surface of the sphere $x^2 + y^2 + z^2 = a^2$, above the xy-plane. 5

6. State Green's theorem in a plane and verify it for $\oint_c \left\{ \left(x^2 + xy \right) dx + x dy \right\}$, where C is the curve enclosing the region bounded by $y = x^2$ and $y = x$. 2+3

Ex/SC/MATH/UG/CORE/TH/09/2023

B. Sc. Mathematics (Hons.) Examination, 2023

(2nd Year, 2nd Semester)

Multivariate Calculus

Paper – Core-9

Time : Two hours Full Marks : 40

Use separate answer script for each Part.

Symbols / Notations have their usual meanings.

Part – I (20 Marks)

Answer ***any four*** questions. 4×5=20

1. a) Find the equation of the tangent plane to $z = 2x^2 + y^2$ at the point (1, 1, 3). 2

   b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x+y}, & \text{when } x + y \neq 0 \\ 0, & \text{when } x + y = 0. \end{cases}$$

   Find $\dfrac{\partial^2 f}{\partial x \partial y} + \dfrac{\partial^2 f}{\partial y \partial x}$ at (0, 0). 3

2. a) Find the maximum rate of change of $f(x, y) = \sqrt{x^2 + y^4}$ at (–2, 3) and the direction in which this maximum rate of change occurs. 2

   b) Check whether the following function

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & \text{when } x^2 + y^2 \neq 0 \\ 0, & \text{when } x^2 + y^2 = 0 \end{cases}$$

   is differentiable at the point (0, 0) or not.

3. Let S be an open subset of $\mathbb{R}^n$ and assume that $f: S \to \mathbb{R}^m$ is differentiable at each point of S. Let **x** and **y** be two points in S such that $L(\mathbf{x},\mathbf{y}) \subseteq S$, where $L(\mathbf{x},\mathbf{y})$ denotes the line segment joining two points **x** and **y** in $\mathbb{R}^n$, that is, $L(\mathbf{x},\mathbf{y}) = \{t\mathbf{x}+(1-t)\mathbf{y} : 0 \le t \le 1\}$. Prove that for every vector **a** in $\mathbb{R}^m$, there is a point **z** in $L(\mathbf{x},\mathbf{y})$ such that $\mathbf{a}\cdot(f(\mathbf{y})-f(\mathbf{x})) = \mathbf{a}\cdot(Df_{\mathbf{z}}(\mathbf{y}-\mathbf{x}))$, where $Df_{\mathbf{z}}$ denotes the derivative of f at the point **z**. 5

4. a) Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function. If $g(u,v) = f(u^2 - v^2)$, then prove that $\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 4(u^2+v^2)f''(u^2-v^2)$. 2

   b) Let $f: E \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function defined in an open set $E \subseteq \mathbb{R}^2$, and $D_1 f$ and $D_{21} f$ exist at every point of E. Suppose $Q \subseteq E$ is a closed rectangle with sides parallel to the coordinate axes, having (a,b) and $(a+h,\ b+k)$ as opposite vertices $(h \ne 0,\ k \ne 0)$. Consider

$$\Delta(f,Q) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b).$$

   Prove that there is a point (x,y) in the interior of Q such that $\Delta(f,Q) = hk(D_{21}f)(x,y)$. 3

5. Find and classify the stationary points of the function $f(x,y) = x^3 - 12xy + 8y^3$. 5

6. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point (3, 1, –1). 5

Part – II (20 Marks)

Answer ***any four*** questions. 4×5=20

1. a) Define curl and divergence of a vector point function.

   b) If the vectors $\vec{A}$ and $\vec{B}$ be irrotational, then show that the vector $\vec{A}\times\vec{B}$ is solenoidal. 2+3

2. a) Show that the greatest rate of change of ϕ takes place in the direction of, and has the magnitude of, the vector $\nabla\phi$. 2+3

   b) Evaluate $I = \iiint_E (y^2z^2 + z^2x^2 + x^2y^2)\,dxdydz$ taken over the domain bounded by the cylinder $x^2+y^2 = 2ax$, and the cone $z^2 = k^2(x^2+y^2)$. 2+3

3. Prove that $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2df}{rdr}$,
   where $r = \sqrt{x^2+y^2+z^2}$. Hence, determine $f(r)$ such that $\nabla^2 f(r) = 0$. 2+3

4. If $\vec{\beta} = (yz)\hat{i} + (xz+z+x)\hat{j} + f(x,y)\hat{k}$ is a vector field, such that *curl* $\vec{\beta} = -\hat{i}+\hat{k}$, find $f(x,y)$. 5

5. If $\vec{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$, using the theory of surface integral, evaluate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$, where S is the