B. Sc. Physics (Hons.) Examination, 2023

(3rd Year, 1st Semester)

Advanced Mathematical Methods

Paper – UG/Sc/DSE/PHY/TH / 01 /A1

Time : Three hours | Full Marks : 75

Use separate Answer Script for each Group

GROUP - A (36 Marks)

Answer any three questions(3x12=36).

1.(a) State the principle of least action. Starting from the Lagrangian $L(q_j, \dot{q}_j, t)$ define the Euler-Lagrangian equation of motion using variational principle.

(b) What are the cyclic co-ordinate ?

(c) Take the Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$, find the conserved quantities of motion. [6+2+4=12]

2.(a) Define the Legendre transformation. Starting from Lagrangian $L(q, \dot{q})$, define the Hamiltonian H of a system. Show that H is a function of q and p only (symbols have usual meaning).

(b) Define the Poission Bracket between two variables f and g. Show that Poission Bracket satisfy the Jacobi identity: $\{f, \{g, h\}\} + \text{cyclic} = 0$

[4+(2+6)= 12]

3.(a) Define a Manifold. A vector in a manifold can be expanded in the tangent basis as $V \equiv V^\mu \partial_\mu$, Show that $\{\partial_\mu\}$ are linearly independent.

(b) Find local maxima and minima of the function $f(x, y) = x^3 - 12xy + 8y^3$.

(c) A geodesic on a given surface is a curve, lying completely on that surface, and along which the distance between two points is the shortest. (For example, on the plane R^2 , a geodesic is the straight line). Now determine the equation of the geodesic on the surface of a ellipsoid, that is for S^2 $:(x, y, z) : | \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. [5+2+5= 12]

4. **(a)** Consider $\mathbb{R}^n$, how many linearly independent m-forms can exist on $\mathbb{R}^n$ $(n > m)$? Define a closed form and an exact form. Hence define the P^{th} cohomology group of $\mathbb{R}^n$.
(b) In high school vector calculus, we learn about the separation of gradient and curl- translate these concepts in the language of differential forms.
(c) The exterior derivative which acts on the space of differential forms as,

$$d(\Omega^n) \to \Omega^{(n+1)}$$

satisfies the fundamental condition $d^2 = 0$. Discuss why this is true by considering an explicit example on $\mathbb{R}^n$. Comment on how this also gives rise to the identities of vector calculus.

GROUP - B (39 Marks)

Answer any three questions(3x13=39).

1. **(a)** What is meant by Self-Contragredient, Self-Adjoint and Complex Conjugate Representations ?
(b) The most familiar illustration of the representation theory is $SU(2)$, or angular momentum theory in quantum mechanics. It is well-known that all integer spin representations, which are odd-dimensional, are potentially real, and indeed their description using Cartesian tensors gives them in real form. However, all half odd integer spin representations which are even dimensional, are pseudo real: the invariant bi-linear form in these cases is antisymmetric. Explain (prove) the statement. [3 + (5+5)]

2. **(a)** How does a state ket change with time ? Write down the infinitesimal time-evolution operator ? Show that it satisfies Schrodinger like equation. Find out time-evolution operator for finite time translation.
(b) Prove that Rotation about different axes do not commute in three dimension however infinitesimal rotation about different axes do commute if second and higher order are ignored. Exploiting the result find out the fundamental commutation relation of angular momentum. [7+6]

3. **(a)** What do you mean by unitary representation of a group?
(b) Consider a matrix $U(g)$ is given by

$$U(g) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1-i & 0 \\ i & 1+i \end{bmatrix}$$

Show that it is not unitary matrix. Consider a nonsingular matrix S is given by $S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, show that under similarity transformation of $U(g)$ i.e. $U'(g)$ becomes unitary.

(c) When you call a group representation is reducible? Suppose you are given a Group G on a vector space of six dimensions. Basis vectors are denoted by $\hat{e}_1, \ldots\ldots, \hat{e}_6$. Under the group action $\hat{e}_1$,$\hat{e}_2$ forms an invariant subspace. Write the matrix for a group element. Say, when this reducible representation can be made decomposable.

(d) The set of translation is an example of continuous group and also abelian. Explain it. [2+3+5+3]

4. (a) For the group $SO(3)$ of real proper orthogonal rotations in 3-dimensions, the one of the possible parametrization is Axis angle parametrization denoted as $R(\hat{n}, \alpha)$. What would be the parameter space for the group elements? How will you get different equivalence class out of it?

(b) $\vec{r}$ is a vector in R^3. It is mapped to $h = \vec{r}.\vec{\sigma}$ where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Make a unitary transformation of h i.e. $h' = uhu^{-1} = \vec{r'} \cdot \vec{\sigma}$ where $u = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$, $u \in SU(2)$. Find Rotation matrix (3×3) in terms of a, a^*, b, b^* which takes $\vec{r}$ to $\vec{r'}$.

(c) $SU(2)$ and $SU(3)$ are locally isomorphic, comment on it. [4+6+3]