Find the tensor algebra $T(M)$ of the $\mathbb{Z}$-module M when (i) $M = \mathbb{Q}/\mathbb{Z}$ and (ii) $M = \mathbb{Z}/n\mathbb{Z}$.

Suppose V is a 2-dimensional vector space over a field F with basis r, r'. Find the exterior algebra $\Lambda(V)$ of V.

2+2+3+3

Ex/M.Sc./Math/22/Unit-4.3/A2.1/2023

M. Sc. Mathematics Examination, 2023

(2nd Year, 2nd Semester)

Advanced Algebra - II

Unit – 4.3 (A 2.1)

Time : Two hours Full Marks : 50

The figures in the margin indicate full marks.

(Unexplained Symbols/Notations have their usual meaning.)

Special credit will be given for precise answer.

Answer any ***Five*** questions. 10×5=50

1. Justify the following with proper arguments.
   i) Ideal of a semisimple ring is semisimple.
   ii) Any matrix ring over a division ring is semisimple.
   iii) Any subdirect product of primitive rings is semisimple.
   iv) If G is a finite p group and F is a field with characteristic p then the group algebra $F[G]$ is not semisimple. 1+3+3+3

2. Let V be a vector space over a division ring D. Prove that the ring $L_D(V)$, of all D-linear transformations from V to itself, is a primitive ring.

   State the Jacobson Density Theorem on primitive rings. Characterize the primitive rings in the class of commutative rings. Hence determine if $R[x]$ is

[Turn over

primitive, where R is a commutative ring with identity. No dense ring of linear operators of an infinite dimensional vector space over a division ring can be Artinian – True or False? Answer with reason.

2+2+3+3

3. Let $R = M_n(D)$ be the matrix ring over a division ring D. Prove that R is Artinian, Simple, Primitive and is isomorphic to $L_D(V)$, the ring of all D-linear transformations from an n-dimensional vector space V over D to itself.

   State the general result related with the above particular case. 10

4. When is a property of rings said to be a radical property? Which radical property defines the Jacobson radical of a ring? Verify that it is actually a radical property of rings and Jacobson radical is the corresponding radical. What is the Jacobson radical of the power series ring $F[[x]]$ over a field F? 2+1+5+2

5. Suppose G be a cyclic group with a as a generator. Check if the representations ρ and σ of G, given by

$$\rho(a)=\begin{pmatrix}\omega & 0\\ 0 & \omega^2\end{pmatrix} \text{ and } \sigma(a)=\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$$

   are equivalent or not (ω is the complex cube root of unity).

State the column orthogonality relations involving the irreducible characters of a finite group. A certain group G has two columns of its character table as follows:

g_i	$g_1 = e$	g_2
$\lvert C_G(g_i)\rvert$	21	7
χ_1	1	1
χ_2	1	1
χ_3	1	1
χ_4	3	y
χ_5	x	$\bar{y}$

Find x and y. 3+2+5

6. Suppose G is a group and V is a vector space over a field F. Suppose $f : G \to GL(V)$ is a group homomorphism. What conditions are satisfied by the corresponding action of G on V. If a group G acts on a vector space V over a field F satisfying similar conditions, is it possible to obtain a group homomorphism from G to $GL(V)$? Determine the regular character of a finite group. Determine the permutation character of S_5. 2+3+3+2

7. Let R be a commutative ring with identity and V be a unitary R-module. What is meant by an n-multilinear alternating form on V (n is a positive integer)? What is meant by an $n \times n$ determinant functions on R?