Ref. No. : $SC/MATH/PG/DSE/TH/03/A2/2023$

Master Of Science Examination - 2023

(Second Year, First Semester)

Mathematics

DSE - 03

($A2$: Advanced Rings and Modules - I)

Full Marks : 40 **Time : 2 Hours**

Answer any four questions.

(Notations / Symbols have their usual meanings)

Let R be a commutative ring with identity 1.

1 (a). Define exact sequence of R-modules and R-homomorphisms. Show that the sequence of R-modules and R-homomorphisms $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is eaxct if and only if f is an isomorphism.

Let $0 \longrightarrow M \longrightarrow 0$ be an exact sequence. Is $M = 0$? Justify. 2+2+1

(b). Consider the short exact sequence of R-modules and R-homomorphisms

$0 \longrightarrow N \longrightarrow M \longrightarrow F \longrightarrow 0$, where F is a free R-module.

Show that the above short exact sequence is a split exact sequence. Hence conclude that $0 \longrightarrow N \longrightarrow M \longrightarrow R \longrightarrow 0$ is a split exact sequence.

Now consider the short exact sequence of R-modules and R-homomorphisms

$0 \longrightarrow F \longrightarrow M \longrightarrow N \longrightarrow 0$, where F is a free R-module.

Is this a split exact sequence? Justify. 2+1+2

2 (a). Define divisible group. Show that an additive abelian group M is divisible if and only if the $\mathbb{Z}$-module M is injective. 1+4

(b). With proper justification, give an example of a (an)

(i) projective module which is not injective

(ii) injective module which is not projective 2+3

[Turn over

[2]

3 (*a*). Define tensor product of two R-modules. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. 2+3

(*b*). Let M be an R-module. Show that $R \otimes_R M \cong M$. Hence conclude that $R \otimes_R R \cong R$. 4+1

4 (*a*). Define local ri Let L be a local ring with identity 1.

(*i*) Show that L has no idempotent element except 0 and 1.

(*ii*) If for all $a, b \in L$, $a + b = 1$ then show that either a is a unit or b is a unit. 1+2+2

(*b*). Define module of fractions. If M is an R-module and S is a multiplicatively closed subset of R then show that $M \otimes_R S^{-1}R \cong S^{-1}M$. 2+3

5 (*a*). Define radical ideal of a ring. Let I be a proper ideal of R. Show that I is a radical ideal of R if and only if R/I has no nonzero nilpotents elements. 1+4

(*b*). Define nil radical of a ring. Let $N(R)$ be the nil radical of R. Show that R has only one prime ideal if and only if $R/N(R)$ is a field. 1+4

6 (*a*). Define irreducible and primary ideal of a ring. Let R be a Noetherian ring. Show that every irreducible ideal of R is a primary ideal of R. 2+3

(*b*). What do you mean by derivation and inner derivation of a ring? Show that a finite field admits no non-zero derivation. 3+2