Ex/SC/MATH/PG/CORE/TH/01/2023

M. Sc. Mathematics Examination, 2023

(1st Year, 1st Semester)

Mathematics

Paper – Core-01

[Abstract Algebra]

Time : Two hours | Full Marks : 40

Unexplained Notations and Symbols have their usual meanings.

Special credit will be given for precise answer.

1. Answer ***any two*** from the following questions: 5×2=10

   a) i) Establish an isomorphism to prove that the fields $\mathbb{Q}\left(\sqrt[3]{2}\right)$ and $\mathbb{Q}\left(\sqrt[3]{2}e^{\frac{2\pi i}{3}}\right)$ are isomorphic. 2

   ii) Suppose F is an extension field of K with degree p (a prime number). Prove that for any $\alpha \in F - K$, $F = K(\alpha)$. 1

   iii) Prove that a field of order p^r (p a prime number) contains a subfield of order p^k if and only if k divides r. 2

   b) i) Prove that if F is a finite extension field of K then F is finitely generated and algebraic over K. Give an example to illustrate that a finitely generated extension need not be an algebraic extension. 2+1

[Turn over

4. Answer ***any two*** from the following questions: 5×2=10

   a) Prove that every nilpotent group is solvable. The converse is not true. Illustrate with an example. 3+2

   b) Give an example with explanation of

   i) a Noetherian ring which is not Artinian; 1

   ii) a ring R such that its every proper ideal is finite and hence it is Artinian but it is not Noetherian. 4

   c) i) Prove that for any field F, any homomorphic image of the polynomial ring $F[x]$ is Noetherian. Is the result true if F is replaced by $\mathbb{Z}$? 1+1

   ii) Prove that any Artinian integral domain is a field. 3

ii) Let p be a prime number such that the regular p-gon can be constructed by ruler and compass. Prove that $p = 2^r + 1$ for some positive integer $r > 0$. 2

c) Prove that the set $\mathbb{A}$ of algebraic numbers

i) is a field and is algebraic over $\mathbb{Q}$; 2

ii) is algebraically closed; 1

iii) is not a finite extension of $\mathbb{Q}$. 2

2. Answer ***any two*** from the following questions: 5×2=10

a) Let K be a field and $K(x)$ be the field of rational functions over K. Prove that

i) x is transcedental over K; 1

ii) x is algebraic over any intermediate field $E \neq K$; 3

iii) for any intermediate field $E \neq K$, $[K(x):E]$ is finite. 1

b) i) Suppose K is a field. Prove that if K is infinite then $K(x)$ is a Galois extension of K. 4

ii) Suppose β is a root of $x^4 + x + 1 \in \mathbb{Q}[x]$ and $\alpha = \sqrt[3]{2}$. Find the degree of $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$. 1

c) Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

i) Prove that F is a normal and separable extension of $\mathbb{Q}$. 2

ii) Prove that the fixed field of the Galois group $\mathrm{Aut}_{\mathbb{Q}}F$ is $\mathbb{Q}$. 1

iii) Applying the Fundamental theorem for Galois theory for finite dimensional extension, determine the number of intermediate fields between $\mathbb{Q}$ and F. 1

iv) Without finding explicitly explain that F has a primitive element over $\mathbb{Q}$. 1

3. Answer ***any two*** from the following questions: 5×2=10

a) i) Give an example, with explanation, of a non separable field extension as well as of a non normal field extension. 2+1

ii) Let n be a positive integer and F be a cyclotomic extension of order n of $\mathbb{Q}$. Then using the fact that the nth cyclotomic polynomial over $\mathbb{Q}$ is irreducible, prove that the degree of F over $\mathbb{Q}$ is $\phi(n)$. 2

b) Stating (no proof is required) the relevant result find the Galois group of $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$. Then show that this group is not solvable. 1+4

c) Define a perfect field. State three equivalent conditions for a field to be perfect. Prove that every finite field is perfect. 5