M. Sc. Mathematics Examination, 2023

(1st Year, 2nd Semester)

Paper – Core-09

Ordinary Differential Equations and Special Functions

Time : Two hours Full Marks : 40

Use separate Answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (Marks: 20)

Answer ***any two*** questions. 10×2=20

1. Find the Green's function of the boundary value problem

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-\frac{6}{x^2}y=\begin{cases} x & \text{for } 0<x<1 \\ 0 & \text{for } x>1 \end{cases}$$

subject to the boundary conditions:

$y \to 0$ as $x \to 0$ and $y \to 0$ as $x \to \infty$.

Using the Green's function, solve the boundary value problem. 7+3

2. a) Show that the operator $L = p\frac{d^2}{dx^2}+q\frac{d}{dx}+r$ is self-adjoint if and only if $\frac{dp}{dx}=q$, where p, q, r are differentiable functions of x.

b) Prove that

$$\int_{-1}^{1} P_m(z)P_n(z)dz = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{2}{2n+1} & \text{for } m = n \end{cases}$$

where $P_m(z)$ is the Legendre polynomial of degree m. 4+6

3. a) Find the analytic solution in the neighbourhood $z=0$ of the equation: $(1-z^2)w'' - z^2 w' + zw = 0$.

b) State and prove Abel's Formula. 6+4

Part – II (Marks: 20)

Answer ***any two*** questions. 10×2=20

1. Legendre function of second kind $Q_n(z)$ is defined by

$$Q_n(z) = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)} z^{-n-1} F\left(\frac{n}{2}+1, \frac{n+1}{2}, n+\frac{3}{2}, z^{-2}\right)$$

If n is not a zero or integer and z is not a real number lying in [–1, 1] then $Q_n(z)$ is also defined as

$$Q_n(z) = \frac{1}{2^{n+2} i \sin(n\pi)} \int_\Gamma \frac{\left(t^2-1\right)^n}{(z-t)^{n+1}} dt$$

where Γ is a contour to be specified by you.

Reconcile the two definitions. 10

2. Consider a second order differential equation of following type

$$L_z(w) = \left(a_0 z^2 + b_0 z + c_0\right)\frac{d^2 w}{dz^2} + \left(a_1 z + b_1\right)\frac{dw}{dz} + a_2 w = 0$$

where $a_i''s$, $b_i''s$ and c_0 are known constants. Find a contour integral solution of the above differential equation in the form

$$w = \int_\Gamma (t-z)^{\lambda+1} v(t)\,dt$$

where the contour Γ, the constant λ and the function $v(t)$ are to be specified by you. 10

3. For the differential equation

$$\frac{d^2 w}{dz^2} = p_1(z)\frac{dw}{dz} + p_2(z)w$$

state how fundamental equation belonging to the singularity z_0 is defined for the choice of $w_1(z)$ and $w_2(z)$ of the fundamental system of solutions. Show that fundamental equation is an invariant. 10