M. Sc. Mathematics Examination, 2023

(2nd Year, 1st Semester)

Mathematics

Paper – DSE-4

[Advanced Differential Geometry]

Time : Two hours Full Marks : 40

Notations / Symbols have their usual meanings

Answer ***any four*** of the followings.

1. a) Check whether the circle $C=\{(x,y)\in \mathbb{R}^2 / x^2+y^2=4\}$ is a differentiable manifold or not.

   b) Prove that

   $[fX,Y]=f[X,Y]-(Yf)X \quad \forall X,Y\in \infty(M)$ and $f\in F(M)$.

   c) Find the integral curve for the vector field $X=x^2\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^2}$ on $\mathbb{R}^2$. 4+3+3

2. a) If an 1-form ω on $\mathbb{R}^2$ is given by $\omega=x^1x^2dx^1+\frac{1}{2}\left\{\left(x^1\right)^2-x^2\right\}dx^2$, check whether ω is closed or not. 4+6

b) Define curvature tensor on a differentiable manifold M and find its local representation with respect to basis $\left\{\frac{\partial}{\partial x^i},\ i=1,2,\ldots,n\right\}$.

3. a) Prove that a differentiable manifold which is Hausdorff and second countable has a Riemannian metric.

b) What is sectional curvature of a Riemannian manifold? When a Riemannian manifold is said to be a manifold of constant curvature? Prove that a Riemannian manifold of constant curvature is an Einstein manifold. 4+6

4. a) When an almost complex manifold is called complex manifold? Give an example of complex manifold.

b) If N is a Nijenhuis tensor on an almost complex manifold, then prove that

i) $N(X,\bar{Y}) = N(\bar{X},Y)$

ii) $N(\bar{X},\bar{Y}) = -N(X,Y)$

c) State and prove the condition for an almost Hermite manifold to be Kähler. 3+4+3

5. a) If in a conformally flat Riemannian manifold M^n, $R(X,Y)A = AR(X,Y)$ holds, then prove that $\left(A^2 - \frac{rA}{n-1}\right)X \wedge X = 0$, where A is a symmetric transformation on M and r is a scalar curvature of M.

b) If a connection is given by $\bar{\nabla}_X Y = \nabla_X Y - T(X,Y)$ on a differentiable manifold, then prove that $\bar{\nabla}$ is a linear connection and $\bar{T} = -T$, where T is torsion tensor with respect to ∇ and $\bar{T}$ is torsion tensor with respect to $\bar{\nabla}$. 6+4

6. a) Prove that every almost contact manifold M admits a Riemannian metric g such that

i) $g(\phi X,\ \phi Y) = g(X,Y) - \eta(X)\eta(Y)$

ii) $g(X,\xi) = \eta(X)$

iii) $g(\phi X,Y) + g(X,\phi Y) = 0$

b) Construct an example of an almost contact manifold. 6+4