MASTER OF SCIENCE EXAMINATION, 2023

(2nd Year, 1st Semester)

MATHEMATICS

UNIT - DSE 04 A30

[NUMBER THEORY]

Time : 2 Hours Full Marks : 40

Symbols have usual meanings, if not mentioned otherwise

Answer ***Q. no. 1*** and any ***three*** from the rest.

1. (a) Show that in any integral domain a prime element is irreducible.

   (b) Show that 3 is divisible by $(1-\omega)^2$ in $\mathbb{Z}[\omega]$.

   (c) Show that $\upsilon(n)$ is odd if and only if n is a square.

   (d) If n is not a prime, show that $(n-1)! \equiv 0(n)$, except when n=4. 2+2+3+3=10

2. (a) Show that the equation $7x^2 + 2 = y^3$ has no solution in integers.

   (b) Solve the system of congruences $x \equiv 1(7), x \equiv 4(9)$, $x \equiv 3(5)$ with the help of Chinese Remainder Theorem.

(c) Let p be an odd prime. Show that a is a primitive root modulo p if and only if $a^{(p-1)/q} \not\equiv 1(p)$ for all prime divisors q of p–1. 2+3+5=10

3. (a) Suppose that a is a primitive root modulo p^n, where p is an odd prime. Show that a is a primitive root modulo p.

(b) Find the solutions to the contruences $x^3 \equiv 1(19)$ and $x^4 \equiv 1(17)$. 5+5=10

4. (a) Show that the minimal polynomial of $\sqrt[3]{2}$ is x^{3-2} over $\mathbb{Q}$.

(b) Use the Jacobi symbol to determine $\left(\frac{215}{761}\right), \left(\frac{514}{1093}\right)$, and $\left(\frac{401}{757}\right)$. 5+5=10

5. (a) Suppose that p is a prime with $p \equiv 1(3)$. Prove that there are integers A and B such that $4p = A^2 + 27B^2$. If $A \equiv 1(3)$, then prove that A is uniquely determined, and $N(x^3 + y^3 = 1) = p - 2 + A$.

(b) Find the number of solutions to the equation $x^3 + y^3 = 1$ in Fp for p = 19, 37, and 97. 5+5=10

6. (a) If π is a prime in D, prove that $x^3 \equiv 2(\pi)$ is solvable if and only if $\pi \equiv 1(2)$.

(b) Suppose p is a rational prime such that $p \equiv 1(3)$. Prove that $x^3 \equiv 2(p)$ is solvable if and only if there are integers C and D such that $p = C^2 + 27D^2$.

(c) Show that

(i) $x^3 \equiv 2(7)$ is not solvable, and

(ii) $x^3 \equiv 2(31)$ is solvable. 3+4+3=10