B. Sc. Mathematics (Hons.) Examination, 2023

(2nd Year, 1st Semester)

Ring Theory and Linear Algebra - I

Paper – Core-6

Time : Two hours | Full Marks : 40

Use separate Answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (20 Marks)

Answer ***any four*** questions. 5×4=20

1. Does there exist a ring R having no identity but a subing S with identity? Justify. Give an example of a ring R with identity 1_R and a subring S with identity 1_S such that $1_R \neq 1_S$. If R is an integral domain then show that $1_R = 1_S$. 2+1+2

2. Define field. Show that an integral domain with finite number of ideals is a field. Hence conclude that every finite integral domain is a field. 1+3+1

3. Let R be a commutative ring with identity and N be the set of all nilpotent elements of R. Show that N forms an ideal of R and the quotient ring R/N has no non-zero nilpotent elements. Is commutativity of the ring R essential in the above result? Justify. (1+2)+2

4. Let R and R' be two rings and $f : R \to R'$ be a mapping. When is f said to be a ring homomorphism? Let I be an

ideal of R. Is $f(I)$ an ideal of R'? Justify.

Let F be a field, R be a ring and $f : F \to R$ be a homomorphism of rings.

i) Show that f is either a monomorphism or a zero homomorphism.

ii) If there is a non zero element $\lambda \in F$ such that $f(\lambda) = 0_R$, then show that f is the zero homomorphism. 1+1+(2+1)

5. State First Isomorphism Theorem for ring. By using this theorem, show that every epimorphism from the ring $\mathbb{Z}$ of integers onto itself is an isomorphism. Is every monomorphism from the ring $\mathbb{Z}$ of integers into itself an isomorphism? Justify. 1+2+2

6. Let R be a ring with identity such that $x^2 = x$ for all $x \in R$. Show that characteristic of R is 2 and R is a commutative ring. Is every prime ideal of R a maximal ideal of R? Justify. (1+2)+2

Part – II (20 Marks)

(Linear Algebra)

1. Answer ***any two*** from the following questions: 3×2=6

a) Define

i) the column rank and the row rank of an $m \times n$ matrix A over a field F.

c) i) Prove that eigenvectors corresponding to two distinct eigenvalues of T are linearly independent.

ii) Suppose V is finite dimensional and $B = \{v_1, v_2, \ldots\ldots, v_n\}$ is a basis of V such that v_i is an eigenvector of T corresponding to an eigenvalue λ_i, for all $i = 1, 2, \ldots\ldots n$. Find the matrix of T with respect to the basis B. 2+1

4. Answer ***any one*** from the following questions : 2×1=2

a) Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x + y, y, z)$. Find the eigenvalues of T and their geometric multiplicities. 2

b) Determine the subspaces of the vector space $\mathbb{R}^2$. 2

b) Suppose the matrix of the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard ordered basis of $\mathbb{R}^3$ is $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$. Find $T(x,y,z)$ for any $(x,y,z) \in \mathbb{R}^3$ and find the matrix B(say) of T with respect to the ordered basis $\{(1,1,0),(0,0,1),(0,1,1)\}$. 2+1

c) Verify that $B = P^{-1}AP$ where A, B are as in (b) and P is as in (a)(ii). 3

3. Answer ***any two*** from the following questions: 3×2=6

a) Let V be a vector space over a field F and $T:V \to V$ be a linear operator on V.

i) Prove that $v \in V$ is an eigenvector of T corresponding to an eigenvalue $\lambda \in F$ if and only if $v \neq 0$ and $v \in \ker(T - \lambda I)$.

ii) Prove that $\lambda \in F$ is an eigenvalue of T if and only if $Null(T - \lambda I) \geq 1$. 2+1

b) If V is finite dimensional then prove that $\lambda \in F$ is an eigenvalue of T if and only if λ is a root of the polynomial $\det(xI_n - A)$ where $n = \dim V$, I_n is the identity matrix of order n over F and A is any matrix of T. 3

ii) the rank and nullity of a linear transformation between two vector spaces. 2+1

b) Suppose F is a field, m, n are positive integers and A is an $m \times n$ matrix over F. Prove that the mapping $L_A : F^n \to F^m$, $X \mapsto AX$ is a linear transformation. Conversely, prove that if $T:F^n \to F^m$ is a linear transformation then there exists an $m \times n$ matrix A over F such that $T = L_A$. 3

c) State and prove the rank nullity theorem for a linear transformation. 3

d) Prove that there cannot exist any onto linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^7$. Generalize this result. 3

2. Answer ***any two*** from the following questions : 3×2=6

a) i) Does there exist a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^4$ such that $T(1,1,0) = (1,2,3-1)$, $T(0,1,1) = (0,1,3,-2)$? Answer with precise reason.

ii) Find the change of basis matrix P(say) for the change from the basis $B_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ to $B_2 = \{(1,1,0),(0,0,1),(0,1,1)\}$. 2+1