b) Define chaotic orbit. Prove that the map $f(x) = 2x(\text{mod}\,1)$ defined on the real line R has chaotic orbit. 6

c) Write a short note on Figenbaum number. 2

B. Sc. Mathematics (Hons.) Examination, 2023

(3rd Year, 2nd Semester)

Dynamical Systems

Paper – DSE-4E

Time : Two hours Full Marks : 40

The figures in the margin indicate full marks.

Symbols / Notations have their usual meanings.

Part – I (Marks: 20)

Answer **Q. No. 5** and ***any two*** from the rest.

1. The canonical form of a linear system is given by :

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where μ_1, μ_2 are real and distinct. Discuss the stability of the origin (0, 0) and draw all possible phase diagrams of the trajectories. 6

2. Define periodic orbit of a dynamical system. Show that the system

$$\dot{x} = y + \frac{1}{4}x\left(1 - 2r^2\right),$$
$$\dot{y} = -x + \frac{1}{2}y\left(1 - 2r^2\right),$$

where $r^2 = x^2 + y^2$, has a periodic orbit. 6

3. a) For a nonlinear system, define the following: (i) Lyapunov stability, (ii) Quasi-asymptotic stability, (iii) Asymptotic stability, (iv) Global stability.

b) For what value of the parameter $p \in \mathbf{R}$ will the zero solution be stable for the following autonomous system?

$$\dot{x} = px + y$$
$$\dot{y} = -x \quad . \qquad 3+3$$

4. State Lyapunov theorem of stability. Use it to show that the trivial equilibrium of the Lorenz system is asymptotically stable:

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = xy - bz,$$

where $\sigma > 0,\ b > 0,\ 0 < r < 1$. 6

5. What do you mean by a bifurcation? State Hopf bifurcation theorem for a planar system. Find the restriction on the parameter $c \in \mathbf{R}$ for which the following system exhibits a Hopf bifurcation around $(0,0)$:

$$\dot{x} = y,$$
$$\dot{y} = -x - \left(x^2 - c\right)y. \qquad 8$$

Part – II (Marks: 20)

Answer ***any two*** questions.

1. a) Derive the condition for which a fixed point of the one dimensional differential equation $\frac{dx}{dt} = f(x)$ is linearly stable. When does the linear theory fail? Discuss the stability of the fixed point x^*, such that $\frac{df}{dx}\left(x^*\right) = 0$. 5

   b) Apply linear stability analysis and geometrical method to find asymptotic behaviour of the solution of the differential equation $\frac{dx}{dt} = ax - x^3$, where a can be positive, negative or zero. Discuss all three cases. 5

2. a) For what values of k does the equation $\frac{d\theta}{dt} = \sin k\theta$ give a well defined vector field on the circle? With these values of k find and classify all the fixed points and sketch the phase portrait on the circle for the vector field. 4

   b) Define box dimension. 1

   c) Consider a unit square. Delete a symmetric cross from the middle, leaving four corner squares with side length 1/3. Repeat this step with each remaining squares. Draw the figures for first two steps. The limiting set of this process is called Sierpinski carpet. Show that the Sierpinski carpet has zero area. Find the box dimension of the limiting set. 5

3. a) Prove that the Lyapunov exponent corresponding to a stable p-cycle is negative. 2