. Consider the linear vector space formed by the 2×2 matrices. Find out one set of basis vectors for this vector space. Can you define an inner product in this vector space? Explain.

b) Starting from the condition of metric compatibility, derive the expression for the Chrystoffel connection, $\Gamma^{\mu}_{\nu\lambda}$. Hence find out the component Γ^{t}_{tx}, of the same for the 2-d metric : $ds^2 = e^{2ax}\left(-dt^2+dx^2\right)$, where a is a constant.

(1.5+1.5+1.5+1.5)+4

2. a) Suppose a group G is given, it has a representation D_1 on linear vector space V_1 and another representation D_2 on V_2. Both are irreducible representations. Assume that V_1 and V_2 are complex

linear vector spaces. Suppose one can construct a linear transformation $T: V_1 \to V_2$ such that the equation $TD_1(g) = D_2(g)T$ is true for all $g \in G$ (i) If $T = 0$ can you draw any conclusion on the relationship between D_1 and D_2? (ii) Consider $T \neq 0$ but T is singular/non-singular. Discuss about the possibility of such situation.

b) i) Show that the set of translation $T = T(a)$ with the parameter of translation $a \in R$ (namely, $-\infty < a < \infty$) satisfy all the properties for defining a group. You may identify $T(a) = I + a\frac{d}{dx}$.

ii) The set of translations is an example of continuous group and is also Abelian. Explain it.

c) Derive the Fourier transform of the Gaussian function : $e^{-\alpha x^2}$ $(\alpha > 0)$. Draw the required contour.

3+4+3

3. a) Find out the following Fourier transform,

$$\int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}.\vec{r}} \ln k$$

where $\vec{k}$ is the 3-momentum and $k = |\vec{k}|$. $\vec{r}$ is the radius vector in 3-dimensions.

b) Discuss how the Poisson equation in electrostatics (in 3-d) can be solved using the Green's function method.

c) Suppose we are interested to compute the Laplace transform of $f(x) = \cosh at$, where a is real and positive. State the condition for this particular transformation to exist.

5+4+1

Part – B

Answer ***any two*** questions.

4. Consider the Legendre differential equation:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0$$

which arises in a wide variety of physical situations. Show that in such cases, physical considerations and mathematical requirements of convergence/finiteness forces λ to be of the form: $\lambda = n(n+1)$, where $n \in \mathbb{Z}$.

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5. Solve the one-dimensional heat/diffusion equation : Find the temperature distribution $\Theta(x,t)$ in an infinite 1-dim solid, given that initially: $\Theta(x,t) = f(x)$, some given function, and $\frac{\partial^2\Theta(x,t)}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\Theta(x,t)}{\partial t}$

How would the distribution look, in the special cases when : (a) $f(x)$ is a point source; and (b) $f(x)$ is a *plane* source?

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