Ex/SC/MATH/UG/CORE/TH/13/2023

B. Sc. Mathematics (Hons.) Examination, 2023

(3rd Year, 2nd Semester)

Group Theory - II

Paper – Core-13

Time : Two hours Full Marks : 40

All questions carry equal marks.

Answer ***any four*** questions. 4×10

Let $\mathbb{N}$ be the set of natural numbers.

1. a) Define the set of all automosphisms $\mathscr{A}(G)$ of a group G. Show that $\mathscr{A}(G)$ is a group for any group G. If G is a finite cyclic group of order n, then determine $\mathscr{A}(G)$.

   b) Prove that every finite group having more than two elements has an automorphism other than the identity map.

2. a) Let $A(X)$ denote the permutation group on a nonempty set X. Let G be a group and H be a subgroup of G. Let S be the set of all left cosets of H in G. Then prove that there exists a homomorphism from G into $A(S)$ whose kernel is the largest normal subgroup of G contained in H.

   b) Let G be a group of order pm, where p is a prime and $p > m \in \mathbb{N}$. Show that every subgroup of order p is normal in G.

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3. a) Let G be a finite group such that a prime p divides the order of G. Show that G has an element of order p.

   b) Let G be a group of order p^n, where p is prime and $n \in \mathbb{N}$. Let $H \neq \{e\}$ be a normal subgroup of G, where e is the identity of G. Then show that $H \cap Z(G) \neq \{e\}$, where $Z(G)$ is the center of G.

4. a) Show that any group of order 99 is abelian. Hence find all non-isomorphic groups of order 99.

   b) Let G be a group of order 231. Prove that G has a normal subgroup of order 11 which is lying in the center of G.

5. a) Define a *simple group*. Prove that A_5, the group of all even permutations on $\{1, 2, 3, 4, 5\}$ is simple.

   b) Let G be a simple group of order 168 and H be subgroup of G of order 7. Show that the order of the normalizer $N_G(H)$ of H in G is 21. Hence show that G has no subgroup of order 14.

6. a) Let $(G, +)$ be an abelian group and r be an integer. Let $G[r] = \{g \in G \mid rg = 0\}$ and $rG = \{rg \mid g \in G\}$. Show that G[r] and rG are subgroups of G and $G / G[r] \cong rG$.

   b) If G is a finite abelian group and $n \in \mathbb{N}$ divides the order of G, then show that the number of solutions of $x^n = e$ in G is a multiple of n.