M. Sc. Mathematics Examination, 2023

Second year First Semester

DSE 03 (A4) [redacted] : Graph Theory I

Full Marks : 40 | Time : 2 Hours

Answer any **four** questions 4×10

1. (a) Define the Ramsey Number $r(m, n)$ for natural numbers m, n. Prove that $r(2, n) = n$ for all $n \geq 2$ and $r(3, 3) = 6$.

   (b) Define the adjacency matrix of a graph. Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and A be the adjacency matrix of G with respect to this ordering of vertices. Prove that the (i, j)-th entry of A^k denotes the number of distinct walks of length k from v_i to v_j in G for all positive integer k.

2. (a) Define a bipartite graph. Prove that a graph is bipartite if and only if it does not contain any cycle of odd length.

   (b) Define a cut edge of a graph. Let $G = (V, E)$ be a connected graph and $e \in E$. Then show that e is a cut edge if and only if e does not belong to any cycle of G.

3. (a) Define a Hamiltonian graph. Let G be a simple graph with $n > 2$ vertices. If degree of each vertex is at least $\frac{n}{2}$, then prove that G is Hamiltonian.

   (b) Define an Eulerian trail of a graph. Prove that a connected graph G has an Eulerian trail if and only if G has only two vertices of odd degree.

4. (a) Define a matching M and an M-augmenting path of a graph. Show that a matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.

   (b) Define a vertex cover of a graph G. Prove that in a bipartite graph G, the maximum size of a matching G is equal to the minimum size of a vertx cover of G.

5. (a) Define the chromatic number $\chi(G)$ of a graph G. If a graph G has a degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then show that $\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\}$.

   (b) Define the chromatic index $\chi'(G)$ of a graph G. Prove that in a bipartite graph G, $\chi'(G) = \Delta(G)$, where $\Delta(G)$ is the maximum degree of vertices of G.

6. (a) Define a chordal graph. Show that a graph G is chordal if and only if it has a perfect elimination ordering.

   (b) Define an interval graph. Show that an interval graph cannot contain an induced 4-cycle.