M. Sc. Mathematics Examination, 2023

(1st Year, 2nd Semester)

Complex Analysis and PDE

Paper – Core-07

Time : Two hours Full Marks : 40

Use separate Answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (Marks: 20)

Answer ***any four*** questions. 4×5=20

1. i) Evaluate by Cauchy's Residue theorem:

$\oint_{\mathcal{C}} \frac{e^{2z}}{(z+1)^4} dz$, where $\mathcal{C}$ is the circle $|z| = 3$.

ii) State and prove Fundamental Theorem of Algebra.

2+3

2. Suppose that f is analytic and bounded on the whole complex plane $\mathbb{C}$. Prove that f is constant. 5

3. i) Let $|F(z)| \le M / R^k$ for $z = \mathrm{Re}^{i\theta}$, where $k > 1$ and M are constants. Prove that $\lim_{R\to\infty} \int_{\Gamma} F(z) dz = 0$, where Γ is the semi-circular arc of radius R.

ii) Using (i) and Cauchy's Residue theorem, find $\int_0^{\infty} \frac{dx}{x^6+1}$. 3+2

4. State Cauchy's residue theorem. As an application of this theorem, show that the trigonometric integral $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{12}$. 5

5. Suppose that f is analytic inside and on simple closed curve $\mathcal{C}$. Then prove that the maximum value of $|f(z)|$ occurs on $\mathcal{C}$, unless f is a constant. 5

6. i) Let the Rectangular region $\mathcal{R}$ in the z-plane be bounded by $x=0, y=0, x=2$ and $y=1$. Determine the region $\mathcal{R}'$ of w-plane into which $\mathcal{R}$ mapped under the transformation $w=\sqrt{2}e^{i\pi/4}z+(1-2i)$.

   ii) Find bilinear transformation that maps points $z=0$, $-i, -1$ into $w=i,1,0$, respectively. 2+3

Part – II (Marks: 20)

Partial Differential Equations

Answer ***any two*** questions.

1. a) Prove that the Cauchy problem for an elliptic PDE described by $u_{xx}+u_{yy}=0$, $-\infty<x<\infty$, $y>0$, subject to $u(x,0)=0$, $u_y(x,0)=0$, $-\infty<x<\infty$, is an all-posed problem. It is given that the solution of the above problem is $\frac{\sinh(ny)\sin(nx)}{n^2}$, if the value of $u_y(x,0)$ is changed from 0 to $\frac{\sin nx}{n}$. 7

   b) Find the solution of the IBVP $\theta_t=\kappa\theta_{xx}$, $0<x<L$, $t>0$ together with $\theta(0,t)=0$, $\frac{\partial\theta}{\partial x}(L,t)=0$, $t>0$, $\theta(x,0)=\theta_0$, $0\le x\le L$, where is a constant. 7

2. a) Prove that, if a harmonic function vanishes everywhere on the boundary of a domain, then it is identically zero everywhere. 2

   b) Derive the Poisson integral solution of the interior Dirichlet problem for a circle given as:

   $\nabla^2\psi=0$, $0\le r<a$, $0\le\theta<2\pi$, along with $\psi(a,\theta)=f(\theta)$, $0\le\theta\le 2\pi$, where $f(\theta)$ is a continuous function of θ. 8

3. a) Find $u(1/2,\ 1/8)$ and $u(1,2)$, where $u(x,t)$ is the solution of the problem given below.

   $u_{tt}=4u_{xx},\ -\infty<x<\infty,\ t>0,$

   $$u(x,0)=\begin{cases} x/2, & 0\le x\le 1/2 \\ (1-x)/2, & 1/2\le x\le 1 \\ 0, & \text{elsewhere,}\end{cases}$$

   $u_t(x,0)=0,\ -\infty<x<\infty$. 2

   b) Find the solution of non-homogeneous wave equation described by

   $u_{tt}-c^2u_{xx}=x,\ -\infty<x<\infty,\ t>0$

   $u(x,0)=0,\ u_t(x,0)=3,\ -\infty<x<\infty$. 8