Ref. No.: SC/MATH/PG/DSE/TH/03/A16/2023

M.Sc. Mathematics 2nd year 1st Semester examination, 2023
Subject: Operator Algebra I
Paper: DSE-03 (A16)

Time: Two hours — Full Marks: 40

Notations: Here, the ground field for all algebras is the complex field $\mathbb{C}$ and $\sigma_{\mathscr{A}}(a)$ denotes the spectrum of the element a with respect to the Banach algebra / C^*-algebra $\mathscr{A}$.

Answer any four questions ($10 \times 4 = 40$)

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1. (a) Let $\mathscr{A}$ be a unital Banach algebra and $a \in \mathscr{A}$. Prove that $\sigma_{\mathscr{A}}(a) \neq \emptyset$.

   (b) Prove that every Banach division algebra is isometrically isomorphic to $\mathbb{C}$.

   (c) Let $\mathscr{A}$ be a unital Banach algebra and $a, b \in \mathscr{A}$. Prove that $\sigma_{\mathscr{A}}(ab) \setminus \{0\} = \sigma_{\mathscr{A}}(ba) \setminus \{0\}$ [5+2+3]

2. (a) If I is a maximal ideal of a unital Banach algebra $\mathscr{A}$, then prove that I is closed.

   (b) Let $\mathscr{A}$ be a unital Banach algebra and $\mu_{\mathscr{A}}$ be the set of characters on $\mathscr{A}$. If $\phi \in \mu_{\mathscr{A}}$, then prove that $||\phi|| = 1$.

   (c) Let $\mathscr{A}$ and $\mathscr{B}$ be unital Banach algebras with identities $1_{\mathscr{A}}, 1_{\mathscr{B}}$ respectively. Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a homomorphism such that $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$. Prove that $\sigma_{\mathscr{B}}(\Phi(a)) \subseteq \sigma_{\mathscr{A}}(a)$. [3+4+3]

3. (a) Let $\mathscr{A}$ be a unital commutative C^*-algebra and $\mu_{\mathscr{A}}$ be the set of characters on $\mathscr{A}$. Prove that the function sending a in $\mathscr{A}$ to $\hat{a}$ in $C(\mu_{\mathscr{A}})$ defined by $\hat{a}(\phi) = \phi(a)$, $\phi \in \mu_{\mathscr{A}}$ is an isometric $*$-isomorphism from $\mathscr{A}$ to $C(\mu_{\mathscr{A}})$, where $C(\mu_{\mathscr{A}})$ denotes the set of continuous functions on $\mu_{\mathscr{A}}$.

   (b) Let $\mathscr{A}$ and $\mathscr{B}$ be two unital C^*-algebras and $\pi : \mathscr{A} \to \mathscr{B}$ be a unital $*$-homomorphism. Prove that π is contractive. [7+3]

4. (a) Let a be a normal element in a unital C^*-algebra $\mathscr{A}$ and $f : \sigma_{\mathscr{A}}(a) \to \mathbb{C}$ be any continuous function. Prove that $\sigma_{\mathscr{A}}(f(a)) = \{f(\lambda) : \lambda \in \sigma_{\mathscr{A}}(a)\}$.

   (b) Let a be a normal element in a unital C^*-algebra $\mathscr{A}$. Prove that a is unitary if and only if $\sigma_{\mathscr{A}}(a) \subseteq \{z \in \mathbb{C} : |z| = 1\}$.

   (c) Let $\mathscr{A}$ be a unital C^*-algebra with identity $1_{\mathscr{A}}$ and ϕ be a linear functional on $\mathscr{A}$ such that $||\phi|| = \phi(1_{\mathscr{A}}) = 1$. Prove that ϕ is a state. [3+3+4]

[Turn over

5. (a) Let a be an element in a unital C^*-algebra $\mathscr{A}$ with identity $1_{\mathscr{A}}$. Prove that $\|a\|^2 \cdot 1_{\mathscr{A}} - a^*a$ is a positive element of $\mathscr{A}$.

   (b) Prove that for a unital C^*-algebra $\mathscr{A}$ there exists a pair (H, π), where H is a complex Hilbert space and $\pi : \mathscr{A} \to B(H)$ is an injective $*$-homomorphism.

[4+6]

6. (a) Let $\mathscr{A}$ be a unital C^*-algebra and $a \in \mathscr{A}$ be in the unit ball of $\mathscr{A}$ such that $||a|| < 1 - \frac{2}{n}$, for some $n \in \mathbb{N}$ with $n \geq 3$. Prove that there exist n unitaries $u_1, u_2, \ldots, u_n$ in $\mathscr{A}$ such that $a = \frac{u_1 + u_2 + \cdots + u_n}{n}$.

   (b) Prove that every element in a unital C^*-algebra is a positive multiple of sum of three unitaries.

   (c) Prove or disprove: If a and b are two positive elements in a unital C^*-algebra $\mathscr{A}$, then $a + b$ is positive. [6+2+2]