M. Sc. Mathematics Examination, 2023

(1st Year, 2nd Semester)

Paper – Core-08

Functional Analysis

Time : Two hours Full Marks : 40

Use separate Answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (Marks: 20)

Answer ***any four*** questions. 4×5=20

1. a) Let X and Y be normal linear spaces. If X is finite dimensional, then prove that every linear operator $A: X \to Y$ is continuous. 3

   b) Show that if X is a Banach space, then every absolutely convergent series in it converges. 2

2. a) Let c be the normal linear space of all convergent scalar sequences with norm $\|\cdot\|_\infty$ and $f(x) = \lim_{n\to\infty} x(n)$, $x \in c$. Show that f is a bounded linear functional on c and find $\|f\|$. 3

   b) Show that c_{00} is not a Banach space under any norm. 2

3. a) Let X be a Banach space, Y be a normed linear space and $\mathscr{A}$ be a subset of $B(X, Y)$. If $\mathscr{A}$ is not uniformly bounded, then prove that there exists a dense subset

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D of X such that for every $x \in D$, $\{A(x) : A \in \mathscr{A}\}$ is not bounded in Y. 3

b) Show that if a normed linear space X is separable and Y is a closed subspace of X, then the quotient space X / Y is separable. 2

4. Let $X = c_{00}$ be a normed linear space with norm $\|\cdot\|_\infty$ and $T : X \to X$ be defined by $T(x)(j) = \frac{1}{j}x(j),\ j = 1,2,3,...$

   Show that T is bijective, linear and bounded, but T^{-1} is unbounded. Does this contradict the bounded inverse theorem? 5

5. Let E be a subset of a normed space X, Y = span E and $a \in X$. Show that $a \in \overline{Y}$ if and only if $f(a) = 0$ whenever $f \in X'$ and f is zero everywhere on E. 5

6. a) Prove or disprove: The space l^1 is not reflexive. 3

   b) Let X be a normed linear space. Show that if $x \in X$ is such that $f(x) = 0$ for every $f \in X'$, then $x = 0$. 2

Part – II (Marks: 20)

Answer ***any four*** questions. 4×5=20

1. Define unitary operator on a Hilbert space H. If T is a linear operator from H to H then prove that the following conditions are equivalent.

   i) T * T = I

   ii) $\langle Tx, Ty \rangle = \langle x, y \rangle\ \forall\ x, y \in H$

   iii) $\|Tx\| = \|x\|\ \forall\ x \in H$

2. a) Prove that an operator T on a Hilbert space is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in H$.

   b) Give an example of an isometric operator on a Hilbert space which is not unitary. 3+2

3. Define Projection operator on a Hilbert space with an example. Prove that for a projection operator P, $P^2 = P$ holds. 3+2

4. a) State and prove Cauchy-Schwarz imperality for an inner product space.

   b) Give an example of a Banach space which is not a Hilbert space with explanation. 3+2

5. a) Define complete orthonormal basis on a Hilbert space with an example. 1+4

   b) If $\{e_i\}$ is a complete orthonormal basis in a Hilbert space H, then for any $x \in H$, prove that $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$. What is the same of this series?

6. If H is a Hilbert space then for any continuous linear functional T on H prove that there exists a unique element $y \in H$ such that $T(x) = \langle x, y \rangle\ \forall\ x \in H$. 5