Ex/SC/MATH/UG/CORE/TH/10/2023

B. Sc. Mathematics (Hons.) Examination, 2023

(2nd Year, 2nd Semester)

Ring Theory and Linear Algebra - II

Paper – Core-10

Time : Two hours Full Marks : 40

Use separate answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I

All questions carry equal marks.

Answer ***any four*** questions. 4×5=20

Let $\mathbb{Z}$ be the set of all Integers.

1. Define prime and irreducible elements in a commutative ring with identity. Give an example of a prime element which is not irreducible and an example of an irreducible element which is not prime. Prove that $a+ib$ is irreducible in $\mathbb{Z}[i]$ if a^2+b^2 is irreducible in $\mathbb{Z}$.

2. Define a principal ideal domain. Show that in a principal ideal domain, a nonzero nonunit element p is irreducible if and only if p is prime.

3. Show that 2 and $1+i\sqrt{5}$ are relatively prime in $\mathbb{Z}[i\sqrt{5}]$.

4. Define a Euclidean domain. Let R be a Euclidean domain with Euclidean norm δ. Let $a,b \in R \setminus \{0\}$. Then show that b is a unit in R if and only if $\delta(a)=\delta(ab)$.

5. Let R be a commutative ring with identity such that $R[x]$ is a principal ideal domain. Show that R is a field.

6. Show that the polynomial $x^7 - 9x^4 + 11$ is irreducible in $\mathbb{Z}[x]$.

Part – II

Answer ***any five*** questions. 5×4=20

1. Find the algebraic and geometric multiplicities of the eigenvalues of the matrix $\begin{bmatrix} -1 & 1 & 1 \\ -3 & 3 & 1 \\ -4 & 3 & 2 \end{bmatrix}$. Hence justify whether the matrix is diagonalizable or not.

2. Let V be a finite-dimensional vector space and T be a linear operator on V. Then show that T is invertible if and only if the constant term in the minimal polynomial of T is non-zero.

3. i) If f is a non-zero linear functional on a vector space V then show that the kernel of f is a hyperspace of V.

   ii) Let $f : R^3 \to R$ be defined by $f(x, y, z) = x + y + z$. Find a basis for the kernel of f.

4. Let $T : R^2(R) \to R^2(R)$ be defined as $T(x, y) = (y, -x)$ and $S : C^2(C) \to C^2(C)$ be defined as $S(z, w) = (w, -z)$. Then find the eigen-values of T and S, if they exist.

5. Using Gram-Schmidt orthogonalisation process construct an orthonormal basis from the basis {(–1, 0, 1), (1, –1, 1), (0, 0, 1)} of R^3.

6. i) Let x, y be eigenvectors corresponding to the distinct eigenvalues λ, μ of a linear operator T defined on an inner product space X. Justify whether the vectors x, y are orthogonal or not.

   ii) Let V be a real inner product space and T be a linear operator on V such that $\langle Tv, v \rangle$ is real for all $v \in V$. Justify whether T is self adjoint or not.

7. State Spectral theorem for a normal operator T on a finite-dimensional inner product space. Using this prove that a normal operator T is unitary if and only if all the eigenvalues of T are of unit modulus.

8. i) Show that for a normal operator T, a scalar λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

   ii) Give an example of an orthogonal operator on R^3 with justification.