4. Define isolated singularity. Prove that a meromorphic function cannot be bounded in the neighbourhood of an isolated singularity. 1+4=5

5. i) Expand $f(z)=\dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for $1<|z|<2$. 3+2=5

   ii) Determine and classify all the singularities of the function $f(z)=z/\left(e^{1/z}-1\right)$.

6. i) Let $f(z)=u(z)+iv(z)$ be an entire function in the complex plane $\mathbb{C}$. If $|u(z)|<M$ for every z in $\mathbb{C}$, where M is a positive constant, then prove that f is a constant function.

   ii) Let f be an entire function such that $|f(z)|\geq 1$ for every z in $\mathbb{C}$. Prove that f is a constant function.

   3+2=5

$|u(z)| < M$

B. Sc. Mathematics (Hons.) Examination, 2023

(3rd Year, 1st Semester)

Metric Space & Complex Analysis

Paper – CORE - 12

Time : Two hours Full Marks : 40

Symbols / Notations have their usual meanings.

Goup – A (20 Marks)

(Metric Space)

Answer ***any four*** questions.

All questions carry equal marks. 4×5=20

1. Let (X,d_X) and (Y,d_Y) be metric spaces and $f:X\to Y$ be a map. Then show that the following statements are equivalent:

   i) f is continuous on X;

   ii) $\overline{f^{-1}(B)}\subseteq f^{-1}(\overline{B})$ for all subset B of Y;

   iii) $f(\overline{A})\subseteq \overline{f(A)}$ for all subset A of X.

2. Define homeomorphism between two metric spaces (X,d_X) and (Y,d_Y). Show that the function $f:\mathbb{R}\to(-1,\ 1)$ defined by $f(x)=x/(1+|x|)$ is a homeomorphism under usual metric. Also, show that f is uniformly continuous. 1+3+1=5

3. Let (X, d) be a metric space.

a) Then the following statements are equivalent:

i) (X, d) is disconnected;

ii) there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0), where $X_0 = \{0,1\}$.

b) Define equivalent metrics. Give examples of two equivalent metrics. Show that the metrics d_1 and d_2 defined on $C[0, 1]$ as

$$d_1(f,g) = \sup\{|f(x) - g(x)| : x \in |0,1|\},$$

$$d_2(f,g) = \int_0^1 |f(x) - g(x)| dx$$

are not equivalent. 2+(1+2)=5

4. Define path connected space. If a metric space (X, d) is path connected, then show that it is connected. Give an example to justify that the converse is not true.

1+3+1=5

5. Let K be defined and continuous on $[a,b]\times[a,b]$, g be continuous on $[a,b]$ and let λ be a real number. Prove that the integral equation

$$f(x) = \lambda\int_a^b K(x,y) f(y) dy + g(x),\ x \in [a,b]$$

has unique solution in $C[a,b]$ for sufficiently small λ.

5

6. i) Define compact metric space.

ii) If Y is a compact subset of (X, d), then prove that Y is closed and bounded.

iii) Prove that continuous image of a compact set is compact. 1+2+2=5

Goup – B (20 Marks)

(Complex Analysis)

Answer ***any four*** questions.

All questions carry equal marks. 4×5=20

1. i) State the Cauchy's integral formula.

ii) a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$,

b) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$

where C is the circle $|z| = 3$. 1+2+2=5

2. Prove that every polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$, where the degree $n \ge 1$ and $a_n \ne 0$, has at least one root.

5

3. Suppose $\tan z$ is expanded into a Laurent series about $z = \pi/2$. Show that:

i) the principal part is $-1/(z - \pi/2)$,

ii) the series converges for $0 < |z - \pi/2| < \pi/2$,

iii) $z = \pi/2$ is a simple pole. 2+2+1=5