Ex/SC/MATH/PG/CORE/TH/06/2023

M. Sc. Mathematics Examination, 2023

(1st Year, 2nd Semester)

Paper – 06

Linear Algebra and Module Theory

Time : Two hours Full Marks : 40

The figures in the margin indicate full marks.

(Unexplained Symbols/Notations have their usual meaning.)

Special credit will be given for precise answer.

Answer **Q. No. 4** and ***any three*** from the rest. 16+(8×3)=40

1. a) Define a torsion module and a finitely generated module. 1

   b) Suppose V is a vector space over a field F and T is a linear operator on V. Then V is an $F[x]$-module via T. Prove that W is an $F[x]$-submodule of V if and only if W is a T-invariant subspace of the vector space V over F. Also prove that if as a vector space over F, $\dim V$ is finite then as an $F[x]$-module V is a finitely generated torsion module. 5

   c) Give an example, with explanation, of a linear operator T on $\mathbb{R}^4$ such that the $\mathbb{R}[x]$-module via T becomes a cyclic module. 2

2. a) Find the Jordan form J of the real matrix $A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$.

Find an invertible real matrix P such that $P^{-1}AP = J$. Compute e^{tJ}. Using e^{tJ} and P solve the differential equation $\frac{dX}{dt} = AX$. Hence find the solution of $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$. 6

b) What are the invariant factors of the 100 × 100 real matrix with each entry 3? 2

3. a) Suppose A is a real matrix with the characteristic polynomial $(x+2)^4(x-5)^5$ and the minimal polynomial $(x+2)^2(x-5)^2$. Find all possible invariant factors, elementary divisors and then find the corresponding rational canonical forms (with respect to both invariant factors and elementary divisors) and Jordan canonical forms of A. 6

b) Let $V = \mathbb{R}[x]/\langle (x-3)^4 \rangle$ be the canonical $\mathbb{R}[x]$-module. Then V is a vector space over $\mathbb{R}$ and $T : V \to V$, $v \mapsto xv$ is a linear operator. Find a suitable basis of the vector space V over $\mathbb{R}$ with respect to which the matrix of T is the elementary Jordan matrix corresponding to 3 of order 4. 2

4. a) Suppose A is a complex $n \times n$ matrix such that $A^k = I$, k is a positive integer. Then A is diagonalizable. True or False? Justify. (I is the identity matrix of order n) 1

b) Suppose B is a real matrix of order 2023. Prove that $x^2 + x + 1$ cannot be the minimal polynomial of A. What happens if the order of B is 2024? Justify your answer. 2

c) Suppose T is a linear operator on a vector space V over a field F such that T has more than one eigenvalues. Prove that V has at least one subspace which is not T-invariant. 2

d) Suppose T be the linear operator on $\mathbb{R}^4$ which is represented in the standard ordered basis by the matrix $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & b & 0 \end{pmatrix}$.

Consider three cases $a = b = 1$; $a = b = 0$; $a = 1, b = 0$ and find, with least possible computation, the characteristic polynomial, the minimal polynomial for T and the geometric multiplicities of the eigenvalues in each case. 2

e) Suppose F is a field. Let A and B be two 3 × 3 matrices over F with the same minimal polynomial and the same characteristic polynomial. Prove that A

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and B are similar. Does the result hold if A and B are of order 4? Justify your answer. 4

f) Suppose F is a field and A, B are two $n \times n$ matrices over F. Then AB and BA have the same minimal polynomial. True or False? Justify. 2

g) The companion matrix of a monic polynomial over a field cannot be triangular. True or False? Justify. 1

h) Suppose T is a diagonalizable linear operator on a finite dimensional vector space V over a field F. Suppose T has a cyclic vector. Then T has n (=dim$V$) distinct eigenvalues. True or False? Justify. 1

i) The Jordan form and the rational canonical form of a matrix of order greater than 1 cannot be identical. True or False? Justify. 1

5. a) The characteristic polynomial of the linear operator T on $\mathbb{R}^3$, defined by

$$T(x, y, z) = (3x + y - z, 2x + 2y - z, 2x + 2y)$$

for all $(x, y, z) \in \mathbb{R}^3$, is $(x-1)(x-2)^2$. Find the minimal polynomial of T with the least possible computation. Then apply the Primary Decomposition Theorem on T to find a diagonalizable operator D on $\mathbb{R}^3$ and a nilpotent operator N on $\mathbb{R}^3$ such that $T + D = N$ and $DN = ND$. Find also the matrices of D and N in the standard ordered basis of $\mathbb{R}^3$. 5

b) Suppose V is a vector space over a field F and T is a linear operator V. Suppose v, w are eigenvectors corresponding to two distinct eigenvalues of T. Prove that $v + w$ cannot be an eigenvector of T. State the general result in this regard. 3

6. a) Suppose T is a linear operator on a vector space V over a field F such that T commutes with every projection of V. Deduce at least four conclusions about T. 3

b) Let $V = M_{n \times n}(F)$ (F is a field). Let $A \in V$. Let T, U be the linear operators on V defined by

$$T(B) = AB$$
$$U(B) = AB - BA \text{ for all } B \in M_{n \times n}(F)$$

i) If A is diagonalizable then T is diagonalizable. True or False? Justify.

ii) If A is diagonalizable then U is diagonalizable. True or False? Justify. 1+4