Ex/SC/MATH/PG/DSE/TH/05/B/2023

M. Sc. Mathematics Examination, 2023

(2nd Year, 2nd Semester)

Mathematics

Paper – DSE-05 (Unit 4·1)

[Advanced Functional Analysis]

Time : Two hours Full Marks : 40

Answer ***any four*** questions.

1. a) Is the real number space $\mathbb{R}$ with cofinite topology a topological vector space? Answer with reasons.

   b) Let K and C be compact and closed subsets of a topological vector space X with $K \cap C = \phi$. Show that there is a nbd V of θ such that $(K+V) \cap (C+V) = \phi$.

   c) In a topological vector space X, for any $A \subset X$ show that $\overline{A} = \bigcap (A+V)$ where V runs over all nbds of θ.

   2+6+2

2. a) Prove that every convex nbd of θ in a topological vector space X contains a convex balanced nbd of θ.

   b) Define a bounded set in a topological vector space. Prove that A subset E of a topological vector space X is bounded iff for any sequence $\{x_n\}_n$ in E and a sequence of scalers $\{\alpha_n\}_n$ with $\alpha_n \to 0$ as $n \to \infty$, $\alpha_n x_n \to \theta$ as $n \to \infty$. 5+5

[Turn over

3. Let Δ be a linear functional on a topological vector space X and assume that $\Delta x \neq 0$ for some $x \in X$. Then prove that following are equivalent.
   i) Δ is continuous.
   ii) The null space N(Δ) is closed.
   iii) N(Δ) is not dense in X.
   iv) Δ is bounded on some nbd V of θ. 10

4. a) Let X and Y be two topological vector spaces, Γ be a collection of continuous linear mappings from X to Y and B is the set of all $x \in X$ whose orbits $\Gamma(x)$ are bounded in Y. If B is of second category in X then prove that B=X and Γ is equicontinuous.

   b) Let X, Y be topological vector spaces and $\{\Delta_n\}_n$ a sequence of continuous linear mappings from X to Y. Let C be the set of all $x \in X$ for which $\{\Delta_n x\}_n$ is a Cauchy sequence in Y. If C is of 2nd category then show that $\overline{C} = X$. 6+4

5. a) Let M be a subspace of a real topological vector space X,
   i) $p : X \to R$ satisfies $p(x+y) \leq p(x) + p(y)$ and $p(tx) = tp(x)$ for all $x, y \in X$ and $t \geq 0$.
   ii) $f : M \to R$ is linear and $f(x) \leq p(x)$ on M.
   Then show that there exists a linear functional

$\Delta : X \to R$ such that $\Delta(x) = f(x)$ for $x \in M$ and $-p(x) \leq \Delta(x) \leq p(x)$ for all $x \in X$.

   b) Give an example to show that sum of two closed sets in a topological vector space may not be closed. 8+2

6. a) Suppose Y is a subspace of a topological vector space X and Y is an F-space in the topology inherited from X. Show that Y is a closed subspace of X.

   b) If *f* is a continuous linear functional on a subspace M of a locally convex topological vector space X, then prove that there exists $\Delta \in X^*$ such that $\Delta = f$ on M. 4+6