Ref. No.: SC/MATH/PG/CORE/TH/11/2023

Master Of Science Examination - 2023

(Second Year, First Semester)

Mathematics

Core 11

(Integral Equation, Integral Transform and Calculus of variation)

Full Marks : 40 Time : 2 Hours

Symbols/Notations have their usual meaning

Use separate answerscripts for each part

Group A

(24 Marks)

Answer Question no 1 and any TWO from the rest

1. Answer any ONE of the following

a) Find the extremal of the functional

$$J[y,z] = \int_0^{\pi/2} [1 + y'^2 + z'^2 + 2yz]dx,$$

subject to $y(0) = 0, \quad y(\pi/2) = 1, \quad z(0) = 0, \quad z(\pi/2) = 1.$ (8)

b) Find the extremal of the functional

$$J[y] = \int_0^1 [1 + y''^2 \,]\; dx,$$

subject to $y(0) = 0, \; y'(0) = 1 \; y(1) = 1, \; y'(1) = 1.$ (8)

2. Find the value(s) of λ for which the unique solution of

$$\phi(x) = x + \lambda \int_0^1 (1 + x + t)\phi(t)dt, \qquad 0 \leq x \leq 1$$

exists. Hence find the solution. (8)

[Turn over

3. Prove the following:

a) For all Hermitian kernels $k(x,t)$, the following hold.

$$(K\phi,\psi) = (\phi, K\psi),$$

where (ϕ,ψ) denotes the inner product of two square-integrable functions ϕ and ψ and $K\phi(x) = \int_a^b k(x,t)\phi(t)dt,\ a \le x \le b$.

b) The eigenfunctions of a symmetric kernel corresponding to different eigenvalues, are orthogonal. 4+4

4. Find the iterated kernels and $k_n(x,t)$ and the resolvent kernel $R(x,t;\lambda)$ for the following integral equation $\phi(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\ \phi(t)\ dt, \quad 0 < x < 1.$ (8)

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Time : 2 Hours **Full Marks : 40**

The figures in the margin indicate full marks.

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Group – B (Marks: 16)

(Integral Transforms)

Answer **any two** questions.

1. (a) Find the Fourier transform of $f(x) = 1,\ -\infty < x < \infty$ by considering the notion of generalized Dirac delta function.

   (b) Show that $H_n[f(r);\rho] = \frac{a^{n+1}}{\rho} J_{n+1}(\rho a)$, where $f(r) = \begin{cases} r^n & r \leq a \\ 0 & r > a \end{cases}$

   and hence using the Parseval relation of Hankel transform to evaluate

   $\int_0^\infty \frac{J_{n+1}(ar)J_{n+1}(br)}{r} dr$ for $n > -\frac{1}{2},\ 0 < a < b.$ [3 + 5]

2. (a) State and prove convolution theorem for Laplace transform.

   (b) Find $L\{erfc(\frac{a}{\sqrt{t}})\}$, where $erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2} dx.$ [4 + 4]

3. (a) Show that $lim_{|s| \to \infty} |F(s)| = 0$, where $F(s) = F\{f(x)\}$.

   (b) Using Fourier transform to solve the partial differential equation

   $\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{a^2} \frac{\partial \phi}{\partial t} = 0,\ -\infty < x < \infty,\ \ t \geq 0$

   with boundary conditions $\phi(x,y),\ \frac{\partial \phi}{\partial x} \to 0$ as $|x| \to \infty$ and initial condition $\phi(x,0) = f(x),\ -\infty < x < \infty$, where $f(x)$ is prescribed function of x and its Fourier transform exists. [2 + 6]