Ex/SC/MATH/UG/CORE/TH/08/2023

B. Sc. Mathematics (Hons.) Examination, 2023

(2nd Year, 2nd Semester)

Riemann Integration & Series of Functions

Paper – Core-8

Time : Two hours Full Marks : 40

The figures in the margin indicate full marks.

Symbols / Notations have their usual meanings.

Use separate answer script for each Part.

Part – I (Marks: 20)

Answer ***any four*** questions. 4×5=20

1. A function *f* is defined by $f(x)=x^2$, $x\in[a,b]$, where $a>0$. Find $\int_a^{\overline{b}} f$ and $\int_{\underline{a}}^{b} f$. Deduce that *f* is integrable on $[a, b]$. 5

2. Prove that $\frac{\pi^2}{9}<\int_{\pi/6}^{\pi/2}\frac{x}{\sin x}dx<\frac{2\pi^2}{9}$. 5

3. Evaluate $\lim\limits_{x\to 0}\frac{\int_0^{x^2} e^{\sqrt{1+t}}dt}{x^2}$. 5

4. Use first Mean value theorem to prove that 5
$$\frac{\pi}{6}\le\int_0^{1/2}\frac{1}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}dx\le\frac{\pi}{6}\cdot\frac{1}{\sqrt{1-k^2/4}},\ k^2<1$$

5. Show that the second Mean value theorem (Weierstrass' form) is applicable to $\int_a^b\frac{\sin x}{x}dx$, where $0<a<b<\infty$.

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Also prove that $\left|\int_a^b \frac{\sin x}{x}dx\right| < 4/a$. 5

6. A function f is defined on [0, 1] by

$$f(x)=\begin{cases}\sin x, & x \text{ is rational}\\ x, & x \text{ is irrational}\end{cases}.$$

i) Evaluate $\int_0^{\overline{\pi/2}} f(x)dx$ and $\int_{\underline{0}}^{\pi/2} f(x)dx$;

ii) Show that f is not integrable on $\left[0, \frac{\pi}{2}\right]$. 3+2=5

Part – II (Marks: 20)

Answer ***any four*** questions. 4×5=20

1. Show that the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if m, n are both positive. 5

2. A sequence of functions $\{f_n\}$ is defined on $[0, a]$, $0<a<1$, by $f_n(x)=x^n$, $x\in[0,a]$. 5

3. For each $n\in\mathbb{N}$, let $f_n(x)=x^{n-1}-x^n$, $x\in[0,1]$. Use Dini's theorem to prove that the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$. 5

4. For the series $\sum_{n=1}^{\infty} f_n(x)$,

where $f_n(x)=n^2xe^{-n^2x^2}-(n-1)^2xe^{-(n-1)^2x^2}$, $x\in[0,1]$.

Show that $\sum_1^{\infty}\int_0^1 f_n(x)dx \neq \int_0^1\left(\sum_1^{\infty} f_n(x)dx\right)$. 5

Is the series $\sum_1^{\infty} f_n(x)$ uniformly convergent on [0, 1]?

5. A function f is defined on $\left(-\frac{1}{3},\frac{1}{3}\right)$ by

$$f(x)=1+2.3x+3.3^2x^2+\cdots+n.3^{n-1}x^{n-1}+\cdots;$$

i) Show that f is continuous on $\left(-\frac{1}{3},\frac{1}{3}\right)$.

ii) Evaluate $\int_0^{\frac{1}{4}} f$. 3+2=5

6. Obtain the Fourier series expansion of the function $f(x)=x\sin x$ on $[-\pi,\pi]$. Hence deduce that $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\cdots$ 4+1