M. Sc. Mathematics Examination, 2023

(2nd Year, 1st Semester)

Mathematics

Paper – DSE-04-A15

[Theory of Semigroup - I]

Time : Two hours | Full Marks : 40

(Symbols / Notations have their usual meanings.)

Answer ***any four*** questions.

1. a) Define rectangular band. Show that a semigroup is isomorphic to a rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup and a right zero semigroup.

   b) Define a congruence ρ on a semigroup S. If $\phi : S \to T$ is a homomorphism such that $\rho \subseteq \ker\phi$, then show that there is a unique homomorphism $\beta : S/\rho \to T$ such that $\operatorname{ran}(\beta) = \operatorname{ran}(\phi)$ and $\phi\beta = \rho^k$, where ρ^k is the natural epimorphism from S onto S/ρ. 5+5

2. Define a perodic semigroup. Define also index and period of S. Show that in a periodic semigroup every element has a power which is idempotent. Explain with an example of such semigroup. 2+2+3+3

[Turn over

3. a) Define the Green's equivalence relations α and R an a semigroup. Prove that $\alpha \circ R = R \circ \alpha$.

   b) Define $\rho \circ \sigma$ and $\rho \vee \sigma$ for the congruence ρ and σ on a semigroup S. If $\rho \circ \sigma = \sigma \circ \rho$, then show that $\rho \vee \sigma = \rho \circ \sigma$. 5+5

4. a) Define the Green's equivalence relation *D* and *J* on a semigroup. Prove that in a periodic semigroup, *D*=*J*.

   b) Define an orthodox semigroup. Let S be a regular semigroup such that if *e* is an idempotent, then every inverse of *e* is idempotent. Prove that *S* is an orthodox semigroup. 5+5

5. a) When is a semigroup said to be embedded said to be embedded in another semigroup? Suppose S be a semigroup with identity (ie, a monoid). Is it embedded in semigroup of transformations J_s? — explain.

   b) Let *a* be an element of a regular D-class *D* in a semigroup S. If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contains idempotent *e* and *f* respectively, then show that H_b contains and inverse a^* such that $aa^* = e$ and $a^* a = f$.

6. a) Show that a semigroup S is completely regular if and only if S is union of groups.

   b) Let *e* and *f* be idempotents in a regular semigroup S. Then prove that S(*e*, *f*) is a subsemigroup of S and also a rectangular band, where

   $S(e,f) = \{g \in E : ge = fg = g,\ egf = ef\}$,

   is a sandwich set of *e* and *f* and E is the set of all idempotents of S. 5+5