Ex/SC/MATH/PG/CORE/TH/10/2023(S)

M. Sc. Mathematics Examination, 2023

(2nd Year, 1st Semester, Special Supplementary)

Dynamical Systems

Paper – Core-10

Time : Two hours | Full Marks : 40

The figures in the margin indicate full marks.

Symbols / Notations have their usual meanings.

(Use separate answer script for each Part)

Part – I (Marks: 20)

Answer ***any TWO*** questions.

1. a) Describe the stability around (0,0) and sketch phase product of the system $\dot{X} = AX$, where

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \alpha, \beta \in \mathbb{R}$$

b) Define center manifold theorem for periodic orbits

c) Determine Hamiltonian function of the system

$$\dot{\theta} = \nu$$
$$\dot{\nu} = -\sin\theta$$

6+2+2

2. a) State Paincore-Bendixson Theorem. Use this Theorem to show that the following system has at least one Periodic orbit.

$$\dot{x} = x - y - z(x^2 + \tfrac{3}{2}y^2),\ \ \dot{y} = x + y(x^2 + \tfrac{1}{2}y^2)$$

b) Consider the nonlinear autonomous system

$\dot{x} = f(x),\ x \in \mathbb{R}^n$ with an isolated equilibrium point

[Turn over

$\overline{x}$. Define Lyapunav stability, asymptotic stability and global stability of the equilibrium point. 6+4

3. a) Prove that any nondegenerate critical point of an analytic Himiltonian system $\dot{x} = H_y\,(x, y)$, $\dot{y} = -H_x\,(x, y)$ is either a saddle or a center. Furthermore, (x_0, y_0) is a saddle iff it is a saddle of the function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a center for $\dot{x} = Hy$, $\dot{y} = -Hx$.

   b) Find the flow operator ϕ^t for the system $\dot{x} = x - x^2$. Verify the group properties of flow. 5+5

Part – II (Marks: 20)

Answer ***any four*** questions. 5×4=20

1. Derive the general solution of the difference equation $x(n+k) + p_1(n)x(n+k-1) + p_2(n)x(n+k-2) + \ldots\ldots$ $p_k(n)x(n) = 0$, when all the characteristic roots are real and distinct. Hence find the solution of all the characteristic roots are real and distinct. Hence find the solution of $x(n+2) - 5x(n+1) + 6x(n) = 0$; $x(3) = 0$, $x(4) = 12$, where n = 3, 4, 5..... 5

2. Find period-2 point(s) of the map $x(n+1) = -(x(n))^3$. Determine the stability of the point(s). Find the stable set if the point is attracting. 5

3. Solve the initial value problem $y(n+2) + y(n) = 10(3^n)$; $y(0) = 0$, $y(1) = 0$. 5

4. Solve the system of equations $x(n+1) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x(n)$, where $x(0) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Draw the phase portrait near origin taking different values of u_1 and u_2. 5

5. Discuss the bifurcation and draw the bifurcation curve for the map $x_{n+1} = rx_n - x_n^3$ at $r = 1$. 5

6. When is a map f said to have sensitive dependence at a point x_0? Derive a quantitative measure for the sensitive dependence. Hence prove that logistic map $x_{n+1} = 4x_n(1 - x_n)$ has chaotic orbit. 5