M. Sc. Mathematics Examination, 2023

(2nd Year, 2nd Semester)

Mathematics

Paper – DSE-07 (B17)

[Operator Theory II]

Time : 2 hours Full Marks : 40

The figures in the margin indicate full marks.

Notations / Symbols have their usual meaning.

Answer any **four** questions.

All questions carry equal marks.

1. i) Let H be a complex Hilbert space and $T : H \to H$ be a bounded self-adjoint linear operator. Show that the spectrum $\sigma(T)$ is real.

   ii) Let $T : H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space H. If $M = \sup_{\|x\|=1} \langle Tx, x \rangle$, then show that M is a spectral value of T. 5+5

2. Let $\{P_n\}$ be a monotone increasing sequence of projections defined on a complex Hilbert space H. Then show that

   i) $\{P_n\}$ is strongly operator convergent and the limit operator P is a projection on H.

   ii) P projects H onto $P(H) = \overline{\bigcup_{n=1}^{\infty} P_n(H)}$.

[Turn over

iii) P has the null space $N(P)=\bigcap_{n=1}^{\infty} N(P_n)$. 10

3. Let S and T be two bounded self adjoint linear operators on a complex Hilbert space H. If both the operators are positive and $ST = TS$ then prove that ST is also positive. 10

4. Let $T: H \rightarrow H$ be a bounded self adjoint linear operator on a complex Hilbert space H and E be the projection from H on to $N(T^{+})$. Then prove that

   i) E commutes with every bounded self adjoint operator that T commutes with.

   ii) $T^{+}E = ET^{+} = 0$, $T^{-}E = ET^{-} = T^{-}$.

   iii) $TE = -T^{-}$, $T(I-E) = T^{+}$.

   iv) $T^{+} \geq 0$.

   v) $T^{-} \geq 0$.

5. i) Show that the spectrum $\sigma(T)$ of an unbounded self adjoint linear operator $T: \mathcal{D}(T) \rightarrow H$ is real and closed, where H is a complex Hilbert space and $\mathcal{D}(T)$ is dense in H.

   ii) What happens to the spectrum if T is bounded.

   8+2=10

6. i) Let $T: H \rightarrow H$ be a bounded linear operator on a complex Hilbert space H and $\mathcal{E}=(E_\lambda)$ be the

corresponding spectral family. Then show that if there is a $\gamma > 0$ suchthat $\mathcal{E}=(E_\lambda)$ is constant on the interval $[\lambda_0-\gamma, \lambda_0+\gamma]$ then $\lambda_0 \in \rho(T)$.

   ii) Show that the differentiation operator $T: \mathcal{D}(T) \rightarrow L^2(-\infty,+\infty)$ defined by $T(x) = ix'$, where $x' = dx/dt$ and $\mathcal{D}(T) \subset L^2(-\infty,+\infty)$ is unbounded