M. Sc. Mathematics Examination, 2023

(2nd Year, 2nd Semester)

Mathematics

Paper – DSE-06

[Advanced Rings and Modules-II]

Time : 2 hours　　　　Full Marks : 40

The figures in the margin indicate full marks.

Special credit will be given for precise answer.

(Unexplained Symbols/Notations have their usual meaning.)

Unless otherwise mentioned, throughout the question any ring is assumed to the with identity $1 \neq 0$ any module is assumed to be a unitary module.

Answer **Q.No. 6** and ***any three*** from the rest.　　10×4=40

1. a) State any one version of Köthe's Conjecture.　1

   b) Define the *Jacobson radical radR* of a ring R which is not necessarily with identity.　1

   c) State and prove Wedderburn Artin Structure theorem for a left semisimple ring. A left semisimple ring is right semisimple and conversely – Justify.　7+1

2. a) What are $\mathbb{Z}$-simple modules and what are $\mathbb{Z}$-semisimple modules?　3

   b) Prove that for a ring R, the intersection of all maximal left ideals and that of all maximal right ideals are the same and hence that it is an ideal of R. Prove that the Jacobson radical radR of a ring R is the largest ideal I of R with the property that $1+I\subseteq U(R)$. 7

3. a) Define a semiprimary ring. 1

b) Prove (Hopkins-Levitzki Theorem) that a ring is left. Artinian if and only if it is left Noetherian and semiprimary. Hence prove that $\mathbb{Z}$ is not semiprimary. 4

c) Obtain a characterization of the Jacobson radical *radR* of a ring *R* in terms of left or right primitive ideals of *R*. 5

4. a) Prove the following chain of implications for a ring. *scmisimple* $\Rightarrow$ *vonNeumann regular* $\Rightarrow$ *J* – *semisimple*. 5

b) Give examples to illustrate that reverse implications in (a) do not hold. State the conditions under which the reverse implications hold. 5

5. a) For a ring R, establish the relation among *Baer's lower nilradical* i.e. *Bear – McCoy radical Nil. R. Upper nilradical Nil*R* and *Jacobson radical radR*. State what happens if (i) R is commutative and (ii) R is left Artinian. 5

b) Prove that a ring *R* is semiprime if and only if *Nil.* $R=0$ and hence prove that any *J-semisimple* i.e., *semiprimitive* ring is semiprime. 3

c) Prove that a simple ring is prime and hence conclude that $M_2(\mathbb{R})$ is a prime ring. Give an example of a prime ring which is not simple. 2

6. a) Give a canonical example of a left primitive ring which is neither left Artinian nor simple. 4

b) Consider the vector space $V=\mathbb{R}^3$. Give an example (with justification) of

i) a linear operator T on V such that the $\mathbb{R}[x]$-module V via T is not semisimple. 2

ii) a linear operator T on V such that the $\mathbb{R}[x]$-module V via T is not semisimple. 2

c) Draw an implication diagram to show the hierarchy of the class of rings viz., (i) simple, (ii) semisimple, (iii) semiprimitive i.e. J-semisimple, (iv) left primitive, (v) prime (vi) semiprime. 2