5. a) If a is a complex number then find Z transform of

$$f(n)=\begin{cases} a^n, & \text{for } n \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

b) Prove that $\mathcal{Z}\left[nf(n)\right]=-z\dfrac{dF(z)}{dz}$. 3+2

6. If $f(t)$ is piecewise continuous and is of exponential order $O\left(e^{ct}\right)$, then prove that its Laplace transform $F(p)$ satisfies

i) $\lim\limits_{|p|\to\infty} pF(p)=\lim\limits_{t\to 0+} f(t)=f(0)$

ii) $\lim\limits_{|p|\to 0} pF(p)=\lim\limits_{t\to\infty} f(t)$. 2+3

7. Find $\mathcal{L}\left[er\ f(t)\right]$ and $\mathcal{L}\left[er\ fc\left(\dfrac{a}{\sqrt{t}}\right)\right]$. 2+3

Ex/SC/MATH/PG/Unit 4.2/2023

M. Sc. Mathematics Examination, 2023

(2nd Year, 2nd Semester)

Mathematics

Paper – 4.2

[Integral Equation and Integral Transform]

Time : 2 hours Full Marks : 50

The figures in the margin indicate full marks.

(Symbols and notations have their usual meanings)

(Use a separate Answer-Script for each Part)

Part – I (Marks: 25)

Answer **Q.No.1** and ***any two*** from the rest.

1. Define symmetric kernel with example. 1

2. a) If f is a continuous function on $[a, b]$ and $k(x,t)$ is a continuous function on $\mathbb{R}=\{(x,t);\ a\leq x,\ t\leq b\}$ and $\phi_0(x)$ is any function continuous on $[a, b]$ and for $x\in[a,b]$, $\phi_n(x)=f(x)+\lambda\int_a^b k(x.t)\phi_{n-1}(t)dt$ $(n=1,2,3,...)$, then show that the sequence $\{\phi_n(x)\}$ converges uniformly to the unique continuous solution of the integral equation $u(x)=f(x)+\lambda\int_a^b k(x,t)u(t)dt$ for finite value of λ provided $|\lambda|M(b-a)<1$,

where $M=\sup\{|k(x,t)|;\ (x,t)\in R\}$.

b) Convert the differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y = 0 \text{ with } b(0)=1,\ b'(0)=0 \text{ to}$$

an integral equation. 8+4

2. a) State and prove Fredholm's first fundamental relation.

b) Find non-trivial solutions of

$$\phi(x) = \lambda\int_0^1\left(3-\frac{3x}{2}\right) t\ \phi(t)dt,\ 0<x<1.$$ 7+5

3. a) Use Hilbert-Schmiidt theorem, to solve the following integral equation

$$\phi(x) = f(x) + \lambda\int_0^1 k(x,t)\phi(t)dt,\ 0\le x\le 1$$

when the kernel $k(x,t) = \begin{cases} x(1-t) & x<t \\ t(1-x) & x\ge t \end{cases}.$

b) Show that every eigenvalue of a symmetric kernel is real.

c) Define index of a root of $D(\lambda)=0$. 7+3+2

Part – II (Marks: 25)

Answer ***any five*** questions.

1. Use Fourier transform suitably to solve the following boundary value problem for $\phi(x,y)$.

$$\phi_{xx} + \phi_{yy} = 0,\ 0<x<\infty,\ y>0,$$

$\phi(0,y)=0$, $\phi(x,0)=f(x)$, where $f(x)$ is some known function of x and $\phi(x,y)\to 0$ as $\sqrt{x^2+y^2}\to\infty$.

Hence find $\phi(x,y)$ when $f(x)=1$. 4+1

2. Let $\mathcal{F}_c[f(x)] \equiv F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$

and $\mathcal{F}_c[g(x)] \equiv G_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx dx$, then show that $\int_0^\infty F_c(s)G_c(s)ds = \int_0^\infty f(t)g(t)dt$.

Hence show that $\int_0^\infty |F_c(s)|^2 ds = \int_0^\infty |f(x)|^2 dx$.

3. a) Derive Mellin Transform together with its inverse from Fourier Integral Theorem.

b) Prove that if $\lim_{x\to\infty} x^8 f(x) = 0$ then

$$M\left[\int_0^x f(t)dt; s\right] = -\frac{F(s+1)}{s}.$$ 3+2

4. a) Find Hankel transform of order zero of $\frac{\sin ar}{r}$, $a>0$.

b) If $F_v(\rho)\equiv\mathcal{H}_v[f(r)]$ and $G_v(\rho)\equiv\mathcal{H}_v[f(r)]$ then prove that

$$\int_0^\infty \rho F_v(\rho)G_v(\rho)d\rho = \int_0^\infty rf(r)g(r)dr.$$ 3+2