M. Sc. Mathematics Examination, 2023

(2nd Year, 2nd Semester)

Mathematics

Paper – 4.1

[Advanced Functional Analysis]

Time : 2 hours Full Marks : 50

Answer ***any five*** questions.

1. a) Is the real number space $\mathbb{R}$ with cofinite topology a topological vector space? Answer with reasons.

   b) Define a bounded subset of a topological vector space X. Prove that B is bounded iff for every sequence $\{x_n\}_n$ in B and any sequence of scalers $\{\alpha_n\}_n$ with $\alpha_n \to 0$ as $n \to \infty$, $\alpha_n x_n \to \theta$ as $n \to \infty$. 3+1+6

2. a) i) For A, $B \subset X$ show that $\overline{A} + \overline{B} \subset \overline{A+B}$.

   ii) If Y is a subspace of X then so is $\overline{Y}$.

   b) Let K and C be compact and closed subsets of a topological vector space X with $K \cap C = \phi$. Show that there is a nbd Y of θ such that $(K+V) \cap (C+V) = \phi$. 3+2+5

3. a) Prove that every convex nbd of θ in a topological vector space X contains a balanced convex nbd of θ.

[Turn over

b) If V is a nbd of θ in a topological vector space X and $0 < r_1 < r_2 < .. < r_n \to \infty$ as $n \to \infty$ then show that $X = \bigcup_{n=1}^{\infty} r_n V$. 6+4

4. Let T be a linear functional on a topological vector space X and assume that $Tx \neq 0$ for some $x \in X$. Then prove that following are equivalent.

   i) T is continuous.

   ii) The null space N(T) is closed.

   iii) N(T) is not dense in X.

   iv) T is bdd in some nbd V of θ. 10

5. a) Prove that every locally compact topological vector space X has finite dimension.

   b) When a topological vector space is called locally compact? Give an example of a topological vector space which is not locally compact. 7+3

6. a) Define sets of first category and second category with two suitable examples.

   b) Let X, Y be two topological vector spaces, Γ be a collection of continuous linear mappings from X to Y and B is the set of all $x \in X$ whose orbits $\Gamma(x)$ are bounded in Y. If B is of second category in X then prove that B=X and Γ is equicontinuous. 4+6

7. Suppose M is a subspace of a topological vector space X, p is a seminorm on X, *f* is a linear functional on M such that $|f(x)| \le p(x)$ for all $x \in M$. Then show that *f* extends to a linear functional Δ on X satisfying $|\Delta x| \le p(x)$ for all $x \in X$. 10

8. a) State Baire's theorem in a locally compact topological vector space.

   b) Suppose that X is a topological vector space and X_1 is a separating vector space of linear functionals on X. Then prove that the X_1-topology τ_1 makes X into a locally convex space whose dual space is X_1. 2+8