Master of Science Examination, 2023

(2nd Year, 1st Semester)

MATHEMATICS

UNIT - DSE 04 A17

[Operator Theory I]

Time : 2 Hours Full Marks : 40

Symbols/Notations have their usual meaning.
Answer any ***four*** questions.

1. (i) Suppose $\{\alpha_n\}$ is a sequence of real numbers dense in [0,1]. Let $T : \ell^2 \to \ell^2$ be defined by

$$T(x_1, x_2, ...) = (\alpha_1 x_1, \alpha_2 x_2, ...)$$

for every $(x_1, x_2, ...) \in \ell^2$. Then find $\sigma(T)$.

(ii) Let $T \in \mathbb{B}(X, X)$ where X is a Banach space. If $\|T\| < 1$, then show that $(I-T)^{-1}$ exists as a bounded linear operator on X and

$$(I-T)^{-1} = I + T + T^2 + ...$$ 6+4

2. (i) If T is a bounded linear operator on a complex Banach space X, then show that $\sigma(T) \neq \phi$.

(ii) If T is a normal operator on a Hilbert space H, then show that the spectral radius $r_\sigma(T) = \|T\|$. 6+4

3. (i) Let T be a compact linear operator on a normed linear space X. Then show that the point spectrum set is at most countable and the only possible point of accumulation is 0.

(ii) Show that $T : \ell^p \to \ell^p, 1 \le p < \infty$ defined by

$$T\left(x_1, x_2, \ldots\right) = \left(\frac{x_1}{2}, \frac{x_2}{2^2}, \ldots.\right)$$

is compact. 6+4

4. (i) Let $T : X \to X$ be a compact linear operator on a normed space X and $\lambda \neq 0$. Then for every $y \in X$, the equation $Tx - \lambda x = y$ has a solution iff the homogeneous equation $Tx - \lambda x = 0$ has only the trivial solution.

(ii) Show that for a compact linear operator on a normed linear space every non-zero spectral value is an eigenvalue. 6+4

5. (i) Show that the numerical range of a bounded linear operator on a Hilbert space is always convex.

(ii) Show that the closure of the numerical range always contains the spectrum. 6+4

6. (i) Let T be a compact normal operator on a Hilbert space H. If $x \perp N(T - \lambda I)$ for every λ, then show that x=0.

(ii) Let T be a compact normal operator on a Hilbert space H. Then show that there exists an eigenvalue λ such that $|\lambda| = \|T\|$. 6+4