Ex/SC/MATH/PG/DSE/TH/01/A/2023

M. Sc. Mathematics Examination, 2023

(1st Year, 2nd Semester)

Paper – DSE-01A

Advanced Topology

Time : Two hours Full Marks : 40

Answer **Q. No. 1** and ***any three*** from the rest.

1. a) Answer with reasons whether $\mathbb{R}$ with lower limit topology is metrizable.

   b) Among all Havsdorff compactifications of a locally compact non-compact Tychonoff space can we find any smallest or largest element?

2. a) Prove that a topological space (X, τ) is Havsdorff iff every convergent net in X has a unique limit.

   b) Define an ultra filter. If $\mathcal{F}$ is a family of sets with finite intersection property then show that there exists an ultrafilter $\mathcal{F}^*$ containing $\mathcal{F}$.

   c) Show that the filter associated with a maximal net is an ultra filter. 5+6+1

3. a) Prove that a locally compact Havsdorff space is completely regular.

   b) Show that the one-point compactification of a non-compact space X is Havsdorff iff X is locally compact Havsdorff. 8+4

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4. a) Give an example to show that in general for a family of sets $\{B_i : i \in \Delta\}$, $\bigcup_{i\in\Delta} \overline{B_i} \subsetneqq \overline{\bigcup_{i\in\Delta} B_i}$. If the family $\{B_i : i \in \Delta\}$ is locally finite then prove that $\bigcup_{i\in\Delta} \overline{B_i} = \overline{\bigcup_{i\in\Delta} B_i}$.

b) Define a paracompact space. Give an example of a paracompact space which is not compact.

c) If every open cover of a topological space X has a closed locally finite refinement then show that X is paracompact. 3+2+7

5. a) Show that $\mathbb{R}^{\omega}$ with product topology is metrizable.

b) Let $\{(X_n, d_n)\}_n$ be a countable family of metrizable spaces. Let $diam(X_n) \leq M$ for all large n and $diam(X_n) \to 0$ as $n \to \infty$.

Define $e(x, y) = \sup_n \{d_n(x_n, y_n)\}$. Prove that τ_e (the topology corresponding to the inelitic e) is equal to the product topology $\prod_n (X_n, \tau_{d_n})$. 8+4

6. a) Let X and Y be topological spaces and let $p : X \to Y$ be a subjective continuous map. Show that following are equivalent.

i) The pair (Y, p) is a quotient space for some equivalence relation is on X.

ii) $U \subset Y$ is open in Y iff $p^{-1}(U)$ is open in X.

iii) $F \subset Y$ is closed in Y iff $p^{-1}(F)$ is closed in Y.

b) Give an example to show that the quotient space of a Havsdorff space may not be Havsdorff.

c) Give an explicit construction of a cylinder as a quotient space of the unit square. 6+3+3