M. Sc. Mathematics Examination, 2023

(2nd Year, 2nd Semester)

Mathematics

Paper – DSE-06 (B16)

[Operator Algebra II]

Time : Two hours Full Marks : 40

Symbols / Notations have their usual meanings.

Here, SOT denotes Strong Operator Topology.

Answer ***any four*** questions. 10×4=40

1. a) Let H be a Hilbert space and $\mathscr{A}$ be a *-closed unital subalgebra of $B(H)$. Prove that $\mathscr{A}$ is SOT dense in $\mathscr{A}''$, where $\mathscr{A}''$ denotes the double commutant of $\mathscr{A}$. 7

   b) Prove or disprove: Let H be a Hilbert space and $\{T_n\}$ be a sequence in $B(H)$. If $\{T_n\}$ converges to T is SOT, then for any fixed operators A, B in $B(H)$, the sequence $\{AT_nB\}$ converges to ATB in SOT. 3

2. Let (Ω, F, μ) be a σ-finite measure space. Prove that $\{M_f : f \in L^\infty\}$ is a von Neumann algebra. 10

3. a) Let Σ be a non-empty compact subset of $\mathbb{C}$ and B_Σ be the Borel σ field of Σ. Suppose μ is a finite positive measure on (Σ, B_Σ). Let

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$\pi : C(\Sigma) \to B(L^2(\mu))$ defined by $\pi(f) = M_f$. Prove that π extends uniquely to

$$\tilde{\pi} : L^\infty(\Sigma, B_\Sigma, \mu) \to B(L^2(\Sigma, B_\Sigma, \mu))$$

satisfying

i) $\tilde{\pi}(f) = \pi(f)$ for $f \in C(\Sigma)$ and

ii) any uniformly norm bounded sequence $\{f_n\}$ in $C(\Sigma)$ converges in measure μ to a function $f \in L^\infty(\mu)$ if and only if $\{\tilde{\pi}(f_n)\}$ converges in SOT to $\tilde{\pi}(f)$. 7

b) Let H be a Hilbert space and N be the set of bounded normal operators on H. Prove that the map $A \to A^*$ is continuous in SOT on N. 3

4. a) Let H be a Hilbert space and E be a spectral measure taking values in B(H) on a measurable space (Ω, F). Prove that for any unit vector $v \in H$, $E_{v,v}$ defined by $E_{v,v}(D) = \langle v, E(D)v \rangle$, $D \in F$ is a probability measure on (Ω, F). 4

b) Let P, Q be two projections on a Hilbert space H and $x \in H$. Check whether the following sequence

$$x,\ Px,\ QPx,\ PQPx,\ QPQPx,\ \ldots$$

is convergent or not. If it is convergent, then find its limit. 6

5. a) Prove that the function $f(t) = \dfrac{t}{1+t^2}$ is strongly continuous. 4

b) Let H be a complex seperable Hilbert space. Let C be a unital C^* subalgebra of $B(H)$ and let $\mathscr{A}$ be the double commutant of C. Prove that the SOT closure of the set of self-adjoint operators in the unit ball of C is the set of self-adjoint elements in the unit ball of $\mathscr{A}$. 6

6. a) Let H, K be Hilbert spaces and $U : H \to K$ be a bounded linear map. If U is a partial isometry then prove that U^*U is a projection. 3

b) Let H be a Hilbert space and T be an element of a von Neumann algebra $\mathscr{A} \subseteq B(H)$. Let $T = U|T|$ be the polar decomposition of T. Show the U and $|T|$ are in $\mathscr{A}$. 7