Ref. No. Ex/Math/PG/Core/TH/10/2023

Master of Science Examination, 2023

(Second Year, First Semester)

MATHEMATICS

CORE 10

(Dynamical System)

Time : Two Hours **Full Marks : 40**

The figures in the margin indicate full marks.

Symbols / Notations have their usual meanings.

(Use separate answer script for each group)

PART I (Marks: 20)

Answer **any two** questions.

1. (a) If $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then show that $e^{At} = e^{at}\begin{pmatrix} cosbt & -sinbt \\ sinbt & cosbt \end{pmatrix}$.

   (b) If (0, 0) is a focus of the Hamiltonian system in R^2 then show that (0, 0) is not a strict local optimum of the Hamiltonian function.

   (c) Find the evolution operator $\phi^t : R^3 \to R^3$ for the system $\dot{x} = f(x)$ with

$$f(x) = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}$$

   and hence show that the set $S = \{(x_1, x_2, x_3) \in R^3; x_3 = -x_1^2/3\}$ is invariant under the flow ϕ^t. [2 + 3 + 5]

2. (a) What do you mean by Heteroclinic bifurcation.

   (b) State and prove fundamental theorem for linear system in R^n.

   (c) Find α-limit set, ω-limit set, limit cycle(s) and attractor of the system

$$\dot{x} = x - y - x(x^2 + y^2)$$

$$\dot{y} = x + y - y(x^2 + y^2).$$

   Draw the phase portrait. [2 + 4 + 4]

3. (a) Define hyperbolic flow for the system $\dot{x} = Ax$, $x \in R^n$.

(b) Define basin of attraction of the equilibrium point for the system $\dot{x} = f(x)$, $x \in R^n$.

(c) State and prove Bendixson's criteria for the system $\dot{x} = f(x)$, $x \in R^2$.

(d) Determine the nature of the equilibrium point (s) of the system $\dot{x} = Ax$ where $A = \begin{pmatrix} -1 & 0 \\ 0 & \alpha \end{pmatrix}$, $x \in R^2$, $\alpha \in R$.
Draw the phase portrait.

[1 + 1 + 4 + 4]

Part-II (Marks: 20)
Answer **any four** questions.

1. If $x_1(n), x_2(n)$ and $x_3(n)$ be three solutions of $x(n+3) + p_1(n)x(n+2) + p_2(n)x(n+1) + p_3(n)x(n) = 0$, for $n \geq n_0$ and $W(n)$ be their Casoratian, then prove that (5)

$$W(n) = (-1)^{3(n-n_0)}\Big(\prod_{i=n_0}^{n-1} p_3(i)\Big)W(n_0).$$

2. Solve the difference equation $x(n+2) + 8x(n+1) + 7x(n) = n2^n$. (5)

3. Suppose that x^* is an equilibrium point of the difference equation $x_{n+1} = f(x_n)$ such that $f'(x^*) = -1$. Prove that x^* is unstable, if $S(f(x^*)) > 0$, where $S(f(x))$ denotes the Schwarzian derivative of $f(x)$. (5)

4. Find general solution of the system of equation (5)

$$\mathbf{x}(n+1) = \begin{pmatrix} -2 & 1 \\ -1 & 3 \end{pmatrix} \mathbf{x}(n).$$

Hence find the asymptotic behaviour ($n \to \infty$) of the solution. Draw the phase portrait near origin.

5. Find bifurcation at $\mu = 0$ in the system of maps $x_{n+1} = \mu + x_n - x_n^2$. Draw the bifurcation diagram. (5)

6. Define the term 'sensitive dependence on initial condition'. Derive a measure for it. (5)