Ex/SC/MATH/PG/CORE/TH/02/2023

# M. Sc. Mathematics Examination, 2023

( 1st Year, 1st Semester )

## Mathematics

## Paper – Core-02

## [ Real Analysis ]

Time : Two hours Full Marks : 40

Answer Question No. **1** and any **four** questions from the rest.

All questions carry equal marks.

Notations and Symbols have their usual meanings.

1. a) Give an example of an uncountable set of Lebesgue measure zero.

   b) If $f$ and $g$ are measurable functions defined on a measurable set $D$ then prove that the set $\{x \in D : f(x) \neq g(x)\}$ is measurable.

   c) Let $\mathbf{R}$ be a ring of subsets of a set $X$. Prove that

   $$\mathbf{C} = \{A \subset X : \text{ either } A \text{ or } A^c \in \mathbf{R}\}$$

   is an algebra of subsets of $X$. 2+3+3

2. a) Define Lebesgue outer measure. Show that Lebesgue outer measure is countably sub-additive.

   b) For a monotonically decreasing sequence of Lebesgue measurable sets $\{E_n\}_n$ where $\mu(E_n) < \infty$ for some $n$, prove that $\mu(\lim_n E_n) = \lim_n \mu(E_n)$. Give an example to show that the condition

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"$\mu(E_n) < \infty$ for some $n$" is essential. 4+4

3. a) Prove that $[0, \infty)$ is Lebesgue measurable and hence show that every open set is Lebesgue measurable.

   b) Prove that a function $f$ on a measurable set which is continuous almost everywhere is measurable. 5+3

4. a) If a sequence of measurable functions $\{f_n\}_n$ is convergent almost everywhere to a measurable function $f$ on a measurable set $E$ with $\mu(E) < \infty$ then show that $f_n$ converges to $f$ in measure.

   b) When a sequence of functions $f_n$ is said to converge almost uniformly. Does convergence in measure implies convergence almost uniformly? Answer with reasons. 5+3

5. a) State and prove bounded convergence theorem.

   b) Give an example to show that if a function $f$ is Riemann integrable in the improper sense then it may not be Lebesgue integrable. 6+2

6. a) Prove that $f$ is a function of bounded variation on $[a,b]$ if and only if $f$ can be expressed as the difference of two increasing functions on $[a,b]$.

   b) If $f:[a,b] \to \mathbb{R}$ is non-decreasing then show that the derivative $f'$ is Lebesgue integrable with $\int_a^b f' d\mu \le f(b) - f(a)$. 4+4



7. a) Prove that for any sequence of non-negative measurable functions $\{f_n\}_n$,

$$\int \liminf f_n d\mu < \liminf \int f_n d\mu .$$

   b) Let $\mu$ be a measure on a ring **R** and let $\mu^*$ be the extension of $\mu$ to an outer measure on **H**(**R**). Let $\overline{S}$ be the collection of $\mu^*$–measurable sets and $\overline{\mu}$ be the extension of $\mu$ to $\overline{S}$. Then prove that every set $E$ of **H**(**R**) with σ-finite outer measure has a measurable cover $F$ such that $\overline{\mu}(F) = \mu^*(E)$. 3+5

8. a) Prove that **S** is a σ-ring if and only if it is a ring and a monotone class.

   b) For a ring **R**, show that **S**(**R**) = **M**(**R**). 2+6