M. Sc. Mathematics Examination, 2023

(2nd Year, 1st Semester)

Mathematics

Paper – DSE-02C

[Algebraic and Differential Topology]

Time : Two hours Full Marks : 40

Answer ***any five*** questions
taking at least two from each group.

Group – A

1. a) Give an example to show that the quotient space of a Hausdorff space need not be so.

   b) Let $\rho \subset X \times X$ be an equivalence relation on a topological space X and let $q : X \to Q = X / \rho$ be the quotient space. If Q is Hausdorff then show that ρ is a closed subspace of X×X. Prove that the converse is true if in addition $q : X \to Q$ is an open map.

   3+5

2. a) Prove that a space X is contractible if and only if given any topological space T, only two continuous maps $f, g : T \to X$ are homotopic.

   b) Show that a contractible space is path connected.

   4+4

3. a) Find $\pi_1\left(D^2, 1\right)$ where

$$D^2 = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\right\}$$

   b) Define a covering map. Show that the continuous map $f : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ is a covering map. 3+5

4. State and prove Brouwer's fixed point theorem. 8

Group – B

5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\tilde{a} \in \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be differentiable at $\tilde{b} = f(\tilde{a}) \in \mathbb{R}^m$. Then prove that $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $\tilde{a} \in \mathbb{R}^n$ and $h'(\tilde{a}) = g'(\tilde{b}) \circ f'(\tilde{a})$. 8

6. Let A be an open subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}^n$ be continuous and has finite partial derivatives $D_j f_i$ on A. If f is |–| on A and $J_f(\tilde{x}) \neq 0 \ \forall \ \tilde{x} \in A$ prove that f(A) is open. 8

7. Prove that the subset

$$H_a = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = a\right\}$$

   is a 2-dimensional manifold for $a > 0$ but is not a manifold for $a = 0$. 8

8. a) Prove local immersion theorem.

   b) Give an example to show that the image of a manifold under a smooth, |–| immersion may not be a manifold. 6+2